Big Image of Galois Representations Associated with Finite Slope p-adic Families of Modular Forms

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Abstract We prove that the Lie algebra of the image of the Galois representation associated with a finite slope family of modular forms contains a congruence subalgebra of a certain level. We interpret this level in terms of congruences with CM forms.

Keywords Modular forms · CM forms · Galois representations · Lie algebras · Relative Sen theory · Eigenvarieties

1 Introduction

Let f be a non-CM cuspidal eigenform and let ℓ be a prime integer. By the work of Ribet [15, 17] and Momose [13], it is known that the ℓ-adic Galois representation $\rho_{f,\ell}$ associated with f has large image for every ℓ and that for almost every ℓ it satisfies

(cong$_\ell$) Im $\rho_{f,\ell}$ contains the conjugate of a principal congruence subgroup $\Gamma(\ell^m)$ of $\mathrm{SL}_2(\mathbb{Z}_\ell)$.

For instance if Im $\rho_{f,\ell}$ contains an element with eigenvalues in $\mathbb{Z}_\ell^\times$ distinct modulo ℓ then (cong$_\ell$) holds.

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In [9], Hida proved an analogous statement for p-adic families of non-CM ordinary cuspidal eigenforms, where p is any odd prime integer. We fix once and for all an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, identifying $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with a decomposition subgroup G_p of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We also choose a topological generator u of $\mathbb{Z}_p^\times$. Let $\Lambda = \mathbb{Z}_p[[T]]$ be the Iwasawa algebra and let $\mathfrak{m} = (p, T)$ be its maximal ideal. A special case of Hida's first main theorem ([9, Theorem I]) is the following.

Theorem 1.1 *Let* $\mathbf{f}$ *be a non-CM Hida family of ordinary cuspidal eigenforms defined over a finite extension* $\mathbb{I}$ *of* Λ *and let* $\rho_{\mathbf{f}} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{I})$ *be the associated Galois representation. Assume that* $\rho_{\mathbf{f}}$ *is residually irreducible and that there exists an element* d *in its image with eigenvalues* $\alpha, \beta \in \mathbb{Z}_p^\times$ *such that* $\alpha^2 \not\equiv \beta^2 \pmod{p}$. *Then there exists a nonzero ideal* $\mathfrak{l} \subset \Lambda$ *and an element* $g \in \mathrm{GL}_2(\mathbb{I})$ *such that*

$$g\Gamma(\mathfrak{l})g^{-1} \subset \mathrm{Im}\,\rho_{\mathbf{f}},$$

where $\Gamma(\mathfrak{l})$ *denotes the principal congruence subgroup of* $\mathrm{SL}_2(\Lambda)$ *of level* $\mathfrak{l}$.

Under mild technical assumptions it is also shown in [9, Theorem II] that if the image of the residual representation of $\rho_{\mathbf{f}}$ contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_p)$ then $\mathfrak{l}$ is trivial or $\mathfrak{m}$-primary, and if the residual representation is dihedral "of CM type" the height one prime factors P of $\mathfrak{l}$ are exactly those of the g.c.d. of the adjoint p-adic L function of $\mathbf{f}$ and the anticyclotomic specializations of Katz's p-adic L functions associated with certain Hecke characters of an imaginary quadratic field. This set of primes is precisely the set of congruence primes between the given non-CM family and the CM families.

In her Ph.D. dissertation (see [12]), J. Lang improved on Hida's Theorem I. Let $\mathbb{T}$ be Hida's big ordinary cuspidal Hecke algebra; it is finite and flat over Λ. Let $\mathrm{Spec}\,\mathbb{I}$ be an irreducible component of $\mathbb{T}$. It corresponds to a surjective Λ-algebra homomorphism $\theta \colon \mathbb{T} \to \mathbb{I}$ (a Λ-adic Hecke eigensystem). We also call θ a Hida family. Assume that it is not residually Eisenstein. It gives rise to a residually irreducible continuous Galois representation $\rho_\theta \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I})$ that is p-ordinary. We suppose for simplicity that $\mathbb{I}$ is normal. Consider the Λ-algebra automorphisms σ of $\mathbb{I}$ for which there exists a finite order character $\eta_\sigma \colon G_{\mathbb{Q}} \to \mathbb{I}^\times$ such that for every prime ℓ not dividing the level, $\sigma \circ \theta(T_\ell) = \eta_\sigma(\ell)\theta(T_\ell)$ (see [12, 17]). These automorphisms form a finite abelian 2-group Γ. Let $\mathbb{I}_0$ be the subring of $\mathbb{I}$ fixed by Γ. Let $H_0 = \bigcap_{\sigma\in\Gamma} \ker\,\eta_\sigma$; it is a normal open subgroup of $G_{\mathbb{Q}}$. One may assume, up to conjugation by an element of $\mathrm{GL}_2(\mathbb{I})$, that $\rho_\theta|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0)$.

Theorem 1.2 [12, Theorem 2.4] *Let* $\theta \colon \mathbb{T} \to \mathbb{I}$ *be a non-CM Hida family such that* $\overline{\rho}_\theta$ *is absolutely irreducible. Assume that* $\overline{\rho}_\theta|_{H_0}$ *is an extension of two distinct characters. Then there exists a nonzero ideal* $\mathfrak{l} \subset \mathbb{I}_0$ *and an element* $g \in \mathrm{GL}_2(\mathbb{I})$ *such that*

$$g\Gamma(\mathfrak{l})g^{-1} \subset \mathrm{Im}\,\rho_\theta,$$

where $\Gamma(\mathfrak{l})$ *denotes the principal congruence subgroup of* $\mathrm{SL}_2(\mathbb{I}_0)$ *of level* $\mathfrak{l}$.

For all of these results it is important to assume the ordinarity of the family, as it implies the ordinarity of the Galois representation and in particular that some element of the image of inertia at p is conjugate to the matrix

$$C_T = \begin{pmatrix} u^{-1}(1+T) & * \\ 0 & 1 \end{pmatrix}.$$

Conjugation by the element above defines a Λ-module structure on the Lie algebra of a pro-p subgroup of $\mathrm{Im}\,\rho_\theta$ and this is used to produce the desired ideal $\mathfrak{l}$. Hida and Lang use Pink's theory of Lie algebras of pro-p subgroups of $\mathrm{SL}_2(\mathbb{I})$.

In this paper we propose a generalization of Hida's work to the finite slope case. We establish analogues of Hida's Theorems I and II. These are Theorems 6.2, 7.1 and 7.4 in the text. Moreover, we put ourselves in the more general setting considered in Lang's work. In the positive slope case the existence of a normalizing matrix analogous to C_T above is obtained by applying relative Sen theory ([19, 21]) to the expense of extending scalars to the completion $\mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}_p$.

More precisely, for every $h \in (0, \infty)$, we define an Iwasawa algebra $\Lambda_h = \mathcal{O}_h[[t]]$ (where $t = p^{-s_h}T$ for some $s_h \in \mathbb{Q} \cap]\frac{1}{p-1}, \infty[$ and $\mathcal{O}_h$ is a finite extension of $\mathbb{Z}_p$ containing p^{s_h} such that its fraction field is Galois over $\mathbb{Q}_p$) and a finite torsion free Λ_h-algebra $\mathbb{T}_h$ (see Sect. 3.1), called an adapted slope $\leqslant h$ Hecke algebra. Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be an irreducible component; it is finite and torsion-free over Λ_h. The notation $\mathbb{I}^\circ$ is borrowed from the theory of Tate algebras, but $\mathbb{I}^\circ$ is not a Tate or an affinoid algebra. We write $\mathbb{I} = \mathbb{I}^\circ[p^{-1}]$. We assume for simplicity that $\mathbb{I}^\circ$ is normal. The finite slope family θ gives rise to a continuous Galois representation $\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$. We assume that the residual representation $\overline{\rho_\theta}$ is absolutely irreducible. We introduce the finite abelian 2-group Γ as above, together with its fixed ring $\mathbb{I}_0$ and the open normal subgroup $H_0 \subset G_\mathbb{Q}$. In Sect. 5.1 we define a ring $\mathbb{B}_r$ (with an inclusion $\mathbb{I}_0 \hookrightarrow \mathbb{B}_r$) and a Lie algebra $\mathfrak{H}_r \subset \mathfrak{sl}_2(\mathbb{B}_r)$ attached to the image of ρ_θ. In the positive slope case CM families do not exist (see Sect. 3.3) hence no "non-CM" assumption is needed in the following. As before we can assume, after conjugation by an element of $\mathrm{GL}_2(\mathbb{I}^\circ)$, that $\rho_\theta(H_0) \subset \mathrm{GL}_2(\mathbb{I}_0^\circ)$. Let $P_1 \subset \Lambda_h$ be the prime $(u^{-1}(1+T)-1)$.

Theorem 1.3 (Theorem 6.2) *Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a positive slope family such that $\overline{\rho}_\theta|_{H_0}$ is absolutely irreducible. Assume that there exists $d \in \rho_\theta(H_0)$ with eigenvalues $\alpha, \beta \in \mathbb{Z}_p^\times$ such that $\alpha^2 \not\equiv \beta^2 \pmod{p}$. Then there exists a nonzero ideal $\mathfrak{l} \subset \mathbb{I}_0[P_1^{-1}]$ such that*

$$\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r) \subset \mathfrak{H}_r.$$

The largest such ideal $\mathfrak{l}$ is called the Galois level of θ.

We also introduce the notion of fortuitous CM congruence ideal for θ (see Sect. 3.4). It is the ideal $\mathfrak{c} \subset \mathbb{I}$ given by the product of the primary ideals modulo which a congruence between θ and a slope $\leqslant h$ CM form occurs. Following the proof of Hida's Theorem II we are able to show (Theorem 7.1) that the set of primes of $\mathbb{I}_0 = \mathbb{I}_0^\circ[p^{-1}]$ containing $\mathfrak{l}$ coincides with the set of primes containing $\mathfrak{c} \cap \mathbb{I}_0$, except possibly for the primes of $\mathbb{I}_0$ above P_1 (the weight 1 primes).

Several generalizations of the present work are currently being studied by one of the authors.[1] They include a generalization of [10], where the authors treated the ordinary case for GSp_4 with a residual representation induced from the one associated with a Hilbert modular form, to the finite slope case and to bigger groups and more types of residual representations.

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2 The Eigencurve

2.1 The Weight Space

Fix a prime integer $p > 2$. We call *weight space* the rigid analytic space over $\mathbb{Q}_p$, $\mathcal{W}$, canonically associated with the formal scheme over $\mathbb{Z}_p$, $\mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$. The $\mathbb{C}_p$-points of $\mathcal{W}$ parametrize continuous homomorphisms $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$.

Let X be a rigid analytic space defined over some finite extension $L/\mathbb{Q}_p$. We say that a subset S of $X(\mathbb{C}_p)$ is Zariski-dense if the only closed analytic subvariety Y of X satisfying $S \subset Y(\mathbb{C}_p)$ is X itself.

For every $r > 0$, we denote by $\mathcal{B}(0, r)$, respectively $\mathcal{B}(0, r^-)$, the closed, respectively open, disc in $\mathbb{C}_p$ of centre 0 and radius r. The space $\mathcal{W}$ is isomorphic to a disjoint union of $p - 1$ copies of the open unit disc $\mathcal{B}(0, 1^-)$ centre in 0 and indexed by the group $\mathbb{Z}/(p-1)\mathbb{Z} = \widehat{\mu}_{p-1}$. If u denotes a topological generator of $1 + p\mathbb{Z}_p$, then an isomorphism is given by

$$\mathbb{Z}/(p-1)\mathbb{Z} \times \mathcal{B}(0, 1^-) \to \mathcal{W}, \quad (i, v) \mapsto \chi_{i,v},$$

where $\chi_{i,v}((\zeta, u^x)) = \zeta^i(1+v)^x$. Here we wrote an element of $\mathbb{Z}_p^\times$ uniquely as a pair (ζ, u^x) with $\zeta \in \mu_{p-1}$ and $x \in \mathbb{Z}_p$. We make once and for all the choice $u = 1 + p$.

We say that a point $\chi \in \mathcal{W}(\mathbb{C}_p)$ is classical if there exists $k \in \mathbb{N}$ and a finite order character $\psi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ such that χ is the character $z \mapsto z^k\psi(z)$. The set of classical points is Zariski-dense in $\mathcal{W}(\mathbb{C}_p)$.

If $\mathrm{Spm}\, R \subset \mathcal{W}$ is an affinoid open subset, we denote by $\kappa = \kappa_R : \mathbb{Z}_p^\times \to R^\times$ its tautological character given by $\kappa(t)(\chi) = \chi(t)$ for every $\chi \in \mathrm{Spm}\, R$. Recall ([3, Proposition 8.3]) that κ_R is r-analytic for every sufficiently small radius $r > 0$ (by which we mean that it extends to a rigid analytic function on $\mathbb{Z}_p^\times\mathcal{B}(1, r)$).

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[1] A. Conti.

2.2 Adapted Pairs and the Eigencurve

Let N be a positive integer prime to p. We recall the definition of the spectral curve Z^N and of the cuspidal eigencurve C^N of tame level $\Gamma_1(N)$. These objects were constructed in [6] for $p > 2$ and $N = 1$ and in [3] in general. We follow the presentation of [3, Part II]. Let $\operatorname{Spm} R \subset \mathcal{W}$ be an affinoid domain and let $r = p^{-s}$ for $s \in \mathbb{Q}$ be a radius smaller than the radius of analyticity of κ_R. We denote by $M_{R,r}$ the R-module of r-overconvergent modular forms of weight κ_R. It is endowed it with a continuous action of the Hecke operators T_ℓ, $\ell \nmid Np$, and U_p. The action of U_p on $M_{R,r}$ is completely continuous, so we can consider its associated Fredholm series $F_{R,r}(T) = \det(1 - U_p T | M_{R,r}) \in R\{\{T\}\}$. These series are compatible when R and r vary, in the sense that there exists $F \in \Lambda\{\{T\}\}$ that restricts to $F_{R,r}(T)$ for every R and r.

The series $F_{R,r}(T)$ converges everywhere on the R-affine line $\operatorname{Spm} R \times \mathbb{A}^{1,\mathrm{a}n}$, so it defines a rigid curve $Z^N_{R,r} = \{F_{R,r}(T) = 0\}$ in $\operatorname{Spm} R \times \mathbb{A}^{1,\mathrm{a}n}$. When R and r vary, these curves glue into a rigid space Z^N endowed with a quasi-finite and flat morphism $w_Z \colon Z^N \to \mathcal{W}$. The curve Z^N is called the spectral curve associated with the U_p-operator. For every $h \geqslant 0$, let us consider

$$Z_R^{N,\leqslant h} = Z_R^N \cap \left(\operatorname{Spm} R \times B(0, p^h)\right).$$

By [3, Lemma 4.1] $Z_R^{N,\leqslant h}$ is quasi-finite and flat over $\operatorname{Spm} R$.

We now recall how to construct an admissible covering of Z^N.

Definition 2.1 We denote by C the set of affinoid domains $Y \subset Z$ such that:

- there exists an affinoid domain $\operatorname{Spm} R \subset \mathcal{W}$ such that Y is a union of connected components of $w_Z^{-1}(\operatorname{Spm} R)$;
- the map $w_Z|_Y \colon Y \to \operatorname{Spm} R$ is finite.

Proposition 2.2 [3, Theorem 4.6] *The covering C is admissible.*

Note in particular that an element $Y \in C$ must be contained in $Z_R^{N,\leqslant h}$ for some h.

For every R and r as above and every $Y \in C$ such that $w_Z(Y) = \operatorname{Spm} R$, we can associate with Y a direct factor M_Y of $M_{R,r}$ by the construction in [3, Sect. I.5]. The abstract Hecke algebra $\mathcal{H} = \mathbb{Z}[T_\ell]_{\ell \nmid Np}$ acts on $M_{R,r}$ and M_Y is stable with respect to this action. Let $\mathbb{T}_Y$ be the R-algebra generated by the image of $\mathcal{H}$ in $\operatorname{End}_R(M_Y)$ and let $C_Y^N = \operatorname{Spm} \mathbb{T}_Y$. Note that it is reduced as all Hecke operators are self-adjoint for a certain pairing and mutually commute.

For every Y the finite covering $C_Y^N \to \operatorname{Spm} R$ factors through $Y \to \operatorname{Spm} R$. The eigencurve C^N is defined by gluing the affinoids C_Y^N into a rigid curve, endowed with a finite morphism $C^N \to Z^N$. The curve C^N is reduced and flat over $\mathcal{W}$ since it is so locally.

We borrow the following terminology from Bellaïche.

Definition 2.3 [1, Definition II.1.8] Let $\mathrm{Spm}\, R \subset \mathcal{W}$ be an affinoid open subset and $h > 0$ be a rational number. The couple (R, h) is called adapted if $Z_R^{N, \leqslant h}$ is an element of $\mathcal{C}$.

By [1, Corollary II.1.13] the sets of the form $Z_R^{N, \leqslant h}$ are sufficient to admissibly cover the spectral curve.

Now we fix a finite slope h. We want to work with families of slope $\leqslant h$ which are finite over a wide open subset of the weight space. In order to do this it will be useful to know which pairs (R, h) in a connected component of $\mathcal{W}$ are adapted. If $\mathrm{Spm}\, R' \subset \mathrm{Spm}\, R$ are affinoid subdomains of $\mathcal{W}$ and (R, h) is adapted then (R', h) is also adapted by [1, Proposition II.1.10]. By [3, Lemma 4.3], the affinoid $\mathrm{Spm}\, R$ is adapted to h if and only if the weight map $Z_R^{N, \leqslant h} \to \mathrm{Spm}\, R$ has fibres of constant degree.

Remark 2.4 Given a slope h and a classical weight k, it would be interesting to have a lower bound for the radius of a disc of centre k adapted to h. A result of Wan ([24, Theorem 2.5]) asserts that for a certain radius r_h depending only on h, N and p, the degree of the fibres of $Z_{\mathcal{B}(k, r_h)}^{N, \leqslant h} \to \mathrm{Spm}\, \mathcal{B}(k, r_h)$ at classical weights is constant. Unfortunately we do not know whether the degree is constant at all weights of $\mathcal{B}(k, r_h)$, so this is not sufficient to answer our question. Estimates for the radii of adapted discs exist in the case of eigenvarieties for groups different than GL_2; see for example the results of Chenevier on definite unitary groups ([4, Sect. 5]).

2.3 *Pseudo-characters and Galois Representations*

Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$. Let X be a rigid analytic variety defined over K. We denote by $\mathcal{O}(X)$ the ring of global analytic functions on X equipped with the coarsest locally convex topology making the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$ continuous for every affinoid $U \subset X$. It is a Fréchet space isomorphic to the inverse limit over all affinoid domains U of the K-Banach spaces $\mathcal{O}(U)$. We denote by $\mathcal{O}(X)^\circ$ the $\mathcal{O}_K$-algebra of functions bounded by 1 on X, equipped with the topology induced by that on $\mathcal{O}(X)$. The question of the compactness of this ring is related to the following property of X.

Definition 2.5 [2, Definition 7.2.10] We say that a rigid analytic variety X defined over K is nested if there is an admissible covering $X = \bigcup X_i$ by open affinoids X_i defined over K such that the maps $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ induced by the inclusions are compact.

We equip the ring $\mathcal{O}(X)^\circ$ with the topology induced by that on $\mathcal{O}(X) = \varprojlim_i \mathcal{O}(X_i)$.

Lemma 2.6 *[2, Lemma 7.2.11(ii)] If X is reduced and nested, then $\mathcal{O}(X)^\circ$ is a compact (hence profinite) $\mathcal{O}_K$-algebra.*

We will be able to apply Lemma 2.6 to the eigenvariety thanks to the following.

Proposition 2.7 [2, Corollary 7.2.12] *The eigenvariety C^N is nested for $K = \mathbb{Q}_p$.*

Given a reduced nested subvariety X of C^N defined over a finite extension K of $\mathbb{Q}_p$ there is a pseudo-character on X obtained by interpolating the classical ones. Let $\mathbb{Q}^{N_p}$ be the maixmal extension of $\mathbb{Q}$ uniamified outside N_p and let $G_{\mathbb{Q}}$, $N_p = Gal(\mathbb{Q}^{N_p}/\mathbb{Q})$.

Proposition 2.8 [1, Theorem IV.4.1] *There exists a unique pseudo-character*

$$\tau\colon G_{\mathbb{Q},Np} \to \mathcal{O}(X)^\circ$$

of dimension 2 *such that for every ℓ prime to Np, $\tau(\mathrm{Frob}_\ell) = \psi_X(T_\ell)$, where ψ_X is the composition of $\psi\colon \mathcal{H} \to \mathcal{O}(C^N)^\circ$ with the restriction map $\mathcal{O}(C^N)^\circ \to \mathcal{O}(X)^\circ$.*

Remark 2.9 One can take as an example of X a union of irreducible components of C^N in which case $K = \mathbb{Q}_p$. Later we will consider other examples where $K \neq \mathbb{Q}_p$.

3 The Fortuitous Congruence Ideal

In this section we will define families with slope bounded by a finite constant and coefficients in a suitable profinite ring. We will show that any such family admits at most a finite number of classical specializations which are CM modular forms. Later we will define what it means for a point (not necessarily classical) to be CM and we will associate with a family a congruence ideal describing its CM points. Contrary to the ordinary case, the non-ordinary CM points do not come in families so the points detected by the congruence ideal do not correspond to a crossing between a CM and a non-CM family. For this reason we call our ideal the "fortuitous congruence ideal".

3.1 The Adapted Slope $\leqslant h$ Hecke Algebra

Throughout this section we fix a slope $h > 0$. Let $C^{N,\leqslant h}$ be the subvariety of C^N whose points have slope $\leqslant h$. Unlike the ordinary case treated in [9] the weight map $w^{\leqslant h}\colon C^{N,\leqslant h} \to \mathcal{W}$ is not finite which means that a family of slope $\leqslant h$ is not in general defined by a finite map over the entire weight space. The best we can do in the finite slope situation is to place ourselves over the largest possible wide open subdomain U of $\mathcal{W}$ such that the restricted weight map $w^{\leqslant h}|_U\colon C^{N,\leqslant h} \times_{\mathcal{W}} U \to U$ is finite. This is a domain "adapted to h" in the sense of Definition 2.3 where only affinoid domains were considered. The finiteness property will be necessary in order to apply going-up and going-down theorems.

Let us fix a rational number s_h such that for $r_h = p^{-s_h}$ the closed disc $B(0, r_h)$ is adapted for h. We assume that $s_h > \frac{1}{p-1}$ (this will be needed later to assure the convergence of the exponential map). Let $\eta_h \in \overline{\mathbb{Q}}_p$ be an element of p-adic valuation s_h. Let K_h be the Galois closure (in $\mathbb{C}_p$) of $\mathbb{Q}_p(\eta_h)$ and let $\mathcal{O}_h$ be its valuation ring. Recall that T is the variable on the open disc of radius 1. Let $t = \eta_h^{-1} T$ and $\Lambda_h = \mathcal{O}_h[[t]]$. This is the ring of analytic functions, with $\mathcal{O}_h$-coefficients and bounded by one, on the wide open disc $\mathcal{B}_h$ of radius p^{-s_h}. There is a natural map $\Lambda \to \Lambda_h$ corresponding to the restriction of analytic functions on the open disc of radius 1, with $\mathbb{Z}_p$ coefficients and bounded by 1, to the open disc of radius r_h. The image of this map is the ring $\mathbb{Z}_p[[\eta t]] \subset \mathcal{O}_h[[t]]$.

For $i \geqslant 1$, let $s_i = s_h + 1/i$ and $\mathcal{B}_i = \mathcal{B}(0, p^{-s_i})$. The open disc $\mathcal{B}_h$ is the increasing union of the affinoid discs $\mathcal{B}_i$. For each i a model for $\mathcal{B}_i$ over K_h is given by Berthelot's construction of $\mathcal{B}_h$ as the rigid space associated with the $\mathcal{O}_h$-formal scheme Spf Λ_h. We recall it briefly following [7, Sect. 7]. Let

$$A_{r_i}^\circ = \mathcal{O}_h\langle t, X_i\rangle/(pX_i - t^i).$$

We have $\mathcal{B}_i = \operatorname{Spm} A_{r_i}^\circ[p^{-1}]$ as rigid space over K_h. For every i we have a morphism $A_{r_{i+1}}^\circ \to A_{r_i}^\circ$ given by

$$X_{i+1} \mapsto X_i t$$

$$t \mapsto t$$

We have induced compact morphisms $A_{r_{i+1}}^\circ[p^{-1}] \to A_{r_i}^\circ[p^{-1}]$, hence open immersions $\mathcal{B}_i \to \mathcal{B}_{i+1}$ defined over K_h. The wide open disc $\mathcal{B}_h$ is defined as the inductive limit of the affinoids $\mathcal{B}_i$ with these transition maps. We have $\Lambda_h = \varprojlim_i A_{r_i}^\circ$.

Since the s_i are strictly bigger than s_h for each i, $\mathcal{B}(0, p^{-s_i}) = \operatorname{Spm} A_{r_i}^\circ[p^{-1}]$ is adapted to h. Therefore for every $r > 0$ sufficiently small and for every $i \geqslant 1$ the image of the abstract Hecke algebra acting on $M_{A_{r_i},r}$ provides a finite affinoid $A_{r_i}^\circ$-algebra $\mathbb{T}_{A_{r_i}^\circ,r}^{\leqslant h}$. The morphism $w_{A_{r_i}^\circ,r}\colon \operatorname{Spm} \mathbb{T}_{A_{r_i}^\circ,r}^{\leqslant h} \to \operatorname{Spm} A_{r_i}^\circ$ is finite. For $i < j$ we have natural open immersions $\operatorname{Spm} \mathbb{T}_{A_{r_j}^\circ,r}^{\leqslant h} \to \operatorname{Spm} \mathbb{T}_{A_{r_i}^\circ,r}^{\leqslant h}$ and corresponding restriction maps $\mathbb{T}_{A_{r_i}^\circ,r}^{\leqslant h} \to \mathbb{T}_{A_{r_j}^\circ,r}^{\leqslant h}$. We call C_h the increasing union $\bigcup_{i\in\mathbb{N},r>0} \operatorname{Spm} \mathbb{T}_{A_{r_i}^\circ,r}^{\leqslant h}$; it is a wide open subvariety of C^N. We denote by $\mathbb{T}_h$ the ring of rigid analytic functions bounded by 1 on C_h. We have $\mathbb{T}_h = \mathcal{O}(C_h)^\circ = \varprojlim_{i,r} \mathbb{T}_{A_{r_i}^\circ,r}^{\leqslant h}$. There is a natural weight map $w_h\colon C_h \to \mathcal{B}_h$ that restricts to the maps $w_{A_{r_i}^\circ,r}$. It is finite because the closed ball of radius r_h is adapted to h.

3.2 The Galois Representation Associated with a Family of Finite Slope

Since $\mathcal{O}(B_h)^\circ = \Lambda_h$, the map w_h gives $\mathbb{T}_h$ the structure of a finite Λ_h-algebra; in particular $\mathbb{T}_h$ is profinite.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_h$. The residue field $k = \mathbb{T}_h/\mathfrak{m}$ is finite. Let $\mathbb{T}_\mathfrak{m}$ denote the localization of $\mathbb{T}_h$ at $\mathfrak{m}$. Since Λ_h is henselian, $\mathbb{T}_\mathfrak{m}$ is a direct factor of

$\mathbb{T}_h$, hence it is finite over Λ_h; it is also local noetherian and profinite. It is the ring of functions bounded by 1 on a connected component of C_h. Let $W = W(k)$ be the ring of Witt vectors of k. By the universal property of W, $\mathbb{T}_\mathfrak{m}$ is a W-algebra. The affinoid domain Spm $\mathbb{T}_\mathfrak{m}$ contains a zarisiki-dense set of points x corresponding to cuspidal eigenforms f_x of weight $w(x) = k_x \geqslant 2$ and level Np. The Galois representations ρ_{f_x} associated with the f_x give rise to a residual representation $\overline{\rho}\colon G_{\mathbb{Q},Np} \to \mathrm{GL}_2(k)$ that is independent of f_x. By Proposition 2.8, we have a pseudo-character

$$\tau_{\mathbb{T}_\mathfrak{m}} : G_{\mathbb{Q},Np} \to \mathbb{T}_\mathfrak{m}$$

such that for every classical point $x\colon \mathbb{T}_\mathfrak{m} \to L$, defined over some finite extension $L/\mathbb{Q}_p$, the specialization of $\tau_{\mathbb{T}_\mathfrak{m}}$ at x is the trace of Lf_x.

Proposition 3.1 *If $\overline{\rho}$ is absolutely irreducible there exists a unique continuous irreducible Galois representation*

$$\rho_{\mathbb{T}_\mathfrak{m}} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{T}_\mathfrak{m}),$$

lifting $\overline{\rho}$ and whose trace is $\tau_{\mathbb{T}_\mathfrak{m}}$.

This follows from a result of Nyssen and Rouquier ([14], [18, Corollary 5.2]), since $\mathbb{T}_\mathfrak{m}$ is local henselian.

Let $\mathbb{I}^\circ$ be a finite torsion-free Λ_h-algebra. We call *family* an irreducible component of Spec $\mathbb{T}_h$ defined by a surjective morphism $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ of Λ_h-algebras. Since such a map factors via $\mathbb{T}_\mathfrak{m} \to \mathbb{I}^\circ$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{T}_h$, we can define a residual representation $\overline{\rho}$ associated with θ. Suppose that $\overline{\rho}$ is irreducible. By Proposition 3.1 we obtain a Galois representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated with θ.

Remark 3.2 If $\eta_h \notin \mathbb{Q}_p$, Λ_h is not a power series ring over $\mathbb{Z}_p$.

3.3 Finite Slope CM Modular Forms

In this section we study non-ordinary finite slope CM modular forms. We say that a family is CM if all its classical points are CM. We prove that for every $h > 0$ there are no CM families with positive slope $\leqslant h$. However, contrary to the ordinary case, every family of finite positive slope may contain classical CM points of weight $k \geqslant 2$. Let F be an imaginary quadratic field, $\mathfrak{f}$ an integral ideal in F, $I_\mathfrak{f}$ the group of fractional ideals prime to $\mathfrak{f}$. Let σ_1, σ_2 be the embeddings of F into $\mathbb{C}$ (say that $\sigma_1 = \mathrm{Id}_F$) and let $(k_1, k_2) \in \mathbb{Z}^2$. A Grössencharacter ψ of infinity type (k_1, k_2) defined modulo $\mathfrak{f}$

is a homomorphism $\psi\colon I_{\mathfrak{f}} \to \mathbb{C}^*$ such that $\psi((\alpha)) = \sigma_1(\alpha)^{k_1}\sigma_2(\alpha)^{k_2}$ for all $\alpha \equiv 1 \pmod{^\times \mathfrak{f}}$. Consider the q-expansion

$$\sum_{\mathfrak{a}\subset O_F,(\mathfrak{a},\mathfrak{f})=1} \psi(\mathfrak{a})q^{N(\mathfrak{a})},$$

where the sum is over ideals $\mathfrak{a} \subset O_F$ and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. Let $F/\mathbb{Q}$ be an imaginary quadratic field of discriminant D and let ψ be a Grössencharacter of exact conductor $\mathfrak{f}$ and infinity type $(k-1, 0)$. By [22, Lemma 3] the expansion displayed above defines a cuspidal newform $f(F, \psi)$ of level $N(\mathfrak{f})D$.

Ribet proved in [16, Theorem 4.5] that if a newform g of weight $k \geqslant 2$ and level N has CM by an imaginary quadratic field F, one has $g = f(F, \psi)$ for some Grössencharacter ψ of F of infinity type $(k-1, 0)$.

Definition 3.3 We say that a classical modular eigenform g of weight k and level Np has CM by an imaginary quadratic field F if its Hecke eigenvalues for the operators T_ℓ, $\ell \nmid Np$, coincide with those of $f(F, \psi)$ for some Grössencharacter ψ of F of infinity type $(k-1, 0)$. We also say that g is CM without specifying the field.

Remark 3.4 For g as in the definition the Galois representations $\rho_g, \rho_{f(F,\psi)}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ associated with g and $f(F, \psi)$ are isomorphic, hence the image of the representation ρ_g is contained in the normalizer of a torus in GL_2.

Proposition 3.5 *Let g be a CM modular eigenform of weight k and level Np^m with N prime to p and $m \geqslant 0$. Then its p-slope is either* 0, $\frac{k-1}{2}$, $k-1$ *or infinite.*

Proof Let F be the quadratic imaginary field and ψ the Grössencharacter of F associated with the CM form g by Definition 3.3. Let $\mathfrak{f}$ be the conductor of ψ.

We assume first that g is p-new, so that $g = f(F, \psi)$. Let a_p be the U_p-eigenvalue of g. If p is inert in F we have $a_p = 0$, so the p-slope of g is infinite. If p splits in F as $\mathfrak{p}\bar{\mathfrak{p}}$, then $a_p = \psi(\mathfrak{p}) + \psi(\bar{\mathfrak{p}})$. We can find an integer n such that $\mathfrak{p}^n$ is a principal ideal (α) with $\alpha \equiv 1 \pmod{^\times \mathfrak{f}}$. Hence $\psi((\alpha)) = \alpha^{k-1}$. Since α is a generator of $\mathfrak{p}^n$ we have $\alpha \in \mathfrak{p}$ and $\alpha \notin \bar{\mathfrak{p}}$; moreover $\alpha^{k-1} = \psi((\alpha)) = \psi(\mathfrak{p})^n$, so we also have $\psi(\mathfrak{p}) \in \mathfrak{p} - \bar{\mathfrak{p}}$. In the same way we find $\psi(\bar{\mathfrak{p}}) \in \bar{\mathfrak{p}} - \mathfrak{p}$. We conclude that $\psi(\mathfrak{p}) + \psi(\bar{\mathfrak{p}})$ does not belong to $\mathfrak{p}$, so its p-adic valuation is 0.

If p ramifies as $\mathfrak{p}^2$ in F, then $a_p = \psi(\mathfrak{p})$. As before we find n such that $\mathfrak{p}^n = (\alpha)$ with $\alpha \equiv 1 \pmod{^\times \mathfrak{f}}$. Then $(\psi(\mathfrak{p}))^n \psi(\mathfrak{p}^n) = \psi((\alpha)) = \alpha^{k-1} = \mathfrak{p}^{n(k-1)}$. By looking at p-adic valuations we find that the slope is $\frac{k-1}{2}$.

If g is not p-new, it is the p-stabilization of a CM form $f(F, \psi)$ of level prime to p. If a_p is the T_p-eigenvalue of $f(F, \psi)$, the U_p-eigenvalue of g is a root of the Hecke polynomial $X^2 - a_pX + \zeta p^{k-1}$ for some root of unity ζ. By our discussion of the p-new case, the valuation of a_p belongs to the set $\left\{0, \frac{k-1}{2}, k-1\right\}$. Then it is easy to see that the valuations of the roots of the Hecke polynomial belong to the same set. □

We state a useful corollary.

Corollary 3.6 *There are no CM families of strictly positive slope.*

Proof We show that the eigencurve C_h contains only a finite number of points corresponding to classical CM forms. It will follow that almost all classical points of a family in C_h are non-CM. Let f be a classical CM form of weight k and positive slope. By Proposition 3.5 its slope is at least $\frac{k-1}{2}$. If f corresponds to a point of C_h its slope must be $\leqslant h$, so we obtain an inequality $\frac{k-1}{2} \leqslant h$. The set of weights $\mathcal{K}$ satisfying this condition is finite. Since the weight map $C_h \to B_h$ is finite, the set of points of C_h whose weight lies in $\mathcal{K}$ is finite. Hence the number of CM forms in C_h is also finite. □

We conclude that, in the finite positive slope case, classical CM forms can appear only as isolated points in an irreducible component of the eigencurve C_h. In the ordinary case, the congruence ideal of a non-CM irreducible component is defined as the intersection ideal of the CM irreducible components with the given non-CM component. In the case of a positive slope family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$, we need to define the congruence ideal in a different way.

3.4 Construction of the Congruence Ideal

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family. We write $\mathbb{I} = \mathbb{I}^\circ[p^{-1}]$.

Fix an imaginary quadratic field F where p is inert or ramified; let $-D$ be its discriminant. Let $\mathfrak{Q}$ be a primary ideal of $\mathbb{I}$; then $\mathfrak{q} = \mathfrak{Q} \cap \Lambda_h$ is a primary ideal of Λ_h. The projection $\Lambda_h \to \Lambda_h/\mathfrak{q}$ defines a point of $\mathcal{B}_h$ (possibly non-reduced) corresponding to a weight $\kappa_{\mathfrak{Q}}\colon \mathbb{Z}_p^* \to (\Lambda_h/\mathfrak{q})^*$. For $r > 0$ we denote by $\mathcal{B}_r$ the ball of centre 1 and radius r in $\mathbb{C}_p$. By [3, Proposition 8.3] there exists $r > 0$ and a character $\kappa_{\mathfrak{Q},r}\colon \mathbb{Z}_p^\times \cdot \mathcal{B}_r \to (\Lambda_h/\mathfrak{q})^\times$ extending $\kappa_{\mathfrak{Q}}$.

Let σ be an embedding $F \hookrightarrow \mathbb{C}_p$. Let r and $\kappa_{\mathfrak{Q},r}$ be as above. For m sufficiently large $\sigma(1 + p^m\mathcal{O}_F)$ is contained in $\mathbb{Z}_p^\times \cdot \mathcal{B}_r$, the domain of definition of $\kappa_{\mathfrak{Q},r}$.

For an ideal $\mathfrak{f} \subset \mathcal{O}_F$ let $I_\mathfrak{f}$ be the group of fractional ideals prime to $\mathfrak{f}$. For every prime ℓ not dividing Np we denote by $a_{\ell,\mathfrak{Q}}$ the image of the Hecke operator T_ℓ in $\mathbb{I}^\circ/\mathfrak{Q}$. We define here a notion of non-classical CM point of θ (hence of the eigencurve C_h) as follows.

Definition 3.7 Let $F, \sigma, \mathfrak{Q}, r, \kappa_{\mathfrak{Q},r}$ be as above. We say that $\mathfrak{Q}$ defines a CM point of weight $\kappa_{\mathfrak{Q},r}$ if there exist an integer $m > 0$, an ideal $\mathfrak{f} \subset \mathcal{O}_F$ with norm $N(\mathfrak{f})$ such that $DN(\mathfrak{f})$ divides N, a quadratic extension $(\mathbb{I}/\mathfrak{Q})'$ of $\mathbb{I}/\mathfrak{Q}$ and a homomorphism $\psi\colon I_{\mathfrak{f}p^m} \to (\mathbb{I}/\mathfrak{Q})'^\times$ such that:

1. $\sigma(1 + p^m\mathcal{O}_F) \subset \mathbb{Z}_p^\times \cdot \mathcal{B}_r$;
2. for every $\alpha \in \mathcal{O}_F$ with $\alpha \equiv 1 \ (\mathrm{mod}^\times \mathfrak{f}p^m)$, $\psi((\alpha)) = \kappa_{\mathfrak{Q},r}(\alpha)\alpha^{-1}$;
3. $a_{\ell,\mathfrak{Q}} = 0$ if L is a prime inert in F and not dividing Np;
4. $a_{\ell,\mathfrak{Q}} = \psi(\mathfrak{l}) + \psi(\bar{\mathfrak{l}})$ if ℓ is a prime splitting as $\mathfrak{l}\bar{\mathfrak{l}}$ in F and not dividing Np.

Note that $\kappa_{\mathfrak{Q},r}(\alpha)$ is well defined thanks to condition 1.

Remark 3.8 If $\mathfrak{P}$ is a prime of $\mathbb{I}$ corresponding to a classical form f then $\mathfrak{P}$ is a CM point if and only if f is a CM form in the sense of Sect. 3.3.

Proposition 3.9 *The set of CM points in* Spec $\mathbb{I}$ *is finite.*

Proof By contradiction assume it is infinite. Then we have an injection $\mathbb{I} \hookrightarrow \prod_{\mathfrak{P}} \mathbb{I}/\mathfrak{P}$ where $\mathfrak{P}$ runs over the set of CM prime ideals of $\mathbb{I}$. One can assume that the imaginary quadratic field of complex multiplication is constant along $\mathbb{I}$. We can also assume that the ramification of the associated Galois characters $\lambda_{\mathfrak{P}}: G_F \to (\mathbb{I}/\mathfrak{P})^\times$ is bounded (in support and in exponents). On the density one set of primes of F prime to $\mathfrak{f}p$ and of degree one, they take values in the image of $\mathbb{I}^\times$ hence they define a continuous Galois character $\lambda: G_F \to \mathbb{I}^\times$ such that $\rho_\theta = \mathrm{Ind}_{G_F}^{G_\mathbb{Q}}\lambda$, which is absurd (by Corallary 3.6 and specialization at non-CM classical points which do exist). $\square$

Definition 3.10 The (fortuitous) congruence ideal $\mathfrak{c}_\theta$ associated with the family θ is defined as the intersection of all the primary ideals of $\mathbb{I}$ corresponding to CM points.

Remark 3.11 (Characterizations of the CM locus)

1. Assume that $\overline{\rho}_\theta = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}\overline{\lambda}$ for a unique imaginary quadratic field K. Then the closed subscheme $V(\mathfrak{c}_\theta) = \mathrm{Spec}\,\mathbb{I}/\mathfrak{c}_\theta \subset \mathrm{Spec}\,\mathbb{I}$ is the largest subscheme on which there is an isomorphism of Galois representations $\rho_\theta \cong \rho_\theta \otimes \left(\frac{K/\mathbb{Q}}{\bullet}\right)$. Indeed, for every artinian $\mathbb{Q}_p$-algebra A, a CM point $x: \mathbb{I} \to A$ is characterized by the conditions $x(T_\ell) = x(T_\ell)\left(\frac{K/\mathbb{Q}}{\ell}\right)$ for all primes ℓ not dividing Np.
2. Note that N is divisible by the discriminant D of K. Assume that $\mathbb{I}$ is N-new and that D is prime to N/D. Let W_D be the Atkin-Lehner involution associated with D. Conjugation by W_D defines an automorphism ι_D of $\mathbb{T}_h$ and of $\mathbb{I}$. Then $V(\mathfrak{c}_\theta)$ coincides with the (schematic) invariant locus $(\mathrm{Spec}\,\mathbb{I})^{\iota_D=1}$.

4 The Image of the Representation Associated with a Finite Slope Family

It is shown by Lang in [12, Theorem 2.4] that, under some technical hypotheses, the image of the Galois representation $\rho: G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated with a non-CM ordinary family $\theta: \mathbb{T} \to \mathbb{I}^\circ$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$, where $\mathbb{I}_0^\circ$ is the subring of $\mathbb{I}^\circ$ fixed by certain "symmetries" of the representation ρ. In order to study the Galois representation associated with a non-ordinary family we will adapt some of the results in [12] to this situation. Since the crucial step ([12, Theorem 4.3]) requires the Galois ordinarity of the representation (as in [9, Lemma 2.9]), the results of this section will not imply the existence of a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ contained in the image of ρ. However, we will prove in later sections the existence of a "congruence Lie subalgebra" of $\mathfrak{sl}_2(\mathbb{I}_0^\circ)$ contained in a suitably defined Lie algebra of the image of ρ by means of relative Sen theory.

For every ring R we denote by $Q(R)$ its total ring of fractions.

4.1 The Group of Self-twists of a Family

We follow [12, Sect. 2] in this subsection. Let $h \geqslant 0$ and $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family of slope $\leqslant h$ defined over a finite torsion free Λ_h-algebra $\mathbb{I}^\circ$. Recall that there is a natural map $\Lambda \to \Lambda_h$ with image $\mathbb{Z}_p[[\eta t]]$.

Definition 4.1 We say that $\sigma \in \mathrm{Aut}_{Q(\mathbb{Z}_p[[\eta t]])}(Q(\mathbb{I}^\circ))$ is a conjugate self-twist for θ if there exists a Dirichlet character $\eta_\sigma\colon G_\mathbb{Q} \to \mathbb{I}^{\circ,\times}$ such that

$$\sigma(\theta(T_\ell)) = \eta_\sigma(\ell)\theta(T_\ell)$$

for all but finitely many primes ℓ.

Any such σ acts on $\Lambda_h = \mathcal{O}_h[[t]]$ by restriction, trivially on t and by a Galois automorphism on $\mathcal{O}_h$. The conjugates self-twists for θ form a subgroup of $\mathrm{Aut}_{Q(\mathbb{Z}_p[[\eta t]])}(Q(\mathbb{I}^\circ))$. We recall the following result which holds without assuming the ordinarity of θ.

Lemma 4.2 [12, Lemma 7.1] *Γ is a finite abelian 2-group.*

We suppose from now on that $\mathbb{I}^\circ$ is normal. The only reason for this hypothesis is that in this case $\mathbb{I}^\circ$ is stable under the action of Γ on $Q(\mathbb{I}^\circ)$, which is not true in general. This makes it possible to define the subring $\mathbb{I}_0^\circ$ of elements of $\mathbb{I}^\circ$ fixed by Γ.

Remark 4.3 The hypothesis of normality of $\mathbb{I}^\circ$ is just a simplifying one. We could work without it by introducing the Λ_h-order $\mathbb{I}^{\circ,\prime} = \Lambda_h[\theta(T_\ell), \ell \nmid Np] \subset \mathbb{I}^\circ$: this is an analogue of the Λ-order $\mathbb{I}'$ defined in [12, Sect. 2] and it is stable under the action of Γ. We would define $\mathbb{I}_0^\circ$ as the fixed subring of $\mathbb{I}^{\circ,\prime}$ and the arguments in the rest of the article could be adapted to this setting.

The subring of Λ_h fixed by Γ is an $\mathcal{O}_{h,0}$ form of Λ_h for some subring $\mathcal{O}_{h,0}$ of $\mathcal{O}_h$. We denote it by $\Lambda_{h,0}$ the field of fractions of $\mathcal{O}_{h,0}$.

Remark 4.4 By definition Γ fixes $\mathbb{Z}_p[[\eta t]]$, so we have $\mathbb{Z}_p[[\eta t]] \subset \Lambda_{h,0}$. In particular it makes sense to speak about the ideal $P_k\Lambda_{h,0}$ for every arithmetic prime $P_k = (1 + \eta t - u^k) \subset \mathbb{Z}_p[[\eta t]]$. Note that $P_k\Lambda_h$ defines a prime ideal of Λ_h if and only if the weight k belongs to the open disc B_h, otherwise $P_k\Lambda_h = \Lambda_h$. We see immediately that the same statement is true if we replace Λ_h by $\Lambda_{h,0}$.

Note that $\mathbb{I}_0^\circ$ is a finite extension of $\Lambda_{h,0}$ because $\mathbb{I}^\circ$ is a finite Λ_h-algebra. Moreover, we have $K_h^\Gamma = K_{h,0}$ (although the inclusion $\Lambda_h \cdot \mathbb{I}_0^\circ \subset \mathbb{I}^\circ$ may not be an equality).

We define two open normal subgroups of $G_\mathbb{Q}$ by:

- $H_0 = \bigcap_{\sigma\in\Gamma} \ker \eta_\sigma$;
- $H = H_0 \cap \ker(\det \overline{\rho})$.

Note that H_0 is an open normal subgroup of $G_\mathbb{Q}$ and that H is a n open normal subgroup of H_0 and $G_\mathbb{Q}$.

4.2 The Level of a General Ordinary Family

We recall the main result of [12]. Denote by $\mathbb{T}$ the big ordinary Hecke algebra, which is finite over $\Lambda = \mathbb{Z}_p[[T]]$. Let $\theta\colon \mathbb{T} \to \mathbb{I}^\circ$ be an ordinary family with associated Galois representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$. The representation ρ is p-ordinary, which means that its restriction $\rho|_{D_p}$ to a decomposition subgroup $D_p \subset G_\mathbb{Q}$ is reducible. There exist two characters $\varepsilon, \delta\colon D_p \to \mathbb{I}^{\circ,\times}$, with δ unramified, such that $\rho|_{D_p}$ is an extension of ε by δ.

Denote by $\mathbb{F}$ the residue field of $\mathbb{I}^\circ$ and by $\overline{\rho}$ the representation $G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ obtained by reducing ρ modulo the maximal ideal of $\mathbb{I}^\circ$. Lang introduces the following technical condition.

Definition 4.5 The p-ordinary representation $\overline{\rho}$ is called H_0-regular if $\overline{\varepsilon}|_{D_p\cap H_0} \neq \overline{\delta}|_{D_p\cap H_0}$.

The following result states the existence of a Galois level for ρ.

Theorem 4.6 [12, Theorem 2.4] *Let $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ be the representation associated with an ordinary, non-CM family $\theta\colon \mathbb{T} \to \mathbb{I}^\circ$. Assume that $p > 2$, the cardinality of $\mathbb{F}$ is not 3 and the residual representation $\overline{\rho}$ is absolutely irreducible and H_0-regular. Then there exists $\gamma \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that $\gamma \cdot \mathrm{Im}\,\rho \cdot \gamma^{-1}$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$.*

The proof relies on the analogous result proved by Ribet [15] and Momose [13] for the p-adic representation associated with a classical modular form.

4.3 An Approximation Lemma

In this subsection we prove an analogue of [10, Lemma 4.5]. It replaces in our approach the use of Pink's Lie algebra theory, which is relied upon in the case of ordinary representations in [9, 12]. Let $\mathbb{I}_0^\circ$ be a local domain that is finite torsion free over Λ_h. It does not need to be related to a Hecke algebra for the moment.

Let N be an open normal subgroup of $G_\mathbb{Q}$ and let $\rho\colon N \to \mathrm{GL}_2(\mathbb{I}_0^\circ)$ be an arbitrary continuous representation. We denote by $\mathfrak{m}_{\mathbb{I}_0^\circ}$ the maximal ideal of $\mathbb{I}_0^\circ$ and by $\mathbb{F} = \mathbb{I}_0^\circ/\mathfrak{m}_{\mathbb{I}_0^\circ}$ its residue field of cardinality q. In the lemma we do not suppose that ρ comes from a family of modular forms. We will only assume that it satisfies the following technical condition:

Definition 4.7 Keep notations as above. We say that the representation $\rho\colon N \to \mathrm{GL}_2(\mathbb{I}_0^\circ)$ is $\mathbb{Z}_p$-regular if there exists $d \in \mathrm{Im}\,\rho$ with eigenvalues $d_1, d_2 \in \mathbb{Z}_p$ such that $d_1^2 \not\equiv d_2^2 \pmod p$. We call d a $\mathbb{Z}_p$-regular element. If N' is an open normal subgroup of N then we say that ρ is $(N', \mathbb{Z}_p)$-regular if $\rho|_{N'}$ is $\mathbb{Z}_p$-regular.

Let $B^\pm$ denote the Borel subgroups consisting of upper, respectively lower, triangular matrices in GL_2. Let $U^\pm$ be the unipotent radical of $B^\pm$.

Proposition 4.8 *Let $\mathbb{I}_0^\circ$ be a finite torsion free $\Lambda_{h,0}$-algebra, N an open normal subgroup of $G_\mathbb{Q}$ and $\rho\colon N \to \mathrm{GL}_2(\mathbb{I}_0^\circ)$ a continuous representation that is $\mathbb{Z}_p$-regular. Suppose (upon replacing ρ by a conjugate) that a $\mathbb{Z}_p$-regular element is diagonal. Let $\mathbf{P}$ be an ideal of $\mathbb{I}_0^\circ$ and $\rho_\mathbf{P}\colon N \to \mathrm{GL}_2(\mathbb{I}_0^\circ/\mathbf{P})$ be the representation given by the reduction of ρ modulo $\mathbf{P}$. Let $U^\pm(\rho)$, and $U^\pm(\rho_\mathbf{P})$ be the upper and lower unipotent subgroups of* $\mathrm{Im}\,\rho$*, and* $\mathrm{Im}\,\rho_\mathbf{P}$*, respectively. Then the natural maps $U^+(\rho) \to U^+(\rho_\mathbf{P})$ and $U^-(\rho) \to U^-(\rho_\mathbf{P})$ are surjective.*

Remark 4.9 The ideal $\mathbf{P}$ in the proposition is not necessarily prime. At a certain point we will need to take $\mathbf{P} = P\mathbb{I}_0^\circ$ for a prime ideal P of $\Lambda_{h,0}$.

As in [10, Lemma 4.5] we need two lemmas. Since the argument is the same for U^+ and U^-, we will only treat here the upper triangular case $U = U^+$ and $B = B^+$.

For $* = U, B$ and every $j \geqslant 1$ we define the groups

$$\Gamma_*(\mathbf{P}^j) = \{x \in \mathrm{SL}_2(\mathbb{I}_0^\circ) \mid x \pmod{\mathbf{P}^j} \in *(\mathbb{I}_0^\circ/\mathbf{P}^j)\}.$$

Let $\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^j)$ be the kernel of the reduction morphism $\pi_j\colon \mathrm{SL}_2(\mathbb{I}_0^\circ) \to \mathrm{SL}_2(\mathbb{I}_0^\circ/\mathbf{P}^j)$. Note that $\Gamma_U(\mathbf{P}^j) = \Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^j)U(\mathbb{I}_0^\circ)$ consists of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, d \equiv 1 \pmod{\mathbf{P}^j}$, $c \equiv 0 \pmod{\mathbf{P}^j}$. Let $K = \mathrm{Im}\,\rho$ and

$$K_U(\mathbf{P}^j) = K \cap \Gamma_U(\mathbf{P}^j), \quad K_B(\mathbf{P}^j) = K \cap \Gamma_B(\mathbf{P}^j).$$

Since $U(\mathbb{I}_0^\circ)$ and $\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P})$ are p-profinite, the groups $\Gamma_U(\mathbf{P}^j)$ and $K_U(\mathbf{P}^j)$ for all $j \geqslant 1$ are also p-profinite. Note that

$$\left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} e & f \\ g & -e \end{pmatrix}\right] = \begin{pmatrix} bg-cf & 2(af-be) \\ 2(ce-ag) & cf-bg \end{pmatrix}.$$

From this we obtain the following.

Lemma 4.10 *If $X, Y \in \mathfrak{sl}_2(\mathbb{I}_0^\circ) \cap \begin{pmatrix} \mathbf{P}^j & \mathbf{P}^k \\ \mathbf{P}^i & \mathbf{P}^j \end{pmatrix}$ with $i \geqslant j \geqslant k$, then $[X, Y] \in \begin{pmatrix} \mathbf{P}^{i+k} & \mathbf{P}^{j+k} \\ \mathbf{P}^{i+j} & \mathbf{P}^{i+k} \end{pmatrix}$.*

We denote by $\mathrm{D}\Gamma_U(\mathbf{P}^j)$ the topological commutator subgroup $(\Gamma_U(\mathbf{P}^j), \Gamma_U(\mathbf{P}^j))$. Lemma 4.10 tells us that

$$\mathrm{D}\Gamma_U(\mathbf{P}^j) \subset \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j). \tag{1}$$

By the $\mathbb{Z}_p$-regularity assumption, there exists a diagonal element $d \in K$ with eigenvalues in $\mathbb{Z}_p$ and distinct modulo p. Consider the element $\delta = \lim_{n\to\infty} d^{p^n}$, which belongs to K since this is p-adically complete. In particular δ normalizes K. It is also diagonal with coefficients in $\mathbb{Z}_p$, so it normalizes $K_U(\mathbf{P}^j)$ and $\Gamma_B(\mathbf{P}^j)$. Since $\delta^p = \delta$, the eigenvalues δ_1 and δ_2 of δ are roots of unity of order dividing $p-1$. They still satisfy $\delta_1^2 \neq \delta_2^2$ as $p \neq 2$.

Set $\alpha = \delta_1/\delta_2 \in \mathbb{F}_p^\times$ and let a be the order of α as a root of unity. We see α as an element of $\mathbb{Z}_p^\times$ via the Teichmüller lift. Let H be a p-profinite group normalized by δ. Since H is p-profinite, every $x \in H$ has a unique a-th root. We define a map $\Delta\colon H \to H$ given by

$$\Delta(x) = [x \cdot \mathrm{ad}(\delta)(x)^{\alpha^{-1}} \cdot \mathrm{ad}(\delta^2)(x)^{\alpha^{-2}} \cdots \mathrm{ad}(\delta^{a-1})(x)^{\alpha^{1-a}}]^{1/a}$$

Lemma 4.11 *If $u \in \Gamma_U(\mathbf{P}^j)$ for some $j \geqslant 1$, then $\Delta^2(u) \in \Gamma_U(\mathbf{P}^{2j})$ and $\pi_j(\Delta(u)) = \pi_j(u)$.*

Proof If $u \in \Gamma_U(\mathbf{P}^j)$, we have $\pi_j(\Delta(u)) = \pi_j(u)$ as Δ is the identity map on $U(\mathbb{I}_0^\circ/\mathbf{P}^j)$. Let $\mathrm{D}\Gamma_U(\mathbf{P}^j)$ be the topological commutator subgroup of $\Gamma_U(\mathbf{P}^j)$. Since Δ induces the projection of the $\mathbb{Z}_p$-module $\Gamma_U(\mathbf{P}^j)/\mathrm{D}\Gamma_U(\mathbf{P}^j)$ onto its α-eigenspace for $\mathrm{ad}(d)$, it is a projection onto $U(\mathbb{I}_0^\circ)\mathrm{D}\Gamma_U(\mathbf{P}^j)/\mathrm{D}\Gamma_U(\mathbf{P}^j)$. The fact that this is exactly the α-eigenspace comes from the Iwahori decomposition of $\Gamma_U(\mathbf{P}^j)$, hence a similar direct sum decomposition holds in the abelianization $\Gamma_U(\mathbf{P}^j)/\mathrm{D}\Gamma_U(\mathbf{P}^j)$.

By (1), we have $\mathrm{D}\Gamma_U(\mathbf{P}^j) \subset \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j)$. Since the α-eigenspace of $\Gamma_U(\mathbf{P}^j)/\mathrm{D}\Gamma_U(\mathbf{P}^j)$ is inside $\Gamma_B(\mathbf{P}^{2j})$, Δ projects $u\Gamma_U(\mathbf{P}^j)$ to

$$\overline{\Delta}(u) \in (\Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j))/\mathrm{D}\Gamma_U(\mathbf{P}^j).$$

In particular, $\Delta(u) \in \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j)$. Again apply Δ. Since $\Gamma_B(\mathbf{P}^{2j})/\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^{2j})$ is sent to $\Gamma_U(\mathbf{P}^{2j})/\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^{2j})$ by Δ, we get $\Delta^2(u) \in \Gamma_U(\mathbf{P}^{2j})$ as desired. □

Proof We can now prove Proposition 4.8. Let $\overline{u} \in U(\mathbb{I}_0^\circ/\mathbf{P}) \cap \mathrm{Im}(\rho_\mathbf{P})$. Since the reduction map $\mathrm{Im}(\rho) \to \mathrm{Im}(\rho_\mathbf{P})$ induced by π_1 is surjective, there exists $v \in \mathrm{Im}(\rho)$ such that $\pi_1(v) = \overline{u}$. Take $u_1 \in U(\mathbb{I}_0^\circ)$ such that $\pi_1(u_1) = \overline{u}$ (this is possible since $\pi_1\colon U(\Lambda_h) \to U(\Lambda_h/P)$ is surjective). Then $vu_1^{-1} \in \Gamma_{\mathbb{I}_0^\circ}(\mathbf{P})$, so $v \in K_U(\mathbf{P})$.

By compactness of $K_U(\mathbf{P})$ and by Lemma 4.11, starting with v as above, we see that $\lim_{m\to\infty} \Delta^m(v)$ converges $\mathbf{P}$-adically to $\Delta^\infty(v) \in U(\mathbb{I}_0^\circ) \cap K$ with $\pi_1(\Delta^\infty(v)) = \overline{u}$. □

Remark 4.12 Proposition 4.8 is true with the same proof if we replace $\Lambda_{h,0}$ by Λ_h and $\mathbb{I}_0^\circ$ by a finite torsion free Λ_h-algebra.

As a first application of Proposition 4.8 we give a result that we will need in the next subsection. Given a representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ and every ideal $\mathbf{P}$ of $\mathbb{I}^\circ$ we define $\rho_\mathbf{P}$, $U^\pm(\rho)$ and $U^\pm(\rho_\mathbf{P})$ as above, by replacing $\mathbb{I}_0^\circ$ by $\mathbb{I}^\circ$.

Proposition 4.13 *Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family of slope $\leqslant h$ and $\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ be the representation associated with θ. Suppose that ρ_θ is $(H_0, \mathbb{Z}_p)$-regular and let*

ρ be a conjugate of ρ_θ such that $\mathrm{Im}\,\rho|_{H_0}$ *contains a diagonal $\mathbb{Z}_p$-regular element. Then $U^+(\rho)$ and $U^-(\rho)$ are both nontrivial.*

Proof By density of classical points in $\mathbb{T}_h$ we can choose a prime ideal $\mathbf{P} \subset \mathbb{I}^\circ$ corresponding to a classical modular form f. The modulo $\mathbf{P}$ representation $\rho_\mathbf{P}$ is the p-adic representation classically associated with f. By the results of [13, 15] and the hypothesis of $(H_0, \mathbb{Z}_p)$-regularity of L, there exists an ideal $\mathfrak{l}_\mathbf{P}$ of $\mathbb{Z}_p$ such that $\mathrm{Im}\,\rho_\mathbf{P}$ contains the congruence subgroup $\Gamma_{\mathbb{Z}_p}(\mathfrak{l}_\mathbf{P})$. In particular $U^+(\rho_\mathbf{P})$ and $U^-(\rho_\mathbf{P})$ are both nontrivial. Since the maps $U^+(\rho) \to U^+(\rho_\mathbf{P})$ and $U^-(\rho) \to U^-(\rho_\mathbf{P})$ are surjective we find nontrivial elements in $U^+(\rho)$ and $U^-(\rho)$. □

We adapt the work in [12, Sect. 7] to show the following.

Proposition 4.14 *Suppose that the representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ is $(H_0, \mathbb{Z}_p)$-regular. Then there exists $g \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that the conjugate representation $g\rho g^{-1}$ satisfies the following two properties:*

1. *the image of $g\rho g^{-1}|_{H_0}$ is contained in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$;*
2. *the image of $g\rho g^{-1}|_{H_0}$ contains a diagonal $\mathbb{Z}_p$-regular element.*

Proof As usual we choose a $\mathrm{GL}_2(\mathbb{I}^\circ)$-conjugate of ρ such that a $\mathbb{Z}_p$-regular element d is diagonal. We still write ρ for this conjugate representation and we show that it also has property (1).

Recall that for every $\sigma \in \Gamma$ there is a character $\eta_\sigma\colon G_\mathbb{Q} \to (\mathbb{I}^\circ)^\times$ and an equivalence $\rho^\sigma \cong \rho \otimes \eta_\sigma$. Then for every $\sigma \in \Gamma$ there exists $\mathbf{t}_\sigma \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that, for all $g \in G_\mathbb{Q}$,

$$\rho^\sigma(g) = \mathbf{t}_\sigma \eta_\sigma(g)\rho(g)\mathbf{t}_\sigma^{-1}. \tag{2}$$

We prove that the matrices $\mathbf{t}_\sigma$ are diagonal. Let $\rho(t)$ be a non-scalar diagonal element in $\mathrm{Im}\,\rho$ (for example d). Evaluating (2) at $g = t$ we find that $\mathbf{t}_\sigma$ must be either a diagonal or an antidiagonal matrix. Now by Proposition 4.13 there exists a nontrivial element $\rho(u^+) \in \mathrm{Im}\,\rho \cap U^+(\mathbb{I}^\circ)$. Evaluating (2) at $g = u^+$ we find that $\mathbf{t}_\sigma$ cannot be antidiagonal.

It is shown in [12, Lemma 7.3] that there exists an extension A of $\mathbb{I}^\circ$, at most quadratic, and a function $\zeta\colon \Gamma \to A^\times$ such that $\sigma \to \mathbf{t}_\sigma\zeta(\sigma)^{-1}$ defines a cocycle with values in $\mathrm{GL}_2(A)$. The proof of this result does not require the ordinarity of ρ. Equation (2) remains true if we replace $\mathbf{t}_\sigma$ with $\mathbf{t}_\sigma\zeta(\sigma)^{-1}$, so we can and do suppose from now on that $\mathbf{t}_\sigma$ is a cocycle with values in $\mathrm{GL}_2(A)$. In the rest of the proof we assume for simplicity that $A = \mathbb{I}^\circ$, but everything works in the same way if A is a quadratic extension of $\mathbb{I}^\circ$ and $\mathbb{F}$ is the residue field of A.

Let $V = (\mathbb{I}^\circ)^2$ be the space on which $G_\mathbb{Q}$ acts via ρ. As in [12, Sect. 7] we use the cocycle $\mathbf{t}_\sigma$ to define a twisted action of Γ on $(\mathbb{I}^\circ)^2$. For $v = (v_1, v_2) \in V$ we denote by v^σ the vector (v_1^σ, v_2^σ). We write $v^{[\sigma]}$ for the vector $\mathbf{t}_\sigma^{-1}v^\sigma$. Then $v \to v^{[\sigma]}$ gives an action of Γ since $\sigma \mapsto \mathbf{t}_\sigma$ is a cocycle. Note that this action is $\mathbb{I}_0^\circ$-linear.

Since $\mathbf{t}_\sigma$ is diagonal for every $\sigma \in \Gamma$, the submodules $V_1 = \mathbb{I}^\circ(1,0)$ and $V_2 = \mathbb{I}^\circ(0,1)$ are stable under the action of Γ. We show that each V_i contains an element fixed by Γ. We denote by $\mathbb{F}$ the residue field $\mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}}^\circ$. Note that the action of Γ on V_i induces an action of Γ on the one-dimensional $\mathbb{F}$-vector space $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$. We show that for each i the space $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$ contains a nonzero element $\overline{v}_i$ fixed by Γ. This is a consequence of the following argument, a form of which appeared in an early preprint of [12]. Let w be any nonzero element of $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$ and let a be a variable in $\mathbb{F}$. The sum

$$S_{aw} = \sum_{\sigma\in\Gamma} (aw)^{[\sigma]}$$

is clearly Γ-invariant. We show that we can choose a such that $S_{aw} \neq 0$. Since $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$ is one-dimensional, for every $\sigma \in \Gamma$ there exists $\alpha_\sigma \in \mathbb{F}$ such that $w^{[\sigma]} = \alpha_\sigma w$. Then

$$S_{aw} = \sum_{\sigma\in\Gamma} (aw)^{[\sigma]} = \sum_{\sigma\in\Gamma} a^\sigma w^{[\sigma]} = \sum_{\sigma\in\Gamma} a^\sigma \alpha_\sigma w = \left(\sum_{\sigma\in\Gamma} a^\sigma \alpha_\sigma a^{-1}\right) aw.$$

By Artin's lemma on the independence of characters, the function $f(a) = \sum_{\sigma\in\Gamma} a^\sigma \alpha_\sigma a^{-1}$ cannot be identically zero on $\mathbb{F}$. By choosing a value of a such that $f(a) \neq 0$ we obtain a nonzero element $\overline{v}_i = S_{aw}$ fixed by Γ.

We show that $\overline{v}_i$ lifts to an element $v_i \in V_i$ fixed by Γ. Let $\sigma_0 \in \Gamma$. By Lemma 4.2 Γ is a finite abelian 2-group, so the minimal polynomial $P_m(X)$ of $[\sigma_0]$ acting on V_i divides $X^{2^k} - 1$ for some integer k. In particular the factor $X - 1$ appears with multiplicity at most 1. We show that its multiplicity is exactly 1. If $\overline{P_m}$ is the reduction of P_m modulo $\mathfrak{m}_{\mathbb{I}^\circ}$ then $\overline{P_m}([\sigma_0]) = 0$ on $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$. By our previous argument there is an element of $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$ fixed by Γ (hence by $[\sigma_0]$) so we have $(X-1) \mid \overline{P_m(X)}$. Since $p > 2$ the polynomial $X^{2^k} - 1$ has no double roots modulo $\mathfrak{m}_{\mathbb{I}^\circ}$, so neither does $\overline{P_m}$. By Hensel's lemma the factor $X - 1$ lifts to a factor $X - 1$ in P_m and $\overline{v}_i$ lifts to an element $v_i \in V_i$ fixed by $[\sigma_0]$. Note that $\mathbb{I}^\circ \cdot v_i = V_i$ by Nakayama's lemma since $\overline{v}_i \neq 0$.

We show that v_i is fixed by all of Γ. Let $W_{[\sigma_0]} = \mathbb{I}^\circ v_i$ be the one-dimensional eigenspace for $[\sigma_0]$ in V_i. Since Γ is abelian $W_{[\sigma_0]}$ is stable under Γ. Let $\sigma \in \Gamma$. Since σ has order 2^k in Γ for some $k \geqslant 0$ and $v_i^{[\sigma]} \in W_{[\sigma_0]}$, there exists a root of unity ζ_σ of order 2^k satisfying $v_i^{[\sigma]} = \zeta_\sigma v_i$. Since $\overline{v}_i^{[\sigma]} = \overline{v}_i$, the reduction of ζ_σ modulo $\mathfrak{m}_{\mathbb{I}^\circ}$ must be 1. As before we conclude that $\zeta_\sigma = 1$ since $p \neq 2$.

We found two elements $v_1 \in V_1$, $v_2 \in V_2$ fixed by Γ. We show that every element of $v \in V$ fixed by Γ must belong to the $\mathbb{I}_0^\circ$-submodule generated by v_1 and v_2. We proceed as in the end of the proof of [12, Theorem 7.5]. Since V_1 and V_2 are Γ-stable we must have $v \in V_1$ or $v \in V_2$. Suppose without loss of generality that $v \in V_1$. Then $v = \alpha v_1$ for some $\alpha \in \mathbb{I}^\circ$. If $\alpha \in \mathbb{I}_0^\circ$ then $v \in \mathbb{I}_0^\circ v_1$, as desired. If $\alpha \notin \mathbb{I}_0^\circ$ then there exists $\sigma \in \Gamma$ such that $\alpha^\sigma \neq \alpha$. Since v_1 is $[\sigma]$-invariant we obtain $(\alpha v_1)^{[\sigma]} = \alpha^\sigma v_1^{[\sigma]} = \alpha^\sigma v_1 \neq \alpha v_1$, so αv_1 is not fixed by $[\sigma]$, a contradiction.

Now (v_1, v_2) is a basis for V over $\mathbb{I}^\circ$, so the $\mathbb{I}_0^\circ$ submodule $V_0 = \mathbb{I}_0^\circ v_1 + \mathbb{I}_0^\circ v_2$ is an $\mathbb{I}_0^\circ$-lattice in V. Recall that $H_0 = \bigcap_{\sigma\in\Gamma} \ker \eta_\sigma$. We show that V_0 is stable under the action of H_0 via $\rho|_{H_0}$, i.e. that if $v \in V$ is fixed by Γ, so is $\rho(h)v$ for every $h \in H_0$. This is a consequence of the following computation, where v and h are as before and $\sigma \in \Gamma$:

$$(\rho(h)v)^{[\sigma]} = \mathbf{t}_\sigma^{-1}\rho(h)^\sigma v^\sigma = \mathbf{t}_\sigma^{-1}\eta_\sigma(h)\rho(h)^\sigma v^\sigma = \mathbf{t}_\sigma^{-1}\mathbf{t}_\sigma\rho(h)\mathbf{t}_\sigma^{-1}v^\sigma = \rho(h)v^{[\sigma]}.$$

Since V_0 is an $\mathbb{I}_0^\circ$-lattice in V stable under $\rho|_{H_0}$, we conclude that $\mathrm{Im}\,\rho|_{H_0} \subset \mathrm{GL}_2(\mathbb{I}_0^\circ)$. □

4.4 Fullness of the Unipotent Subgroups

From now on we write ρ for the element in its $\mathrm{GL}_2(\mathbb{I}^\circ)$ conjugacy class such that $\rho|_{H_0} \in \mathrm{GL}_2(\mathbb{I}_0^\circ)$. Recall that H is the open subgroup of H_0 defined by the condition $\det\overline{\rho}(h) = 1$ for every $h \in H$. As in [12, Sect. 4] we define a representation $H \to \mathrm{SL}_2(\mathbb{I}_0^\circ)$ by

$$\rho_0 = \rho|_H \otimes (\det\rho|_H)^{-\frac{1}{2}}\,.$$

We can take the square root of the determinant thanks to the definition of H. We will use the results of [12] to deduce that the $\Lambda_{h,0}$-module generated by the unipotent subgroups of the image of ρ_0 is big. We will later deduce the same for ρ.

We fix from now on a height one prime $P \subset \Lambda_{h,0}$ with the following properties:

1. there is an arithmetic prime $P_k \subset \mathbb{Z}_p[[\eta t]]$ satisfying $k > h+1$ and $P = P_k\Lambda_{h,0}$;
2. every prime $\mathfrak{P} \subset \mathbb{I}^\circ$ lying above P corresponds to a non-CM point.

Such a prime always exists. Indeed, by Remark 4.4 every classical weight $k > h+1$ contained in the disc B_h defines a prime $P = P_k\Lambda_{h,0}$ satisfying (1), so such primes are Zariski-dense in $\Lambda_{h,0}$, while the set of CM primes in $\mathbb{I}^\circ$ is finite by Proposition 3.9.

Remark 4.15 Since $k > h+1$, every point of $\mathrm{Spec}\,\mathbb{T}_h$ above P_k is classical by [5, Theorem 6.1]. Moreover the weight map is étale at every such point by [11, Theorem 11.10]. In particular the prime $P\mathbb{I}_0^\circ = P_k\mathbb{I}_0^\circ$ splits as a product of distinct primes of $\mathbb{I}_0^\circ$.

Make the technical assumption that the order of the residue field $\mathbb{F}$ of $\mathbb{I}^\circ$ is not 3. For every ideal $\mathbf{P}$ of $\mathbb{I}_0^\circ$ over P we let $\pi_\mathbf{P}$ be the projection $\mathrm{SL}_2(\mathbb{I}_0^\circ) \to \mathrm{SL}_2(\mathbb{I}_0^\circ/\mathbf{P})$. We still denote by $\pi_\mathbf{P}$ the restricted maps $U^\pm(\mathbb{I}_0^\circ) \to U^\pm(\mathbb{I}_0^\circ/\mathbf{P})$.

Let $G = \mathrm{Im}\,\rho_0$. For every ideal $\mathbf{P}$ of $\mathbb{I}_0^\circ$ we denote by $\rho_{0,\mathbf{P}}$ the representation $\pi_\mathbf{P}(\rho_0)$ and by $G_\mathbf{P}$ the image of $\rho_\mathbf{P}$, so that $G_\mathbf{P} = \pi_\mathbf{P}(G)$. We state two results from Lang's work that come over unchanged to the non-ordinary setting.

Proposition 4.16 [12, Corollary 6.3] *Let $\mathfrak{P}$ be a prime of $\mathbb{I}_0^\circ$ over P. Then $G_\mathfrak{P}$ contains a congruence subgroup $\Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}) \subset \mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$. In particular $G_\mathfrak{P}$ is open in* $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$.

Proposition 4.17 [12, Proposition 5.1] *Assume that for every prime $\mathfrak{P} \subset \mathbb{I}_0^\circ$ over P the subgroup $G_\mathfrak{P}$ is open in $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$. Then the image of G in $\prod_{\mathfrak{P}|P} \mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$ through the map $\prod_{\mathfrak{P}|P} \pi_\mathfrak{P}$ contains a product of congruence subgroups $\prod_{\mathfrak{P}|P} \Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}_\mathfrak{P})$.*

Remark 4.18 The proofs of Propositions 4.16 and 4.17 rely on the fact that the big ordinary Hecke algebra is étale over Λ at every arithmetic point. In order for these proofs to adapt to the non-ordinary setting it is essential that the prime P satisfies the properties above Remark 4.15.

We let $U^\pm(\rho_0) = G \cap U^\pm(\mathbb{I}_0^\circ)$ and $U^\pm(\rho_\mathbf{P}) = G_\mathbf{P} \cap U^\pm(\mathbb{I}_0^\circ/\mathbf{P})$. We denote by $U(\rho_\mathbf{P})$ either the upper or lower unipotent subgroups of $G_\mathbf{P}$ (the choice will be fixed throughout the proof). By projecting to the upper right element we identify $U^+(\rho_0)$ with a $\mathbb{Z}_p$-submodule of $\mathbb{I}_0^\circ$ and $U^+(\rho_{0,\mathbf{P}})$ with a $\mathbb{Z}_p$-submodule of $\mathbb{I}_0^\circ/\mathbf{P}$. We make analogous identifications for the lower unipotent subgroups. We will use Propositions 4.17 and 4.8 to show that, for both signs, $U^\pm(\rho)$ spans $\mathbb{I}_0^\circ$ over $\Lambda_{h,0}$.

First we state a version of [12, Lemma 4.10], with the same proof. Let A and B be Noetherian rings with B integral over A. We call *A-lattice* an A-submodule of B generated by the elements of a basis of $Q(B)$ over $Q(A)$.

Lemma 4.19 *Any A-lattice in B contains a nonzero ideal of B. Conversely, every nonzero ideal of B contains an A-lattice.*

We prove the following proposition by means of Proposition 4.8. We could also use Pink theory as in [12, Sect. 4].

Proposition 4.20 *Consider $U^\pm(\rho_0)$ as subsets of $Q(\mathbb{I}_0^\circ)$. For each choice of sign the $Q(\Lambda_{h,0})$-span of $U^\pm(\rho_0)$ is $Q(\mathbb{I}_0^\circ)$. Equivalently the $\Lambda_{h,0}$-span of $U^\pm(\rho_0)$ contains a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$.*

Proof Keep notations as above. We omit the sign when writing unipotent subgroups and we refer to either the upper or lower ones (the choice is fixed throughout the proof). Let P be the prime of $\Lambda_{h,0}$ chosen above. By Remark 4.15 the ideal $P\mathbb{I}_0^\circ$ splits as a product of distinct primes in $\mathbb{I}_0^\circ$. When $\mathfrak{P}$ varies among these primes, the map $\bigoplus_{\mathfrak{P}|P} \pi_\mathfrak{P}$ gives embeddings of $\Lambda_{h,0}/P$-modules $\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ \hookrightarrow \bigoplus_{\mathfrak{P}|P} \mathbb{I}_0^\circ/\mathfrak{P}$ and $U(\rho_{P\mathbb{I}_0^\circ}) \hookrightarrow \bigoplus_{\mathfrak{P}|P} U(\rho_\mathfrak{P})$. The following diagram commutes:

$$\begin{array}{ccc} U(\rho_{P\mathbb{I}_0^\circ}) & \xhookrightarrow{\bigoplus_{\mathfrak{P}|P}\pi_\mathfrak{P}} & \bigoplus_{\mathfrak{P}|P} U(\rho_\mathfrak{P}) \\ \big\downarrow & & \big\downarrow \\ \mathbb{I}_0^\circ/P\mathbb{I}_0^\circ & \xhookrightarrow{\bigoplus_{\mathfrak{P}|P}\pi_\mathfrak{P}} & \bigoplus_{\mathfrak{P}|P} \mathbb{I}_0^\circ/\mathfrak{P} \end{array} \tag{3}$$

By Proposition 4.17 there exist ideals $\mathfrak{a}_\mathfrak{P} \subset \mathbb{I}_0^\circ/\mathfrak{P}$ such that $(\bigoplus_{\mathfrak{P}|P} \pi_\mathfrak{P})(G_{P\mathbb{I}_0^\circ}) \supset \bigoplus_{\mathfrak{P}|P} \Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}_\mathfrak{P})$. In particular $(\bigoplus_{\mathfrak{P}|P} \pi_\mathfrak{P})(U(\rho_{P\mathbb{I}_0^\circ})) \supset \bigoplus_{\mathfrak{P}|P}(\mathfrak{a}_\mathfrak{P})$. By Lemma 4.19 each ideal $\mathfrak{a}_\mathfrak{P}$ contains a basis of $Q(\mathbb{I}_0^\circ/\mathfrak{P})$ over $Q(\Lambda_{h,0}/P)$, so that the

$Q(\Lambda_{h,0}/P)$-span of $\bigoplus_{\mathfrak{P}|P}\mathfrak{a}_{\mathfrak{P}}$ is the whole $\bigoplus_{\mathfrak{P}|P} Q(\mathbb{I}_0^\circ/\mathfrak{P})$. Then the $Q(\Lambda_{h,0}/P)$-span of $(\bigoplus_{\mathfrak{P}|P}\pi_{\mathfrak{P}})(G_{\mathfrak{P}}\cap U(\rho_{\mathfrak{P}}))$ is also $\bigoplus_{\mathfrak{P}|P} Q(\mathbb{I}_0^\circ/\mathfrak{P})$. By commutativity of diagram (3) we deduce that the $Q(\Lambda_{h,0}/P)$-span of $G_P\cap U(\rho_{P\mathbb{I}_0^\circ})$ is $Q(\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ)$. In particular $G_{P\mathbb{I}_0^\circ}\cap U(\rho_{P\mathbb{I}_0^\circ})$ contains a $\Lambda_{h,0}/P$-lattice, hence by Lemma 4.19 a nonzero ideal $\mathfrak{a}_P$ of $\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ$.

Note that the representation $\rho_0\colon H\to \mathrm{SL}_2(\mathbb{I}_0^\circ)$ satisfies the hypotheses of Proposition 4.8. Indeed we assumed that $\rho\colon G_{\mathbb{Q}}\to \mathrm{GL}_2(\mathbb{I})$ is $(H_0,\mathbb{Z}_p)$-regular, so the image of $\rho|_{H_0}$ contains a diagonal $\mathbb{Z}_p$-regular element d. Since H is a normal subgroup of H_0, $\rho(H)$ is a normal subgroup of $\rho(H_0)$ and it is normalized by d. By a trivial computation we see that the image of $\rho_0=\rho|_H\otimes(\det\rho|_H)^{-1/2}$ is also normalized by d.

Let $\mathfrak{a}$ be an ideal of $\mathbb{I}_0^\circ$ projecting to $\mathfrak{a}_P\subset U(\rho_{0,P\mathbb{I}_0^\circ})$. By Proposition 4.8 applied to ρ_0 we obtain that the map $U(\rho_0)\to U(\rho_{0,P\mathbb{I}_0^\circ})$ is surjective, so the $\mathbb{Z}_p$-module $\mathfrak{a}\cap U(\rho_0)$ also surjects to $\mathfrak{a}_P$. Since $\Lambda_{h,0}$ is local we can apply Nakayama's lemma to the $\Lambda_{h,0}$-module $\Lambda_{h,0}(\mathfrak{a}\cap U(\rho_0)$ to conclude that it coincides with $\mathfrak{a}$. Hence $\mathfrak{a}\subset\Lambda_{h,0}\cdot U(\rho_0)$, so the $\Lambda_{h,0}$-span of $U(\rho_0)$ contains a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$ by lemma 4.19. □

We show that Proposition 4.20 is true if we replace ρ_0 by $\rho|_H$. This will be a consequence of the description of the subnormal sugroups of $\mathrm{GL}_2(\mathbb{I}^\circ)$ presented in [23], but we need a preliminary step because we cannot induce a $\Lambda_{h,0}$-module structure on the unipotent subgroups of G. For a subgroup $\mathcal{G}\subset\mathrm{GL}_2(\mathbb{I}_0^\circ)$ define $\mathcal{G}^p=\{g^p,\ g\in G\}$ and $\widetilde{\mathcal{G}}=\mathcal{G}^p\cap(1+p\mathrm{M}_2(\mathbb{I}_0^\circ))$. Let $\widetilde{\mathcal{G}}^{\Lambda_{h,0}}$ be the subgroup of $\mathrm{GL}_2(\mathbb{I}^\circ)$ generated by the set $\{g^\lambda\colon g\in\widetilde{\mathcal{G}},\lambda\in\Lambda_{h,0}\}$ where $g^\lambda=\exp(\lambda\log g)$. We have the following.

Lemma 4.21 *The group $\widetilde{\mathcal{G}}^{\Lambda_{h,0}}$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if both of the unipotent subgroups $\mathcal{G}\cap U^+(\mathbb{I}_0^\circ)$ and $\mathcal{G}\cap U^-(\mathbb{I}_0^\circ)$ contain a basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$.*

Proof It is easy to see that $\mathcal{G}\cap U^+(\mathbb{I}_0^\circ)$ contains the basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$ if and only if the same is true for $\widetilde{\mathcal{G}}\cap U^+(\mathbb{I}_0^\circ)$. The same is true for U^-. By a standard argument, used in the proofs of [9, Lemma 2.9] and [12, Proposition 4.2], $\mathcal{G}^{\Lambda_{h,0}}\subset\mathrm{GL}_2(\mathbb{I}_0^\circ)$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if both its upper and lower unipotent subgroup contain an ideal of $\mathbb{I}_0^\circ$. We have $U^+(\mathbb{I}_0^\circ)\cap\mathcal{G}^{\Lambda_{h,0}}=\Lambda_{h,0}(\mathcal{G}\cap U^+(\mathbb{I}_0^\circ))$, so by Lemma 4.19 $U^+(\mathbb{I}_0^\circ)\cap\mathcal{G}^{\Lambda_{h,0}}$ contains an ideal of $\mathbb{I}_0^\circ$ if and only if $\mathcal{G}\cap U^+(\mathbb{I}_0^\circ)$ contains a basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$. We proceed in the same way for U^-. □

Now let $G_0=\mathrm{Im}\,\rho|_H$, $G=\mathrm{Im}\,\rho_0$. Note that $G_0\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of G. Let $f\colon\mathrm{GL}_2(\mathbb{I}_0^\circ)\to\mathrm{SL}_2(\mathbb{I}_0^\circ)$ be the homomorphism sending g to $\det(g)^{-1/2}g$. We have $G=f(G_0)$ by definition of ρ_0. We show the following.

Proposition 4.22 *The subgroups $G_0\cap U^\pm(\mathbb{I}_0^\circ)$ both contain the basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$ if and only if $G\cap U^\pm(\mathbb{I}_0^\circ)$ both contain the basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$.*

Proof Since $G=f(G_0)$ we have $\widetilde{G}=f(\widetilde{G_0})$. This implies that $\widetilde{G}^{\Lambda_{h,0}}=f(\widetilde{G_0}^{\Lambda_{h,0}})$. We remark that $\widetilde{G_0}^{\Lambda_{h,0}}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of $\widetilde{G}^{\Lambda_{h,0}}$. Indeed

$\widetilde{G_0}^{\Lambda_{h,0}} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is normal in $\widetilde{G_0}^{\Lambda_{h,0}}$, so its image $f(G_0^{\Lambda_{h,0}} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)) = G_0^{\Lambda_{h,0}} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is normal in $f(G_0^{\Lambda_{h,0}}) = \widetilde{G}^{\Lambda_{h,0}}$.

By [23, Corollary 1] a subgroup of $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if it is subnormal in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ and it is not contained in the centre. We note that $\widetilde{G_0}^{\Lambda_{h,0}} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ) = (\widetilde{G_0} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ))^{\Lambda_{h,0}}$ is not contained in the subgroup $\{\pm 1\}$. Otherwise also $\widetilde{G_0} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ would be contained in $\{\pm 1\}$ and $\mathrm{Im}\,\rho \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ would be finite, since $\widetilde{G_0}$ is of finite index in G_0^p. This would give a contradiction: indeed if $\mathfrak{P}$ is an arithmetic prime of $\mathbb{I}^\circ$ of weight greater than 1 and $\mathfrak{P}' = \mathfrak{P} \cap \mathbb{I}_0^\circ$, the image of ρ modulo $\mathfrak{P}'$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P}')$ by the result of [15].

Since $\widetilde{G_0}^{\Lambda_{h,0}} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of $\widetilde{G}^{\Lambda_{h,0}}$, we deduce by [23, Corollary 1] that $\widetilde{G_0}^{\Lambda_{h,0}} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ (hence $\widetilde{G_0}^{\Lambda_{h,0}}$) contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if $\widetilde{G}^{\Lambda_{h,0}}$ does. By applying Lemma 4.21 to $\mathcal{G} = G_0$ and $\mathcal{G} = G$ we obtain the desired equivalence. □

By combining Propositions 4.20 and 4.22 we obtain the following.

Corollary 4.23 *The $\Lambda_{h,0}$-span of each of the unipotent subgroups* $\mathrm{Im}\,\rho \cap U^\pm$ *contains a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$.*

Unlike in the ordinary case we cannot deduce from the corollary that $\mathrm{Im}\,\rho$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$, since we are working over $\Lambda_h \neq \Lambda$ and we cannot induce a Λ_h-module structure (not even a Λ-module structure) on $\mathrm{Im}\,\rho \cap U^\pm$. The proofs of [9, Lemma 2.9] and [12, Proposition 4.3] rely on the existence, in the image of the Galois group, of an element inducing by conjugation a Λ-module structure on $\mathrm{Im}\,\rho \cap U^\pm$. In their situation this is predicted by the condition of Galois ordinarity of ρ. In the non-ordinary case we will find an element with a similar property via relative Sen theory. In order to do this we will need to work with a suitably defined Lie algebra rather than with the group itself.

5 Relative Sen Theory

We recall the notations of Sect. 3.1. In particular $r_h = p^{-s_h}$, with $s_h \in \mathbb{Q}$, is the h-adapted radius (which we also take smaller than $p^{-\frac{1}{p-1}}$), η_h is an element in $\mathbb{C}_p$ of norm r_h, K_h is the Galois closure in $\mathbb{C}_p$ of $\mathbb{Q}_p(\eta_h)$ and $\mathcal{O}_h$ is the ring of integers in K_h. The ring Λ_h of analytic functions bounded by 1 on the open disc $\mathcal{B}_h = \mathcal{B}(0, r_h^-)$ is identified to $\mathcal{O}_h[[t]]$. We take a sequence of radii $r_i = p^{-s_h - 1/i}$ converging to r_h and denote by $A_{r_i} = K_h\langle t, X_i\rangle/(pX_i - t^i)$ the K_h-algebra defined in Sect. 3.1 which is a form over K_h of the $\mathbb{C}_p$-algebra of analytic functions on the closed ball $\mathcal{B}(0, r_i)$ (its Berthelot model). We denote by $A_{r_i}^\circ$ the $\mathcal{O}_h$-subalgebra of functions bounded by 1. Then $\Lambda_h = \varprojlim_i A_{r_i}^\circ$ where $A_{r_j}^\circ \to A_{r_i}^\circ$ for $i < j$ is the restriction of analytic functions.

We defined in Sect. 4.1 a subring $\mathbb{I}_0^\circ \subset \mathbb{I}^\circ$, finite over $\Lambda_{h,0} \subset \Lambda_h$. For r_i as above, we write $A_{0,r_i}^\circ = \mathcal{O}_{h,0}\langle t, X_i\rangle/(pX_i - t^i)$ with maps $A_{0,r_j}^\circ \to A_{0,r_i}^\circ$ for $i < j$, so that $\Lambda_{h,0} = \varprojlim_i A_{0,r_i}^\circ$. Let $\mathbb{I}_{r_i}^\circ = \mathbb{I}^\circ \widehat{\otimes}_{\Lambda_h} A_{r_i}^\circ$ and $\mathbb{I}_{0,r_i}^\circ = \mathbb{I}_0^\circ \widehat{\otimes}_{\Lambda_{h,0}} A_{0,r_i}^\circ$, both endowed with their p-adic topology. Note that $(\mathbb{I}_{r_i}^\circ)^\Gamma = \mathbb{I}_{r_i,0}^\circ$.

Consider the representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated with a family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. We observe that ρ is continuous with respect to the profinite topology of $\mathbb{I}^\circ$ but not with respect to the p-adic topology. For this reason we fix an arbitrary radius r among the r_i defined above and consider the representation $\rho_r\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}_r^\circ)$ obtained by composing ρ with the inclusion $\mathrm{GL}_2(\mathbb{I}^\circ) \hookrightarrow \mathrm{GL}_2(\mathbb{I}_r^\circ)$. This inclusion is continuous, hence the representation ρ_r is continuous with respect to the p-adic topology on $\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)$.

Recall from Proposition 4.14 that, after replacing ρ by a conjugate, there is an open normal subgroup $H_0 \subset G_\mathbb{Q}$ such that the restriction $\rho|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ and is $(H_0, \mathbb{Z}_p)$-regular. Then the restriction $\rho_r|_{H_0}$ gives a representation $H_0 \to \mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)$ which is continuous with respect to the p-adic topology on $\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)$.

5.1 Big Lie Algebras

Recall that $G_p \subset G_\mathbb{Q}$ denotes our chosen decomposition group at p. Let G_r and G_r^{loc} be the images respectively of H_0 and $G_p \cap H_0$ under the representation $\rho_r|_{H_0}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)$. Note that they are actually independent of r since they coincide with the images of H_0 and $G_p \cap H_0$ under ρ.

For every ring R and ideal $I \subset R$ we denote by $\Gamma_{\mathrm{GL}_2(R)}(I)$ the GL_2-congruence subgroup consisting of elements $g \in \mathrm{GL}_2(R)$ such that $g \equiv \mathrm{Id}_2 \pmod I$. Let $G_r' = G_r \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)}(p)$ and $G_r^{\prime,\mathrm{loc}} = G_r^{\mathrm{loc}} \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)}(p)$, so that G_r' and $G_r^{\prime,\mathrm{loc}}$ are pro-p groups. Note that the congruence subgroups $\Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r})}(p^m)$ are open in $\mathrm{GL}_2(\mathbb{I}_{0,r})$ for the p-adic topology. In particular G_r' and $G_r^{\prime,\mathrm{loc}}$ can be identified with the images under ρ of the absolute Galois groups of finite extensions of $\mathbb{Q}$ and respectively $\mathbb{Q}_p$.

Remark 5.1 We remark that we can choose an arbitrary r_0 and set, for every r, $G_r' = G_r \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r_0}^\circ)}(p)$. Then G_r' is a pro-p subgroup of G_r for every r and it is independent of r since G_r is. This will be important in Theorem 7.1 where we will take projective limits over r of various objects.

We set $A_{0,r} = A_{0,r}^\circ[p^{-1}]$ and $\mathbb{I}_{0,r} = \mathbb{I}_{0,r}^\circ[p^{-1}]$. We consider from now on G_r' and $G_r^{\prime,\mathrm{loc}}$ as subgroups of $\mathrm{GL}_2(\mathbb{I}_{0,r})$ through the inclusion $\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ) \hookrightarrow \mathrm{GL}_2(\mathbb{I}_{0,r})$.

We want to define big Lie algebras associated with the groups G_r' and $G_r^{\prime,\mathrm{loc}}$. For every nonzero ideal $\mathfrak{a}$ of the principal ideal domain $A_{0,r}$, we denote by $G_{r,\mathfrak{a}}'$ and $G_{r,\mathfrak{a}}^{\prime,\mathrm{loc}}$ the images respectively of G_r' and $G_r^{\prime,\mathrm{loc}}$ under the natural projection $\mathrm{GL}_2(\mathbb{I}_{0,r}) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$. The pro-$p$ groups $G_{r,\mathfrak{a}}'$ and $G_{r,\mathfrak{a}}^{\prime,\mathrm{loc}}$ are topologically of finite type so we can define the corresponding $\mathbb{Q}_p$-Lie algebras $\mathfrak{H}_{r,\mathfrak{a}}$ and $\mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}}$ using the p-adic logarithm map: $\mathfrak{H}_{r,\mathfrak{a}} = \mathbb{Q}_p \cdot \mathrm{Log}\, G_{r,\mathfrak{a}}'$ and $\mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}} = \mathbb{Q}_p \cdot \mathrm{Log}\, G_{r,\mathfrak{a}}^{\prime,\mathrm{loc}}$. They are closed Lie subalgebras of the finite dimensional $\mathbb{Q}_p$-Lie algebra $\mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$.

Let $B_r = \varprojlim_{(\mathfrak{a},P_1)=1} A_{0,r}/\mathfrak{a}A_{0,r}$ where the inverse limit is taken over nonzero ideals $\mathfrak{a} \subset A_{0,r}$ prime to $P_1 = (u^{-1}(1+T)-1)$ (the reason for excluding P_1 will become clear later). We endow B_r with the projective limit topology coming from the p-adic topology on each quotient. We have a topological isomorphism of $K_{h,0}$-algebras

$$B_r \cong \prod_{P \neq P_1} \widehat{(A_{0,r})}_P,$$

where the product is over primes P and $\widehat{(A_{0,r})}_P = \varprojlim_{m \geqslant 1} A_{0,r}/P^m A_{0,r}$ denotes the $K_{h,0}$-Fréchet space inverse limit of the finite dimensional $K_{h,0}$-vector spaces $A_{0,r}/P^m A_{0,r}$. Similarly, let $\mathbb{B}_r = \varprojlim_{(\mathfrak{a},P_1)=1} \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$, where as before $\mathfrak{a}$ varies over all nonzero ideals of $A_{0,r}$ prime to P_1. We have

$$\mathbb{B}_r \cong \prod_{P \neq P_1} \widehat{(\mathbb{I}_{0,r})}_{P\mathbb{I}_{0,r}} \cong \prod_{\mathfrak{P} \nmid P_1} \widehat{(\mathbb{I}_{0,r})}_{\mathfrak{P}} \cong \varprojlim_{(\mathfrak{Q},P_1)=1} \mathbb{I}_{0,r}/\mathfrak{Q},$$

where the second product is over primes $\mathfrak{P}$ of $\mathbb{I}_{0,r}$ and the projective limit is over primary ideals $\mathfrak{Q}$ of $\mathbb{I}_{0,r}$. Here $\widehat{(\mathbb{I}_{0,r})}_{\mathfrak{P}}$ denotes the projective limit of finite dimensional $K_{h,0}$-algebras (endowed with the p-adic topology). The last isomorphism follows from the fact that $\mathbb{I}_{0,r}$ is finite over $A_{0,r}$, so that there is an isomorphism $\mathbb{I}_{0,r} \otimes \widehat{(A_{0,r})}_P = \prod_{\mathfrak{P}} \widehat{(\mathbb{I}_{0,r})}_{\mathfrak{P}}$ where P is a prime of $A_{0,r}$ and $\mathfrak{P}$ varies among the primes of $\mathbb{I}_{0,r}$ above P. We have natural continuous inclusions $A_{0,r} \hookrightarrow B_r$ and $\mathbb{I}_{0,r} \hookrightarrow \mathbb{B}_r$, both with dense image. The map $A_{0,r} \hookrightarrow \mathbb{I}_{0,r}$ induces an inclusion $B_r \hookrightarrow \mathbb{B}_r$ with closed image. Note however that $\mathbb{B}_r$ is not finite over B_r. We will work with $\mathbb{B}_r$ for the rest of this section, but we will need B_r later.

For every $\mathfrak{a}$ we have defined Lie algebras $\mathfrak{H}_{r,\mathfrak{a}}$ and $\mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}}$ associated with the finite type Lie groups $G'_{r,\mathfrak{a}}$ and $G'^{,\mathrm{loc}}_{r,\mathfrak{a}}$. We take the projective limit of these algebras to obtain Lie subalgebras of $\mathrm{M}_2(\mathbb{B}_r)$.

Definition 5.2 The Lie algebras associated with G'_r and $G'^{,\mathrm{loc}}_r$ are the closed $\mathbb{Q}_p$-Lie subalgebras of $\mathrm{M}_2(\mathbb{B}_r)$ given respectively by

$$\mathfrak{H}_r = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}_{r,\mathfrak{a}}$$

and

$$\mathfrak{H}_r^{\mathrm{loc}} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}},$$

where as usual the products are taken over nonzero ideals $\mathfrak{a} \subset A_{0,r}$ prime to P_1.

For every ideal $\mathfrak{a}$ prime to P_1, we have continuous homomorphisms $\mathfrak{H}_r \to \mathfrak{H}_{r,\mathfrak{a}}$ and $\mathfrak{H}_r^{\mathrm{loc}} \to \mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}}$. Since the transition maps are surjective these homomorphisms are surjective.

Remark 5.3 The limits in Definition 5.2 can be replaced by limits over primary ideals of $\mathbb{I}_{0,r}$. Explicitly, let $\mathfrak{Q}$ be a primary ideal of $\mathbb{I}_{0,r}$. Let $G'_{r,\mathfrak{Q}}$ be the image of G'_r via the natural projection $\mathrm{GL}_2(\mathbb{I}_{0,r}) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{Q})$ and let $\mathfrak{H}_{r,\mathfrak{Q}}$ be the Lie algebra associated with $G'_{r,\mathfrak{Q}}$ (which is a finite type Lie group). We have an isomorphism of topological Lie algebras

$$\mathfrak{H}_r = \varprojlim_{(\mathfrak{Q},P_1)=1} \mathfrak{H}_{r,\mathfrak{Q}},$$

where the limit is taken over primary ideals $\mathfrak{Q}$ of $\mathbb{I}_{0,r}$. This is naturally a subalgebra of $\mathrm{M}_2(\mathbb{B}_r)$ since $\mathbb{B}_r \cong \varprojlim_{(\mathfrak{Q},P_1)=1} \mathbb{I}_{0,r}/\mathfrak{Q}$. The same goes for the local algebras.

5.2 The Sen Operator Associated with a Galois Representation

Recall that there is a finite extension $K/\mathbb{Q}_p$ such that $G_r'^{,\mathrm{loc}}$ is the image of $\rho|_{\mathrm{Gal}(\overline{K}/K)}$ and, for an ideal $P \subset A_{0,r}$ and $m \geqslant 1$, $G_{r,P^m}'^{,\mathrm{loc}}$ is the image of $\rho_{r,P^m}|_{\mathrm{Gal}(\overline{K}/K)}$. Following [19, 21] we can define a Sen operator associated with $\rho_r|_{\mathrm{Gal}(\overline{K}/K)}$ and $\rho_{r,P^m}|_{\mathrm{Gal}(\overline{K}/K)}$ for every ideal $P \subset A_{0,r}$ and every $m \geqslant 1$. We will see that these operators satisfy a compatibility property. We write for the rest of the section ρ_r and ρ_{r,P^m} while implicitly taking the domain to be $\mathrm{Gal}(\overline{K}/K)$.

We begin by recalling the definition of the Sen operator associated with a representation $\tau\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathcal{R})$ where $\mathcal{R}$ is a Banach algebra over a p-adic field L. We follow [21]. We can suppose $L \subset K$; if not we just restrict τ to the open subgroup $\mathrm{Gal}(\overline{K}/KL) \subset \mathrm{Gal}(\overline{K}/K)$.

Let L_∞ be a totally ramified $\mathbb{Z}_p$-extension of L. Let γ be a topological generator of $\Gamma = \mathrm{Gal}(L_\infty/L)$, $\Gamma_n \subset \Gamma$ the subgroup generated by γ^{p^n} and $L_n = L_\infty^{\gamma^{p^n}}$, so that $L_\infty = \cup_n L_n$. Let $L'_n = L_n K$ and $G'_n = \mathrm{Gal}(\overline{L}/L'_n)$. If $\mathcal{R}^m$ is the $\mathcal{R}$-module over which $\mathrm{Gal}(\overline{K}/K)$ acts via τ, define an action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathcal{R}\widehat{\otimes}_L \mathbb{C}_p$ by letting $\sigma \in \mathrm{Gal}(\overline{K}/K)$ map $x \otimes y$ to $\tau(\sigma)(x) \otimes \sigma(y)$. Then by the results of [19, 21] there is a matrix $M \in \mathrm{GL}_m\left(\mathcal{R}\widehat{\otimes}_L \mathbb{C}_p\right)$, an integer $n \geqslant 0$ and a representation $\delta\colon \Gamma_n \to \mathrm{GL}_m(\mathcal{R} \otimes_L L'_n)$ such that for all $\sigma \in G'_n$

$$M^{-1}\tau(\sigma)\sigma(M) = \delta(\sigma).$$

Definition 5.4 The Sen operator associated with τ is

$$\phi = \lim_{\sigma \to 1} \frac{\log\bigl(\delta(\sigma)\bigr)}{\log(\chi(\sigma))} \in \mathrm{M}_m(\mathcal{R}\widehat{\otimes}_L \mathbb{C}_p).$$

The limit exists as for σ close to 1 the map $\sigma \mapsto \dfrac{\log\bigl(\delta(\sigma)\bigr)}{\log(\chi(\sigma))}$ is constant. It is proved in [21, Sect. 2.4] that ϕ does not depend on the choice of δ and M.

If $L = \mathcal{R} = \mathbb{Q}_p$, we define the Lie algebra $\mathfrak{g}$ associated with $\tau(\mathrm{Gal}(\overline{K}/K))$ as the $\mathbb{Q}_p$-vector space generated by the image of the logarithm map in $\mathrm{M}_m(\mathbb{Q}_p)$. In this situation the Sen operator ϕ associated with τ has the following property.

Theorem 5.5 [19, Theorem 1] *For a continuous representation $\tau\colon G_K \to \mathrm{GL}_m(\mathbb{Q}_p)$, the Lie algebra $\mathfrak{g}$ of the group $\tau(\mathrm{Gal}(\overline{K}/K))$ is the smallest $\mathbb{Q}_p$-subspace of $\mathrm{M}_m(\mathbb{Q}_p)$ such that $\mathfrak{g}\otimes_{\mathbb{Q}_p}\mathbb{C}_p$ contains ϕ.*

This theorem is valid in the absolute case above, but relies heavily on the fact that the image of the Galois group is a finite dimensional Lie group. In the relative case it is doubtful that its proof can be generalized.

5.3 The Sen Operator Associated with ρ_r

Set $\mathbb{I}_{0,r,\mathbb{C}_p} = \mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p$. It is a Banach space for the natural norm. Let $\mathbb{B}_{r,\mathbb{C}_p} = \mathbb{B}_r\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p$; it is the topological $\mathbb{C}_p$-algebra completion of $\mathbb{B}_r \otimes_{K_{h,0}} \mathbb{C}_p$ for the (uncountable) set of nuclear seminorms $p_\mathfrak{a}$ given by the norms on $\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}$ via the specialization morphisms $\pi_\mathfrak{a}\colon \mathbb{B}_r \otimes_{K_{h,0}} \mathbb{C}_p \to \mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}$. Let $\mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p} = \mathfrak{H}_{r,\mathfrak{a}} \otimes_{K_{h,0}} \mathbb{C}_p$ and $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p} = \mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},} \otimes_{K_{h,0}} \mathbb{C}_p$. Then we define $\mathfrak{H}_{r,\mathbb{C}_p} = \mathfrak{H}_r\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p$ as the topological $\mathbb{C}_p$-Lie algebra completion of $\mathfrak{H}_r \otimes_{K_{0,h}} \mathbb{C}_p$ for the (uncountable) set of seminorms $p_\mathfrak{a}$ given by the norms on $\mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p}$ and similar specialization morphisms $\pi_\mathfrak{a}\colon \mathfrak{H}_{r,} \otimes_{K_{h,0}} \mathbb{C}_p \to \mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p}$. We define in the same way $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$ in terms of the norms on $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$. Note that by definition we have

$$\mathfrak{H}_{r,\mathbb{C}_p} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p}, \text{ and } \mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}.$$

We apply the construction of the previous subsection to $L = K_{h,0}, \mathcal{R} = \mathbb{I}_{0,r}$ which is a Banach L-algebra with the p-adic topology, and $\tau = \rho_r$. We obtain an operator $\phi_r \in \mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$. Recall that we have a natural continuous inclusion $\mathbb{I}_{0,r} \hookrightarrow \mathbb{B}_r$, inducing inclusions $\mathbb{I}_{0,r,\mathbb{C}_p} \hookrightarrow \mathbb{B}_{r,\mathbb{C}_p}$ and $\mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p}) \hookrightarrow \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$. We denote all these inclusions by $\iota_{\mathbb{B}_r}$ since it will be clear each time to which we are referring to. We will prove in this section that $\iota_{\mathbb{B}_r}(\phi_r)$ is an element of $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$.

Let $\mathfrak{a}$ be a nonzero ideal of $A_{0,r}$. Let us apply Sen's construction to $L = K_{h,0}$, $\mathcal{R} = \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$ and $\tau = \rho_{r,\mathfrak{a}}\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$; we obtain an operator $\phi_{r,\mathfrak{a}} \in \mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p)$.

Let

$$\pi_\mathfrak{a}\colon \mathrm{M}_2(\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p) \to \mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p)$$

and

$$\pi_\mathfrak{a}^\times\colon \mathrm{GL}_2(\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p)$$

be the natural projections.

Proposition 5.6 *We have* $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$ *for all* $\mathfrak{a}$.

Proof Recall from the construction of ϕ_r that there exist $M \in \mathrm{GL}_2\left(\mathbb{I}_{0,r,\mathbb{C}_p}\right)$, $n \geqslant 0$ and $\delta\colon \Gamma_n \to \mathrm{GL}_2(\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}} K'_{h,0,n})$ such that for all $\sigma \in G'_n$ we have

$$M^{-1}\rho_r(\sigma)\sigma(M) = \delta(\sigma) \tag{4}$$

and

$$\phi_r = \lim_{\sigma\to 1} \frac{\log(\delta(\sigma))}{\log(\chi(\sigma))}. \tag{5}$$

Let $M_{\mathfrak{a}} = \pi_{\mathfrak{a}}^{\times}(M) \in \mathrm{GL}_2(\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p})$ and

$$\delta_{\mathfrak{a}} = \pi_{\mathfrak{a}}^{\times} \circ \delta\colon \Gamma_n \to \mathrm{GL}_2((\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\widehat{\otimes}_{K_{h,0}} K'_{h,0,n}).$$

Denote by $\phi_{r,\mathfrak{a}} \in \mathrm{M}_2((\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\widehat{\otimes}_{K_{h,0}} K'_{h,0,n})$ the Sen operator associated with $\rho_{r,\mathfrak{a}}$. Now (4) gives

$$M_{\mathfrak{a}}^{-1}\rho_{r,\mathfrak{a}}(\sigma)\sigma(M_{\mathfrak{a}}) = \delta_{\mathfrak{a}}(\sigma) \tag{6}$$

so we can calculate $\phi_{r,\mathfrak{a}}$ as

$$\phi_{r,\mathfrak{a}} = \lim_{\sigma\to 1} \frac{\log(\delta_{\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))} \in \mathrm{M}_2(\mathcal{R}\widehat{\otimes}_L \mathbb{C}_p). \tag{7}$$

By comparing this with (5) we see that $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$. □

Let $\phi_{r,\mathbb{B}_r} = \iota_{\mathbb{B}_r}(\phi_r)$. For a nonzero ideal $\mathfrak{a}$ of $A_{0,r}$ let $\pi_{\mathbb{B}_r,\mathfrak{a}}$ be the natural projection $\mathbb{B}_r \to \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$. Clearly $\pi_{\mathbb{B}_r,\mathfrak{a}}(\phi_{r,\mathbb{B}_r}) = \pi_{\mathfrak{a}}(\phi_r)$ and $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$ by Proposition 5.6, so we have $\phi_{r,\mathbb{B}_r} = \varprojlim_{(\mathfrak{a},P_1)=1} \phi_{r,\mathfrak{a}}$.

We apply Theorem 5.5 to show the following.

Proposition 5.7 *Let* $\mathfrak{a}$ *be a nonzero ideal of* $A_{0,r}$ *prime to* P_1. *The operator* $\phi_{r,\mathfrak{a}}$ *belongs to the Lie algebra* $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$.

Proof Let n be the dimension over $\mathbb{Q}_p$ of $\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$; by choosing a basis $(\omega_1,\ldots,\omega_n)$ of this algebra as a $\mathbb{Q}_p$-vector space, we can define an injective ring homomorphism $\alpha\colon \mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{Q}_p)$ and an injective group homomorphism $\alpha^{\times}\colon \mathrm{GL}_2(\mathbb{I}_{0,r}/\alpha\mathbb{I}_{0,r}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{Q}_p)$. In fact, an endomorphism f of the $(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$-module $(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})^2 = (\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\cdot e_1 \oplus (\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\cdot e_2$ is $\mathbb{Q}_p$-linear, so it induces an endomorphism $\alpha(f)$ of the $\mathbb{Q}_p$-vector space $(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})^2 = \bigoplus_{i,j}\mathbb{Q}_p\cdot\omega_i e_j$; furthermore if α is an automorphism then $\alpha(f)$ is one too. In particular $\rho_{r,\mathfrak{a}}$ induces a representation $\rho^{\alpha}_{r,\mathfrak{a}} = \alpha^{\times}\circ\rho_{r,\mathfrak{a}}\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2n}(\mathbb{Q}_p)$. The image of $\rho^{\alpha}_{r,\mathfrak{a}}$ is the group $G^{\mathrm{loc},\alpha}_{r,\mathfrak{a}} = \alpha^{\times}(G^{\mathrm{loc}}_{r,\mathfrak{a}})$. We consider its Lie algebra $\mathfrak{H}^{\mathrm{loc},\alpha}_{r,\mathfrak{a}} = \mathbb{Q}_p\cdot \mathrm{Log}\,(G^{\mathrm{loc},\alpha}_{r,\mathfrak{a}}) \subset \mathrm{M}_{2n}(\mathbb{Q}_p)$. The p-adic logarithm commutes with α in the sense that $\alpha(\mathrm{Log}\,x) = \mathrm{Log}\,(\alpha^{\times}(x))$ for every $x \in \Gamma_{\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}}(p)$, so we have $\mathfrak{H}^{\mathrm{loc},\alpha}_{r,\mathfrak{a}} = \alpha(\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a}})$ (recall that $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a}} = \mathbb{Q}_p\cdot\mathrm{Log}\,G^{\mathrm{loc}}_{r,\mathfrak{a}}$).

Let $\phi^{\alpha}_{r,\mathfrak{a}}$ be the Sen operator associated with $\rho^{\alpha}_{r,\mathfrak{a}}\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2n}(\mathbb{Q}_p)$. By Theorem 5.5 we have $\phi^{\alpha}_{r,\mathfrak{a}} \in \mathfrak{H}^{\mathrm{loc},\alpha}_{r,\mathfrak{a},\mathbb{C}_p} = \mathfrak{H}^{\mathrm{loc},\alpha}_{r,\mathfrak{a}}\widehat{\otimes}\mathbb{C}_p$. Denote by $\alpha_{\mathbb{C}_p}$ the map $\alpha\widehat{\otimes}1\colon \mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{C}_p)$. We show that $\phi^{\alpha_{\mathbb{C}_p}}_{r,\mathfrak{a}} = \alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$, from which it follows that $\phi_{r,\mathfrak{a}} \in \mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$ since $\mathfrak{H}^{\mathrm{loc},\alpha_{\mathbb{C}_p}}_{r,\mathfrak{a},\mathbb{C}_p} = \alpha_{\mathbb{C}_p}(\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p})$ and $\alpha_{\mathbb{C}_p}$ is injective. Now let $M_{\mathfrak{a}}$, $\delta_{\mathfrak{a}}$ be as in (6) and $M^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}} = \alpha_{\mathbb{C}_p}(M_{\mathfrak{a}})$, $\delta^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}} = \alpha_{\mathbb{C}_p} \circ \delta_{\mathfrak{a}}$. By applying α_C to (4) we obtain $(M^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}})^{-1}\rho^{\alpha_{\mathbb{C}_p}}_{r,\mathfrak{a}}(\sigma)\sigma(M^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}}) = \delta^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}}(\sigma)$ for every $\sigma \in G'_n$, so we can calculate

$$\phi^{\alpha_{\mathbb{C}_p}}_{r,\mathfrak{a}} = \lim_{\sigma\to 1} \frac{\log(\delta^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))},$$

which coincides with $\alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$. □

Proposition 5.8 *The element $\phi_{r,\mathbb{B}_r}$ belongs to $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$, hence to $\mathfrak{H}_{r,\mathbb{C}_p}$.*

Proof By definition of the space $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$ as completion of the space $\mathfrak{H}^{\mathrm{loc}}_r \otimes_{K_{h,0}} \mathbb{C}_p$ for the seminorms $p_{\mathfrak{a}}$ given by the norms on $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$, we have $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$. By Proposition 5.6, we have $\phi_{r,\mathbb{B}_r} = \varprojlim_{\mathfrak{a}} \phi_{r,\mathfrak{a}}$ and by Proposition 5.7 we have, for every $\mathfrak{a}$, $\phi_{r,\mathfrak{a}} \in \mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$. We conclude that $\phi_{r,\mathbb{B}_r} \in \mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$. □

Remark 5.9 In order to prove that our Lie algebras are "big" it will be useful to work with primary ideals of A_r, as we did in this subsection. However, in light of Remark 5.3, all of the results can be rewritten in terms of primary ideals $\mathfrak{Q}$ of $\mathbb{I}_{0,r}$. This will be useful in the next subsection, when we will interpolate the Sen operators corresponding to the attached to the classical modular forms representations.

From now on we identify $\mathbb{I}_{0,r,\mathbb{C}_p}$ with a subring of $\mathbb{B}_{r,\mathbb{C}_p}$ via $\iota_{\mathbb{B}_r}$, so we also identify $\mathrm{M}_2(\mathbb{I}_{0,r})$ with a subring of $\mathrm{M}_2(\mathbb{B}_r)$ and $\mathrm{GL}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ with a subgroup of $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$. In particular we identify ϕ_r with $\phi_{r,\mathbb{B}_r}$ and we consider ϕ_r as an element of $\mathfrak{H}_{r,\mathbb{C}_p} \cap \mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$.

5.4 The Characteristic Polynomial of the Sen Operator

Sen proved the following result.

Theorem 5.10 *Let L_1 and L_2 be two p-adic fields. Assume for simplicity that L_2 contains the normal closure of L_1. Let $\tau\colon \mathrm{Gal}(\overline{L}_1/L_1) \to \mathrm{GL}_m(L_2)$ be a continuous representation. For each embedding $\sigma\colon L_1 \to L_2$, there is a Sen operator $\phi_{\tau,\sigma} \in \mathrm{M}_m(\mathbb{C}_p \otimes_{L_1,\sigma} L_2)$ associated with τ and σ. If τ is Hodge-Tate and its Hodge-Tate weights with respect to σ are $h_{1,\sigma}, \dots, h_{m,\sigma}$ (with multiplicities, if any), then the characteristic polynomial of $\phi_{\tau,\sigma}$ is $\prod_{i=1}^{m}(X - h_{i,\sigma})$.*

Now let $k \in \mathbb{N}$ and $P_k = (u^{-k}(1+T)-1)$ be the corresponding arithmetic prime of $A_{0,r}$. Let $\mathfrak{P}_f$ be a prime of $\mathbb{I}_r$ above P, associated with the system of Hecke eigenvalues of a classical modular form f. Let $\rho_r\colon \mathbb{G}_Q \to \mathrm{GL}_2(\mathbb{I}_r)$ be as usual. The specialization of ρ_r modulo $\mathfrak{P}$ is the representation $\rho_f\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}_r/\mathfrak{P})$ classically associated with f, defined over the field $K_f = \mathbb{I}_r/\mathfrak{P}_f\mathbb{I}_r$. By a theorem of Faltings [8], when the weight of the form f is k, the representation ρ_f is Hodge-Tate of Hodge-Tate weights 0 and $k-1$. Hence by Theorem 5.10 the Sen operator ϕ_f associated with ρ_f has characteristic polynomial $X(X-(k-1))$. Let $\mathfrak{P}_{f,0} = \mathfrak{P}_f \cap \mathbb{I}_{0,r}$. With the notations of the previous subsection, the specialization of ρ_r modulo $\mathfrak{P}_{f,0}$ gives a representation $\rho_{r,\mathfrak{P}_{f,0}}\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{P}_{f,0})$, which coincides with $\rho_f|_{\mathrm{Gal}(\overline{K}/K)}$. In particular the Sen operator $\phi_{r,\mathfrak{P}_{f,0}}$ associated with $\rho_{r,\mathfrak{P}_{f,0}}$ is ϕ_f.

By Proposition 5.6 and Remark 5.9, the Sen operator $\phi_r \in \mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ specializes modulo $\mathfrak{P}_{f,0}$ to the Sen operator $\phi_{r,\mathfrak{P}_{f,0}}$ associated with $\rho_{r,\mathfrak{P}_{f,0}}$, for every f as above. Since the primes of the form $\mathfrak{P}_{f,0}$ are dense in $\mathbb{I}_{0,r,\mathbb{C}_p}$, the eigenvalues of $\phi_{r,Q}$ are given by the unique interpolation of those of $\rho_{r,\mathfrak{P}_{f,0}}$. This way we will recover an element of $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$ with the properties we need.

Given $f \in A_{0,r}$ we define its p-adic valuation by $v'_p(f) = \inf_{x\in\mathcal{B}(0,r)} v_p(f(x))$, where v_p is our chosen valuation on $\mathbb{C}_p$. Then if $v'(f-1) \leqslant p^{-\frac{1}{p-1}}$ there are well-defined elements $\log(f)$ and $\exp(\log(f))$ in $A_{0,r}$, and $\exp(\log(f)) = f$.

Let $\phi'_r = \log(u)\phi_r$. Note that ϕ'_r is a well-defined element of $\mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ since $\log(u) \in \mathbb{Q}_p$. Recall that we denote by C_T the matrix $\mathrm{diag}(u^{-1}(1+T),1)$. We have the following.

Proposition 5.11 *1. The eigenvalues of ϕ'_r are $\log(u^{-1}(1+T))$ and 0. In particular the exponential $\Phi_r = \exp(\phi'_r)$ is defined in $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Moreover Φ'_r is conjugate to C_T in $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$.*
2. The element Φ'_r of part (1) normalizes $\mathfrak{H}_{r,\mathbb{C}_p}$.

Proof For every $\mathfrak{P}_{f,0}$ as in the discussion above, the element $\log(u)\phi_r$ specializes to $\log(u)\phi_{r,\mathfrak{P}_{f,0}}$ modulo $\mathfrak{P}_{f,0}$. If $\mathfrak{P}_{f,0}$ is a divisor of P_k, the eigenvalues of $\log(u)\phi_{r,\mathfrak{P}_{f,0}}$ are $\log(u)(k-1)$ and 0. Since $1+T = u^k$ modulo $\mathfrak{P}_{f,0}$ for every prime $\mathfrak{P}_{f,0}$ dividing P_k, we have $\log(u^{-1}(1+T)) = \log(u^{k-1}) = (k-1)\log(u)$ modulo $\mathfrak{P}_{f,0}$. Hence the eigenvalues of $\log(u)\phi_{r,\mathfrak{P}_{f,0}}$ are interpolated by $\log(u^{-1}(1+T))$ and 0.

Recall that in Sect. 3.1 we chose r_h smaller than $p^{-\frac{1}{p-1}}$. Since $r < r_h$, $v'_p(T) < p^{-\frac{1}{p-1}}$. In particular $\log(u^{-1}(1+T))$ is defined and $\exp(\log(u^{-1}(1+T))) = u^{-1}(1+T)$, so $\Phi_r = \exp(\phi'_r)$ is also defined and its eigenvalues are $u^{-1}(1+T)$ and 1. The difference between the two is $u^{-1}(1+T)-1$; this elements belongs to P_1, hence it is invertible in $\mathbb{B}_r$. This proves (1).

By Proposition 5.8, $\phi_r \in \mathfrak{H}_{r,\mathbb{C}_p}$. Since $\mathfrak{H}_{r,\mathbb{C}_p}$ is a $\mathbb{Q}_p$-Lie algebra, $\log(u)\phi_r$ is also an element of $\mathfrak{H}_{r,\mathbb{C}_p}$. Hence its exponential Φ'_r normalizes $\mathfrak{H}_{r,\mathbb{C}_p}$. □

6 Existence of the Galois Level for a Family with Finite Positive Slope

Let $r_h \in p^{\mathbb{Q}} \cap]0, p^{-\frac{1}{p-1}}[$ be the radius chosen in Sect. 3. As usual we write r for any one of the radii r_i of Sect. 3.1. Recall that $\mathfrak{H}_r \subset \mathrm{M}_2(\mathbb{B}_r)$ is the Lie algebra attached to the image of ρ_r (see Definition 5.2) and $\mathfrak{H}_{r,\mathbb{C}_p} = \mathfrak{H}_r \widehat{\otimes}\mathbb{Q}_p\mathbb{C}_p$. Let $\mathfrak{u}^{\pm}$ and $\mathfrak{u}^{\pm}_{\mathbb{C}_p}$ be the upper and lower nilpotent subalgebras of $\mathfrak{H}_r$, and $\mathfrak{H}_{r,\mathbb{C}_p}$ respectively.

Remark 6.1 The commutative Lie algebra $\mathfrak{u}^{\pm}$ is independent of r because it is equal to $\mathbb{Q}_p \cdot \mathrm{Log}(U(\mathbb{I}_0^{\circ}) \cap G'_r)$ which is independent of r, provided $r_1 \leqslant r < r_h$.

We fix $r_0 \in p^{\mathbb{Q}} \cap]0, r_h[$ arbitrarily and we work from now on with radii r satisfying $r_0 \leqslant r < r_h$. As in Remark 5.1 this fixes a finite extension of $\mathbb{Q}$ corresponding to the inclusion $G'_r \subset G_r$. For $r < r'$ we have a natural inclusion $\mathbb{I}_{0,r'} \hookrightarrow \mathbb{I}_{0,r}$. Since $\mathbb{B}_r = \varprojlim_{(\mathfrak{a}P_1)=1} \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$ this induces an inclusion $\mathbb{B}_{r'} \hookrightarrow \mathbb{B}_r$. We will consider from now on $\mathbb{B}_{r'}$ as a subring of $\mathbb{B}_r$ for every $r < r'$. We will also consider $\mathrm{M}_2(\mathbb{I}_{0,r',\mathbb{C}_p})$ and $\mathrm{M}_2(\mathbb{B}_{r'})$ as subsets of $\mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ and $\mathrm{M}_2(\mathbb{B}_r)$ respectively. These inclusions still hold after taking completed tensor products with $\mathbb{C}_p$.

Recall the elements $\phi'_r = \log(u)\phi_r \in \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ and $\Phi'_r = \exp(\phi'_r) \in \mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$ defined at the end of the previous section. The Sen operator ϕ_r is independent of r in the following sense: if $r < r' < r_h$ and $\mathbb{B}_{r',\mathbb{C}_p} \to \mathbb{B}_{r,\mathbb{C}_p}$ is the natural inclusion then the image of $\phi_{r'}$ under the induced map $\mathrm{M}_2(\mathbb{B}_{r',\mathbb{C}_p}) \to \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ is ϕ_r. We deduce that ϕ'_r and Φ'_r are also independent of r (in the same sense).

By Proposition 5.11, for every $r < r_h$ there exists an element $\beta_r \in \mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$ such that $\beta_r\Phi'_r\beta_r^{-1} = C_T$. Since Φ'_r normalizes $\mathfrak{H}_{r,\mathbb{C}_p}$, $C_T = \beta_r\Phi'_r\beta_r^{-1}$ normalizes $\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}$.

We denote by $\mathfrak{U}^{\pm}$ the upper and lower nilpotent subalgebras of $\mathfrak{sl}_2$. The action of C_T on $\mathfrak{H}_{r,\mathbb{C}_p}$ by conjugation is semisimple, so we can decompose $\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}$ as a sum of eigenspaces for C_T:

$$\begin{aligned}&\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\\ &\quad= \left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[1] \oplus \left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u^{-1}(1+T)] \oplus \left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u(1+T)^{-1}]\end{aligned}$$

with $\left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u^{-1}(1+T)] \subset \mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$ and $\left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u(1+T)^{-1}] \subset \mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})$.

Moreover, the formula

$$\begin{pmatrix} u^{-1}(1+T) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^{-1}(1+T) & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & u^{-1}(1+T)\lambda \\ 0 & 1 \end{pmatrix}$$

shows that the action of C_T by conjugation coincides with multiplication by $u^{-1}(1+T)$. By linearity this gives an action of the polynomial ring $\mathbb{C}_p[T]$ on $\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1} \cap \mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$, compatible with the action of $\mathbb{C}_p[T]$ on $\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$ given by the inclusions

$\mathbb{C}_p[T] \subset \Lambda_{h,0,\mathbb{C}_p} \subset B_{r,\mathbb{C}_p} \subset \mathbb{B}_{r,\mathbb{C}_p}$. Since $\mathbb{C}_p[T]$ is dense in $A_{h,0,\mathbb{C}_p}$ for the p-adic topology, it is also dense in $B_{r,\mathbb{C}_p}$. Since $\mathfrak{H}_{r,\mathbb{C}_p}$ is a closed Lie subalgebra of $\mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$, we can define by continuity a $B_{r,\mathbb{C}_p}$-module structure on $\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1} \cap \mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$, compatible with that on $\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$. Similarly we have

$$\begin{pmatrix} u^{-1}(1+T) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} u^{-1}(1+T) & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ u(1+T)^{-1}\mu & 1 \end{pmatrix}.$$

We note that $1+T$ is invertible in $A_{0,r}$ since $T = p^{s_h} t$ where $r_h = p^{-s_h}$. Therefore C_T is invertible and by twisting by $(1+T) \mapsto (1+T)^{-1}$ we can also give $\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1} \cap \mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})$ a structure of $B_{r,\mathbb{C}_p}$-module compatible with that on $\mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})$.

By combining the previous remarks with Corollary 4.23, we prove the following "fullness" result for the big Lie algebra $\mathfrak{H}_r$.

Theorem 6.2 *Suppose that the representation ρ is $(H_0, \mathbb{Z}_p)$-regular. Then there exists a nonzero ideal $\mathfrak{l}$ of $\mathbb{I}_0$, independent of $r < r_h$, such that for every such r the Lie algebra $\mathfrak{H}_r$ contains $\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r)$.*

Proof Since $U^{\pm}(\mathbb{B}_r) \cong \mathbb{B}_r$, we can and shall identify $\mathfrak{u}^+ = \mathbb{Q}_p \cdot \mathrm{Log}\, G_r' \cap \mathfrak{U}^+(\mathbb{B}_r)$ with a $\mathbb{Q}_p$-vector subspace of $\mathbb{B}_r$ (actually of $\mathbb{I}_0$), and $\mathfrak{u}^+_{\mathbb{C}_p}$ with a $\mathbb{C}_p$-vector subspace of $\mathbb{B}_{r,\mathbb{C}_p}$. We repeat that these spaces are independent of r since G_r' is, provided that $r_0 \leqslant r < r_h$ (see Remark 5.1). By Corollary 4.23, $\mathfrak{u}^{\pm} \cap \mathbb{I}_0$ contains a basis $\{e_{i,\pm}\}_{i\in I}$ for $Q(\mathbb{I}_0)$ over $Q(\Lambda_{h,0})$. The set $\{e_{i,+}\}_{i\in I} \subset \mathfrak{u}^+$ is a basis for $Q(\mathbb{I}_0)$ over $Q(\Lambda_{h,0})$, so $\mathfrak{u}^+$ contains the basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0$. By Lemma 4.19 we deduce that $\Lambda_{h,0}\mathfrak{u}^+$ contains a nonzero ideal $\mathfrak{a}^+$ of $\mathbb{I}_0$. Hence we also have $B_{r,\mathbb{C}_p}\mathfrak{u}^+_{\mathbb{C}_p} \supset B_{r,\mathbb{C}_p}\mathfrak{a}^+$. Now $\mathfrak{a}^+$ is an ideal of $\mathbb{I}_0$ and $B_{r,\mathbb{C}_p}\mathbb{I}_{0,\mathbb{C}_p} = \mathbb{B}_{r,\mathbb{C}_p}$, so $B_{r,\mathbb{C}_p}\mathfrak{a}^+ = \mathfrak{a}^+\mathbb{B}_{r,\mathbb{C}_p}$ is an ideal in $\mathbb{B}_{r,\mathbb{C}_p}$. We conclude that $B_{r,\mathbb{C}_p} \cdot \mathfrak{u}^+ \supset \mathfrak{a}^+\mathbb{B}_{r,\mathbb{C}_p}$ for a nonzero ideal $\mathfrak{a}^+$ of $\mathbb{I}_0$. We proceed in the same way for the lower unipotent subalgebra, obtaining $B_{r,\mathbb{C}_p} \cdot \mathfrak{u}^- \supset \mathfrak{a}^-\mathbb{B}_{r,\mathbb{C}_p}$ for some nonzero ideal $\mathfrak{a}^-$ of $\mathbb{I}_0$.

Consider now the Lie algebra $B_{r,\mathbb{C}_p}\mathfrak{H}_{\mathbb{C}_p} \subset \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Its nilpotent subalgebras are $B_{r,\mathbb{C}_p}\mathfrak{u}^+$ and $B_{r,\mathbb{C}_p}\mathfrak{u}^-$, and we showed $B_{r,\mathbb{C}_p}\mathfrak{u}^+ \supset \mathfrak{a}^+\mathbb{B}_{r,\mathbb{C}_p}$ and $B_{r,\mathbb{C}_p}\mathfrak{u}^- \supset \mathfrak{a}^-\mathbb{B}_{r,\mathbb{C}_p}$. Denote by $\mathfrak{t} \subset \mathfrak{sl}_2$ the subalgebra of diagonal matrices over $\mathbb{Z}$. By taking the Lie bracket, we see that $[\mathfrak{U}^+(\mathfrak{a}^+\mathbb{B}_{r,\mathbb{C}_p}), \mathfrak{U}^-(\mathfrak{a}^-\mathbb{B}_{r,\mathbb{C}_p})]$ spans $\mathfrak{a}^+ \cdot \mathfrak{a}^- \cdot \mathfrak{t}(\mathbb{B}_{r,\mathbb{C}_p})$ over $B_{r,\mathbb{C}_p}$. We deduce that $B_{r,\mathbb{C}_p}\mathfrak{H}_{\mathbb{C}_p} \supset \mathfrak{a}^+ \cdot \mathfrak{a}^- \cdot \mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Let $\mathfrak{a} = \mathfrak{a}^+ \cdot \mathfrak{a}^-$. Now $\mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$ is a $\mathbb{B}_{r,\mathbb{C}_p}$-Lie subalgebra of $\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Recall that $\beta_r \in \mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$; hence by stability by conjugation we have $\beta_r \left(\mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})\right) \beta_r^{-1} = \mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Thus, we constructed $\mathfrak{a}$ such that $B_{r,\mathbb{C}_p}\left(\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1}\right) \supset \mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. In particular, if $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}$ denote the unipotent subalgebras of $\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1}$, we have $B_{r,\mathbb{C}_p}\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p} \supset \mathfrak{a}\mathbb{B}_{r,\mathbb{C}_p}$ for both signs. By the discussion preceding the proposition the subalgebras $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}$ have a structure of $B_{r,\mathbb{C}_p}$-modules, which means that $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p} = B_{r,\mathbb{C}_p}\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}$. We conclude that $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p} \supset \beta_r \left(\mathfrak{a} \cdot \mathfrak{U}^{\pm}(\mathbb{B}_{r,\mathbb{C}_p})\right) \beta_r^{-1}$ for both signs. By the usual argument of taking the bracket, we obtain $\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1} \supset \mathfrak{a}^2 \cdot \mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. We can untwist by the invertible matrix β_r to conclude that, for $\mathfrak{l} = \mathfrak{a}^2$, we have $\mathfrak{H}_{r,\mathbb{C}_p} \supset \mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$.

Let us get rid of the completed extension of scalars to $\mathbb{C}_p$. For every ideal $\mathfrak{a} \subset \mathbb{I}_{0,r}$ not dividing P_1, let $\mathfrak{H}_{r,\mathfrak{a}}$ be the image of $\mathfrak{H}_r$ in $\mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$. Consider the two finite dimensional $\mathbb{Q}_p$-vector spaces $\mathfrak{H}_{r,\mathfrak{a}}$ and $\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$. Note that they are both subspaces of the finite dimensional $\mathbb{Q}_p$-vector space $\mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$. After extending scalars to $\mathbb{C}_p$, we have

$$\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}) \otimes \mathbb{C}_p \subset \mathfrak{H}_{r,\mathfrak{a}} \otimes \mathbb{C}_p. \tag{8}$$

Let $\{e_i\}_{i \in I}$ be an orthonormal basis of the Banach space $\mathbb{C}_p$ over $\mathbb{Q}_p$, with I some index set, such that $1 \in \{e_i\}_{i \in I}$. Let $\{v_j\}_{j=1,\ldots,n}$ be a $\mathbb{Q}_p$-basis of $\mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$ such that, for some $d \leqslant n$, $\{v_j\}_{j=1,\ldots,d}$ is a $\mathbb{Q}_p$-basis of $\mathfrak{H}_{r,\mathfrak{a}}$.

Let v be an element of $\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$. Then $v \otimes 1 \in \mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}) \otimes \mathbb{C}_p$ and by (8) we have $v \otimes 1 \in \mathfrak{H}_{r,\mathfrak{a}} \otimes \mathbb{C}_p$. As $\{v_j \otimes e_i\}_{1 \leqslant j \leqslant d, i \in I}$, and $\{v_j \otimes e_i\}_{1 \leqslant j \leqslant n, i \in I}$ are orthonormal bases of $\mathfrak{H}_{r,\mathfrak{a}} \otimes \mathbb{C}_p$, and $\mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}) \otimes \mathbb{C}_p$ over $\mathbb{Q}_p$, respectively there exist $\lambda_{j,i} \in \mathbb{Q}_p$, $(j, i) \in \{1, 2, ..., d\} \times I$ converging to 0 in the filter of complements of finite subsets of $\{1, 2, ..., d\} \times I$ such that $v \otimes 1 = \sum_{j=1,\ldots,d;\, i \in I} \lambda_{j,i}(v_j \otimes e_i)$.

But $v \otimes 1 \in \mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}) \otimes 1 \subset \mathrm{M}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}) \otimes \mathbb{C}_p$ and therefore $v \otimes 1 = \sum_{1 \leqslant j \leqslant n} a_j(v_j \otimes 1)$, for some $a_j \in \mathbb{Q}_p$, $j = 1, ..., n$. By the uniqueness of a representation of an element in a $\mathbb{Q}_p$-Banach space in terms of a given orthonormal basis we have

$$v \otimes 1 = \sum_{j=1}^{d} a_j(v_j \otimes 1), \quad \text{i.e.} \quad v = \sum_{j=1}^{d} a_j v_j \in \mathfrak{H}_{r,\mathfrak{a}}.$$

By taking the projective limit over $\mathfrak{a}$, we conclude that

$$\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r) \subset \mathfrak{H}_r.$$

□

Definition 6.3 The Galois level of the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ is the largest ideal $\mathfrak{l}_\theta$ of $\mathbb{I}_0[P_1^{-1}]$ such that $\mathfrak{H}_r \supset \mathfrak{l}_\theta \cdot \mathfrak{sl}_2(\mathbb{B}_r)$ for all $r < r_h$.

It follows by the previous remarks that $\mathfrak{l}_\theta$ is nonzero.

7 Comparison Between the Galois Level and the Fortuitous Congruence Ideal

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a slope $\leqslant h$ family. We keep all the notations from the previous sections. In particular $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ is the Galois representation associated with θ. We suppose that the restriction of ρ to H_0 takes values in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$. Recall that

$\mathbb{I} = \mathbb{I}^\circ[p^{-1}]$ and $\mathbb{I}_0 = \mathbb{I}_0^\circ[p^{-1}]$. Also recall that P_1 is the prime of $\Lambda_{h,0}$ generated by $u^{-1}(1+T) - 1$. Let $\mathfrak{c} \subset \mathbb{I}$ be the congruence ideal associated with θ. Set $\mathfrak{c}_0 = \mathfrak{c} \cap \mathbb{I}_0$ and $\mathfrak{c}_1 = \mathfrak{c}_0\mathbb{I}_0[P_1^{-1}]$. Let $\mathfrak{l} = \mathfrak{l}_\theta \subset \mathbb{I}_0[P_1^{-1}]$ be the Galois level of θ. For an ideal $\mathfrak{a}$ of $\mathbb{I}_0[P_1^{-1}]$ we denote by $V(\mathfrak{a})$ the set of prime ideals of $\mathbb{I}_0[P_1^{-1}]$ containing $\mathfrak{a}$. We prove the following.

Theorem 7.1 *Suppose that*

1. *ρ is $(H_0, \mathbb{Z}_p)$-regular;*
2. *there exists no pair (F, ψ), where F is a real quadratic field and $\psi\colon \mathrm{Gal}(\overline{F}/F) \to \mathbb{F}^\times$ is a character, such that $\overline{\rho}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F}) \cong \mathrm{Ind}_F^\mathbb{Q}\psi$.*

Then we have $V(\mathfrak{l}) = V(\mathfrak{c}_1)$.

Before giving the proof we make some remarks. Let P be a prime of $\mathbb{I}_0[P_1^{-1}]$ and Q be a prime factor of $P\mathbb{I}[P_1^{-1}]$. We consider ρ as a representation $G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}[P_1^{-1}])$ by composing it with the inclusion $\mathrm{GL}_2(\mathbb{I}) \hookrightarrow \mathrm{GL}_2(\mathbb{I}[P_1^{-1}])$. We have a representation $\rho_Q\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}[P_1^{-1}]/Q)$ obtained by reducing ρ modulo Q. Its restriction $\rho_Q|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/(Q \cap \mathbb{I}_0[P_1^{-1}])) = \mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/P)$ and coincides with the reduction ρ_P of $\rho|_{H_0}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}])$ modulo P. In particular $\rho_Q|_{H_0}$ is independent of the chosen prime factor Q of $P\mathbb{I}[P_1^{-1}]$.

We say that a subgroup of $\mathrm{GL}_2(A)$ for some algebra A finite over a p-adic field K is *small* if it admits a finite index abelian subgroup. Let P, Q be as above, G_P be the image of $\rho_P\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/P)$ and G_Q be the image of $\rho_Q\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}[P_1^{-1}]/Q)$. By our previous remark ρ_P coincides with the restriction $\rho_Q|_{H_0}$, so G_P is a finite index subgroup of G_Q for every Q. In particular G_P is small if and only if G_Q is small for all prime factors Q of $P\mathbb{I}[P_1^{-1}]$.

Now if Q is a CM point the representation ρ_Q is induced by a character of $\mathrm{Gal}(F/\mathbb{Q})$ for an imaginary quadratic field F. Hence G_Q admits an abelian subgroup of index 2 and G_P is also small.

Conversely, if G_P is small, $G_{Q'}$ is small for every prime Q' above P. Choose any such prime Q'; by the argument in [16, Proposition 4.4] $G_{Q'}$ has an abelian subgroup of index 2. It follows that $\rho_{Q'}$ is induced by a character of $\mathrm{Gal}(\overline{F}_{Q'}/F_{Q'})$ for a quadratic field $F_{Q'}$. If $F_{Q'}$ is imaginary then Q' is a CM point. In particular, if we suppose that the residual representation $\bar{\rho}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ is not induced by a character of $\mathrm{Gal}(\overline{F}/F)$ for a real quadratic field $F/\mathbb{Q}$, then $F_{Q'}$ is imaginary and Q' is CM. The above argument proves that G_P is small if and only if all points $Q' \subset \mathbb{I}[P_1^{-1}]$ above P are CM.

Proof We prove first that $V(\mathfrak{c}_1) \subset V(\mathfrak{l})$. Fix a radius $r < r_h$. By contradiction, suppose that a prime P of $\mathbb{I}_0[P_1^{-1}]$ contains $\mathfrak{c}_0$ but P does not contain $\mathfrak{l}$. Then there exists a prime factor Q of $P\mathbb{I}[P_1^{-1}]$ such that $\mathfrak{c} \subset Q$. By definition of $\mathfrak{c}$ we have that Q is a CM point in the sense of Sect. 3.4, hence the representation $\rho_{\mathbb{I}[P_1^{-1}],Q}$ has small image in $\mathrm{GL}_2(\mathbb{I}[P_1^{-1}]/Q)$. Then its restriction $\rho_{\mathbb{I}[P_1^{-1}],Q}|_{H_0} = \rho_P$ also has small image in $\mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/P)$. We deduce that there is no nonzero ideal $\mathfrak{I}_P$ of $\mathbb{I}_0[P_1^{-1}]/P$ such that the Lie algebra $\mathfrak{H}_{r,P}$ contains $\mathfrak{I}_P \cdot \mathfrak{sl}_2(\mathbb{I}_0[P_1^{-1}]/P)$.

Now by definition of $\mathfrak{l}$ we have $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B}_r)\subset\mathfrak{H}_r$. Since reduction modulo P gives a surjection $\mathfrak{H}_r\to\mathfrak{H}_{r,P}$, by looking at the previous inclusion modulo P we find $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r}[P_1^{-1}]/P\mathbb{I}_{0,r}[P_1^{-1}])\subset\mathfrak{H}_{r,P}$. If $\mathfrak{l}\not\subset P$ we have $\mathfrak{l}/P\neq 0$, which contradicts our earlier statement. We deduce that $\mathfrak{l}\subset P$.

We prove now that $V(\mathfrak{l})\subset V(\mathfrak{c}_1)$. Let $P\subset\mathbb{I}_0[P_1^{-1}]$ be a prime containing $\mathfrak{l}$. Recall that $\mathbb{I}_0[P_1^{-1}]$ has Krull dimension one, so $\kappa_P=\mathbb{I}_0[P_1^{-1}]/P$ is a field. Let Q be a prime of $\mathbb{I}[P_1^{-1}]$ above P. As before ρ reduces to representations $\rho_Q\colon G_\mathbb{Q}\to\mathrm{GL}_2(\mathbb{I}[P_1^{-1}]/Q)$ and $\rho_P\colon H_0\to\mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/P)$. Let $\mathfrak{P}\subset\mathbb{I}_0[P_1^{-1}]$ be the P-primary component of $\mathfrak{l}$ and let $\mathfrak{A}$ be an ideal of $\mathbb{I}_0[P_1^{-1}]$ containing $\mathfrak{P}$ such that the localization at P of $\mathfrak{A}/\mathfrak{P}$ is one-dimensional over κ_P. Choose any $r<r_h$. Let $\mathfrak{s}=\mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r}[P_1^{-1}]/\mathfrak{P})\cap\mathfrak{H}_{r,\mathfrak{P}}$, that is a Lie subalgebra of $\mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r}[P_1^{-1}]/\mathfrak{P})$.

We show that $\mathfrak{s}$ is stable under the adjoint action $\mathrm{Ad}(\rho_Q)$ of $G_\mathbb{Q}$. Let $\mathfrak{Q}$ be the Q-primary component of $\mathfrak{l}\cdot\mathbb{I}[P_1^{-1}]$. Recall that $\mathfrak{H}_{r,\mathfrak{P}}$ is the Lie algebra associated with the pro-p group $\mathrm{Im}\,\rho_{r,\mathfrak{Q}}|_{H_0}\cap\Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r_0}[P_1^{-1}]/\mathfrak{P})}(p)\subset\mathrm{GL}_2(\mathbb{I}_{0,r}[P_1^{-1}]/\mathfrak{P})$. Since this group is open in $\mathrm{Im}\,\rho_{r,\mathfrak{Q}}\subset\mathrm{GL}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{Q})$, the Lie algebra associated with $\mathrm{Im}\,\rho_{r,\mathfrak{Q}}$ is again $\mathfrak{H}_{r,\mathfrak{P}}$. In particular $\mathfrak{H}_{r,\mathfrak{P}}$ is stable under $\mathrm{Ad}(\rho_Q)$. Since $\mathfrak{H}_{r,\mathfrak{P}}\subset\mathfrak{sl}_2(\mathbb{I}_{0,r}[P_1^{-1}]/\mathfrak{P})$ we have $\mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r}[P_1^{-1}]/\mathfrak{P})\cap\mathfrak{H}_{r,\mathfrak{P}}=\mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{Q})\cap\mathfrak{H}_{r,\mathfrak{P}}$. Now $\mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{Q})$ is clearly stable under $\mathrm{Ad}(\rho_Q)$, so the same is true for $\mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{Q})\cap\mathfrak{H}_{r,\mathfrak{P}}$, as desired.

We consider from now on $\mathfrak{s}$ as a Galois representation via $\mathrm{Ad}(\rho_Q)$. By the proof of Theorem 6.2 we can assume, possibly considering a sub-Galois representation, that $\mathfrak{H}_r$ is a $\mathbb{B}_r$-submodule of $\mathfrak{sl}_2(\mathbb{B}_r)$ containing $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B}_r)$ but not $\mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_r)$ for any $\mathfrak{a}$ strictly bigger than $\mathfrak{l}$. This allows us to speak of the localization $\mathfrak{s}_P$ of $\mathfrak{s}$ at P. Note that, since $\mathfrak{P}$ is the P-primary component of $\mathfrak{l}$ and $\mathfrak{A}_P/\mathfrak{P}_P\cong\kappa_P$, when P-localizing we find $\mathfrak{H}_{r,P}\supset\mathfrak{P}_P\cdot\mathfrak{sl}_2(\mathbb{B}_{r,P})$ and $\mathfrak{H}_{r,P}\not\supset\mathfrak{A}_P\cdot\mathfrak{sl}_2(\mathbb{B}_{r,P})$.

The localization at P of $\mathfrak{a}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r}[P_1^{-1}]/\mathfrak{P})$ is $\mathfrak{sl}_2(\kappa_P)$, so $\mathfrak{s}_P$ is contained in $\mathfrak{sl}_2(\kappa_P)$. It is a κ_P-representation of $G_\mathbb{Q}$ (via $\mathrm{Ad}(\rho_Q)$) of dimension at most 3. We distinguish various cases following its dimension.

We cannot have $\mathfrak{s}_P=0$. By exchanging the quotient with the localization we would obtain $(\mathfrak{A}_P\cdot\mathfrak{sl}_2(\mathbb{B}_{r,P})\cap\mathfrak{H}_{r,P})/\mathfrak{P}_P=0$. By Nakayama's lemma $\mathfrak{A}_P\cdot\mathfrak{sl}_2(\mathbb{B}_{r,P})\cap\mathfrak{H}_{r,P}=0$, which is absurd since $\mathfrak{A}_P\cdot\mathfrak{sl}_2(\mathbb{B}_{r,P})\cap\mathfrak{H}_{r,P}\supset\mathfrak{P}_P\cdot\mathfrak{sl}_2(\mathbb{B}_{r,P})\neq 0$.

We also exclude the three-dimensional case. If $\mathfrak{s}_P=\mathfrak{sl}_2(\kappa_P)$, by exchanging the quotient with the localization we obtain $(\mathfrak{A}_P\cdot\mathfrak{sl}_2(\mathbb{B}_{r,P})\cap\mathfrak{H}_{r,P})/\mathfrak{P}_P=(\mathfrak{A}_P\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r,P}[P_1^{-1}]))/\mathfrak{P}_P\mathbb{I}_{0,r,P}[P_1^{-1}]$, because we have $\mathfrak{A}_P\mathbb{I}_{0,r,P}[P_1^{-1}]/\mathfrak{P}_P\mathbb{I}_{0,r,P}[P_1^{-1}]=\left(\mathbb{I}_{0,r,P}[P_1^{-1}]/\mathfrak{P}_P\mathbb{I}_{0,r,P}[P_1^{-1}]\right)$ and this is isomorphic to κ_P. By Nakayama's lemma we would conclude that $\mathfrak{H}_{r,P}\supset\mathfrak{A}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,P})$, which is absurd.

We are left with the one and two-dimensional cases. If $\mathfrak{s}_P$ is two-dimensional we can always replace it by its orthogonal in $\mathfrak{sl}_2(\kappa_P)$ which is one-dimensional; indeed the action of $G_\mathbb{Q}$ via $\mathrm{Ad}(\rho_Q)$ is isometric with respect to the scalar product $\mathrm{Tr}(XY)$ on $\mathfrak{sl}_2(\kappa_P)$.

Suppose that $\mathfrak{sl}_2(\kappa_P)$ contains a one-dimensional stable subspace. Let ϕ be a generator of this subspace over κ_P. Let $\chi\colon G_\mathbb{Q}\to\kappa_P$ denote the character satisfying $\rho_Q(g)\phi\rho_Q(g)^{-1}=\chi(g)\phi$ for all $g\in G_\mathbb{Q}$. Now ϕ induces a nontrivial morphism of representations $\rho_Q\to\rho_Q\otimes\chi$. Since ρ_Q and $\rho_Q\otimes\chi$ are irreducible, by Schur's lemma ϕ must be invertible. Hence we obtain an isomorphism $\rho_Q\cong\rho_Q\otimes\chi$. By taking determinants we see that χ must be quadratic. If $F_0/\mathbb{Q}$ is the quadratic extension fixed by $\ker\chi$, then ρ_Q is induced by a character ψ of $\mathrm{Gal}(\overline{F_0}/F_0)$. By assumption the residual representation $\rho_{\mathfrak{m}_{\mathbb{I}}}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ is not of the form $\mathrm{Ind}_F^{\mathbb{Q}}\psi$ for a real quadratic field F and a character $\mathrm{Gal}(\overline{F}/F) \to \mathbb{F}^\times$. We deduce that F_0 must be imaginary, so Q is a CM point by Remark 3.11(1). By construction of the congruence ideal $\mathfrak{c} \subset Q$ and $\mathfrak{c}_0 \subset Q \cap \mathbb{I}_0[P_1^{-1}] = P$. □

We prove a corollary.

Corollary 7.2 *If the residual representation* $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ *is not dihedral then* $\mathfrak{l} = 1$.

Proof Since $\overline{\rho}$ is not dihedral there cannot be any CM point on the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. By Theorem 7.1 we deduce that $\mathfrak{l}$ has no nontrivial prime factor, hence it is trivial. □

Remark 7.3 Theorem 7.1 gives another proof of Proposition 3.9. Indeed the CM points of a family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ correspond to the prime factors of its Galois level, which are finite in number.

We also give a partial result about the comparison of the exponents of the prime factors in $\mathfrak{c}_1$ and $\mathfrak{l}$. This is an analogous of what is proved in [9, Theorem 8.6] for the ordinary case; our proof also relies on the strategy there. For every prime P of $\mathbb{I}_0[P_1^{-1}]$ we denote by $\mathfrak{c}_1^P$ and $\mathfrak{l}^P$ the P-primary components of $\mathfrak{c}_1$ and $\mathfrak{l}$ respectively.

Theorem 7.4 *Suppose that* $\overline{\rho}$ *is not induced by a character of* G_F *for a real quadratic field* $F/\mathbb{Q}$*. We have* $(\mathfrak{c}_1^P)^2 \subset \mathfrak{l}^P \subset \mathfrak{c}_1^P$.

Proof The inclusion $\mathfrak{l}^P \subset \mathfrak{c}_1^P$ is proved in the same way as the first inclusion of Theorem 7.1.

We show that the inclusion $(\mathfrak{c}_1^P)^2 \subset \mathfrak{l}^P$ holds. If $\mathfrak{c}_1^P$ is trivial this reduces to Theorem 7.1, so we can suppose that P is a factor of $\mathfrak{c}_1$. Let Q denote any prime of $\mathbb{I}[P_1^{-1}]$ above P. Let $\mathfrak{c}_1^Q$ be a Q-primary ideal of $\mathbb{I}[P_1^{-1}]$ satisfying $\mathfrak{c}_1^Q \cap \mathbb{I}_0[P_1^{-1}] = \mathfrak{c}_1^P$. Since P divides $\mathfrak{c}_1$, Q is a CM point, so we have an isomorphism $\rho_P \cong \mathrm{Ind}_F^{\mathbb{Q}}\psi$ for an imaginary quadratic field $F/\mathbb{Q}$ and a character $\psi\colon G_F \to \mathbb{C}_p^\times$. Choose any $r < r_h$. Consider the κ_P-vector space $\mathfrak{s}_{\mathfrak{c}_1^P} = \mathfrak{H}_r \cap \mathfrak{c}_1^P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r})/\mathfrak{H}_r \cap \mathfrak{c}_1^P P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r})$. We see it as a subspace of $\mathfrak{sl}_2(\mathfrak{c}_1^P/\mathfrak{c}_1^P P) \cong \mathfrak{sl}_2(\kappa_P)$. By the same argument as in the proof of Theorem 7.1, $\mathfrak{s}_{\mathfrak{c}_1^P}$ is stable under the adjoint action $\mathrm{Ad}(\rho_{\mathfrak{c}_1^Q Q})\colon G_{\mathbb{Q}} \to \mathrm{Aut}(\mathfrak{sl}_2(\kappa_P))$.

Let $\chi_{F/\mathbb{Q}}\colon G_{\mathbb{Q}} \to \mathbb{C}_p^\times$ be the quadratic character defined by the extension $F/\mathbb{Q}$. Let $\varepsilon \in G_{\mathbb{Q}}$ be an element projecting to the generator of $\mathrm{Gal}(F/\mathbb{Q})$. Let $\psi^\varepsilon\colon G_F \to \mathbb{C}_p^\times$ be given by $\psi^\varepsilon(\tau) = \psi(\varepsilon\tau\varepsilon^{-1})$. Set $\psi^- = \psi/\psi^\varepsilon$. Since $\rho_Q \cong \mathrm{Ind}_F^{\mathbb{Q}}\psi$, we have a decomposition $\mathrm{Ad}(\rho_Q) \cong \chi_{F/\mathbb{Q}} \oplus \mathrm{Ind}_F^{\mathbb{Q}}\psi^-$, where the two factors are irreducible. Now we have three possibilities for the Galois isomorphism class of $\mathfrak{s}_{\mathfrak{c}_1^P}$: it is either that of $\mathrm{Ad}(\rho_Q)$ or that of one of the two irreducible factors.

If $\mathfrak{s}_{\mathfrak{c}_1^P} \cong \mathrm{Ad}(\rho_Q)$, then as κ_P-vector spaces $\mathfrak{s}_{\mathfrak{c}_1^P} = \mathfrak{sl}_2(\kappa_P)$. By Nakayama's lemma $\mathfrak{H}_r \supset \mathfrak{c}_1^P \cdot \mathfrak{sl}_2(\mathbb{B}_r)$. This implies $\mathfrak{c}_1^P \subset \mathfrak{l}^P$, hence $\mathfrak{c}_1^P = \mathfrak{l}^P$ in this case.

If $\mathfrak{s}_{\mathfrak{c}_1^P}$ is one-dimensional then we proceed as in the proof of Theorem 7.1 to show that $\rho_{\mathfrak{c}_1^Q Q}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{c}_1^Q Q\mathbb{I}_r[P_1^{-1}])$ is induced by a character $\psi_{\mathfrak{c}_1^Q Q}\colon G_F \to \mathbb{C}_p^\times$. In particular the image of $\rho_{\mathfrak{c}_1^P P}\colon H \to \mathrm{GL}_2(\mathbb{I}_{0,r}[P_1^{-1}]/\mathfrak{c}_1^P P\mathbb{I}_{0,r})$ is small. This is a contradiction, since $\mathfrak{c}_1^P$ is the P-primary component of $\mathfrak{c}_1$, hence it is the smallest P-primary ideal $\mathfrak{A}$ of $\mathbb{I}_{0,r}[P_1^{-1}]$ such that the image of $\rho_{\mathfrak{A}}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{A}\mathbb{I}_r[P_1^{-1}])$ is small.

Finally, suppose that $\mathfrak{s}_{\mathfrak{c}_1^P} \cong \mathrm{Ind}_F^{\mathbb{Q}}\psi^-$. Let $d = \mathrm{diag}(d_1, d_2) \in \rho(G_{\mathbb{Q}})$ be the image of a $\mathbb{Z}_p$-regular element. Since d_1 and d_2 are nontrivial modulo the maximal ideal of $\mathbb{I}_0^\circ$, the image of d modulo $\mathfrak{c}_1^Q Q$ is a nontrivial diagonal element $d_{\mathfrak{c}_1^Q Q} = \mathrm{diag}(d_{1,\mathfrak{c}_1^Q Q}, d_{2,\mathfrak{c}_1^Q Q}) \in \rho_{\mathfrak{c}_1^Q Q}(G_{\mathbb{Q}})$. We decompose $\mathfrak{s}_{\mathfrak{c}_1^P}$ in eigenspaces for the adjoint action of $d_{\mathfrak{c}_1^Q Q}$: we write $\mathfrak{s}_{\mathfrak{c}_1^P} = \mathfrak{s}_{\mathfrak{c}_1^P}[a] \oplus \mathfrak{s}_{\mathfrak{c}_1^P}[1] \oplus \mathfrak{s}_{\mathfrak{c}_1^P}[a^{-1}]$, where $a = d_{1,\mathfrak{c}_1^Q Q}/d_{2,\mathfrak{c}_1^Q Q}$. Now $\mathfrak{s}_{\mathfrak{c}_1^P}[1]$ is contained in the diagonal torus, on which the adjoint action of $G_{\mathbb{Q}}$ is given by the character $\chi_{F/\mathbb{Q}}$. Since $\chi_{F/\mathbb{Q}}$ does not appear as a factor of $\mathfrak{s}_{\mathfrak{c}_1^P}$, we must have $\mathfrak{s}_{\mathfrak{c}_1^P}[1] = 0$. This implies that $\mathfrak{s}_{\mathfrak{c}_1^P}[a] \neq 0$ and $\mathfrak{s}_{\mathfrak{c}_1^P}[a^{-1}] \neq 0$. Since $\mathfrak{s}_{\mathfrak{c}_1^P}[a] = \mathfrak{s}_{\mathfrak{c}_1^P} \cap \mathfrak{u}^+(\kappa_P)$ and $\mathfrak{s}_{\mathfrak{c}_1^P}[a^{-1}] = \mathfrak{s}_{\mathfrak{c}_1^P} \cap \mathfrak{u}^-(\kappa_P)$, we deduce that $\mathfrak{s}_{\mathfrak{c}_1^P}$ contains nontrivial upper and lower nilpotent elements $\overline{u^+}$ and $\overline{u^-}$. Then $\overline{u^+}$ and $\overline{u^-}$ are the images of some elements u^+ and u^- of $\mathfrak{H}_r \cap \mathfrak{c}_1^P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}[P_1^{-1}])$ nontrivial modulo $\mathfrak{c}_1^P P$. The Lie bracket $t = [u^+, u^-]$ is an element of $\mathfrak{H}_r \cap \mathfrak{t}(\mathbb{I}_{0,r}[P_1^{-1}])$ (where $\mathfrak{t}$ denotes the diagonal torus) and it is nontrivial modulo $(\mathfrak{c}_1^P)^2 P$. Hence the κ_P-vector space $\mathfrak{s}_{(\mathfrak{c}_1^P)^2} = \mathfrak{H}_r \cap (\mathfrak{c}_1^P)^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r,\mathbb{C}_p}[P_1^{-1}])/\mathfrak{H}_r \cap (\mathfrak{c}_1^P)^2 P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r,\mathbb{C}_p}[P_1^{-1}])$ contains nontrivial diagonal, upper nilpotent and lower nilpotent elements, so it is three-dimensional. By Nakayama's lemma we conclude that $\mathfrak{H}_r \supset (\mathfrak{c}_1^P)^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}[P_1^{-1}])$, so $(\mathfrak{c}_1^P)^2 \subset \mathfrak{l}^P$. □

References

1. Bellaïche, J.: Eigenvarieties and adjoint p-adic L-functions (preprint)
2. Bellaïche, J., Chenevier, G.: Families of Galois Representations and Selmer groups. Astérisque 324, Soc. Math, France (2009)
3. Buzzard, K.: Eigenvarieties, in L-functions and Galois representations. In: Proceedings of Conference Durham: LMS Lect. Notes Series 320. Cambridge University Press 2007, pp. 59–120 (2004)
4. Chenevier, G.: Familles p-adiques de formes automorphes pour GL(n). J. Reine Angew. Math. **570**, 143–217 (2004)
5. Coleman, R.: Classical and overconvergent modular forms. Invent. Math. **124**, 214–241 (1996)
6. Coleman, R., Mazur, B.: The eigencurve. In: Galois Representations in Arithmetic Algebraic Geometry, London Math. Soc. Lecture Note Ser. 254, Cambridge University Press, Cambridge, pp. 1–113 (1998)
7. de Jong, A.J.: Crystalline Dieudonné theory via formal and rigid geometry. Publ. Math. Inst. Hautes Études Sci. **82**(1), 5–96 (1995)
8. Faltings, G.: Hodge-Tate structures and modular forms. Math. Ann. **278**, 133–149 (1987)
9. Hida, H.: Big Galois representations and p-adic L-functions. Compos. Math. **151**, 603–654 (2015)
10. Hida, H., Tilouine, J.: Big image of Galois representations and congruence ideals. In: Dieulefait, L., Heath-Brown, D.R., Faltings, G., Manin, Y.I., Moroz, B.Z., Wintenberger, J.-P. (eds.)

Arithmetic Geometry, Proceedings of Workshop on Serre's Conjecture, Hausdorff Inst. Math., Bonn. Cambridge University Press, pp. 217–254 (2015)
11. Kisin, M.: Overconvergent modular forms and the Fontaine-Mazur conjecture. Invent. Math. **153**, 363–454 (2003)
12. Lang, J.: On the image of the Galois representation associated to a non-CM Hida family. Algebra Number Theory **10**(1), 155–194 (2016)
13. Momose, F.: On the ℓ-adic representations attached to modular forms. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28**(1), 89–109 (1981)
14. Nyssen, L.: Pseudo-representations. Math. Ann. **306**(2), 257–284 (1996)
15. Ribet, K.: On ℓ-adic representations attached to modular forms. Invent. Math. **28**, 245–275 (1975)
16. Ribet, K.: Galois representations attached to modular forms with Nebentypus. In: Modular functions of one variable V, Lecture Notes in Math., vol. 601, pp. 17–51. Springer (1977)
17. Ribet, K.: On ℓ-adic representations attached to modular forms. II. Glasgow Math. J. **27**, 185–194 (1985)
18. Rouquier, R.: Caractérisation des caractères et pseudo-caractères. J. Algebra **180**, 571–586 (1996)
19. Sen, S.: Lie algebras of Galois groups arising from Hodge-Tate modules. Ann. Math. **97**(1), 160–170 (1973)
20. Sen, S.: Continuous cohomology and p-adic Hodge theory. Invent. Math. **62**(1), 89–116 (1980)
21. Sen, S.: An infinite dimensional Hodge-Tate theory. Bull. Soc. Math. France **121**, 13–34 (1993)
22. Shimura, G.: On elliptic curves with complex multiplication as factors of the Jacobians of modular function fields. Nagoya Math. J. **43**, 199–208 (1971)
23. Tazhetdinov, S.: Subnormal structure of two-dimensional linear groups over local rings. Algebra i Logika **22**(6), 707–713 (1983)
24. Wan, D.: Dimension variation of classical and p-adic modular forms. Invent. Math. **133**, 449–463 (1998)