Chapter 4
Misconceptions and Learning Algebra

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Abstract Rather than exclusively focus on mastery of procedural skills, mathematics educators are encouraged to cultivate conceptual understanding in their classrooms. However, mathematics learners hold many faulty conceptual ideas—or misconceptions—at various points in the learning process. In the present chapter, we first describe the common misconceptions that students hold when learning algebra. We then explain why these misconceptions are problematic and detail a potential solution with the capability to help students build correct conceptual knowledge while they are learning new procedural skills. Finally, we discuss other potential implications from the existence of algebraic misconceptions which require further study. In general, preventing and remediating algebraic misconceptions may be necessary for increasing student success in algebra and, subsequently, more advanced mathematics classes.

Keywords Misconceptions • Worked examples • Learning from errors • Conceptual knowledge • Self-explanation

4.1 Common Algebraic Misconceptions

Over the past several decades, researchers in mathematics education and educational psychology have identified a number of misconceptions that students tend to hold about algebraic content. While not an exhaustive list, a few of the most widely studied, including those dealing with equality/inequality, negativity, variables, fractions, order of operations, and functions, are discussed below.

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4.1.1 Equality/Inequality

Students at all levels have been found to hold misconceptions about the equal sign, including those enrolled in college calculus (Clement, Narode, & Rosnick, 1981). Often students have an operational understanding of the equal sign—the belief that the equal sign indicates where the answer should go—rather than a relational understanding, the belief that the equal sign indicates equivalence (Baroudi, 2006; Cheng-Yao, Yi-Yin, & Yu-Chun, 2014; Falkner, Levi, & Carpenter, 1999; Kieran, 1980, 1981; Van Dooren, Verschaffel, & Onghena, 2002). For example, of 375 sixth and seventh grade students, 58 % gave definitions for the equal sign that insinuated that the equal sign connects the answer to the problem (operational understanding), while only 29 % gave definitions that insinuated that the equal sign shows that what is to the left and the right of the sign mean the same thing (relational understanding) (Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008). While this type of arithmetic thinking may be sufficient during the early years, it causes major problems once students are asked to think algebraically (Booth & Koedinger, 2008; Knuth, Stephens, McNeil, & Alibali, 2006). Having a correct understanding of the meaning of the equal sign is imperative in order to manipulate and solve algebraic equations (Carpenter, Franke, & Levi, 2003; Kieran, 1981).

Some children believe that the equal sign cannot be used in an equation that does not have an operator symbol (i.e., $3=3$). These same students also believe that all operators must be on the left side of the equal sign. For instance, $5+2=3+4$ should be rewritten as $5+2=7$ and $3+4=7$ (Behr, Erlwanger, & Nichols, 1980). Furthermore, younger students tend to believe that the number immediately to the right of the equal sign must be the answer (Alibali, 1999; Falkner et al., 1999; Li, Ding, Capraro, & Capraro, 2008). For instance, in one particular study, all 145 sixth grade students incorrectly completed with number sentence $8+4=\_\_\_\_+5$ by filling in a 12 or 17 (Falkner et al., 1999). A second study found that about 76 % of 105 sixth graders were unable to correctly complete the first blank in the number sentence, $\_\_\_\_+3=5+7=\_\_\_\_$; however only about 13 % of those students were unable to answer the second (Li et al., 2008).

A similar misconception is one surrounding the concept of inequality. Similar to the equal sign, students at all levels tend to have difficulties with inequalities (Rowntree, 2009). Some students treat inequalities as equalities (Blanco & Garrote, 2007; Vaiyavutjamai & Clements, 2006). Others have a narrow understanding of the terms *more* or *less* (Warren, 2006). Finally, some students have major difficulties interpreting inequality solutions (Tsamir & Bazzini, 2004; Vaiyavutjamai & Clements, 2006).

4.1.2 Negativity

Another category of algebraic misconceptions is dealing with negativity. Those with an incorrect or incomplete understanding of the negative sign are more likely to use incorrect strategies when solving algebraic equations (Booth & Koedinger, 2008). Due to the abstract nature of negativity, this concept is especially difficult for students moving from arithmetic to algebraic thinking (Linchevski & Williams, 1999). These students tend to only link the negative sign with the binary operation of subtraction. For instance, Vlassis (2002, 2004) found that most eighth graders can easily interpret the meaning of negative nine within the expression $n-9$, but have trouble when -9 is presented alone.

Difficulties with the negative sign persist into the college years. Cangelosi, Madrid, Cooper, Olson, and Hartter (2013) found that college students have difficulty manipulating exponential expressions when a negative sign is included as part of the base, preceding the base, or as part of the exponent. For instance, students often misinterpret $-9^{3/2}$ as $(-9)^{3/2}$ (Cangelosi et al., 2013).

4.1.3 Variables

Misconceptions dealing with the use of variables are also widely studied. One of the more common misunderstandings is the belief that the letter in a number sentence stands for an actual object or is a label (Asquith, Stephens, Knuth, & Alibali, 2007; Clement, 1982; MacGregor & Stacey, 1997; McNeil et al., 2010; Stacey & MacGregor, 1997; Usiskin, 1988). This misinterpretation can be seen in the classic error to the “student and professor” problem. When students are asked to write a number sentence to represent the phrase, six times as many students as professors, the most common error is $6\text{s} = \text{p}$ (Clement, Lochhead, & Monk, 1981; Rosnick, 1981). Students believe that s was a label for students, rather than a variable representing the number of students (Rosnick, 1981).

Alternatively, some students will ignore the variables altogether. For instance, when asked to solve $(n+5)+4$, 20 % of students incorrectly give the answer of 9, ignoring the n (Kuchemann, 1978). Others believe that the letter is associated with its position in the alphabet (Asquith et al., 2007; Herscovics & Kieran, 1980; MacGregor & Stacey, 1997; Watson, 1990). Furthermore, students have trouble understanding that the same letter seen multiple times in a number sentence must represent that same number (Kieran, 1985) or that different letters within a number sentence can also represent the same number (Stephens, 2005; Swan, 2000). On a similar note, students also often misunderstand the meaning of operational symbols when paired with variables. For instance, since students are used to joining two terms when they see the addition symbol (i.e., $2 + ½ = 2½$), they will mistakenly believe that $2+x$ is the same as $2x$ (Booth, 1986).

4.1.4 Fractions

While introduced before algebraic concepts, fraction misconceptions also greatly influence students' acquisition of algebra knowledge. The National Mathematics Advisory Panel (2008) suggests that one of the most important types of knowledge necessary for algebra learning is knowledge of rational numbers or fractions. Fractions can be seen in algebra as coefficients/slope, constants, and solutions (Wu, 2001). Brown and Quinn (2006) assessed Algebra I students' fraction knowledge and found that students have trouble writing a fraction to represent the shaped part of a figure, simplifying fractions to lowest terms, adding and subtraction fractions, and multiplying and dividing fractions. Specifically, students often misused the cross multiplying algorithm when attending to multiply fractions, failed to use the inverse operations to solve equations, and failed to even attempt the problem.

4.1.5 Order of Operations

Another type of misconception affecting students of all ages deals with the order of operations and use of brackets (Kieran, 1985; Pinchback, 1991). Many students do not see the need to adhere to the order of operations rules and resort to solving the expression from left to right (Gardella, 2009; Kieran, 1979). Furthermore, many students fail to realize that brackets can be used to both groups together as well as signal multiplication (i.e., $(20 - 7) = 13$ and $-(20 - 7) = -13$; Linchevski, 1995).

4.1.6 Functions

Lastly, students often misinterpret the meaning of algebraic functions. For instance, some students treat a graph as a picture of a given scenario (i.e., a graph comparing speed and time) (Clement, 1989). Furthermore, both students and adults tend to believe that a linear function must be proportional simply because it increases or decreases at a constant rate (Pugalee, 2010; Van de Walle, Karp, & Bay-Williams, 2013); however this is only true when the function passes through the origin.

4.2 Why Should We Be Concerned About Misconceptions?

The previous section described a number of misconceptions students tend to have when learning algebra. It is well established and documented that such misconceptions exist. But why is having these misconceptions problematic? In this section, we describe a number of ways in which having misconceptions, or flawed conceptual knowledge of algebra, might impact students' performance and learning.

4.2.1 Relation to Procedural Skills

Having good procedural skills, or the ability to carry out procedures to solve problems (Rittle-Johnson, Siegler, & Alibali, 2001), is arguably a critical component of success in mathematics (Kilpatrick, Swafford, & Findell, 2001). It has been well established that conceptual knowledge and procedural skill are related (Rittle-Johnson & Siegler, 1998), and some researchers maintain that the two in fact fall on a single continuum (Star, 2005). Though the two develop iteratively and one or the other may come first depending on the particular content (Rittle-Johnson & Siegler, 1998), for many mathematics domains, it is necessary to have correct conceptual knowledge in order to develop correct procedural skills.

Work in algebra has established that students with stronger conceptual knowledge are better at solving equations and are able to learn new procedures more easily than peers with flawed conceptual knowledge (e.g., Booth, Koedinger, & Siegler, 2007; Sweller & Cooper, 1985). In particular, students who hold misconceptions about the equal sign or negative signs solve fewer equations correctly and have greater difficulty learning how to solve equations (Booth & Koedinger, 2008). Correction of these misconceptions can lead to improvements in equation-solving skills (Booth & Koedinger, 2008).

4.2.2 Relation to Problem Encoding

The ability to correctly encode a problem, or perceptually process the important features of the problem and create an internal representation that can be used later (Chase & Simon, 1973), has been repeatedly shown to be important for problem-solving success (Alibali, Phillips, & Fischer, 2009; Booth & Davenport, 2013; Rittle-Johnson & Alibali, 1999; Siegler, 1976). Prior knowledge necessarily impacts how a learner encodes a problem. For example, students are better at encoding equations that are familiar and tend to misencode problem features in unfamiliar equations as if they follow the structure of more familiar problems (McNeil & Alibali, 2004).

Conceptual knowledge also impacts learners' encoding of problems. Experts in a domain encode problems more accurately than novices (Chase & Simon, 1973; Chi, Feltovich, & Glaser, 1981), and algebra students with more correct conceptual knowledge have been shown to have higher encoding accuracy (Booth & Davenport, 2013). This is, perhaps, not surprising, as correct encoding requires noticing the important features in a problem and conceptual knowledge helps students determine what features are important (Crooks & Alibali, 2013;

Prather, 2012; Rittle-Johnson & Alibali, 1999). In other words, when students have flawed conceptual knowledge, they may not be able to correctly determine which features to focus on and/or may not consider those features in a meaningful way (Booth & Davenport, 2013).

4.2.3 Relation to Specific Problem-Solving Errors

4.2.3.1 Misconceptions and Related Errors

Algebraic misconceptions that students hold predict the types of errors students make during problem-solving (Booth & Koedinger, 2008). Durkin and Rittle-Johnson (2015) demonstrate that errors made with high confidence during problem-solving are representative of strongly held misconceptions that are more difficult to overcome with instruction. Oftentimes, these errors arise when students are learning a new topic and attempt unsuccessfully to relate it to something they've learned prior. Although this can sometimes be a useful strategy, when rules or strategies are overgeneralized, this can lead to struggles as well (Stagylidou & Vosniadou, 2004; Vamvakoussim & Vosniadou, 2004), making students particularly resistant to conceptual change in mathematics (McNeil, 2014).

One common example of when students struggle to learn and apply altering rules during problem-solving when moving to higher levels of mathematics is when they transition from dealing with solely natural numbers to all rational numbers (Van Dooren, Lehtinen, & Vershcaffel, 2015). A natural number bias can often lead students to make errors when dealing with fractions and decimals. Another is when students are asked to understand and use the equal sign as a symbol or equivalence between two expressions in algebra rather than the more commonly used form of seemingly signaling that the student should carry out an operation. This can often lead students to making the error of performing the given operation on all given numbers, regardless of where the numbers are located within the equation (McNeil & Alibali, 2004).

Errors that persist are often an indication that a student holds an underdeveloped understanding of a particular underlying concept (Cangelosi et al., 2013). Analyzing errors that students make during problem-solving is one useful method for learning more about the particular misconceptions that students hold (Clement, 1982; Corder, 1982; Liebenburg, 1997).

4.2.3.2 Persistence of Errors

Certain errors that students make in mathematics are quite persistent and lead to troubles at different levels of mathematics. Most of the misconceptions addressed within this chapter are expressed in algebra. It is vital to understand these misconceptions as Algebra I is considered a gatekeeper course to higher-level STEM courses (Adelman, 2006). However, understanding of algebra is arguably built upon early arithmetic knowledge (Bodin & Capponi, 1996), so it is important to consider how misconceptions in earlier stages of mathematics can lead to errors made later on in algebra. For example, Mazzocco and colleagues (Mazzocco, Murphy, Brown, Rinne, & Herold, 2013) found that errors made in second and third grade are predictive of not only specific types of errors made in eighth grade but also speed during problem-solving. Specifically, students who made particular errors in a symbolic number task in second or third grade were slower and made more errors when completing addition and multiplication computations in eighth grade.

Algebra I is most commonly taken in the eighth or ninth grade. However, some errors made during secondary mathematics have been found to persist even into postsecondary levels of mathematics. Negative sign errors have been found to be quite common and quite persistent at varying levels of mathematics (Booth, Barbieri, Eyer, & Paré-Blagoev, 2014; Seng, 2010). Being able to manipulate integers is a subordinate skill in algebra and higher levels of mathematics. Therefore, it is clear as to why misconceptions about the negative sign (as well as the equal sign) have been found to interfere with students' learning of how to solve algebraic equations (Booth & Koedinger, 2008). This applies to students who may stereotypically be considered advanced or students who manage to complete school standards for Algebra I as well. Negative sign errors are common and interfere with learning at varying levels of mathematics (Kieran, 2007). In a cross-sectional study, Cangelosi and colleagues found that negative sign errors made in College Algebra (e.g., incorrectly simplifying negative numbers with a rational exponent) persist through Calculus II (Cangelosi et al., 2013).

4.2.3.3 Relation of Errors to Learning

Conceptual change is undoubtedly a slow and gradual process (McNeil & Alibali, 2005; Vamvakoussi & Vosniadou, 2010). While some misconceptions seem to persist as demonstrated in the errors students make all the way through college, other misconceptions change in prevalence and persistence based upon the content to be learned (Booth et al., 2014). For example, Durkin and Rittle-Johnson (2015) explored changes in misconceptions when judging the magnitude of decimals over the course of a 1-month period of instruction. *Whole number errors* and *role of zero errors* started off prevalent but declined over time. Whole number errors were classified as those that indicate treating a decimal as if they are whole numbers and believing more numbers to the right of the decimal means a larger number. The role of zero errors were classified as those that indicate treating a decimal with a zero in the tenths place as if the following digit is actually in the tenths place. These errors were considered to be representations of a whole number bias (Ni & Zhou, 2005). However, fraction errors, in which students try to relate the length of the decimal to its magnitude, increased over time. Durkin and Rittle-Johnson suggest that this change in prevalence of types of errors indicates change in conceptual thinking about number. However, how the prevalence and persistence of these errors predicted later achievement was not addressed.

Booth and colleagues (Booth et al., 2014) conducted a similar analysis upon errors over the course of an academic year and found that making certain types of errors while learning particular content in algebra is indicative of detriment to mathematics achievement. For example, students who made variable errors at the beginning of the academic year while taking Algebra I on arguably what would be simpler content demonstrated lower mathematics achievement scores at the end of the academic year. Students who made more errors related to mathematical properties (i.e., inappropriately applying the distributive, commutative, or associative properties) or who conducted the wrong operations during the beginning and middle of the year also struggled on the end of year achievement test. Students who made more errors involving equality and inequality at the middle and end of year also demonstrated lower achievement. Lastly, negative sign and arithmetic errors at the end of the year, when content was presumably most difficult, were indicative of low mathematics achievement. Results from this study emphasize the importance of considering how errors stemming from misconceptions align with particular content. Understanding not only the prevalence and persistence of mathematical errors in relation to particular content but also what these errors indicate about the misconceptions students hold and how these impact future learning are vital first steps when considering designing appropriate interventions that address student misconceptions.

4.3 How Can We Address Student Misconceptions?

A number of interventions exist which aim to improve students' conceptual understanding in algebra, including those focused on reteaching fundamental concepts and principles (Ma, 1999), having students compare multiple solution methods (Rittle-Johnson & Star, 2007), or completely reforming mathematics curricula to be contextualized in real-world problems (Hiebert et al., 1996) or conceptual models (Xin, Wiles, & Lin, 2008). In this chapter, we describe one particular method which has proved to be effective at both improving student's conceptual understanding and procedural skill in algebra. This approach stems from three scientific principles on how people learn: self-explanation, worked examples, and cognitive dissonance. Each of these three principles is described below, before we explain how they have been combined and review findings on the effectiveness of this combination.

4.3.1 Self-Explanation

Self-explanation is defined as explaining information to oneself while reading or studying (Chi, 2000). Early evidence revealed that better learners do this naturally (Chi, Bassok, Lewis, Reimann, & Glaser, 1989), and follow-up studies examined the effectiveness of prompting all students to explain. The self-explanation principle maintains that there are a number of benefits for learning when students are asked to explain information to themselves while reading or studying (Chi, 2000). Some of these benefits include improvement in the degree to which students integrate new information with their prior knowledge, make the newly learned knowledge explicit, and, subsequently, notice gaps in their knowledge and draw inferences to fill those gaps (Chi, 2000; Roy & Chi, 2005).

4.3.2 Worked Examples

Traditional instruction, particularly in science, technology, engineering, and mathematics (STEM) domains, involves demonstrating the procedures for solving problems (on the blackboard, on the smart board, in the textbook) and then having students practice solving those types of problems on their own. However, a large body of work from laboratory studies suggests that these worked examples should not just occur at the beginning of the lesson—they should be interleaved within the practice sessions as well (e.g., Cooper & Sweller, 1987; Sweller & Cooper, 1985; Trafton & Reiser, 1993). The worked example principle maintains that replacing some (or even half) of the practice problems with worked-out solutions for students to study can increase learning of the procedures to solve problems, even though the students have less practice solving those problems themselves (Sweller, 1999). Benefits of focusing students' limited cognitive capacities on understanding the concepts and procedures necessary for problem-solving (rather than on attempting to apply procedures by rote) include faster mastery of instructed procedures (Clark & Mayer, 2003; Schwonke et al., 2009) and increased transfer of procedural skills to solve more difficult problems (Catrambone, 1996, 1998; Cooper & Sweller, 1987).

4.3.3 Cognitive Dissonance

The idea of cognitive dissonance stems from a theory purported by Festinger (1957), which maintains that humans naturally seek consistency between their beliefs and the reality observed in the world and that a clash between belief and reality leads to an unpleasant feeling and a drive to resolve the discrepancy. In other words, if one is presented with information that conflicts with their own beliefs, they will work to make sense of the differences so they can return to a harmonious state. Creating such cognitive disequilibrium is thus proposed to be an effective technique for producing change in thinking (e.g., Graesser, 2009).

One method of promoting cognitive dissonance is through the presentation of errors for students to consider and study. Learning from errors is thought to be effective because it prompts students to identify features of problems that make the demonstrated procedure incorrect, which in turn can help students correct their own misconceptions (Ohlsson, 1996). An additional benefit of studying and explaining errors is that it may help learners acknowledge that the demonstrated procedure is wrong and make it less likely they will utilize that procedure themselves when solving problems (Siegler, 2002).

4.3.4 Combining Self-Explanation, Worked Examples, and Learning from Errors

These principles, which have been well tested in laboratory settings, have been combined into a single effective intervention: explaining correct and incorrect worked examples during problem-solving practice. Essentially, for some of the items in practice assignments, students are shown an example of a fictitious learner's problem solution—solved either correctly or incorrectly and clearly marked as such—and asked to explain the example in response to one or more prompts about particular features in the problems, about particular errors made in solutions, or about how the fictitious learner might be thinking about the problem.

Prior research had established that, compared to studying correct worked examples, *explaining* correct examples increased students' conceptual knowledge (Hilbert, Renkl, Schworm, Kessler, & Reiss, 2008) and their ability to solve both similar and more difficult problems (Renkl, Stark, Gruber, & Mandl, 1998). Further research suggested that explaining correct *and incorrect examples* further increased learning benefits for building correct conceptual understanding (Adams et al., 2014; Booth et al., 2015; Booth, Lange, Koedinger, & Newton, 2013) and decreasing student misconceptions (Durkin & Rittle-Johnson, 2012). Recently, in a randomized controlled trial in real-world classrooms across an entire Algebra 1 curriculum, this combination led to robust improvements on conceptual and procedural skills as well as skills specifically measured by standardized achievement tests (Booth et al., 2015); benefits for conceptual understanding were even stronger for students who were struggling with the material (Booth, Oyer, et al., 2015).

4.4 Practical Implications of the Existence and Persistence of Algebraic Misconceptions

By now, we can hopefully agree that algebraic misconceptions are a problem and that traditional algebra instruction is not doing enough to remedy the problem. We have offered one suggestion of how to change algebra instruction to better target and fix student misconceptions and allow them to move forward productively with learning more difficult algebraic content. This is certainly not the only option for how to alter algebra instruction; any interventions geared toward improving conceptual understanding (while still building procedural skill) may be good candidates for instruction.

However, full remediation may require looking backward as well. Misconceptions don't typically develop out of the blue; they develop them because children are trying to make sense of the world around them by using the information made available (Vosniadou & Brewer, 1992). What information are we making available in younger grades that lead to students developing algebraic misconceptions? One line of work suggests that the way we teach earlier math can have a profound effect on students' understanding of algebraic concepts. For example, McNeil and Alibali (2005) showed that elementary school students' knowledge of arithmetic operation patterns (e.g., operations = answer) hinders their ability to learn from a lesson on solving equations; unfortunately, mathematics textbooks rarely present the equal sign in a context that would encourage a relational understanding—most presentations are the standard operations = answer format (e.g., $6+2=8$) that hinders learning (McNeil et al., 2006). Giving children more practice, solving problems in this format also makes it less likely that they will build a correct concept of mathematical equivalence (McNeil, 2008). One could imagine similar consequences for early presentations and practice (or lack thereof) with negative signs and variables.

How can we prevent such ingrained misconceptions from developing? One possibility may be a combination of systemic changes to early mathematics instruction and materials and the approach described in this chapter. We must change the way we introduce algebraic problem features and concepts in the first place. Recent recommendations stress focusing on such concepts earlier in the mathematics curriculum (e.g., CCSSI, 2010). We must always think about how we are presenting information to young children and whether it will help them build a correct concept. Second, teachers can have students explain correct and incorrect worked examples in earlier grades to help them focus on building a correct conceptual foundation as well as the necessary procedural skills. This may help prevent formation and entrenchment of these misconceptions early on. By preventing and/or quickly remediating misconceptions, we can help future generations have a smoother transition to—and greater success in—learning algebra.

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