Chapter 10
Rethinking Algebra: A Versatile Approach Integrating Digital Technology

Mike Thomas

Abstract Many have thought deeply about the construction of the school algebra curriculum, but the question remains as to why we teach the topics we do in the manner we do, stressing manipulations of symbols, and why some other avenues are ignored. In this chapter we consider the basic constructs in the school algebra curriculum and the procedural approach often taken to learning them and suggest some reasons why certain topics may be excluded. We examine how particular tasks, including some that integrate digital technology into student activity, could be used to rethink the algebra curriculum content with a view to motivating students and promoting versatile thinking. Some reasons why these topics have often not yet found their way into the curriculum are discussed.

Keywords Versatile thinking • Algebra • Tertiary • Digital technology • Representations

The aim of this chapter is to rethink both the content of secondary school algebra and the manner of its delivery and to ask: Should either, or both, be changed in order to improve understanding of algebra? There seems little doubt about two crucial statements:

- Algebra (including the school algebra of generalised arithmetic) is of fundamental importance in mathematics.
- Many students find most of school algebra either difficult or impossible to comprehend.

These two statements are linked together by the fact that school algebra is a semiotic system. It is the signs or representations of this system that at one and the same time make algebra so useful and yet so difficult for many. Consider, for example, the compressive power in a relatively simple symbolism

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$$\sum_{i=0}^{i=2} w^i$$

where w is a cube root of unity. Suspending for a moment the fact that this summation comes to zero and ignoring simplifications of w^2, if we fully expand the symbolisation we get

$$\begin{aligned}\sum_{i=0}^{i=2} w^i = 1 + w + w^2 &= 1 + \frac{-1 + i\sqrt{3}}{2} + \left(\frac{-1 + i\sqrt{3}}{2}\right)^2 \\ &= 1 + \frac{-1 + \sqrt{-1}\sqrt{3}}{2} + \left(\frac{-1 + \sqrt{-1}\sqrt{3}}{2}\right)^2 \\ &= 1 + \frac{-1 + \sqrt{-1}\sqrt{3}}{2} + \left(\frac{-1 + \sqrt{-1}\sqrt{3}}{2}\right)\left(\frac{-1 + \sqrt{-1}\sqrt{3}}{2}\right)\end{aligned}$$

Mason (1987) agrees that a semiotic problem, concerning the relationship between the sign and the signified, or the symbol and the symbolised, is at the root of algebraic difficulties. This semiotic difficulty is not surprising when we consider how long it took for the symbolism to settle down into our modern version. For example, Struik (1969) gives these examples.

(a) What must be the amount of a square, which, when twenty-one dirhams are added to it becomes equal to the equivalent of ten roots of that square? Al-Khwarizmi ca. 825 AD
(b) cubus p : 6 rebus aequalis 20 Cardan ca. 1545 AD
(c) aaa - 3bba = +2ccc Harriot ca. 1610 AD

The triadic model of Peirce describes how signs, constructed through thoughts and ideas, comprise three components: the representamen [or the external material entity]; the object referred to; and the interpretant, or the sense made of the entity. Unlike icons and indexes, symbols, including those used in mathematics, have become associated with their meaning by accepted usage (Peirce, 1898). The grouping of these symbols into systems (sometimes called a representation system), such as the algebra of generalised arithmetic considered here, requires more than a set of symbols; it also needs rules for their production and transformation, and a set of relationships between the signs and their meanings (see Ernest, 2006). Student activity, both within such a system and converting between systems (Duval, 2006), can lead to key epistemological aspects and understanding, of mathematical objects, contributing to the goal of helping students attain *versatile thinking* in mathematics, which according to Thomas (2008a, 2008b), involves at least three abilities:

- To switch at will in any given representational system between a perception of a particular mathematical entity as a process and the perception of the entity as an object

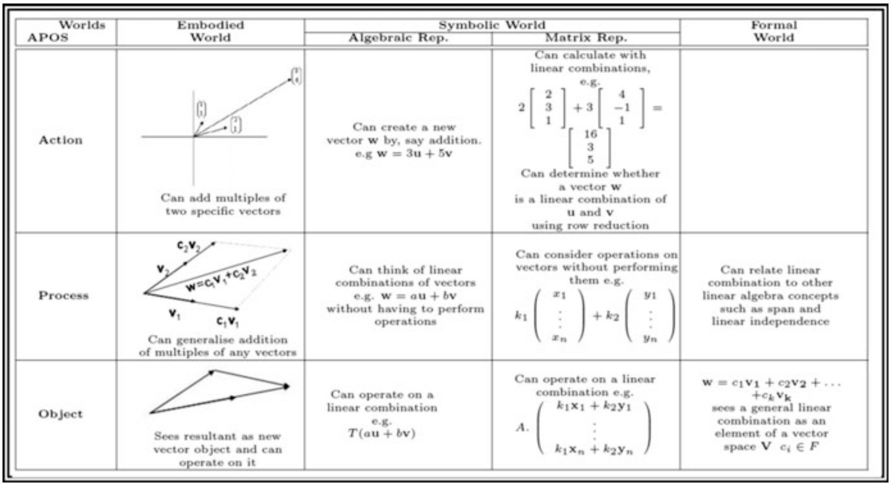
- To exploit the power of visual schemas by linking them to relevant logico/analytic schemas
- To work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations

Thus a versatile view (Graham, Pfannkuch, & Thomas, 2009; Graham & Thomas, 2000, 2005; Tall & Thomas, 1991; Thomas, 1988, 2002, 2008a, 2008b) of the semiotic system of school algebra requires more than the ability to transform symbols according to the rules of the system; it also means making sense of them as processes and objects, and the ability to relate them to other systems. However, much of what happens in school algebra comprises activity aimed at transformations according to the rules of the system with much less effort addressed to considering sense making or conversions. Such standard manipulation algebra (Thomas & Tall, 2001) often leads to what Skemp (1976) described as instrumental understanding, or applying rules without clear reasons.

In order to be able to operate on an entity within a further process, such as when manipulating symbolic literals in algebra, APOS theory (Dubinsky, 1991) tells us that students need an object view of the symbols (although what kind of object they perceive is often open to question—see Tall, Thomas, Davis, Gray, & Simpson, 2000). While in the higher level mathematics of formal world thinking (Tall, 2004, 2008) objects can be brought into being through a definition, which specifies their properties, in school algebra students are often left to abstract properties of objects such as variable, expression, equation, function and polynomial for themselves by learning and repeating procedural actions on symbols. In this chapter I suggest that more attention could be paid to relating the algebraic symbols to other representations and investigating the properties of the objects of algebra. I also propose ways that this could be achieved by harnessing the investigative power of digital technology (DT).

10.1 A Theoretical Framework

In other papers we have proposed a Framework for Advanced Mathematical Thinking (FAMT) (Stewart & Thomas, 2010; Thomas & Stewart, 2011) that combines orthogonally elements of the action-process-object-schema (APOS) framework for studying learning, presented by Dubinsky and others (Dubinsky, 1991; Dubinsky & McDonald, 2001) with each of Tall's (2004, 2008) Three Worlds of Mathematical Thinking. APOS theory describes how mental objects may be constructed from actions and processes via reflective abstraction, while Tall's framework suggests that mathematical thinking can involve an embodied world, with its *visual* and *enactive* aspects, a symbolic world of semiotic symbols, and the formal world of axiomatic and deductive mathematics. The FAMT is based on the principle that each mathematical concept can be examined in terms of action, process and object types of thinking in each of the embodied and symbolic and



Worlds / APOS	Embodied World	Symbolic World: Algebraic Rep.	Symbolic World: Matrix Rep.	Formal World
Action	Can add multiples of two specific vectors	Can create a new vector **w** by, say addition. e.g $\mathbf{w} = 3\mathbf{u} + 5\mathbf{v}$	Can calculate with linear combinations, e.g. $2\begin{bmatrix}2\\3\\1\end{bmatrix} + 3\begin{bmatrix}4\\-1\\1\end{bmatrix} = \begin{bmatrix}16\\3\\5\end{bmatrix}$ Can determine whether a vector **w** is a linear combination of **u** and **v** using row reduction	
Process	$c_2\mathbf{v}_2$, $\mathbf{v}_2$, $\mathbf{w}=c_1\mathbf{v}_1+c_2\mathbf{v}_2$, $\mathbf{v}_1$, $c_1\mathbf{v}_1$ Can generalise addition of multiples of any vectors	Can think of linear combinations of vectors e.g. $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ without having to perform operations	Can consider operations on vectors without performing them e.g. $k_1\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} + k_2\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}$	Can relate linear combination to other linear algebra concepts such as span and linear independence
Object	Sees resultant as new vector object and can operate on it	Can operate on a linear combination e.g. $T(a\mathbf{u} + b\mathbf{v})$	Can operate on a linear combination e.g. $A.\begin{pmatrix}k_1\mathbf{x}_1 + k_2\mathbf{y}_1\\\vdots\\k_1\mathbf{x}_n + k_2\mathbf{y}_n\end{pmatrix}$	$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$ sees a general linear combination as an element of a vector space **V** $c_i \in F$

Fig. 10.1 The Framework for Advanced Mathematical Thinking (FAMT) applied to linear combination

formal worlds of mathematics. Hence, a matrix of cells may be produced with each cell targeting student thinking and understanding in one area, such as an embodied process. While we have found FAMT particularly useful for analysing student thinking in university mathematics, namely linear algebra (see the example in Fig. 10.1), the underlying principles may also prove useful in school level mathematics and we will consider this below.

Providing tasks that enable students to engage in activity that encourages them to think in the manner described by as many of the cells of the framework as possible for a given mathematical construct and to construct meaningful links between them, is one way to promote versatile thinking. This is a key tenet of the ideas presented here.

We will now look at some of the key ideas in school algebra and ask how DT might assist students to construct versatile thinking about them.

10.2 Variables and Expressions in School Algebra

The concept of variable is not an easy one for students to construct. Even Bertrand Russell found the notion of variable problematic.

> 6. Mathematical propositions are not only characterized by the fact that they assert implications, but also by the fact that they contain *variables*. The notion of the variable is one of the most difficult with which logic has to deal. For the present, I openly wish to make it plain that there are variables in all mathematical propositions, even where at first sight they might seem to be absent…We shall find always, in all mathematical propositions, that

the words *any* or *some* occur; and these words are the marks of a variable and a formal implication. (Russell, 1903, pp. 5, 6)

It has been known for well over 35 years now that students have problems understanding the use of symbolic literals or letters in algebra (Küchemann, 1981; Wagner, 1981). That these problems in understanding are persistent was shown by Küchemann's (1981) investigation into children's understanding of the use of letters in algebra, as part of the wide-ranging CSMS study, with four or five years of algebra teaching making very little difference to their understanding of the subject. Around 30 years later in a follow-up study (Hodgen, Brown, Küchemann, & Coe, 2010; Hodgen, Coe, Brown, & Küchemann, 2014) the group concluded that attainment had not changed very much, and

- Algebra results show fewer students reaching the higher Levels 3 and 4, which is the point at which students begin to understand the key algebraic concepts of variable and generalised number (Hodgen et al., 2010, p. 6)

Rosnick and Clement (1980) too showed that even college students had similar problems, such as confusing the use of letter as variable with the use as a label or unit. One of the factors causing this situation is the multiplicity of uses of letters in mathematics, with Wagner (1981) listing placeholder, index, specific unknown, generalised number, indeterminate, independent or dependent variable, constant and parameter as possible uses. She also pointed out that this complexity is increased by the fact that different letters can be used to represent the same thing, and the same letter can be used to represent different things. It still often seems to be the case that, as Skemp (1971, p. 227) noted, 'The idea of a variable is in fact a key concept in algebra—although many elementary texts do not explain or even mention it'. This omission of explaining what a variable is still extends to many classrooms. Hence, expecting students to abstract all the subtle complexities of symbolic literals simply from procedural use of letters appears to be a step too far.

The difficulties students experience with use of letters clearly impinges on the way they view symbols such as '$x + 3$'. Many will not accept this kind of expression as an answer because they expect a number (Küchemann, 1981). To be able to cope with such a symbol requires not only that it be given a meaning, but that the meaning should allow the student the versatility of thought to see it as a procept, representing both as a process (of evaluation when x is known) and also an object that can be operated on. Often students who are used to working in the symbolic actions and symbolic process cells of FAMT see the symbol $x + 3$ solely as a process and not as a mental object; further it is a process they cannot carry out because they do not know what x is.

In two previous papers we have described (Graham & Thomas, 2000; Tall & Thomas, 1991) how DT might be used to help students construct a versatile perspective on the use of letters as generalised number. The basis of the approach used was to use digital technology to give students a symbolisation enabling an embodied view of the use of letter. This embodied, enactive perspective comprises a store with a label and a value that can be changed, as seen in Fig. 10.2, which is

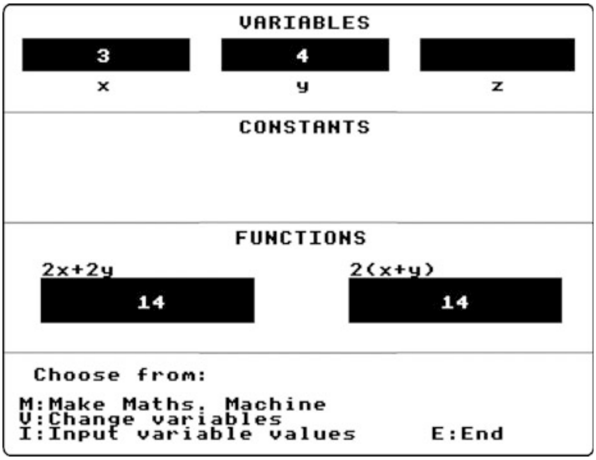


Fig. 10.2 The embodied symbolisation of a variable in the 'Maths Machine'

You can use letters as stores for numbers. Try the following:

Press	See	Explanation
4 [STO▸] [ALPHA] A [ENTER]	4 → A 4	The value 4 is stored in A.
[CLEAR]		This clears the display.
[ALPHA] **A** [ENTER]	A 4	This confirms that the number stored in A *is* 4.

Fig. 10.3 An example of the layout in Graham and Thomas (2000) algebra module

taken from Tall and Thomas (1991). Here students can engage in embodied actions, entering numbers into variable stores, predicting outcomes about algebraic objects and testing these predictions. In a second paper (Graham & Thomas, 2000) we changed the technology from computers to graphic calculators, which intrinsically employ variables with a large number of inbuilt stores labelled by the use of capital letters and where the embodied actions of storing and retrieving numbers from these lettered stores provides a direct correspondence to letter use in early algebra. The same basic embodied model was used here, the graphic calculator's lettered stores as a model of a variable. Each store is represented by a box in which changing values of the variable come and go, and next to which sits its label. Figure 10.3 shows a brief early section from the module used.

Both controlled experiments showed that the students using the DT were more versatile in their thinking than the students following a traditional course. They were significantly better at interpreting symbols, demonstrated an improved understanding of the use of letters as specific unknown and generalised number and were more likely to think of expressions as objects, without losing any procedural facility.

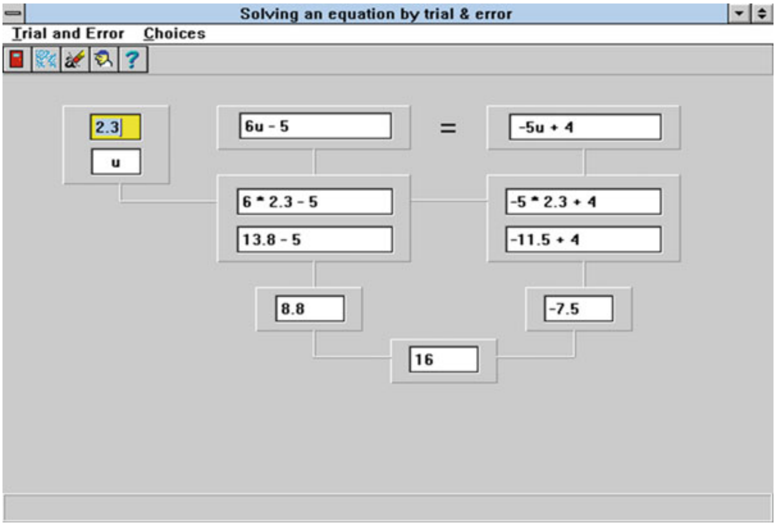


Fig. 10.4 A screen from the Dynamic Algebra programme

In terms of semiotics the research shows that by extending the sign or symbol used to represent a variable, from a single letter to a box plus a letter, students can be assisted to make improved sense of the object represented. Following from this approach a *Dynamic Algebra* programme was developed that enables investigative activity with expressions and equations based on the same mental model. In Fig. 10.4 we see an example of how this programme employs embodied actions such as giving the variable u a value to see the effect on two expressions $6u - 5$ and $-5u + 4$, to see when they reach equality. This is an example of an approach to the hardest type of linear equation at this level.

10.3 Equations

While the '=' sign is now ubiquitous in mathematics, making sense of the meaning of the sign appears not to be straightforward for students, and is often context dependent. For example, many have an operational, process-oriented perspective of the sign as a signal to perform some action (Crowley, Thomas, & Tall, 1994; Godfrey & Thomas, 2008; Kieran, 1981; Thomas, 1994). For these students there is a difference between, say,

$$2x + 1 = 5 \text{ and } 3 = 5x + 2$$

$$\frac{dy}{dx} = 2x + 5 \text{ and } 2x + \frac{dy}{dx} = 3$$

I have found that even among mathematics graduates and teachers of mathematics we have some discussion in my master's courses on what constitutes an equation. For example, when asked whether the following are equations not all agree.

$$\frac{2x+1}{3x-2}=1,\quad f(x)=2x+5,\quad 4=4,\quad k=5,\quad (x-1)(x+3)=x^2+2x-3$$

In their research Hansson and Grevholm (2003) found that very few pre-service teachers considered $y=x+5$ to be an equation, instead tending to a numerical interpretation of $y=x+5$. Others I have asked say that it's an assignment rather than an equation. Indeed in computer science, and other areas, the sign := is reserved for such an assignment to a function or variable, possibly removing an overlap in meaning. We can see that some issues with equations involve whether the statement has to be true, whether it can include an assignment, does there have to be 'something to do' and can it be always true. The following set of three examples may help to illustrate some of these issues in the mind of the reader.

$$x^2+3x-1=x^2+3x+1,\qquad (x-1)(x+3)=x^2+3x-3,\qquad (x-1)(x+3)=x^2+2x-3$$

Addressing this the Collins mathematics dictionary (Borowski & Borwein, 1989) distinguishes between an *identical equation* (or identity), which is true for any values of the variables, and a *conditional equation*, which is only true for certain values of the variables. This distinction seems to be a useful one and it might help if more use were made of the symbol for equivalence (in an identical equation, true for all values of the variables), $\equiv$, that was more commonly used years ago.

In our study on understanding of equation (Godfrey & Thomas, 2008) we found that for Year 10 students (age 14–15 years) many appear to be using the criteria that an equation needs an = sign and an operation to carry out (see examples in Fig. 10.5). On this basis 65.6 % of them rejected $k=5$ as an equation while 72.4 % accepted $7w-w$ as an equation.

In this same study, for those in Year 13 (17–18 years old), the last year of school, 27.6 % still accepted $7w-w$ as an equation, while 56.6 % were unwilling to accept $a=5$ and 61.8 % did not see $a=a$ as an equation. Overall 53.9 % of these students still wanted an equation to have an operation to carry out, and 14.5 % of these

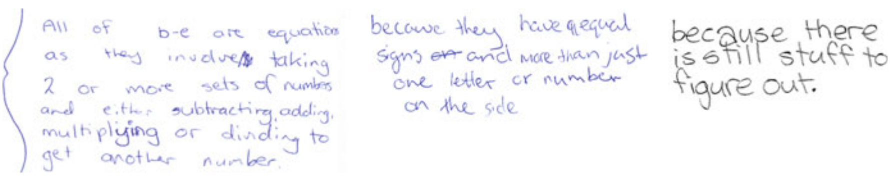


Fig. 10.5 Examples of 14-year-old students criteria for an equation

rejected anything that was an identity or an assignment. In our group of first year university students studying Engineering Science, which has a very high entry requirement, 20.6 % still emphasised the solution aspect of an equation (e.g. 'An equation is a mathematical formula formed by some unknown variables and numbers. And it is those unknown variables we are trying to find a value/answer to it' and 'Statement given to solve unknown variables in order to equate the right hand side is equal to the left hand side'). However, 60 % now accepted $a = a$ as an equation, although 26.7 % did not see $a + b = b + a$ as an equation.

Student understanding of the use of equality often appears not to be predicated on an explicit construction of properties of equations, including the reflexive, symmetric and transitive nature of the '=' sign, that will eventually lead to the idea of equivalence relations. Hence, activities that might allow them to construct some of these properties could be of value.

In addition what could we say to a student who produces this argument?

$$\begin{aligned}
&4x^2 - 5x - 6 = 0\\
&(4x + 3)(x - 2) = 0\\
&4x + 3 = 0, \quad x - 2 = 0\\
&4x + 3 = x - 2 = 0\\
&4x + 3 = x - 2\\
&3x = -5\\
&x = -\frac{5}{3}
\end{aligned}$$

Here the transitive property has been applied to $4x + 3 = 0,\ x - 2 = 0$ as if it reads $4x + 3 = 0$ *and* $x - 2 = 0$. Compare this with $a = b$ and $b = c$ implies $a = c$. However, the line actually should read $4x + 3 = 0$ *or* $x - 2 = 0$, and this might give a teacher the chance to discuss the important logical difference between 'and' and 'or' in mathematical statements. In this way a crucial link between symbolic algebra and logic using natural language could be made.

What about if we are working through an example where we are trying to find the intersection of two graphs, whose equations are $y_1 = x + 6$ and $y_2 = 3x + 1$? Is it necessary to explain how we get from line 1 to line 2 or how we have used the symmetric property that $a = b \Rightarrow b = a$ to get from line 4 to 5?

$$\begin{aligned}
&y_1 = y_2\\
&x + 6 = 3x + 1\\
&6 = 2x + 1\\
&5 = 2x\\
&2x = 5\\
&x = 2\frac{1}{2}
\end{aligned}$$

Or when solving $y_2 = 0$ and then using $3x + 1 = 0$ to do so, would we invoke the transitive property ($y_2 = 3x + 1$ and $y_2 = 0 \Rightarrow 3x + 1 = 0$)?

Equations of the type $ax + b = cx + d$ have been well known to be a cut-off point for those who will make good progress in the learning of algebra and the obstacle has been called the didactic cut (Filloy & Rojano, 1984) or cognitive gap (Herscovics & Linchevski, 1994). It is only at this point in the solving of equations that one has to operate on the variable. One approach when solving $ax + b = cx + d$, or similar equations such as those involving quadratic functions, is to assist student understanding of properties of equations, such as what a solution is and when it is invariant. For example, they might see that there is a difference between what I have called legitimate and productive transformations of an equation (see Hong, Thomas, & Kwon, 2000; Thomas, 2008a). A legitimate transformation of a linear equation adds $\pm k$ or $\pm kx$ for all real k to both sides, but a productive transformation that moves one quickly towards an algebraic solution, is one of the type $\pm ax$, $\pm cx$, $\pm b$, and $\pm d$, taken from the infinite number of legitimate transformations. It is important to understand that the solution remains invariant under both types of transformations. It may be that DT could be employed to help students see some properties of equations by linking the algebraic representation to the graphical one. Clearly adding $\pm k$ to both sides of the equation does not change the solutions because graphically we are translating both graphs parallel to the y-axis by $\pm k$. However, the effect of adding $\pm kx$ to both sides may not be so obvious. In Fig. 10.6a, which was constructed using GeoGebra, we can see that adding $\pm kx$ to both sides of the equation $2x + 2 = 5x - 3$ appears to rotate the graphs of the function on either side of the equation about the point of intersection with the y-axis, although the x-value of the point of intersection, the solution of the equation, remains invariant.

The angle a straight line $y = mx + c$ makes with the x-axis is given by $\tan\theta = m$, where θ is the angle with the x-axis, and adding kx will change it from $\theta = \tan^{-1}(m)$ to $\theta = \tan^{-1}(m + k)$, which may appear to indicate a rotational effect. However, while the angle the line makes with the horizontal changes the individual points do not rotate. Instead, in a move that encourages versatile thinking, we might utilise another area of mathematics; one that is sometimes less often employed in school mathematics, although it is essential for university studies in mathematics. The idea of a transformation of the plane represented in matrix form is very useful here. Linking mathematical ideas across representations in this way is very important (Duval, 2006) and is a way to promote representational versatility (Thomas, 2002, 2008a, 2008b). In essence adding kx to $f(x) = mx + c$ is a shear of the graph of the function by a factor k parallel to the y-axis. Using matrices and vectors we can represent this linear transformation as follows:

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} x \\ f(x) \end{pmatrix} = \begin{pmatrix} x \\ f(x) + kx \end{pmatrix} = \begin{pmatrix} x \\ mx + c + kx \end{pmatrix} = \begin{pmatrix} x \\ (m + k)x + c \end{pmatrix}$$

Now, since every point on the straight line (and in the plane), apart from those on the y-axis, which are all invariant, is moved parallel to the y-axis (giving the

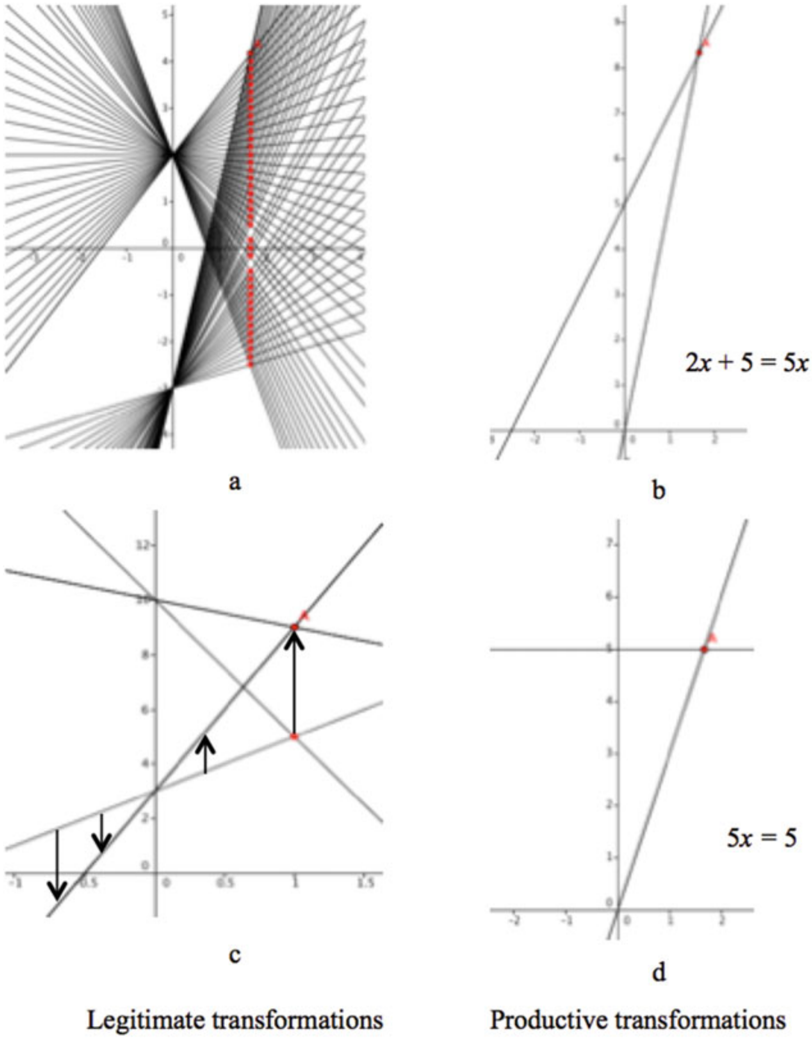


Fig. 10.6 Legitimate and productive transformations of linear equations

appearance of a rotation), we can see that this is also true of the point of intersection of two straight lines. We see this in Fig. 10.6c (we assume $k > 0$ here), and note that when $x < 0$ the points move in the opposite direction, since for $k > 0$, $kx < 0$. Thus the point of intersection ends up with the same x value as before, our invariant solution to the equation. In terms of the FAMT this process has linked a symbolic algebra process with an embodied graphical process and a symbolic matrix process. Further, we have managed to link a pointwise approach to a translation to a global perspective (Hong & Thomas, 2014; Vandebrouck, 2011).

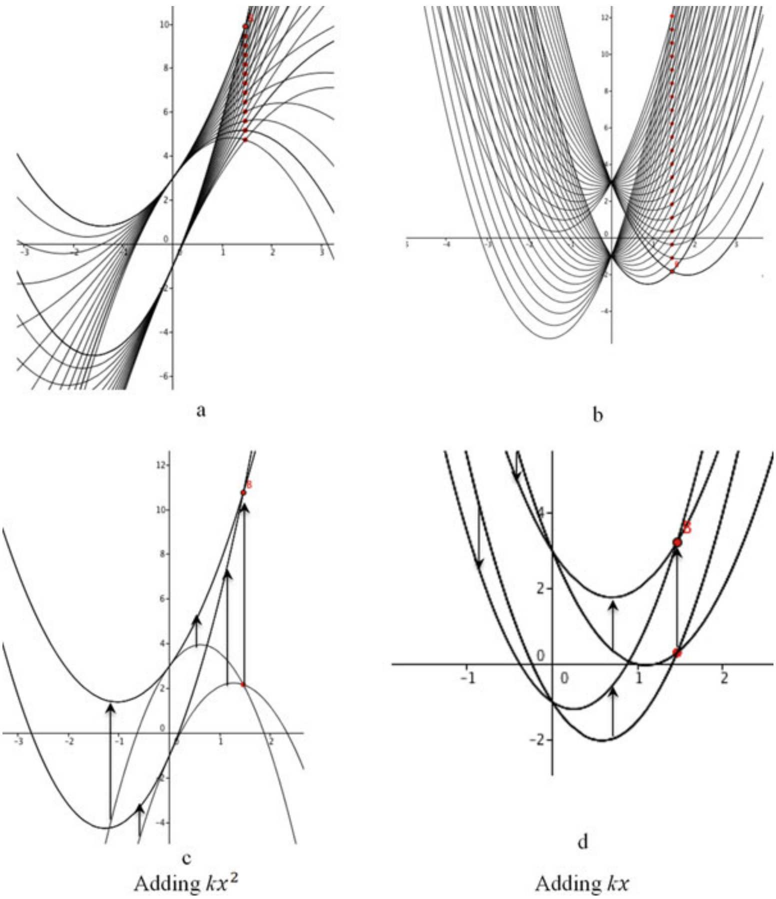


Fig. 10.7 Invariant solution under legitimate transformations of quadratic equations

Having established these basic principles we could now consider what happens with quadratic equations. Of course, the cases of adding a constant and adding a multiple kx of x can be analysed in exactly the same way as above, although the picture, again from GeoGebra, is quite different (see Fig. 10.7a, b). We can see that the case of adding kx^2 to both functions (see Fig. 10.7c) can be viewed in a similar manner to that of adding kx. The translation is again parallel to the y-axis and for $k > 0$, $kx^2 > 0$. Once again the point of intersection remains on each graph, the y-translation is by the same amount and the x-value is unchanged by adding to the value of the function. Hence, the solution is invariant.

10.4 Polynomial Functions

The concept of function, one of the most fundamental ideas in the whole of mathematics, is often given only cursory attention in school mathematics. Hence, it is not surprising that research has shown that students' perspectives on function differ considerably from those of mathematicians. For example, Williams (1998) used function concept maps to compare conceptions of students and professors and found that the students emphasised minor details and the idea that functions are equations. In contrast none of the professors thought of a function as an equation, preferring the idea of a correspondence, a mapping, a pairing or a rule. In a study with trainee mathematics teachers Chinnappan and Thomas (2003) found the teachers had a strong tendency to think of functions graphically and procedurally, and often even separated algebra from functions in their thinking. In Fig. 10.8 we see how a teacher, unable to decide on whether an ordered pair could represent a function, moves from the ordered pair representation to a graph and then to an explicit algebraic formula in order to say that this is a function.

The expectation that a function will have an explicit algebraic formula was prominent in Thomas' (2003) study. In Fig. 10.9a we see an example of how one

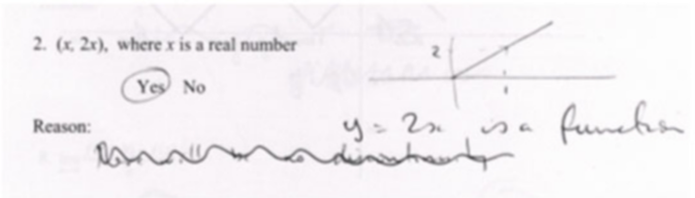


Fig. 10.8 A teacher's use of a graph and an algebraic formula for a function

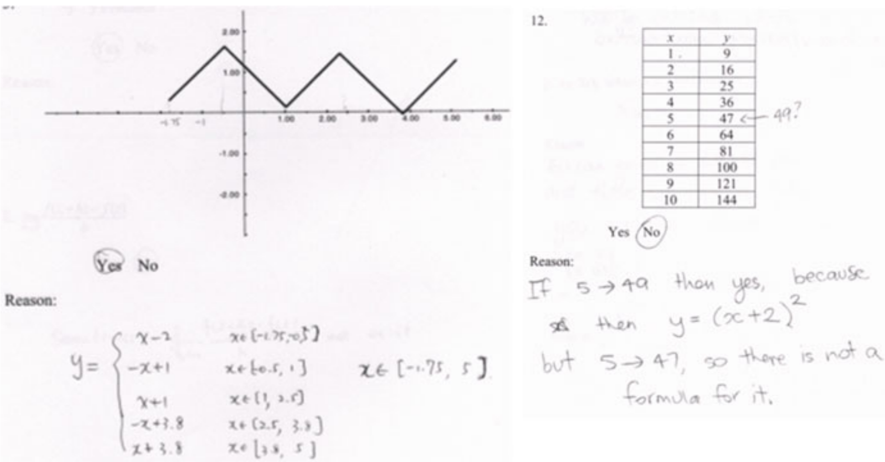


Fig. 10.9 Two teachers' view that functions require an explicit algebraic formula

teacher responded to the question of whether the given graph could represent a function by finding the explicit algebraic formula for each straight line section of the graph in order to be able to respond ‘yes’. The second example in Fig. 10.9b shows the reverse. A teacher rejects the table of values as representing a function because the value at $x = 5$ deviates from the formula $y = (x + 2)^2$ that all the others fit.

This research suggests that for many teachers the graphical representation of function can become so dominant in thinking about function that it could hinder a growth in inter-representational understanding.

In terms of the FAMT framework it would appear that at least some students have a tendency to move between the embodied and symbolic worlds with respect to function. An emphasis on symbolic actions and processes may be behind the desire for an explicit formula and the use of the vertical line test embodied action/process may encourage a graphical perspective on function. This movement between embodied actions and symbolic actions is generally to be encouraged but abstracting the notion of a function from graphical and algebraic expressions exemplars appears to be difficult (Akkoc & Tall, 2002). As Thompson (1994) has pointed out, ‘the core concept of ‘function’ is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance’ (p. 39). Student (and teacher) difficulties with abstracting the invariance from graphs and algebraic formulations implies that the idea of function may be one area where formal actions could be added to student experiences as a means of testing given constructs against a definition of function. Of course, simply giving students a formal definition, such as that in Akkoc and Tall (2002)—see Fig. 10.10—and expecting them to be able to use it will probably not work. In their study Akkoc and Tall (2002) found that some students were unable to see and apply the fundamental (simple) definition of function, instead relying on almost arbitrary aspects of examples they focussed on. Hence, the simplicity of the core function concept eluded most of their students.

Instead Akkoc and Tall used a four part colloquial definition to assist students to focus on essential properties of a function followed by experience of functions in different representations as set diagrams, ordered pairs, graphs and formulas. Employing a colloquial definition, such as each and every element of one set (the

Formal Definition

Let A and B two sets other than empty set, and f is a relation defined from A to B. If

i) $\forall x \in A$, $\exists y \in B$ s.t. $(x,y) \in f$,

ii) $(x,y_1) \in f$ and $(x,y_2) \in f \Rightarrow y_1 = y_2$

Then the relation f from A to B is called a *function* and denoted by f: $A \rightarrow B$ or $A \xrightarrow{f} B$. A is called the domain and B is called the range.

Fig. 10.10 A possible formal world definition of function domain) is mapped to or related to one and only one element of the second set (or codomain) and then testing this with formal actions in the four representations used by Akkoc and Tall along with tables of values may be a way forward. Although as they found, this is not the complete solution.

These difficulties with thinking about the concept of function are further exemplified when students meet the idea of a polynomial. When asked what a polynomial is (see Chinnappan & Thomas, 2003) some trainee teachers responded:

- An equation which has more than 1 x variable whose power is bigger than 1
- An equation that has a power of x other than 1
- An equation with a power of x greater than one
- When I am talking about functions, I am not talking about polynomials and vice versa, I find it very difficult to um.. interchange the words
- If somebody said 'is that straight line relation a polynomial?', my gut reaction would be to say no. Just because a polynomial, poly being many.

So we can see an apparently common misconception here that linear functions, and by extension constant functions, are not polynomials and that the set of polynomials is not a subset of functions. Polynomials are perceived as beginning with the quadratic function, since that is probably where the term was first met. This view is reinforced by the natural language prefix 'poly', seen in other places in mathematics, such as polygon (where the number of sides has to be three or more). Confirming this are the kinds of responses received to the question of whether $3 - x$ is a polynomial.

- No, linear
- No—The powers of x is low
- Yes—Not sure! Maybe it's not!
- Yes—because for each value of x, there is 1 corresponding y value

Once again, the idea of a polynomial (function) may be an area where it would be beneficial to add formal actions based on a definition to student experiences so that carefully chosen examples and non-examples of polynomials could be used to build the construct. For example, one could define a real polynomial of degree n as an expression of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ where x is a real variable, n is a non-negative integer and each a_i is a real number (later we may define polynomials, for example, over the complex numbers) with $a_n \neq 0$. We can then use a formal action of testing against a definition to determine whether we have a polynomial or not, such as: Is $x + 1$ a polynomial? Is $x^{5/2}$ a polynomial? Is 0 a polynomial?

When it comes to a consideration of the properties of some low order polynomials it would appear that, for cubic functions, a number of interesting areas for study have been often overlooked and would repay attention. I suggest one or two of these below that are accessible with DT.

10.5 Investigating Cubic Functions

One of the reasons for deciding what polynomial properties are studied in school algebra may be whether the properties are considered to be accessible to students through procedural calculations. However, with the advent of DT we can now investigate properties that may have previously been in the domain of 'higher' mathematics.

10.5.1 Symmetry

To simplify matters we will limit our discussion to monic cubic functions of the form $x^3 + ax^2 + bx + c$ with little loss of generality since $ax^3 + bx^2 + cx + d = a\left(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right)$ when $a \neq 0$. For the function $x^3 + ax^2 + bx + c$ we note without proof here that the transformation $f\left(x - \frac{a}{3}\right)$ always removes the x^2 term (Why this works is an important question and CAS DT will confirm this). For example, if we have a function f with $f(x) = x^3 - 3x^2 + x - 5$, then $f\left(x - \frac{-3}{3}\right) = f(x+1) = (x+1)^3 - 3(x+1)^2 + (x+1) - 5$, which reduces to $x^3 - 2x - 6$. You might want to reach for your DT device to verify the above!

While this is an interesting property in its own right, it leads to two other interesting ideas. Firstly, if we draw the graph of the two functions, $f(x) = x^3 - 3x^2 + x - 5$ and $g(x) = x^3 - 2x - 6$, what do we find? Look at Fig. 10.11.

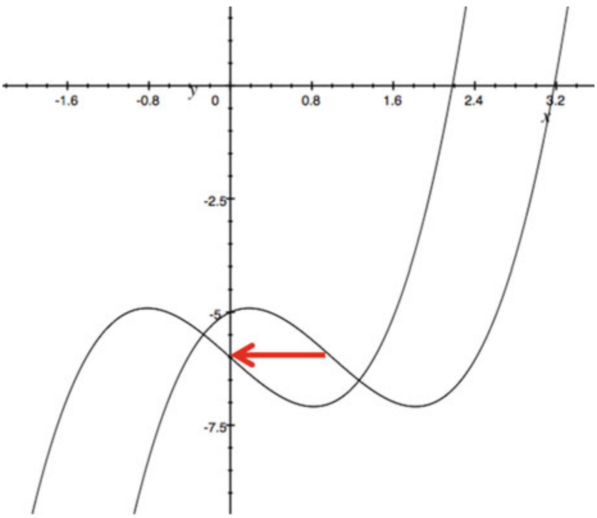


Fig. 10.11 An example of the graphical transformation of the cubic function for $f\left(x - \frac{a}{3}\right)$

Since $f'(x) = 3x^2 - 6x + 1$[1] and $f''(x) = 6x - 6 = 6(x - 1)$ the cubic has a point of inflection at $x = 1$ and since $f(x + 1)$ represents a translation of –1 parallel to the x-axis the point of inflection (1, –6) is mapped to (0, –6), on the y-axis. In general the point of inflection for the function j, where $j(x) = x^3 + ax + b$, will be mapped to $(0, b)$. Looking at this transformation in general we note that for $p(x) = x^3 + ax^2 + bx + c$, $p''(x) = 2(3x + a)$, giving a point of inflection at $x = -\frac{a}{3}$. Hence, all cubic graphs have a point of inflection and the translation $p\left(x - \frac{a}{3}\right)$ moves the point of inflection to the y-axis. Of course, if $a = 0$ there is no x^2 term and the point of inflection is already on the axis.

Turning back to the function f we can move the point of inflection to the origin by adding 6 to $g(x) = x^3 - 2x - 6$, giving the function h, where $h(x) = x^3 - 2x$. Clearly h is an odd function (since $h(-x) = -h(x)$ for all x) and hence h has 180° rotational symmetry about the origin. The point is that this whole process generalises, so that translating $j(x) = x^3 + ax + b$ by $\frac{a}{3}$ parallel to the x-axis and then by $p\left(-\frac{a}{3}\right) = \dfrac{2a^3 - 9ab + 27c}{27}$ parallel to the y-axis the graph's point of inflection will be moved to the origin. Hence, we always end up with the odd function $x^3 - \dfrac{(a^2 - 3b)}{3}x$, showing that all cubic polynomials have rotational symmetry of 180° about the point of inflection $\left(-\dfrac{a}{3}, \dfrac{2a^3 - 9ab + 27c}{27}\right)$. Finding this general property can be made accessible to some students with the assistance of DT, as seen in Fig. 10.12, which was produced using TI-Nspire software.

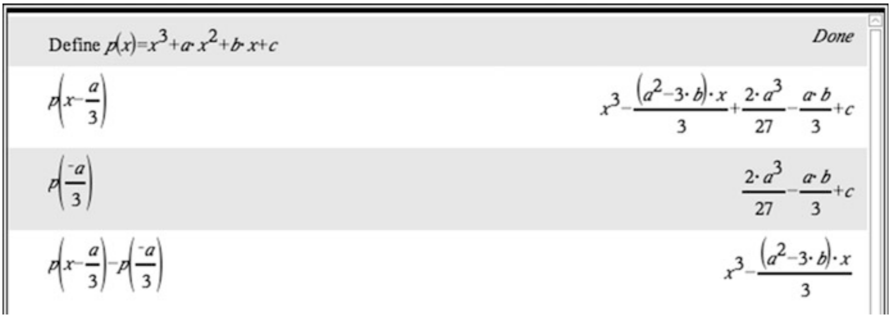


Fig. 10.12 Using TI-Nspire software to show cubic symmetry

[1] In this chapter we make some use of calculus differentiation techniques. While calculus is usually not studied in school in the USA, many countries do include it in the curriculum from age 16 or 17. Since the primary aim of school algebra is to lead to calculus some minimal use seems reasonable.

10.5.2 Solving Equations

Interestingly the first step above, removing the term in x^2, was also the first step in the Tartaglia-Cardano method for solving cubic equations. If we then consider solutions to equations of the form $x^3 + ax + b = 0$, using Vieta's substitution, $x = z - \frac{a}{3z}$ enables us to solve the equation. For ease of calculation, although this is not crucial with DT, consider the equation $x^3 + 9x + 8 = 0$, where a is divisible by 3. We make the substitution $x = z - \frac{3}{z}$ and this gives rise to a 'disguised' quadratic that can easily be solved for z and hence x is found using $x = z - \frac{3}{z}$. Once again we show this process in Fig. 10.13, using TI-Nspire software. There are some things to note here. In Fig. 10.14 we move representations and draw the graph of the function f where $f(x) = x^3 + 9x + 8$, noting that the point of inflection appears on the y-axis as expected. This enables us to ask whether there is only one real root to the equation. We are trying to avoid calculus in this discussion where possible, since it lies beyond the remit of school algebra in the USA (see footnote 1), but note that since $f'(x) = 3x^2 + 9 > 0$ for all x the function is strictly (or monotone) increasing and so there is only one zero and hence only one real root of our equation. Other possible questions worth considering are whether this method always works (and if not when does it fail) and how we might find the complex roots.

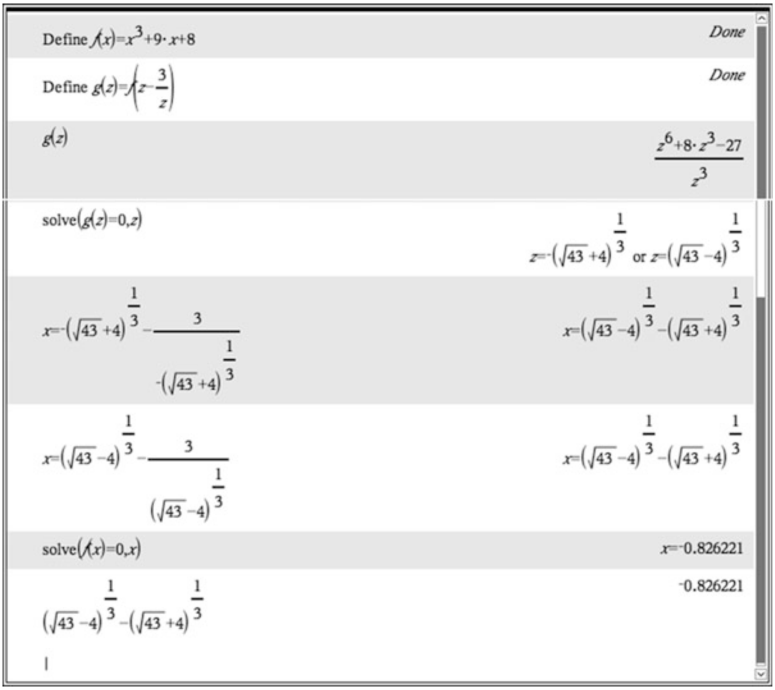


Fig. 10.13 Using TI-Nspire software to find exact solutions of cubic equations

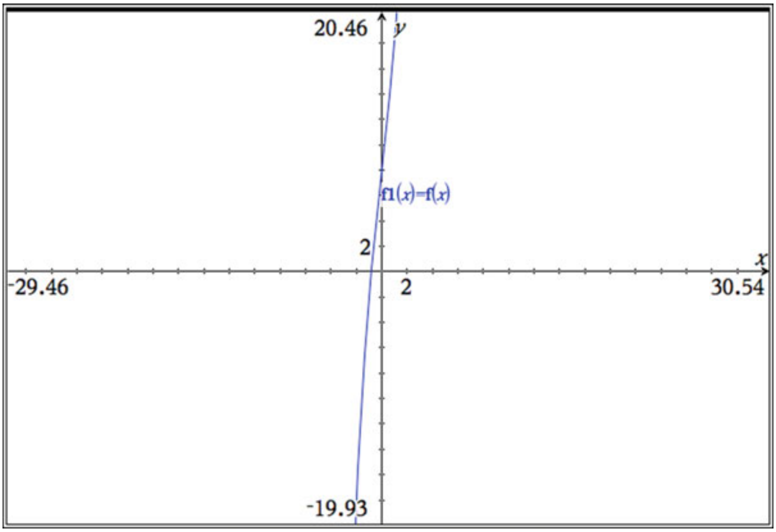


Fig. 10.14 Using TI-Nspire software to graph a solution to a cubic equation

10.5.3 Touching Graphs

Another task involving polynomials that could be given to students is:

Can we find quadratic functions whose graphs touch at a given point (p, q) with gradient k? How many possible graphs are there? Is there a general solution to the problem?

This task involves polynomials of degree 2 and links algebraic and graphical representations. While students can relatively easily find simple solutions, such as the graphs of polynomials x^2 and $-x^2$ that meet at (0, 0) with gradient 0, it is not so easy to solve more general cases by trial and error. However, once again this task is more approachable with DT. If we take a general quadratic function $f(x) = ax^2 + bx + c$ then we require the graph to pass through (p, q) and the gradient of the graph of the function at that point to be k. These two conditions can be written:

$$f(p) = ap^2 + bp + c = q \quad \text{and} \quad f'(p) = 2ap + b = k$$

In Fig. 10.15 we see the TI-Nspire software again employed to solve these equations simultaneously. The solution here is given in terms of a parameter c and, of course, p, q and k.

Choosing values for the point (p, q) and the gradient k gives a and b in terms of c, and we note that $c + kp - q \neq 0$ (since then we don't have a quadratic function) and $p \neq 0$. Figure 10.16 shows some of the possible solutions for the point (2, 3) and gradient 3 drawn using TI-Nspire. It is good practice to check these solutions, of course. For example, with $p = 2, q = 3,\ k = 1$ if we choose $c = -2$ then our function

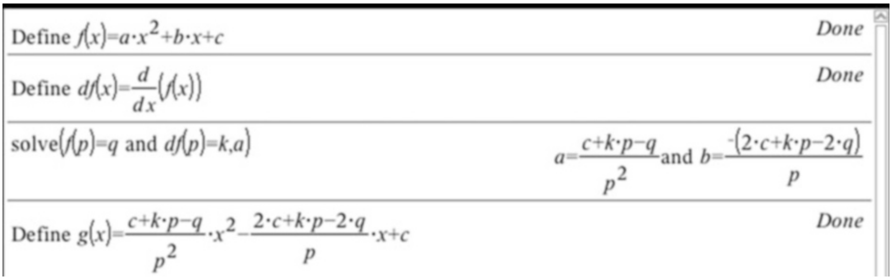


Fig. 10.15 Using the TI-Nspire software to find a general solution

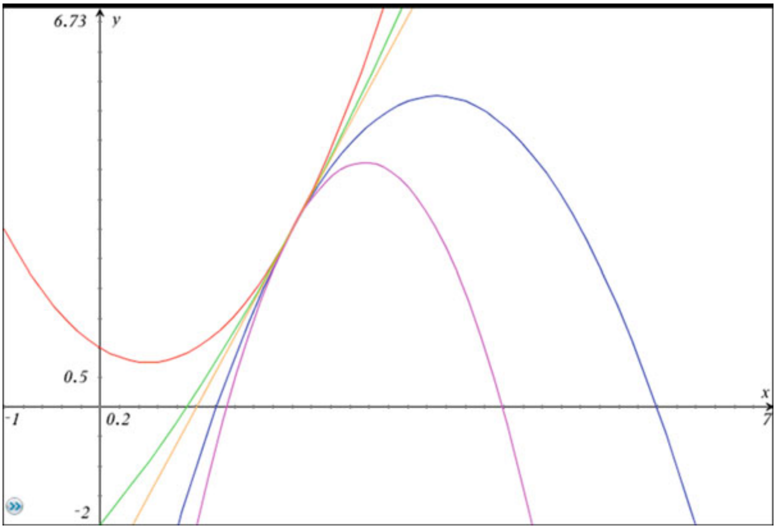


Fig. 10.16 Using the TI-Nspire software to show graphs of possible solution functions

is given by $f(x) = \frac{1}{4}x^2 + 2x - 2$, which passes through (2, 3) and the gradient there is $\frac{1}{2}(2) + 2 = 3$, as required.

A further question for investigation that arises is: does the latter condition $p \neq 0$ for the general solution mean that it is not possible to find graphs that meet on the y-axis with the same gradient? Well we have already seen that x^2 and $-x^2$ meet at (0, 0) with gradient 0, and in general so does kx^2, $k \neq 0$, k real. But what about other points not at the origin and whose gradient at $x = 0$ is not zero? Well it certainly appears to be possible to find some, as Fig. 10.17 shows, but students will have to engage with how we might find these solutions. It is hoped that ways to structure interesting tasks for students that promote understanding of properties of polynomials will become apparent.

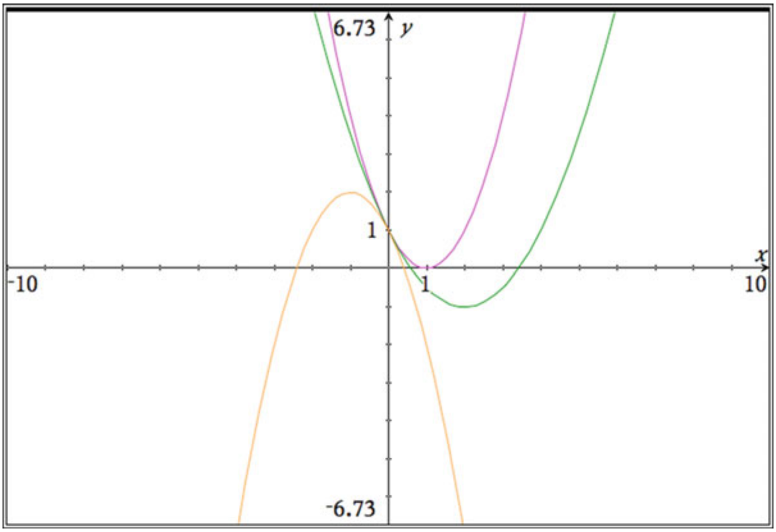


Fig. 10.17 Graphs of possible solution functions with $p = 0$

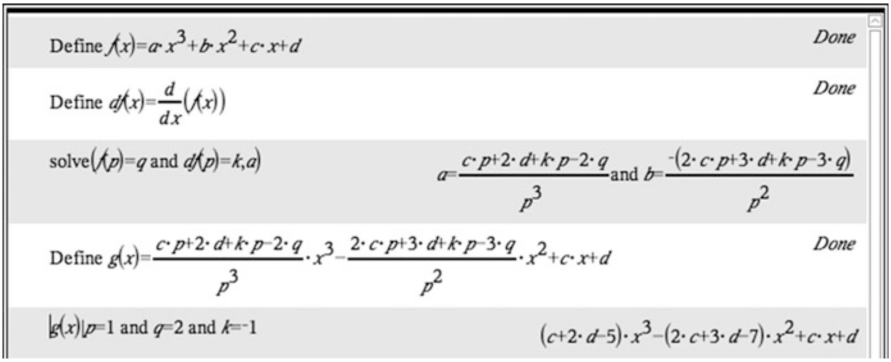


Fig. 10.18 Using the TI-Nspire software to find the general solution for cubic polynomials

Since generalising is always a key aim in mathematics, a possible next step is to try to extend these ideas further. One question we might ask is: Can we do the same for cubic polynomial functions? Using the DT again, as Fig. 10.18 shows, two parameters, c and d, are needed, where $(c + k)p + 2(d - q) \neq 0$ (since then we don't have a cubic) and $p \neq 0$, and Fig. 10.19 shows examples of the graphs of some polynomials of degree three meeting at the point (1, 2) with gradient –1 ($c + 2d - 5 \neq 0$ here).

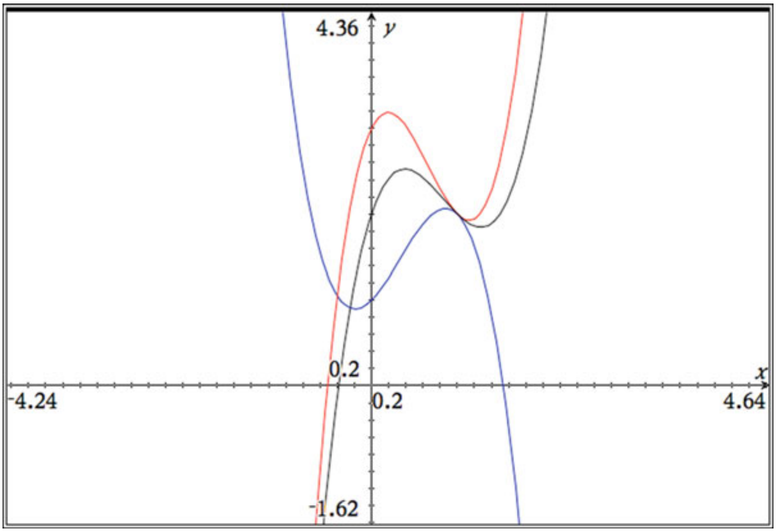


Fig. 10.19 Using the TI-Nspire software to show graphs of possible solution functions

10.5.4 Tangents to Cubic Polynomials

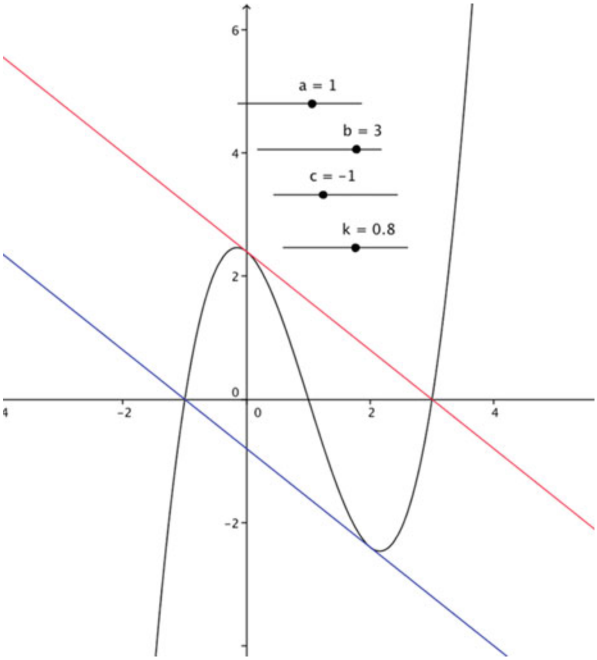
If we consider a cubic polynomial with three distinct real zeros then they have an interesting property related to their tangents that could be investigated (see de Alwis, 2012). We will consider a particular case first. The graph of the cubic function f where $f(x) = (x + 1)(x - 1)(x - 3)$ is shown in Fig. 10.20, which is drawn using GeoGebra. It is reasonably clear that the graph meets the x-axis at the three points (–1, 0), (1, 0) and (3, 0). Let's take the point on the curve where $x = \frac{1+3}{2} = 2$, the mean of the x values of the last two points of intersection, and find the equation of the tangent to the graph there. We have $f(2) = (3)(1)(-1) = -3$ and since

$$f(x) = (x + 1)(x - 1)(x - 3) = (x^2 - 1)(x - 3) = x^3 - 3x^2 - x + 3$$
$$f'(x) = 3x^2 - 6x - 1$$

and $f'(2) = -1$. So the equation of the tangent is $y + 3 = -1(x - 2)$ or $y + x + 1 = 0$ and when $y = 0$ for this tangent $x = -1$. So the tangent at the mean value of two points of intersection passes through the third point of intersection. Figure 10.20 also shows the tangent at the point where $x = \frac{-1+1}{2} = 0$ passing through the point (3, 0).

Of course, the tangent at the point where $x = \frac{-1+3}{2} = 1$ passes through the point (1, 0) here since it's a special case where the zeros are equally spaced. So the question is does this result generalise? Is it always true for cubics? One way to investigate it using GeoGebra is to use sliders for the function coefficients.

Fig. 10.20 The graph of the cubic function



$f(x) = (x + 1)(x - 1)(x - 3)$

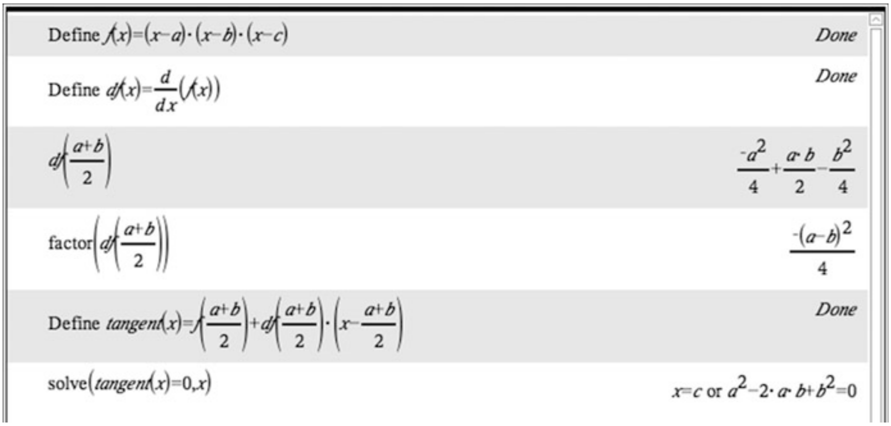


Fig. 10.21 Using TI-Nspire to demonstrate the generality of the tangent property of the cubic function $f(x) = (x - a)(x - b)(x - c)$

Algebraically, consider the monic polynomial function f where $f(x) = (x - a)(x - b)(x - c)$, and without loss of generality consider the tangent at the point M where $x = \frac{a+b}{2}$. This could be done by hand but once again the symbolic process can be left to the DT, in this with case TI-Nspire as shown in Fig. 10.21. The function df is the derivative of f and we note that the DT does not automatically factorise the result, although this is not crucial in this example. Using *df* we can find the gradient of the tangent at the point where $x = \frac{a+b}{2}$ and hence the equation of the tangent using the well-known equation $y - y_1 = m(x - x_1)$, where m is the gradient and (x_1, y_1) a point on the line (nb $y = y_1 + m(x - x_1)$ is used here). Then solving for where the tangent is zero gives $x = c$ or, interestingly, $a^2 - 2ab + b^2 = 0$ but then $(a - b)^2 = 0,\ a = b$, which would contradict our requirement that f have three distinct real zeros. So the tangent at $x = \frac{a+b}{2}$ does indeed pass through the point $(c, 0)$.

Once more the DT has allowed us to make some crucial links, this time between embodied actions and processes involving graphs and tangents and symbolic processes in order to find a solution for the task. Often we make the link by encouraging embodied views of symbolic expressions, so it is good to have an example that links the representations in the other direction.

10.6 Polynomials in Two Variables

Students at school often consider Pythagoras' theorem and its solutions, and while the theorem does not generalise to higher powers, as Fermat's last theorem states, solutions to other Diophantine equations are in reach if we use DT. One of these that can be approached, that I have described elsewhere (see Heid, Thomas, & Zbiek, 2013), is $x^2 + y^2 = z^3$, a special case of the general equation $x^n + y^n = z^{n+1}$, whose solutions have been outlined by, for example, Hoehn (1989). As I previously suggested, in a structured task students could be encouraged to use a DT spreadsheet listing values of n^2 and n^3 to try to find two of the squares that add up to a cube (for example, $x = 2$, $y = 2$ and $z = 2$ may be seen immediately). In this way $x = 5$, $y = 10$ and $z = 5$ might also be found. Hence, there are solutions. Further, if we substitute $x = ka$ and $y = kb$ in the equation $x^2 + y^2 = z^3$ we obtain $k^2\left(a^2 + b^2\right) = z^3$ and although this substitution is not obvious this last equation gives a big leap forward to finding solutions, since setting $k = a^2 + b^2$ will produce a solution $z = k = a^2 + b^2$. As an example, if we let $a = 2,\ b = 3$ then $k = 13$ and $x = 26$, $y = 39$ and $z = a^2 + b^2 = 26^2 + 39^2 = 2197 = 13^3$. In Fig. 10.22 we can see how the DT might be used to investigate the problem by introducing a function of two variables (we can also see this as a polynomial in two variables), an idea that will be very important later in mathematics. Hence, this constitutes an example of mathematics at the horizon in the mathematical knowledge for teaching framework (Ball, Hill, & Bass, 2005; Hill & Ball, 2004).

Extending the same method to a general equation $x^n + y^n = z^{n+1}$ could be too difficult for most school students, but the method above does generalise and this can be seen using DT, as in Fig. 10.23. Interestingly, as shown, the factorisation of $(a(a^n + b^n))^n + (b(a^n + b^n))^n$ seems beyond this DT program, but those students who have been taught to 'see' algebraic factors may be able to work as follows:

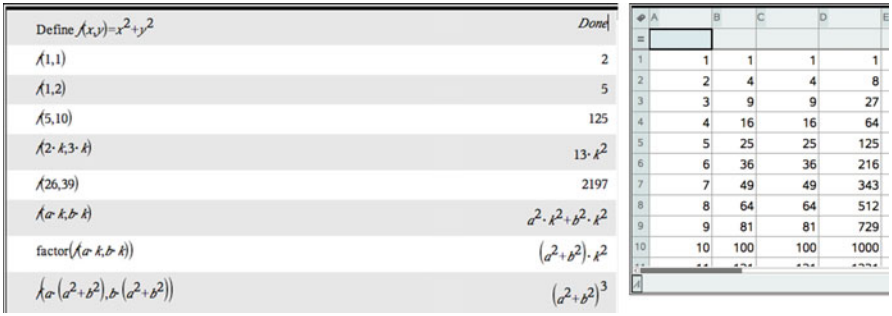


Fig. 10.22 Linking representations to find solutions to $x^2 + y^2 = z^3$

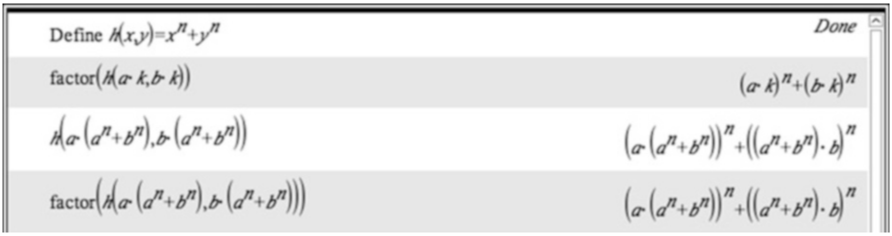


Fig. 10.23 Using DT to find solutions to $x^n + y^n = z^{n+1}$

$$
\begin{aligned}
(a(a^n + b^n))^n + (b(a^n + b^n))^n &= a^n(a^n + b^n)^n + b^n(a^n + b^n)^n \\
&= (a^n + b^n)^n\{a^n + b^n\} = (a^n + b^n)^{n+1}
\end{aligned}
$$

and hence this leads to a solution with $x = a(a^n + b^n)$, $y = b(a^n + b^n)$ and $z = a^n + b^n$.

10.7 Concluding Remarks

In a standard algebra curriculum students are involved in a great deal of what we have called manipulation algebra (Thomas & Tall, 2001). The outcome of this practice is that students may learn a lot about manipulating symbolic literals but far less about the nature of the objects they represent, such as polynomial functions, and their properties. Stressing the value of enactive and iconic thinking (Bruner, 1966) through visualisation encourages students to engage in the inter-representational conversions (Duval, 2006) that are a crucial constituent of building versatile thinking. Central to that inter-representational thinking is the DT, which, if it is used thoughtfully, can take on the role of epistemic mediator in order to help students to abstract properties of objects and the structure related to them and even to generalise to other sets of objects.

It has to be acknowledged first that some of the examples looked at above are at the top end of the difficulty scale for students in secondary school or college. Further, as I have noted elsewhere (Thomas & Palmer, 2013), while DT can provide many opportunities there are also a number of obstacles to be overcome in order to make good use of it. A major issue surrounds the role of the teacher in using DT in the manner described here. Some of the factors involved are extrinsic to the teacher, such as provision of suitable hardware. However, considering intrinsic teacher factors influencing use of DT led me (Hong & Thomas, 2006; Thomas & Hong, 2005) to propose an emerging framework for *pedagogical technology knowledge* (PTK) as a construct that could be an indicator of teacher progress in implementation of technology use. A teacher's PTK incorporates the principles, conventions, and techniques required to teach mathematics through DT. While the teacher has to be a proficient user of the technology she must also understand what is required to build tasks and situations that incorporate it, in order to enable mathematical learning through the technology. The essential teacher factors that combine to produce PTK include: instrumental genesis; mathematical knowledge for teaching; orientations and goals (Schoenfeld, 2011), especially beliefs about the value of technology and the nature of learning mathematical knowledge; and other affective aspects, such as confidence in teaching with DT.

In spite of these reservations I suggest that a rethink of the algebra curriculum and the dominance of the symbol manipulation approach usually employed might pay dividends in terms of stimulating versatile thinking by students and hence improve understanding of algebra.

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