Chapter 7
Algebra as Part of an Integrated High School Curriculum

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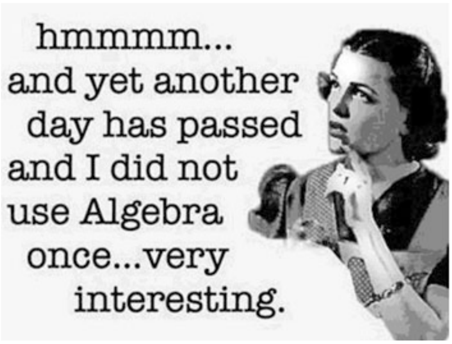
Abstract Traditional high school mathematics curricula in the United States devote 2 years almost exclusively to development of student proficiency in the symbolic manipulations required for solving algebraic equations and generating equivalent algebraic expressions. However, recent design experiments have shown that a focus on functions, mathematical modeling, and computer algebra tools enables effective integration of algebra with the other core strands of high school mathematics.

Keywords Integrated curriculum • School algebra • Functions • Problem-based learning • Mathematical modeling • Applications



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The figure at the left is a meme circulating on the Internet that summarizes nicely the public perception of the importance of algebra as it is taught in most American schools. The statement may be true of every other school subject save English, but it gets a chuckle only for algebra.

Long-standing tradition in American education calls for organization of the high school mathematics core curriculum in a layer cake of 3 year-long single-subject courses—elementary algebra, geometry, and advanced algebra. The algebra courses that dominate this curriculum emphasize training students in what Robert Davis once characterized as a ‘dance of symbols’—a collection of procedures for manipulating symbolic expressions, equations, and inequalities. The applications of those symbol manipulation rules are commonly limited to an array of classic word problems of dubious authenticity, and precious little attention is given to topics with easily demonstrated practical importance such as probability, statistics, and modern discrete mathematics, much less the process of mathematical modeling that is central to contemporary applied mathematics.

This dominant structure of American high school mathematics curricula appears to have evolved during the late nineteenth and early twentieth century as mathematics courses that had been in the curricula of prominent colleges were transformed into admission prerequisites for those institutions (Jones & Coxford, 1970; Kilpatrick & Izsak, 2008). We, the authors, were high school students in the 1950s, when it could be argued that “classical” algebra and geometry were the subjects that could be taught in secondary schools because they could be mastered with the tools then available: paper, pencil, ruler and compass (for geometry), and brain power. But only a small percentage of students were highly successful in these courses, and they tended to be “us,” the people who became college and university faculty members in mathematics, engineering, science, or education.

Half a century later, the world is a very different place. The problems to be solved are more challenging, our brain-extender tools are much more sophisticated, our school and college populations are more diverse, and our knowledge-based economy is no longer dependent primarily on agriculture and manufacturing. But in most places our school mathematics curriculum has evolved only marginally, “allowing” use of electronic calculators and no longer relying solely on Euclid as the definition of geometry. While other disciplines have moved on,[1] the “standard” secondary mathematics curriculum has much in common with the Saber-Tooth Curriculum (Benjamin, 1939/2004).

Prompted by striking findings from a series of recent international studies of mathematics teaching and learning, American mathematics educators have explored different ways of thinking about the high school curriculum. In particular, efforts such as the Interactive Mathematics Project (Fendel, Resek, Alper, & Fraser, 2015) and the Core-Plus Mathematics Project (Hirsch, Fey, Schoen, Hart, & Watkins, 2014) have developed and tested American versions of the most common

[1] In the 1950s, the importance of DNA was not well known in biology, black holes were not known to exist, and tectonic plates were still considered heresy by earth scientists.

international model that advances student understanding of all major content strands in each year of high school study. These so-called integrated or standards-based[2] mathematics curricula give significant attention to a broader range of mathematical topics than traditional algebra-centric curricula, and they also pay explicit attention to developing student understanding and skill in mathematical processes such as problem solving, communication, reasoning, and connection of ideas.[3]

Proposals to broaden and integrate topics in high school mathematics challenge the centrality of traditional algebraic content. The focus of school algebra on formal procedures for manipulating expressions, equations, and inequalities is also challenged by emergence of technological tools such as graphing calculators and computer algebra systems. If every algebraic symbol manipulation can be performed quickly and accurately by universally available computer software, is it still important for students to spend 2 full years of high school study in pursuit of what is inevitably incomplete and fragile mastery of those same routines? What is the right agenda of algebra learning goals for students today?

7.1 New Perspective: Function as Fundamental Concept

One curricular approach that has shown great promise in several innovative integrated curriculum projects replaces the traditional focus on formal manipulation of symbolic expressions and equations with an emphasis on a different fundamental mathematical idea—*functions*. To see what a focus on functions might look like and how it can lead to productive development of still essential algebraic understandings and skills, consider a problem that requires mathematical modeling and reasoning:

A new professional sports league has a business problem:

What average ticket price will maximize operating profit of the league all-star game?

The situation involves several key variables—number of tickets sold, income from ticket sales, income from concession sales, operating costs, and operating profits—most of which depend ultimately on average ticket price. Market research and other business analyses could lead to function models for those dependencies. For example, we might come up with functions such as these:

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[2] The term *standards-based* generally refers to curricula that embody recommendations of the 1989 National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for Teaching Mathematics* and the 2000 *Principles and Standards for School Mathematics*.

[3] The notion that proficiency in mathematics includes certain *habits of mind*, as well as knowledge of specific facts, concepts, and procedural skills, has been reflected in all professional curriculum guidelines over the past quarter-century, most recently in the *Common Core State Standards for Mathematics*.

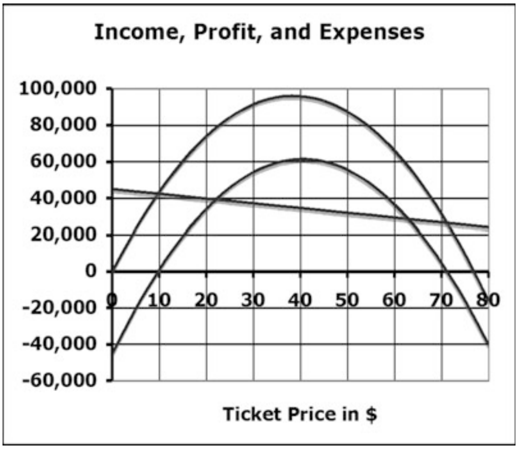
$$\begin{aligned}
\text{Demand}: n(x) &= 5000 - 65x \\
\text{Income}: I(x) &= 5000x - 65x^2 \\
\text{Expenses}: E(n) &= 4n + 25{,}000 \\
E(x) &= 45{,}000 - 260x \\
\text{Profit}: P(x) &= -65x^2 + 5260x - 45{,}000
\end{aligned}$$

Algebraic notation and symbol manipulation are very useful in expressing such problem conditions, in calculations for finding derivatives, and in solving equations to find specific numeric values such as optimum ticket and break-even prices. In fact, for a conventional treatment of this problem, algebraic skills are essential.

But using the numeric, graphic, and symbolic tools provided by calculators and computers, one has access to very effective approximation strategies for solving equations and inequalities and even finding maximum or minimum values for functions. Furthermore, contemporary computer algebra systems will actually perform all required exact calculations.

$$\frac{d}{dx}\left(-65x^2 + 5260x - 45{,}000, x\right) \quad 5260 - 130x$$

$$\text{solve}(5260 - 130x = 0, x) \quad x = \frac{526}{13} \quad \text{or} \quad x = 40.46$$



Price	Income	Expense	Profit
0	0	45,000	–45,000
10	43,500	42,400	1100
20	74,000	39,800	34,200
30	91,500	37,200	54,300
40	96,000	34,600	61,400
50	87,500	32,000	55,500
60	66,000	29,400	36,600
70	31,500	26,800	4700
80	–16,000	24,200	–40,200

The business analysis problem posed by planning for a sports league all-star game is typical of tasks encountered by students in calculus for the management sciences—a course for which high school algebra is assumed to be essential preparation. Success in the conventional version of that course certainly does require proficiency in writing and manipulating algebraic expressions and in solving equations. But the central concepts of calculus are functions and rates of change, and applying those concepts to realistic problems requires thinking about *variables*, *expressions*, and *equations* in different ways than the traditional approaches to elementary algebra emphasize. In applications of calculus, variables represent quantities that change over time or in response to change of other related variables. Equations show how variables are related. Expressions show how to calculate values of dependent variables. So instead of thinking about algebra as only a collection of symbol manipulation techniques for discovering fixed but unknown values of x, it makes sense to think of algebra as a way of expressing and reasoning about relationships between changing quantities. Techniques for solving equations and inequalities are helpful in finding answers to specific questions situated in the context of broader quantitative relationships.

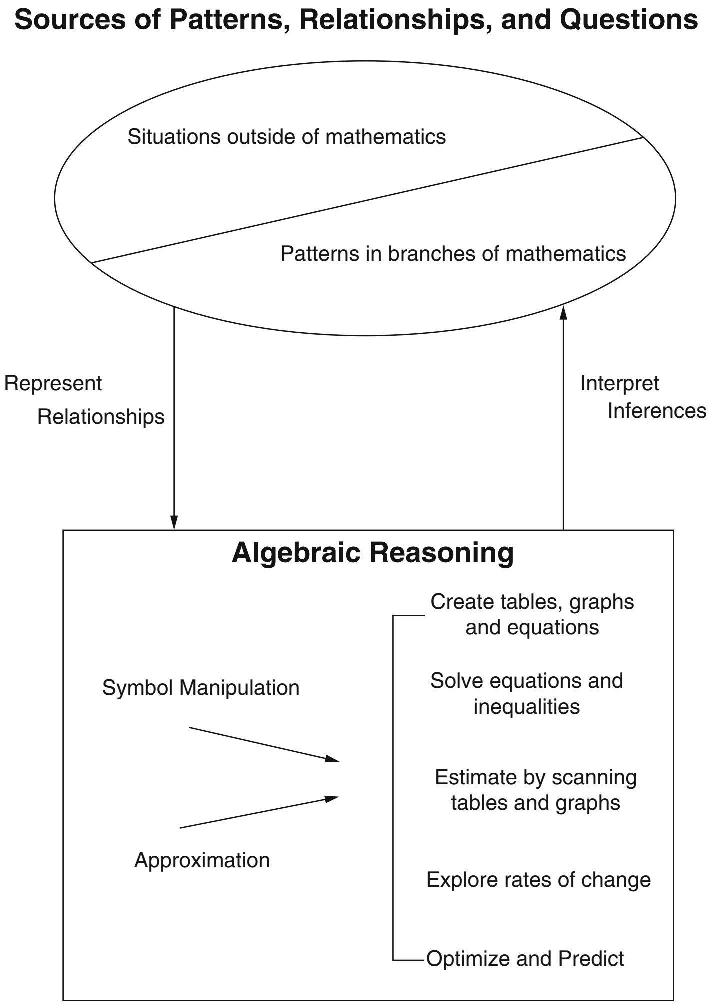
Using functions—rather than symbolic expressions, equations, and manipulation—as the central organizing concepts for high school mathematics has a number of important payoffs.

- As the preceding example shows, viewing algebraic expressions and equations as functions encourages use of numeric and graphic representations that provide insight into how specific points of interest fit into the overall relationships of variables. For example, as one examines the graph and table of values for the profit function in a neighborhood around the maximum point, it becomes clear that moderate changes in ticket price will have very little effect on event profit.
- The concept of function is central to calculus. So having students encounter functions in their introduction to algebra lays important conceptual foundation for later studies.
- Functions in algebra connect naturally and effectively to transformations in geometry that students will use to reason about congruence and similarity.
- Statistical methods for data analysis and modeling lead naturally to functions as representations of relationships between correlated variables.
- Iteratively defined functions play a fundamental role in many applications of discrete mathematics to questions in finance and population dynamics.

Algebraic notation is valuable for representing what we know or what we want to find out. Algebraic procedures for manipulating symbolic expressions and equations into alternative equivalent forms are useful for gaining insight into relationships between variables. But there are now many powerful tools for doing that work. So algebra courses focused on developing skill in formal symbol manipulation are a poor use of valuable instructional time for all but a few students.

7.2 A Conceptual Framework for School Algebra

A more useful conceptual framework for thinking about school algebra can be expressed with a diagram that has become common in discussions of the mathematics curriculum. As students explore a numeric pattern or problem, they find ways to represent relationships between variables. They use these representations to reason about the situation in a variety of ways—solving equations and inequalities, understanding relationships, making predictions, and verifying patterns. Then when their mathematical model has suggested new insights into the problem situation, those ideas have to be evaluated to see how they play out back in the real world.



To operate in this mathematical arena, students need several key dispositions, understandings, and specific technical skills from algebra:

- **Disposition to look** for quantitative variables in problem situations and for relationships among variables that reflect *cause-and-effect*, *change over time*, or *pure number* patterns.
- A **repertoire** of significant and common patterns to look for—linear, quadratic, exponential, inverse variation, and periodic functions.
- Ability to **represent** relationships between variables in words, graphs, data tables and plots, and in appropriate symbolic expressions.

- Ability to **draw inferences** from represented relationships by estimation from tables and graphs, by exact reasoning using symbolic manipulations, and by insightful interpretation of symbolic forms.
- **Disposition to interpret** mathematical deductions in the original problem situations, with sensitivity to limitations of the modeling process.

These goals suggest a presentation of school algebra that begins by drawing students' attention to the many interesting situations in the worlds of science, business, engineering, and technology where quantities change naturally over time or in response to changes in other related quantities. The symbolic notation of algebra can be introduced naturally to make precise and efficient representations of observed patterns. Then students can learn, with a very modest amount of introductory personal symbolic reasoning, how to use the widely available array of computing tools to answer questions about the observed situations. For those students who ultimately need sophisticated and efficient personal skills for symbolic work and understanding of algebra that includes the formal structural aspects of the subject, we are now in a position to provide personal skill development when it appears essential, rather than as the first step toward proficiency in algebra-assisted reasoning.

In some sense this way of thinking about school algebra turns the traditional sequence of mathematical ideas, skills, and applications upside down—developing concepts and problem solving before personal symbol manipulation skills. But, in addition to providing broadly useful mathematical understandings and technology-assisted skills, the function-oriented development provides students with intuitions about variables, expressions, and equations that are a very effective concrete grounding for later development of the formal aspects of algebra. The syntactic rules of symbolic algebra become procedures that just make sense, rather than formal logical consequences of abstract field axioms.

7.3 A Sample Function-Oriented Curriculum

The curriculum projects mentioned above (IMP and CPMP) show how the proposed development of algebra in the context of functions can be accomplished. For example, the algebra/functions strand in Core-Plus Mathematics includes 15 units over the course of four high school years, units that are woven together with topics in other content strands. Each unit develops fundamental understandings and skills in use of functions and algebraic reasoning to solve problems in mathematics and its applications to science, business, and everyday life.

- *Patterns of Change* focuses on quantitative variables using data tables, coordinate graphs, and symbolic expressions.
- *Linear Functions* focuses on relationships between variables characterized by constant additive rate of change, straight line graphs, and equations in the general form $y = mx + b$.

- *Exponential Functions* focuses on growth and decay patterns characterized by constant multiplicative rate of change and expressed by the general form $y = Ab^x$.
- *Quadratic Functions* focuses on relations between variables expressed by the general form $y = ax^2 + bx + c$.
- *Functions, Equations, and Systems* focuses on relationships between two or more variables that can be expressed as inverse variations $y = \frac{k}{x}$, power functions $y = kx^r$, and systems of linear equations with two independent variables.
- *Matrix Methods* develops concepts and operations on matrices to represent and solve multivariable problems in algebra, geometry, and discrete mathematics.
- *Nonlinear Functions and Equations* introduces and develops formal symbolic methods for reasoning about quadratic functions, expressions, and equations, as well as logarithms for reasoning about exponential equations.
- *Inequalities and Linear Programming* focuses on algebraic and graphical reasoning about linear inequalities and systems.
- *Polynomial and Rational Functions* develops familiar concepts and skills in work with polynomials and rational expressions in the context of functions and their graphs.
- *Recursion and Iteration* develops properties of sequences as iteratively defined discrete functions, with special attention to arithmetic and geometric sequences and their connections to linear and exponential functions.
- *Inverse Functions* develops the inverse concept with special attention to logarithms and inverse trigonometric functions.[4]
- *Families of Functions* reviews and integrates student understanding of core function types and their representation in symbols, data tables, and graphs with a focus on transformation of basic function forms to model complex scientific relationships.
- *Algebraic Functions and Equations* develops core results in theory of equations and work with rational functions and equations.
- *Exponential Functions and Data Modeling* extends prior work with exponential and logarithmic functions and their expressions to the case of natural exponential and logarithmic functions, including use of logarithms for data linearization and modeling of patterns.
- *Concepts of Calculus* builds on prior work with functions, graphs, and rates of change to introduce core understandings about derivatives and integrals and their most common applications.

Note that the earlier units in this sequence don't attempt to teach students everything we know about any one topic. Each topic is revisited as necessary in later units and often in units that are not part of the algebra/functions strand.

[4] The trigonometric functions are developed in an earlier geometry/trigonometry unit titled *Circles and Circular Functions*. This is one example of the integration of strands in Core-Plus Mathematics that are roughly categorized as algebra/functions, geometry/trigonometry, statistics/probability, and discrete mathematics.

7.4 Challenges to the Function-Oriented Algebra Proposal

Teachers and mathematicians reacting to the proposed function-oriented view of school algebra raise a number of plausible questions about the approach.

Challenge: This is not really *algebra*. So much of the traditional content of algebra courses (such as factoring, expanding, and simplifying expressions, and solving equations) seems to be omitted or at least moved to the background.

Response: Whether a curriculum that highlights functions and moves formal symbol manipulation to the background is or is not *algebra*, is not the core question for consideration in school mathematics. The heart of the matter is whether functions make more sense as the mathematical spine of a secondary school curriculum than the long-standing approach that emphasizes formal manipulation of abstract expressions, equations, and inequalities.

Challenge: There are many mathematical problems and reasonings that are not well served by the focus on functions.

Response: While we can all imagine some interesting mathematical and applied problems that use facets of algebra not naturally developed through a focus on functions, we think functions and mathematical modeling are the place to start with most secondary school students. Furthermore, nothing proposed in the function-centric development rules out training students in more standard algebraic principles and skills as an extension of the focus on functions.

Challenge: Even the impressive capabilities of computer algebra systems lack essential symbolic flexibility capabilities like those that well-developed personal symbol manipulation skills can provide.

Response: It is certainly true that current computer algebra systems do not include the kind of subtle mathematician's intuition that can tell which equivalent form of a symbolic expression might be most useful in answering a specific algebraic question, nor do they have the flexibility to make nuanced variations on standard options. So relying on CAS for core symbol manipulation tasks places some inherent limitations on student algebraic reasoning performance. The standard response to this challenge is to argue, "Since we don't know which students will need highly developed symbol sense and skill, we should aim high for all students." However, as with all inclusion/exclusion decisions of curriculum design, there is an important cost-benefit calculation to be made. Is the time required to develop admittedly desirable symbol manipulation skill and intuition really time well spent? Evidence from long experience with algebra teaching suggests that the answer for most students is, "Probably not."

Challenge: The concepts-before-skills developmental sequence is not an effective learning trajectory—procedural skill takes a long time to develop, and one learns best by acquiring procedural skills and then having the structure of that skill domain become clear at a later point.

Response: 25 years ago there was little evidence that a change in priorities and developmental approaches to emphasize functions first and foremost would work with real students and teachers. But the intervening years have yielded a great deal of existence-proof evidence that those ideas are not so far-fetched. Furthermore, the power, access, and ease of use of calculating and computing tools have increased in dramatic ways from the days of the first graphing calculators and personal computers, and this trajectory seems certain to only accelerate in the near future. Thus if we aim to provide the kind of mathematical understandings and skills that will be useful and attractive to most students, a development of algebra that emphasizes functions and their applications can make a very strong claim for priority in school mathematics.

Challenge: Finally, professionals with knowledge of the history of mathematics education can argue fairly that proposals for integrated curricula and emphasis on functions have been around for over a century, but they never seem to take hold in practice.

Response: Even casual reading in the history of mathematics education reveals recommendations from many individuals and professional advisory groups to integrate topics in the high school curriculum[5] and to emphasize function as a central unifying idea.[6] However, neither recommendation had much impact on conventions in American mathematics education. The broadening of mathematical sciences during the twentieth century, especially the growth of probability, statistics, and computer science, makes the case for a broader and more unified school curriculum with new urgency. With respect to the recommendations about centrality of functions, we argue that the emergence of digital technologies, especially graphing calculators and computer algebra systems, has changed conditions for mathematics education in ways that make teaching about functions more natural and effective than ever before. As indicated in the ticket-price example, access to graphing tools makes it natural and insightful to look for solutions of equations such as $5000x - 65x^2 = 0$ by scanning the graph of $I(x) = 5000x - 65x^2$ for x-intercepts. Use of computer algebra systems to find exact solutions (and other complex symbol manipulations) should be quite appropriate skill for most students.

[5] For example, in his famous 1902 retiring presidential address to the American Mathematical Society, E. H. Moore urged schools to "abolish the 'watertight compartments' in which algebra, geometry, and physics were taught." Similar recommendations appeared in the 1912 *Report of the American Commissioners of the International Commission on the Teaching of Mathematics*, the 1923 Mathematical Association of American National Committee on Mathematical Requirements' *The Reorganization of Mathematics in Secondary Education* (Jones & Coxford Jr, 1970; Kilpatrick & Izsak, 2008).

[6] Emphasis on functions and interrelationships within mathematics had been made as early as the middle of the nineteenth century by the distinguished German mathematician, Felix Klein. That same thematic recommendation was picked up by curriculum advisory reports in the United States throughout the twentieth century.

7.5 Summary

We believe that development of important algebraic concepts and techniques through an approach emphasizing functions offers very attractive opportunities to provide powerful mathematical understandings and skills as part of an integrated curriculum. The necessary tools and textbooks and teaching strategies all exist, and we owe it to students of the twenty-first century to see that they are adequately equipped for the world in which they live.

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