Chapter 6
Output Tracking Control of Constrained Switched Nonlinear Systems

6.1 Background and Motivation

Control systems often suffer from various limits or constraints in the operation space [1, 2], that may arise out of performance requirements or physical constraints imposed on the system by its environmental regulations. For instance, the restoring torque of an aircraft certainly has a maximum value, as has the armature of a DC motor [3]. If the constraints are destroyed during operation, then serious consequences causing performance degradation, hazards or system damage will happen. Therefore, tackling constraints in control design has attracted much attention from various fields in science and engineering.

In the study of constrained linear or nonlinear systems, different approaches have been presented over the last a few years. To handle both state and input constraints in linear systems, many techniques have been developed (see, e.g., [4–6]), most of which are based on the notions of set invariance and admissible set control [7, 8]. Model Predictive Control that represents an effective control design methodology for handling both constraints and performance issues has been investigated in [9, 10]. In addition, reference governors have also been proposed to tackle the problem of constraints for nonlinear systems in [11]. The approaches mentioned above are numerical in nature or depend heavily on computationally intensive algorithms to solve the control problems.

It is worth pointing out that Barrier Lyapunov Functions (BLFs), which have been proposed in [12, 13], can be used to handle constraints. In the method, output constraints are handled directly during the controller design procedure. The proposed design procedure is flexible and can handle bounded uncertainties in the system. However, a resulting problem is that the constructed asymmetric BLF is of a switching type, a C^1 function. Consequently, the subsequent stabilizing functions must be of a high power. Furthermore, p-times differentiable unbounded functions are first introduced in [14] to handle the output tracking error constraints for a class of nonlinear systems in a lower triangular form. The advantage of the p-times differentiable

X. Zhao et al., *Control Synthesis of Switched Systems*, Studies in Systems, Decision and Control 80, DOI 10.1007/978-3-319-44830-5_6

unbounded function method is that in the controller design procedure, switching is not needed despite the asymmetrical limit range.

Note that control problems for switched systems with constraints have been investigated recently. Time optimal control for a class of integrator switched systems with state constraints was considered in [15]. A predictive control framework for a class of nonlinear switched systems subject to state and control constraints was presented in [16].

In this chapter, we aim at the problem of output tracking control for a class of constrained nonlinear switched systems in lower triangular form. By ensuring boundedness of the employed BLFs in the closed-loop, we assure that the constraints are not exceeded. Under the simultaneous domination assumption, we construct continuous feedback controllers for the switched system, which render that asymptotic output tracking is achieved, the limits are not transgressed and all closed-loop signals keep bounded. Moreover, we also explore the use of p-times differentiable unbounded functions to deal with asymmetric output constraints.

Notations: We use the following notations throughout this chapter. $\mathbb{R}_+$ denotes the set of nonnegative real numbers, $\mathbb{R}^n$ represents the n-dimensional real Euclidean vector space and $\| \bullet \|$ stands for the Euclidean vector norm. For positive integers i, j, we also denote $\bar{x}_i = [x_1, x_2, \ldots, x_i]^T$, $\bar{z}_i = [z_1, z_2, \ldots, z_i]^T$, $z_{i:j} = [z_i, z_{i+1}, \ldots, z_j]^T$, $\tilde{y}_{d_i} = [y_d, y_d^{(1)}, y_d^{(2)}, \ldots, y_d^{(i)}]^T$, $\tilde{b}_1^{(i)} = [b_1, b_1^{(1)}, b_1^{(2)}, \ldots, b_1^{(i)}]^T$ and $\tilde{b}_2^{(i)} = [b_2, b_2^{(1)}, b_2^{(2)}, \ldots, b_2^{(i)}]^T$, respectively.

6.2 Barrier Lyapunov Functions-Based Control Design

6.2.1 Problem Formulation and Preliminaries

Consider a class of switched nonlinear systems described by:

$$
\begin{aligned}
\dot{x}_1 &= f_1^{\sigma(t)}(x_1) + x_2, \\
&\cdots \\
\dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i) + x_{i+1}, \\
&\cdots \\
\dot{x}_{n-1} &= f_{n-1}^{\sigma(t)}(\bar{x}_{n-1}) + x_n, \\
\dot{x}_n &= f_n^{\sigma(t)}(\bar{x}_n) + g^{\sigma(t)}(\bar{x}_n)u, \\
y &= x_1,
\end{aligned}
\tag{6.1}
$$

where $x_1, x_2, \ldots, x_n$ are the states, $u = [u_1, u_2, \ldots, u_q]^T \in \mathbb{R}^q$ and $y \in \mathbb{R}$ are the input and output, respectively. $\sigma(t)$ is the switching signal, which takes its values in a finite set $I_m = \{1, 2, \ldots, m\}$ where $m > 1$ is the number of subsystems. $\forall i =$

$1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$; functions f_i^k, g^k are smooth with $g^k(\bar{x}_n) \neq 0, \forall \bar{x}_n \in R^n$. The output is required to satisfy certain constraints that are specified later.

For system (6.1), we design a feedback controller by using the information of all the states and a desired trajectory $y_d(t)$ such that $\lim_{t\to\infty}(y(t) - y_d(t)) = 0$ under arbitrary switchings.

The control objective is to solve the output tracking control problem guaranteeing all closed-loop signals to be bounded without exceeding the constraints.

To avoid the violation of the constraints, we employ a BLF with the following definition.

Definition 6.1 ([13]) A BLF is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region D containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of D, has the property $V(x) \to \infty$ as x approaches the boundary of D, and satisfies $V(x) \leq b, \forall t \geq 0$ along the solution $\dot{x} = f(x)$ for $x(0) \in D$ and some positive constant b.

It is worth pointing out that the Lyapunov function $V(x)$ in Definition 6.1 can be extended to be time-varying when the constraints are time-varying.

The following lemma that establishes a result of barrier function is first proposed for the subsequent developments.

Lemma 6.1 *For any positive constants $b_i, i = 1, 2, \ldots, n$, let $Z = \{\bar{z}_n \in \mathbb{R}^n : |z_i| < b_i, i = 1, \ldots, n\} \subset \mathbb{R}^n$ be an open set. Consider the switched system:*

$$\dot{\bar{z}}_n = h_{\sigma(t)}(t, \bar{z}_n), \tag{6.2}$$

where $\sigma(t)$ is the same as in (6.1); $h_i : \mathbb{R}_+ \times Z \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in η, uniformly in t, on $\mathbb{R}_+ \times Z$. We assume that the state of the system (6.2) does not jump at the switching instants. Let $Z_i = \{z_i \in \mathbb{R} : |z_i| < b_i\} \subset \mathbb{R}$. Suppose that there exist functions $V_i : z_i \to \mathbb{R}_+, i = 1, 2, \ldots, n$ continuously differentiable and positive definite in their respective domains, such that

$$V_i(z_i) \to \infty, \quad \text{as } z_i \to -b_i \text{ or } z_i \to b_i. \tag{6.3}$$

Let $\overline{V}(\bar{z}_n) = \sum_{i=1}^{n} V_i(z_i)$ and $z_i(0) \in Z_i$. If the inequality

$$\dot{\overline{V}}(\bar{z}_n) = \frac{\partial \overline{V}(\bar{z}_n)}{\partial \bar{z}_n} h_i(t, \bar{z}_n) < 0, \quad \forall \bar{z}_n \neq 0, i \in I_m \tag{6.4}$$

holds, then under arbitrary switchings, $z_i(t) \in Z_i, \forall t \in [0, \infty)$.

Proof The conditions on h_i and the trajectory of the system (6.2) is continuous at the switching instants ensuring the existence and uniqueness of a maximal solution $\bar{z}_n(t)$ on the time interval $[0, \tau_{\max})$. This implies that $\overline{V}(\bar{z}_n(t))$ exists for $\forall t \in [0, \tau_{\max})$.

From the fact that $z_i(0) \in Z_i$ and $V_i(z_i(0)), i = 1, 2, \ldots, n$ are known, we have that $\overline{V}(z_n(0))$ exists. Since $\overline{V}(\bar{z}_n)$ is positive definite and $\dot{\overline{V}}(\bar{z}_n) < 0$, therefore we obtain that $\overline{V}(\bar{z}_n(t)) < \overline{V}(\bar{z}_n(0))$ for $\forall t \in [0, \tau_{\max})$. Because $\overline{V}(\bar{z}_n) = \sum_{i=1}^{n} V_i(z_i)$ and the fact that $V_i(z_i)$ are positive functions, it is clear that each $V_i(z_i)$ is also bounded for $\forall t \in [0, \tau_{\max})$. Thus, we conclude from (6.3) that $z_i \neq -b_i$ and $z_i \neq b_i$. Given $-b_i < z_i(0) < b_i$, we know that $z_i(t)$ remains in the set Z_i for $\forall t \in [0, \tau_{\max})$.

Therefore, there is a compact subset $K \subseteq Z$ such that the maximal solution of (6.2) satisfies $\bar{z}_n(t) \in K$ for $\forall t \in [0, \tau_{\max})$. As a direct consequence of [38, p.481 Proposition C.3.6], we can infer that $\bar{z}_n(t) \in K$ is established for $\forall t \in [0, \infty)$. It follows that $|z_i(t)| \in Z_i, \forall t \in [0, \infty)$. In addition, it is clear that $V(\bar{z}_n)$ is a common Lyapunov function for the system (6.2), then the result holds under arbitrary switchings. □

Lemma 6.2 (Barbalat's Lemma) *Consider a differentiable function* $h(t)$. *If* $\lim_{t\to\infty} h(t)$ *is finite and* $\dot{h}(t)$ *is uniformly continuous, then* $\lim_{t\to\infty} \dot{h}(t) = 0$.

6.2.2 Control Design for Full State Constraints

We consider the full state constraints in the following; that is, for system (1), $x_i(t)$ is required to remain in the set $|x_i| \leq c_i, \forall t \geq 0$, where c_i are positive constants, for all $i = 1, 2, \ldots, n$. The controller is designed to achieve asymptotic output tracking while ensuring that the full state constraints are not violated.

First, the following assumptions are used in the backstepping design procedures.

Assumption 6.1 For any $c_1 > 0$, there exist positive constants $\underline{B}_0$, $\overline{B}_0$, A_0, B_1, B_2, $\ldots$, B_n satisfying $\max\{\underline{B}_0, \overline{B}_0\} \leq A_0 < c_1$ such that the desired trajectory $y_d(t)$ and its time derivatives satisfy $-\underline{B}_0 \leq y_d(t) \leq \overline{B}_0$, $|\dot{y}_d(t)| < B_1$, $|\ddot{y}_d(t)| < B_2, \ldots,$ $|y_d^{(n)}(t)| < B_n, \forall t \geq 0$.

Assumption 6.2 The functions $g^k(\bar{x}_n) = [g^{k,1}(\bar{x}_n), g^{k,2}(\bar{x}_n), \ldots, g^{k,q}(\bar{x}_n)]$, $k = 1, 2, \ldots, m$ are known. Furthermore, for $\forall j \in \{1, 2, \ldots, q\}$, assume that $\min_{k\in\{1,2,\ldots,m\}} g^{k,j}(\bar{x}_n) \geq 0, \forall \overline{x}_n \in \mathbb{R}^n$ or $\max_{k\in\{1,2,\ldots,m\}} g^{k,j}(\bar{x}_n) \leq 0, \forall \overline{x}_n \in \mathbb{R}^n$. For ease of analysis, denote

$$
\begin{aligned}
M &= \{j \in \{1, 2, \ldots, q\} | min_{k\in\{1,2,\ldots,m\}} g^{k,j}(\bar{x}_n) \geq 0\}, \\
F &= \{j \in \{1, 2, \ldots, q\} | j \notin M\}.
\end{aligned}
\tag{6.5}
$$

In what follows, the control design is proposed based on the simultaneous domination assumption with a barrier function in each step of the backstepping procedure.

Denote $z_1 = x_1 - y_d$ and $z_i = x_i - \phi_{i-1}, i = 2, \ldots, n$. Consider the Lyapunov function candidate:

$$\overline{V}_i(\bar{z}_i) = \sum_{l=1}^{i} V_l(z_l), \quad V_i(z_i) = \frac{1}{2}\log\frac{b_i^2}{b_i^2 - z_i^2}, \quad i = 1, 2, \ldots, n, \tag{6.6}$$

where $\phi_{i-1}, i = 2, \ldots, n$ stand for virtual controls, $\log(\bullet)$ denotes the natural logarithm of $\bullet$, $b_1 = c_1 - A_0$ and $b_i, i = 2, \ldots, n$ are positive constants. It is easy to know that $\overline{V}_n(\bar{z}_n) = \sum_{i=1}^{n} V_i(z_i)$ is positive definite and continuously differentiable in the set $|z_i| < b_i$ for all $i = 1, 2, \ldots, n$.

Step 1. Consider the following collection of auxiliary first-order subsystems.

$$\dot{z}_1 = f_1^k(x_1) + x_2 - \dot{y}_d, \qquad k = 1, 2, \ldots, m. \tag{6.7}$$

With the candidate Lyapunov function $V_1(z_1)$ and taking x_2 as the virtual control, we say that these first-order subsystems are simultaneously dominant if there exists a differentiable feedback law $\phi_1(x_1, z_1, \tilde{y}_{d_1}) = \phi_1^*(x_1, y_d) + \dot{y}_d$ such that, along the solutions of the subsystems in (6.7),

$$\dot{V}(z_1) = \frac{z_1\dot{z}_1}{b_1^2 - z_1^2} = \frac{z_1(\phi_1^*(x_1, y_d) + f_1^k(x_1))}{b_1^2 - z_1^2} < 0, \quad \forall z_1 \neq 0, k = 1, 2, \ldots, m. \tag{6.8}$$

Define

$$d_1^k(x_1, z_1, \tilde{y}_{d_1}) = \frac{z_1(\phi_1^*(x_1, y_d) + f_1^k(x_1))}{b_1^2 - z_1^2}, \qquad k = 1, 2, \ldots, m. \tag{6.9}$$

With $V_1(z_1)$, the control design for the first step is completed if a simultaneously dominating feedback law $x_2 = \phi_1(x_1, z_1, \tilde{y}_{d_1})$ is found.

Step i (for $i = 2, \ldots, n-1$). Consider the collection of auxiliary ith-order subsystems:

$$\begin{aligned} \dot{z}_1 &= f_1^k(x_1) + z_2 + \phi_1^*(x_1, y_d), \\ &\cdots \\ \dot{z}_i &= f_i^k(\bar{x}_i) + x_{i+1} - \sum_{j=1}^{i-1} \tfrac{\partial \phi_{i-1}}{\partial x_j}(x_{j+1} + f_j^k(\bar{x}_j)) - \sum_{j=0}^{i-1} \tfrac{\partial \phi_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)}, \\ k &= 1, 2, \ldots, m. \end{aligned} \tag{6.10}$$

With the candidate Lyapunov function $\overline{V}_i(\bar{z}_i)$ and taking x_{i+1} as the virtual control, we say that the ith-order subsystems are simultaneously dominatable if there exists a continuously differentiable feedback law $x_{i+1} = \phi_i(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i})$ such that, along the solutions of the subsystems in (6.10),

$$\dot{\overline{V}}_i(\bar{z}_i) = \frac{z_1\dot{z}_1}{b_1^2 - z_1^2} + \sum_{j=2}^{i} \frac{z_j\dot{z}_j}{b_j^2 - z_j^2} = \sum_{j=1}^{i} d_j^k(\bar{x}_j, \bar{z}_j, \tilde{y}_{d_j}) < 0, \quad \forall \bar{z}_i \neq 0, k = 1, 2, \ldots, m, \tag{6.11}$$

where, for $j = 2, \ldots, i$,

$$d_j^k(\bar{x}_j, \bar{z}_j, \tilde{y}_{d_j}) = z_j \left[\frac{z_{j-1}}{b_{j-1}^2 - z_{j-1}^2} + \frac{1}{b_j^2 - z_j^2} \left(\phi_j + f_j^k(\bar{x}_j) \right. \right. \tag{6.12}$$

$$\left. \left. - \sum_{l=1}^{j-1} \frac{\partial \phi_{j-1}}{\partial x_l}(x_{l+1} + f_l^k(\bar{x}_l)) - \sum_{l=0}^{j-1} \frac{\partial \phi_{j-1}}{\partial y_d^{(l)}} y_d^{(l+1)} \right) \right]. \tag{6.13}$$

With the constructed $\overline{V}_i(\bar{z}_i)$, the control design for the ith step is completed if a simultaneously dominating feedback law $x_{i+1} = \phi_i(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i})$ is found.

By using repeatedly the inductive argument above, we say that the subsystems of (6.1) are simultaneously dominant if the control design for the $(n-1)$th step can be completed. Then, we construct a controller for the final step.

Step n. The derivative of $\overline{V}_n(\bar{z}_n)$ in (6.6) along the trajectory of the kth subsystem is

$$\begin{aligned}
\dot{\overline{V}}_n =& \frac{z_1\dot{z}_1}{b_1^2 - z_1^2} + \sum_{i=2}^{n} \frac{z_i\dot{z}_i}{b_i^2 - z_i^2} \\
=& \sum_{i=1}^{n-1} d_i^k(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i}) + z_n \left[\frac{z_{n-1}}{b_{n-1}^2 - z_{n-1}^2} + \frac{1}{b_n^2 - z_n^2} \left(f_n^k(\bar{x}_n) \right. \right. \\
& \left. \left. + g^k(\bar{x}_n)u - \sum_{j=1}^{n-1} \frac{\partial \phi_{n-1}}{\partial x_j}(x_{j+1} + f_j^k(\bar{x}_j)) - \sum_{j=0}^{n-1} \frac{\partial \phi_{n-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \right) \right] \\
=& a_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) + b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})u,
\end{aligned} \tag{6.14}$$

where

$$\begin{aligned}
a_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) =& \sum_{i=1}^{n-1} d_{i,k}(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i}) + z_n \left[\frac{z_{n-1}}{b_{n-1}^2 - z_{n-1}^2} + \frac{1}{b_n^2 - z_n^2} \left(f_n^k(\bar{x}_n) \right. \right. \\
& \left. \left. - \sum_{j=1}^{n-1} \frac{\partial \phi_{n-1}}{\partial x_j}(x_{j+1} + f_j^k(\bar{x}_j)) - \sum_{j=0}^{n-1} \frac{\partial \phi_{n-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \right) \right],
\end{aligned} \tag{6.15}$$

$$b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) = \frac{z_n}{b_n^2 - z_n^2} g^k(\bar{x}_n). \tag{6.16}$$

In view of the above discussions and the simultaneous domination condition, a controller for systems (6.1) can be established:

$$u(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) = [u_1(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), u_2(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), \ldots, u_q(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})]^T, \quad (6.17)$$

where

$$u_j = \begin{cases} \min_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right\}, & if\ z_n > 0, \\ \max_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right\}, & if\ z_n < 0, \quad for\ j \in M, \\ 0, & if\ z_n = 0, \end{cases} \quad (6.18)$$

and

$$u_j = \begin{cases} \max_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right\}, & if\ z_n > 0, \\ \min_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right\}, & if\ z_n < 0, \quad for\ j \in F, \\ 0, & if\ z_n = 0, \end{cases} \quad (6.19)$$

with

$$\begin{aligned} p_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) &= \left[p_{k,1}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), p_{k,2}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), \ldots, p_{k,q}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right]^T \\ &= \begin{cases} -b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) \frac{\max\left\{a_k(\bar{x}_n,\bar{z}_n,\tilde{y}_{d_n})+b_k(\bar{x}_n,\bar{z}_n,\tilde{y}_{d_n})b_k^T(\bar{x}_n,\bar{z}_n,\tilde{y}_{d_n}),0\right\}}{b_k(\bar{x}_n,\bar{z}_n,\tilde{y}_{d_n})b_k^T(\bar{x}_n,\bar{z}_n,\tilde{y}_{d_n})}, if\ z_n \neq 0, \\ 0, if\ z_n = 0. \end{cases} \end{aligned} \quad (6.20)$$

Lemma 6.3 *Consider switched system (6.1). Suppose that the subsystems of (6.1) are simultaneously dominatable. Then, the continuous controller (6.17) can be constructed such that, along the solutions of all the closed-loop subsystems,*

$$\dot{\overline{V}}_n(\bar{z}_n) < 0, \quad \forall \bar{z}_n \neq 0, \quad (6.21)$$

where $\overline{V}_n(\bar{z}_n)$ is the Lyapunov function obtained in (6.6).

Proof For the sake of simplicity, we rewrite the system (6.1) as

$$\dot{\bar{x}}_n = \hat{f}_k(\bar{x}_n) + \hat{g}_k(\bar{x}_n)u, \quad k \in I_m. \quad (6.22)$$

In what follows, we will show that, $\forall k = 1, 2, \ldots, m$,

$$\begin{aligned} &\frac{\partial \overline{V}_n(\bar{z}_n)}{\partial \bar{z}_n}(\hat{f}_k(\bar{x}_n) + \hat{g}_k(\bar{x}_n)u(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})) \\ =&a_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) + b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})u(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) < 0, \quad \forall \bar{z}_n \neq 0, \end{aligned} \quad (6.23)$$

where $u(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})$ is the controller presented in (6.17).

For simplicity, we shall omit the dependence on $\bar{x}_n, \bar{z}_n$ and $\tilde{y}_{d_n}$ for functions wherever no confusion will be caused. 1. Consider $z_n = 0$. In this case, $b_k = 0$, $u = 0$, and

$$a_k + b_k u = a_k < 0, \quad k = 1, 2, \ldots, m. \tag{6.24}$$

2. Consider $z_n > 0$. In this case, by the definitions of (6.18) and (6.19), we have

$$u_j = \begin{cases} \min_{i \in \{1,2,\ldots,m\}} \left\{ p_{i,j} \right\}, j \in M, \\ \max_{i \in \{1,2,\ldots,m\}} \left\{ p_{i,j} \right\}, j \in F. \end{cases} \tag{6.25}$$

If $j \in M$, then $b_{k,j} \geq 0$. Therefore, we have $b_{k,j} u_j = b_{k,j} \min_{i \in \{1,2,\ldots,m\}} \left\{ p_{i,j} \right\} \leq b_{k,j} p_{k,j}$. Similarly, if $j \in F$, we have $b_{k,j} u_j \leq b_{k,j} p_{k,j}$. Therefore,

$$\begin{aligned} a_k + b_k u &= a_k + \sum_{i \in M} b_{k,i} u_i + \sum_{j \in F} b_{k,j} u_j \leq a_k + \sum_{j=1}^{q} b_{k,j} p_{k,j} \\ &= a_k + b_k p_k = \begin{cases} -b_k b_k^T, & if \quad a_k + b_k b_k^T \geq 0 \\ a_k, & if \quad a_k + b_k b_k^T < 0 \end{cases} \\ &< 0, \quad k = 1, 2, \ldots, m. \end{aligned} \tag{6.26}$$

3. Consider $z_n < 0$. Similarly, in this case we can show that

$$u_j = \begin{cases} \max_{i \in \{1,2,\ldots,m\}} \left\{ p_{i,j} \right\}, j \in M, \\ \min_{i \in \{1,2,\ldots,m\}} \left\{ p_{i,j} \right\}, j \in F. \end{cases} \tag{6.27}$$

and

$$a_k + b_k u_k < 0, \quad k = 1, 2, \ldots, m. \tag{6.28}$$

Therefore, we conclude that, $\forall k = 1, 2, \ldots, m$, (6.23) is true. Thus, $\overline{V}_n(\bar{z}_n)$ is a common Lyapunov function for all subsystems of (1). □

Based on the above discussions, we are now in a position to give the following result.

Theorem 6.1 *Consider the closed-loop system (6.1), (6.17) under Assumptions 6.1–6.2. Let A_i be an upper bound for ϕ_i in compact set Ω_i:*

$$A_i \geq \sup_{\left(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i}\right) \in \Omega_i} \left| \phi_i \left(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i} \right) \right|, \quad i = 1, \ldots, n-1, \tag{6.29}$$

where $\Omega_i = \{\bar{x}_i \in R^i, \bar{z}_i \in R^i, \tilde{y}_{d_i} \in R^{i+1} : |x_j| \le D_{z_j} + A_{j-1}, |z_j| \le D_{z_j}, |y_d| < A_0, |y_d^{(j)}| \le B_j, j = 1, \ldots, i\}$, $\quad D_{z_j} = b_j\sqrt{1 - \frac{\Pi_{k=1}^n (b_k^2 - z_k^2(0))}{\Pi_{k=1}^n b_k^2}}, i = 1, \ldots, n-1.$
Given that the following conditions are satisfied,

(1) $c_{i+1} > A_i + b_{i+1}$ *holds for* $\forall i = 1, 2, \ldots, n-1$.

(2) The initial conditions $\bar{z}_n(0)$ *belong to the set* $\Omega_{z_0} = \{ \bar{z}_n \in R^n : |z_i| < b_i, i = 1, \ldots, n \}$.

Under arbitrary switching signals, closed-loop system (1) has the following properties:

(i) The signals $z_i(t), i = 1, 2, \ldots, n$, *remain in the compact set* $\Omega_z = \{\bar{z}_n \in R^n : |z_i| < D_{z_i}, i = 1, 2, \ldots, n\}$.

(ii) $x_i(t)$ *remains in the set* $\Omega_x = \{\bar{x}_n \in R^n : |x_i| < D_{z_i} + A_{i-1} < c_i, i = 1, \ldots, n\}, \forall t \ge 0$; *i.e., the full state constraints are never violated.*

(iii) All closed-loop signals are bounded.

(iv) The output tracking error $z_1(t)$ *asymptotically converges to zero, i.e.,* $y(t) \to y_d(t)$ *as* $t \to \infty$.

Proof (i) By $\dot{\overline{V}}_n < 0$, it is clear that $\overline{V}_n(t) < \overline{V}_n(0), \forall t \ge 0$. Because $z_i^2(0) < b_i^2$ from condition (2), we have that $\overline{V}_n(0) < \sum_{i=1}^n \frac{1}{2} \log \frac{b_i^2}{b_i^2 - z_i^2(0)}$, which means

$$\frac{1}{2} \log \frac{b_i^2}{b_i^2 - z_i^2} < \sum_{i=1}^n \frac{1}{2} \log \frac{b_i^2}{b_i^2 - z_i^2(0)} \tag{6.30}$$

for $i = 1, \ldots, n$. Because $\log a + \log b = \log ab$, we rewrite (6.30) as

$$\log \frac{b_i^2}{b_i^2 - z_i^2} < \log \frac{\Pi_{i=1}^n b_i^2}{\Pi_{i=1}^n \left(b_i^2 - z_i^2(0)\right)} \tag{6.31}$$

for $i = 1, \ldots, n$. Furthermore, we obtain from Lemma 1 that $b_i^2 - z_i^2(t) > 0, \forall t \ge 0$. Then, (6.31) is equivalent to $|z_i(t)| < D_{z_i}, \forall t \ge 0$.

(ii) Because $|z_1(t)| < D_{z_1} < c_1 - A_0$, we obtain

$$|x_1(t)| < D_{z_1} + |y_d(t)| < c_1 - A_0 + |y_d(t)|. \tag{6.32}$$

Noting that $|y_d(t)| < A_0$, we thus conclude from Assumption 6.1 that $|x_1(t)| < D_{z_1} + A_0 < c_1, \forall t \ge 0$.

To show that $|x_2(t)| < c_2$, we first verify that there exists a positive constant A_1 such that $|\phi_1(t)| \le A_1, \forall t \ge 0$. Because $|x_1(t)| < D_{z_1} + A_0$, $|z_1(t)| \le D_{z_1}$ and $|\dot{y}_d(t)| \le B_1$, it is clear that $\left(x_1(t), z_1(t), \tilde{y}_{d_1}(t)\right) \in \Omega_1$, and thus, the stabilizing function ϕ_1 is bounded because it is a continuous function. As a result, $\sup_{(x_1, z_1, \tilde{y}_{d_1}) \in \Omega_i} \left|\phi_1\left(x_1, z_1, \tilde{y}_{d_1}\right)\right|$ exists, and an upper bound A_1 can be found. Then, we can see from $|z_2(t)| \le D_{z_2} < b_2$ that

$$|x_2(t)| \le D_{z_2} + |\phi_1(t)| < b_2 + |\phi_1(t)|. \tag{6.33}$$

Since $|\phi_1(t)| < A_1$, therefore we deduce that $|x_2(t)| \le D_{z_2} + A_1 < b_2 + A_1 < c_2, \forall t \ge 0$.

We can get that $|x_{i+1}(t)| \le c_{i+1}, i = 2, \ldots, n-1$, after verifying that there exist positive constants A_i such that $|\phi_i(t)| \le A_i, \forall t \ge 0$. Because $|x_i(t)| \le D_{z_i} + A_{i-1}, |z_i(t)| \le D_{z_i}$ and $\left|y_d^{(i)}(t)\right| \le Y_i$, it is clear that $\left(\bar{x}_i(t), \bar{z}_i(t), \tilde{y}_{d_i}(t)\right) \in \Omega_i$, and thus, the stabilizing function ϕ_i is bounded because it is a continuously differentiable function. As a result, we have that $\sup_{(\bar{x}_i, \bar{z}_i, \tilde{y}_i) \in \Omega_i} |\phi_i(\bar{x}_i, \bar{z}_i, \tilde{y}_i)|$ exists, and an upper bound A_i can be found. Then, from $|z_{i+1}(t)| \le D_{z_{i+1}} < b_{i+1}$, we can show that $|x_{i+1}(t)| < D_{z_{i+1}} + |\phi_i(t)| < b_{i+1} + |\phi_i(t)|$. Because $|\phi_i(t)| \le A_i$, therefore we have that $|x_{i+1}(t)| < D_{z_{i+1}} + A_i < b_{i+1} + A_i < c_{i+1}, \forall t \ge 0$.

(iii) By virtue of the boundedness of $\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}$, it is clear that stabilizing functions $\phi_i(\bar{x}_i, \bar{z}_i, \tilde{y}_i)$ and control $u_n\left(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}\right)$ are bounded. Therefore, all closed-loop signals are bounded.

(iv) Based on the fact that $\bar{x}_i(t), \bar{z}_i(t), i = 1, 2, \ldots, n$ are bounded, it can be obtained that $\ddot{V}$ is bounded, which means that $\dot{V}$ is uniformly continuous. Then, by Lemma 6.2, we obtain that $z_i(t) \to 0$ as $t \to 0$. Because $z_1(t) = x_1(t) - y_d(t)$ and $y(t) = x_1(t)$, we finally have $y(t) \to y_d(t)$ as $t \to \infty$. □

6.2.3 Control Design for Time-Varying Output Constraints

In this section, we consider the case that the output is required to satisfy $-\bar{c}_1(t) < y(t) < \bar{c}_2(t), \forall t \ge 0$, where $\bar{c}_1(t), \bar{c}_2(t)$ are positive-valued time-varying functions. By incorporating an appropriate barrier function in the backstepping design, we show that the output constraints are not violated at any time and asymptotic output tracking is realized while ensuring boundedness of all closed-loop signals.

Assumption 6.3 There exist positive constants $K_l^i, i = 0, 1, \ldots, n, l = 1, 2$ such that the time-varying functions $\bar{c}_l(t)$ and their time derivatives satisfy $\bar{c}_l(t) \le K_l^0, \bar{c}_l^{(i)}(t) \le K_l^i, i = 1, 2, \ldots, n, l = 1, 2, \forall t \ge 0$.

Assumption 6.4 There exist functions $B_l : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $B_l(t) < \bar{c}_l(t), l = 1, 2, \forall t \ge 0$ and positive constants $B_l^i, i = 1, 2, \ldots, n$ such that the desired trajectory $y_d(t)$ and its time derivatives satisfy $-B_1(t) \le y_d(t) \le B_2(t), -B_1^i < y_d^{(i)}(t) < B_2^i, i = 1, 2, \ldots, n, \forall t \ge 0$.

Lemma 6.4 *For any positive constants a_0, b_0, let $\Pi = \{\xi \in \mathbb{R} : -a_0 < \xi < b_0\} \subset \mathbb{R}$ and $X = \mathbb{R}^v \times \Pi \subset \mathbb{R}^{v+1}$ be open sets. Consider the switched system:*

$$\dot{\eta} = h_{\sigma(t)}(t, \eta), \tag{6.34}$$

where $\eta := [\xi, z] \in X, z \in \mathbb{R}^v$, $\sigma(t)$ is the same as in (1), and $h_i : \mathbb{R}_+ \times X \to \mathbb{R}^{v+1}$ is piecewise continuous in t and locally Lipschitz in η, uniformly in t, on $\mathbb{R}_+ \times X$.

We also assume that the state of the switched system (6.34) does not jump at switching instants. Suppose that there exist functions $V_1 : \Pi \to \mathbb{R}_+$ *and* $V_2 : \mathbb{R}^v \to \mathbb{R}_+$ *continuously differentiable and positive definite in their individual domains, such that*

$$V_1(\xi) \to \infty, \quad as\, \xi \to -a_0\, or\;\; \xi \to b_0, \tag{6.35}$$

$$\gamma_1(\|z\|) \le V_2(z) \le \gamma_2(\|z\|), \tag{6.36}$$

where γ_1 *and* γ_2 *are class* K_∞ *functions. Let* $V(\eta) = V_1(\xi) + V_2(z)$, *and* $\xi(0)$ *belong to the set* $(-a_0, b_0)$. *If the inequality*

$$\dot{V}(\eta) = \frac{\partial V(\eta)}{\partial \eta} h_i(t, \eta) < 0, \quad \forall \eta \ne 0, i \in I_m \tag{6.37}$$

holds, then under arbitrary switchings, $\xi(t)$ *remains in the open set* $(-a_0, b_0)$, $\forall t \in [0, \infty)$.

Proof The proof is similar to Lemma 6.1. □

Noting that the output constraints are asymmetric and time-varying, we construct the following asymmetric barrier function, which explicitly depends on time.

$$V_1(z_1(t), b_1(t)) = \frac{1}{2}(1 - q(z_1(t))) \log \frac{b_1^2(t)}{b_1^2(t) - z_1^2(t)} + \frac{1}{2} q(z_1(t)) \log \frac{b_2^2(t)}{b_2^2(t) - z_1^2(t)}, \tag{6.38}$$

where $z_1 = x_1 - y_d$, $b_1(t) = \bar{c}_1(t) - B_1(t)$ and $b_2(t) = \bar{c}_2(t) - B_2(t)$ are the constraints on z_1; that is, we require $-b_1(t) < z_1(t) < b_2(t)$, and

$$q(\bullet) = \begin{cases} 0, & if \quad \bullet \le 0, \\ 1, & if \quad \bullet > 0. \end{cases} \tag{6.39}$$

Lemma 6.5 *The Lyapunov function candidate* V_1 *in (6.38) is positive definite and* C^1 *in the set* $(-b_1(t), b_2(t))$.

Proof For $-b_1(t) < z_1(t) < b_2(t)$, we have that $V_1 \ge 0$ and $V_1 = 0$ if and only if $z_1(t) = 0$. This means that V_1 is positive definite. Furthermore, V_1 is piecewise smooth among intervals $z_1(t) \in (-b_1(t), 0]$ and $z_1(t) \in (0, b_2(t))$. Noting that $\lim_{z_1 \to 0^-} \frac{dV_1}{dz_1} = \lim_{z_1 \to 0^+} \frac{dV_1}{dz_1} = 0$, we conclude that V_1 is C^1. This completes the proof. □

Then, to remove the explicit dependence on time in (6.38), we use a coordinate transformation:

$$\xi_1 = \frac{z_1(t)}{b_1(t)}, \quad \xi_2 = \frac{z_1(t)}{b_2(t)}, \quad \xi = (1 - q(z_1))\xi_1 + q(z_1)\xi_2. \tag{6.40}$$

Therefore, we can rewrite V_1 in (6.38) as

$$V_1(\xi) = \frac{1}{2}\log\frac{1}{1-\xi^2}. \tag{6.41}$$

It is clear that $V_1(\xi)$ is positive definite and continuously differentiable in the set $|\xi| < 1$.

Now, consider the Lyapunov function candidate:

$$\overline{V}_i(\xi, \bar{z}_{2:i}) = V_1(\xi) + \sum_{l=2}^{i} V_l(z_l),\ V_i(z_i) = \frac{1}{2}z_i^2, \quad i = 2, 3, \ldots, n, \tag{6.42}$$

where $z_i = x_i - \phi_{i-1}, i = 2, \ldots, n$, and $\phi_1 = (1 - q(z_1))\phi_1^1(x_1, \xi_1, z_1, \tilde{b}_1^{(1)}, \tilde{y}_{d_1}) + q(z_1)\phi_1^2(x_1, \xi_2, z_1, \tilde{b}_2^{(1)}, \tilde{y}_{d_1}), \phi_j = \phi_j(\bar{x}_j, \xi_1, \xi_2, \bar{z}_j, \tilde{b}_1^{(j)}, \tilde{b}_2^{(j)}, \tilde{y}_{d_j}), j = 2, \ldots, n-1$ are the virtual controls.

Using the backstepping design technique in Sect. 6.3, we can then get

$$d_1^k \tag{6.43}$$
$$= (1 - q(z_1))\frac{\xi_1(\phi_1^1 + f_1^k(x_1) - \dot{y}_d - \xi_1\dot{b}_1)}{b_1(1-\xi_1^2)} + q(z_1)\frac{\xi_2(\phi_1^2 + f_1^k(x_1) - \dot{y}_d - \xi_2\dot{b}_2)}{b_2(1-\xi_2^2)}.$$

$$d_2^k \tag{6.44}$$
$$= (1 - q(z_1))z_2\left(\frac{\xi_1}{b_1(1-\xi_1^2)} + \phi_2 + f_2^k(\bar{x}_2) - \frac{\partial\phi_1}{\partial t} - \frac{\partial\phi_1}{\partial x_1}(x_2 + f_2^k(\bar{x}_2)) - \sum_{l=0}^{1}\frac{\partial\phi_1}{\partial y_d^l}y_d^{l+1}\right)$$
$$+ q(z_1)z_2\left(\frac{\xi_2}{b_2(1-\xi_2^2)} + \phi_2 + f_2^k(\bar{x}_2) - \frac{\partial\phi_1}{\partial t} - \frac{\partial\phi_1}{\partial x_1}(x_2 + f_2^k(\bar{x}_2)) - \sum_{l=0}^{1}\frac{\partial\phi_1}{\partial y_d^l}y_d^{l+1}\right),$$

$$d_j^k \tag{6.45}$$
$$= z_j\left(z_{j-1} + \phi_j + f_j^k(\bar{x}_j) - \frac{\partial\phi_{j-1}}{\partial t} - \sum_{l=1}^{j-1}\frac{\partial\phi_{j-1}}{\partial x_l}(x_{l+1} + f_l^k(\bar{x}_l)) - \sum_{l=0}^{j-1}\frac{\partial\phi_{j-1}}{\partial y_d^l}y_d^{l+1}\right).$$

$$k = 1, 2, \ldots, m,\ j = 3, 4, \ldots, n-1. \tag{6.46}$$

and

$$a_k = \sum_{i=1}^{n-1} d_{i,k} + z_n\left(z_{n-1} + f_n^k(\bar{x}_n) - \frac{\partial\phi_{n-1}}{\partial t}\right. \tag{6.47}$$
$$\left. - \sum_{j=1}^{n-1}\frac{\partial\phi_{n-1}}{\partial x_j}(x_{j+1} + f_j^k(\bar{x}_j)) - \sum_{j=0}^{n-1}\frac{\partial\phi_{n-1}}{\partial y_d^{(j)}}y_d^{(j+1)}\right),$$
$$b_k = z_n g^k(\bar{x}_n). \tag{6.48}$$

We design the following controller for the system (6.1).

$$\begin{aligned} & u(\bar{x}_n, \xi_1, \xi_2, \bar{z}_n, \tilde{b}_1^{(n)}, \tilde{b}_2^{(n)}, \tilde{y}_{d_n}) \\ =&[u_1(\bar{x}_n, \xi_1, \xi_2, \bar{z}_n, \tilde{b}_1^{(n)}, \tilde{b}_2^{(n)}, \tilde{y}_{d_n}), u_2(\bar{x}_n, \xi_1, \xi_2, \bar{z}_n, \tilde{b}_1^{(n)}, \tilde{b}_2^{(n)}, \tilde{y}_{d_n}), \ldots, \\ & u_q(\bar{x}_n, \xi_1, \xi_2, \bar{z}_n, \tilde{b}_1^{(n)}, \tilde{b}_2^{(n)}, \tilde{y}_{d_n})]. \end{aligned} \tag{6.49}$$

Applying Lemma 6.3, we can conclude that $\dot{\overline{V}}_n(\xi, \bar{z}_{2:n}) < 0, \forall(\xi, z_{2:n})^T \neq 0$ along the solutions of closed-loop system (6.1).

Based on the above discussions, we can obtain the following theorem.

Theorem 6.2 *Consider the switched system (6.1) under Assumptions 6.2–6.4. If the subsystems are simultaneously dominatable with the controller (6.49), then the closed-loop system (6.1) has the following properties under arbitrary switching, where the initial conditions are* $\bar{z}_n(0) \in \Omega_{z_0} = \{\bar{z}_n \in R^n : -b_1(0) < z_1(0) < b_2(0)\}$.

(i) The signals $\xi_1(t), \xi_2(t)$ *and* $z_i(t), i = 1, 2, \ldots, n$ *are bounded, for* $\forall t \geq 0$, *as follows.*

$$\begin{aligned} & -\sqrt{1 - e^{-2V_n(0)}} < \xi_1(t) < 0, \\ & 0 \leq \xi_2(t) < \sqrt{1 - e^{-2V_n(0)}}, \\ & -b_1(t) < -\underline{D}_{z_1}(t) < z_1(t) < \overline{D}_{z_1}(t) < b_2(t), \\ & \|\bar{z}_{2:n}(t)\| < \sqrt{2V_n(0)}, \end{aligned} \tag{6.50}$$

where $\underline{D}_{z_1}(t) = b_1(t)\left(1 - e^{-2V_n(0)}\right)^{\frac{1}{2}}, \overline{D}_{z_1}(t) = b_2(t)\left(1 - e^{-2V_n(0)}\right)^{\frac{1}{2}}$.

(ii) The output $y(t)$ *remains in the set* $\Omega_y = \{y \in R : -\bar{c}_1(t) < -b_2(t) - B_2(t) < y(t) < b_1(t) + B_1(t) < \bar{c}_2(t)\}$*; i.e., the output constraints are never violated.*

(iii) All closed-loop signals are bounded.

(iv) The output tracking error asymptotically converges to zero; i.e., $y(t) \to y_d(t)$ *as* $t \to \infty$.

Proof (i) Applying Lemma 6.4 yields that $|\xi_i(t)| < 1, i = 1, 2$, from which we have that $-b_1(t) < z_1(t) < b_2(t), \forall t \geq 0$. Furthermore, it follows from $\overline{V}_n(t) < \overline{V}_n(0), \forall t \geq 0$, that

$$\overline{V}_n(0) > \begin{cases} \log \frac{b_1^2(t)}{b_1^2(t) - z_1^2(t)}, & -b_1(t) < z_1(t) < 0, \\ \log \frac{b_2^2(t)}{b_2^2(t) - z_1^2(t)}, & 0 \leq z_1(t) < b_2(t). \end{cases} \tag{6.51}$$

Then, we get that

$$z_1^2(t) < \begin{cases} b_1^2(t)\left(1 - e^{-2\overline{V}_n(0)}\right), & -b_1(t) < z_1(t) < 0, \\ b_2^2(t)\left(1 - e^{-2\overline{V}_n(0)}\right), & 0 \leq z_1(t) < b_2(t). \end{cases} \tag{6.52}$$

This implies that $z_1(t) > -b_1(t)\left(1 - e^{-2\overline{V}_n(0)}\right)^{\frac{1}{2}}$ for negative $z_1(t)$, and $z_1(t) < b_2(t)\left(1 - e^{-2\overline{V}_n(0)}\right)^{\frac{1}{2}}$ for nonnegative $z_1(t)$. Therefore, it is obvious that $-\underline{D}_{z_1}(t) < z_1(t) < \overline{D}_{z_1}(t), \forall t \geq 0$.

Similarly, from the fact that $\frac{1}{2}\sum_{j=2}^{n} z_j^2(t) \leq \overline{V}_n(0)$, we can obtain that $|z_{2:n}(t)| \leq \sqrt{2\overline{V}_n(0)}, \forall t \geq 0$.

(ii) Because $y(t) = z_1(t) + y_d$, $-\underline{D}_{z_1}(t) < z_1(t) < \overline{D}_{z_1}(t)$, and $|y_d(t)| \leq B_l(t), l = 1, 2, \forall t \geq 0$. Then, we can conclude that

$$-\underline{D}_{z_1}(t) - B_1(t) < z_1(t) + y_d(t) < \overline{D}_{z_1}(t) + B_2(t). \tag{6.53}$$

$\underline{D}_{z_1}(t) < b_1(t)$ and $\overline{D}_{z_1}(t) < b_2(t)$, therefore we know that

$$\begin{aligned} \underline{D}_{z_1}(t) + y_d(t) &< b_1(t) + B_1(t) = \bar{c}_1(t), \\ \overline{D}_{z_1}(t) + y_d(t) &< b_2(t) + B_2(t) = \bar{c}_2(t). \end{aligned} \tag{6.54}$$

Hence, we can deduce that $y(t) \in \Omega_y, \forall t \geq 0$.

(iii) From (i), we know that the error signals $z_1(t), z_2(t), \ldots, z_n(t)$ are bounded. The boundedness of $z_1(t)$ and $y_d(t)$ implies that the state $x_1(t)$ is bounded. From (6.38), we see that $\dot{b}_i(t)$ are bounded, because $\dot{\bar{c}}_i(t) \leq K_i^1$ and $|\dot{y}_d(t)| \leq B_i^1, i = 1, 2$, where K_i^1 and B_i^1 are some positive constants. Therefore, the virtual control ϕ_1 is also bounded. This leads to the boundedness of $x_2(t)$, because $x_2 = z_2 + \phi_1$. Furthermore, it is not hard to check that all variables of continuous function ϕ_2 are bounded, and thus we get that ϕ_2 is bounded. This leads to the boundedness of state $x_3(t)$, because $x_3 = z_3 + \phi_2$. Following the same procedures, one can know that each ϕ_i, for $i = 3, \ldots, n-1$, is bounded. Hence, the boundedness of state $x_{i+1}(t)$ can be ensured. With $\bar{x}_n(t)$ and $\bar{z}_n(t)$ being bounded, and $-b_1(t) < z_1(t) < b_2(t), \forall t \geq 0$, we deduce that the control $u(t)$ is bounded. Thus, all closed-loop signals are bounded.

(iv) Let $d_1 = d_1^k, k = 1, 2, \ldots, m$, which is differentiable in the set $|\xi| < 1$. Because $|\xi(t)| < 1, \forall t \geq 0$ from Lemma 6.1, we can integrate both sides of $\dot{\overline{V}}_n = a_k + b_k u$ with the controller (6.49) to obtain

$$\lim_{t\to\infty}\int_0^t d_1(\tau)\,d\tau < V(0) < \infty, \forall k = 1, 2, \ldots, m. \tag{6.55}$$

Meanwhile, one can also derive from d_1^k that $\dot{d}_1(t)$ is bounded, which implies that $d_1(t)$ is uniformly continuous. By Lemma 6.2, one can get that $d_1(t) \to 0,\ as\ t \to \infty$, which means $\xi_i(t) \to 0,\ as\ t \to \infty$, because $\xi_i(t) = z_1(t)/b_i(t)$ and $b_i(t) > 0, i = 1, 2, \forall t \geq 0$. Subsequently, one can obtain $z_1(t) \to 0,\ as\ t \to \infty$. Therefore, we finally have $y(t) \to y_d(t),\ as\ t \to \infty$. □

6.2.4 Simulation Results

In this section, two examples are presented to demonstrate the effectiveness of the obtained results.

Consider the following switched nonlinear system,

$$\begin{aligned} \dot{x}_1 &= g_1^{\sigma(t)}(x_1)\,x_2, \\ \dot{x}_2 &= f_2^{\sigma(t)}(\bar{x}_2, d(t)) + g_2^{\sigma(t)}(\bar{x}_2)\,u_{\sigma(t)}, \\ y &= x_1, \end{aligned} \tag{6.56}$$

where $\sigma : [0, +\infty) \to \{1, 2\}$, $g_1^1(x_1) = g_1^2(x_1) = 1$, $f_2^1(\bar{x}_2, d(t)) = \theta x_2^2, \theta \in [0.4, 0.8]$, $f_2^2(\bar{x}_2, d(t)) = x_2 \cos\left(2x_1 x_2^2\right)$. The control objective is to design a state feedback controller such that the output x_1 of the system can track the given signal $y_c = 0.2$, and does not destroy a symmetric constraint $\underline{L} = \overline{L} = 0.25$.

Due to the symmetric constraint $\underline{L} = \overline{L} = 0.25$, one can set $z_1 = \Psi(x_{1d}, -0.25, 0.25) = tan[2\pi(x_1 - 0.2)]$ and $V_1(z_1) = \frac{1}{2}z_1^2$. By using the proposed method, the common stabilizing function $\phi_1(z_1)$ can be obtained for each subsystem at the initial step:

$$\phi_1(z_1) = -z_1\left[1 + \frac{2}{2\pi \sec^2[2\pi(x_1 - 0.2)]}\right]. \tag{6.57}$$

Next, set $z_2 = x_2 - \phi_1(z_1)$, and $\overline{V}_2(\bar{z}_2) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ is the CLF for system (6.56). We can give the following state feedback controller.

$$\begin{aligned} u(\bar{z}_2) = -z_2\Bigg[&1.6\pi \sec^2[2\pi(x_1 - 0.2)] + \left(1 + \frac{2}{2\pi \sec^2[2\pi(x_1 - 0.2)]}\right)^4 \\ &+ \left(1 + \frac{2}{2\pi \sec^2[2\pi(x_1 - 0.2)]}\right)^2 (1 + x_2^2) + (1 + x_2^2)^{\frac{1}{2}} \\ &+ \left(1 + \frac{2}{2\pi \sec^2[2\pi(x_1 - 0.2)]}\right) + 1\Bigg]. \end{aligned} \tag{6.58}$$

Choose the initial values as $x_1(0) = 0.449$, $x_2(0) = -2.2$. Figure 6.1 demonstrates that asymptotic tracking performance can be achieved under a randomly generated switching signal in Fig. 6.2. From Fig. 6.3, it can be seen that the output tracking error x_{1d} converges to zero while remaining in the set $(-0.25, 0.25)$. Finally, the state response of the p-times differentiable unbounded function $z_1 = tan[2\pi(x_1 - 0.2)]$ is shown in Fig. 6.4 demonstrating the validity of the designed state feedback controller (6.58).

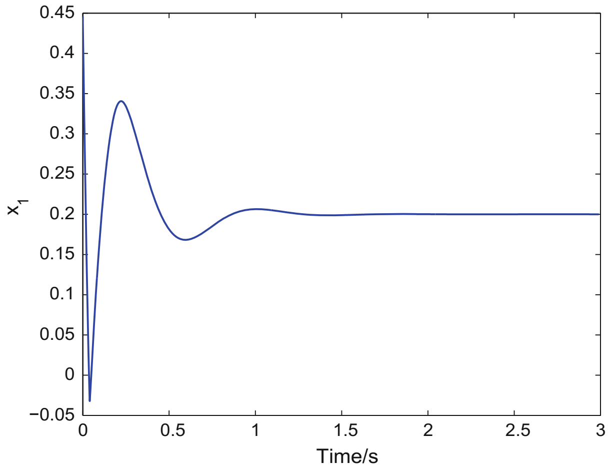

Fig. 6.1 Output tracking for the desired signal $y_d = 0.2$

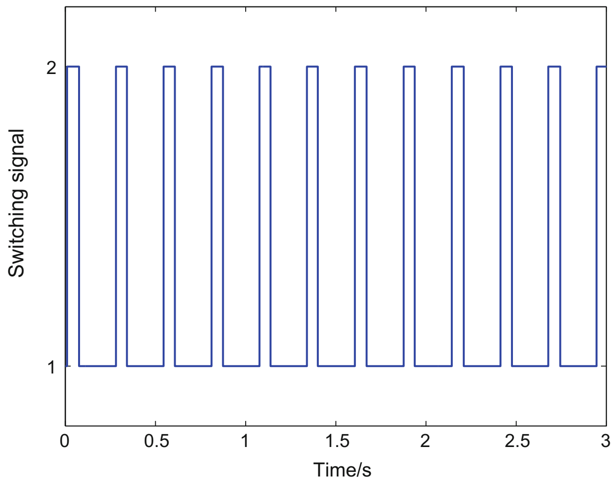

Fig. 6.2 The given switching signal for the system (6.56)

6.2.5 Conclusions

Based on the BLF approach, a control design method for constrained nonlinear switched systems in lower triangular form has been developed to achieve the output tracking objective. By guaranteeing the boundedness of the BLF in the closed-loop, the restrictions are not transgressed. Furthermore, asymptotic output tracking is achieved without violating the constraints, and all closed-loop signals are bounded.

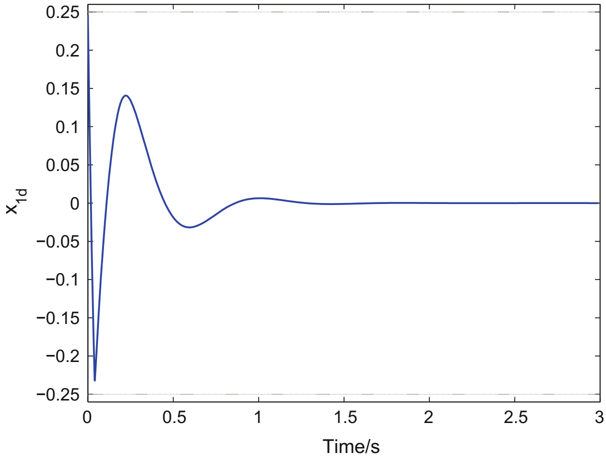

Fig. 6.3 The state response of the output tracking error x_{1d}

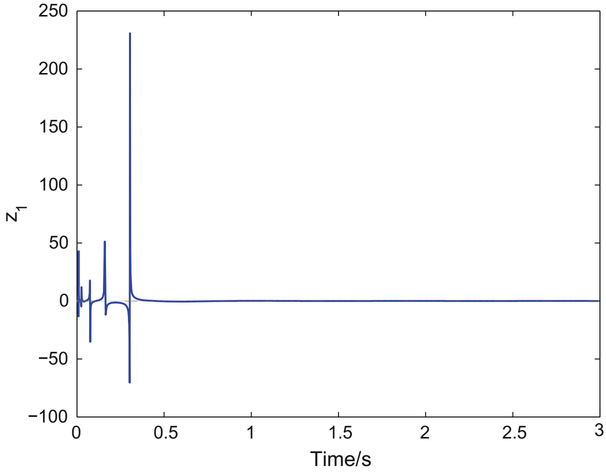

Fig. 6.4 The state response of the p-times differentiable unbounded function z_1

In particular, the issue of output tracking control with full state constraints and asymmetric time-varying output constraints are considered for switched nonlinear systems.

6.3 p-Times Differentiable Unbounded Functions-Based Control Design

6.3.1 Problem Formulation and Preliminaries

Consider uncertain switched nonlinear systems with the following lower triangular form,

$$\begin{aligned}\dot{x}_1 &= g_1^{\sigma(t)}(x_1)\,x_2,\\ \dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i, d(t)) + g_i^{\sigma(t)}(\bar{x}_i)\,x_{i+1},\ i = 2, 3, \ldots, n-1,\\ \dot{x}_n &= f_n^{\sigma(t)}(\bar{x}_n, d(t)) + g_n^{\sigma(t)}(\bar{x}_n)\,u_{\sigma(t)},\\ y &= x_1,\end{aligned} \tag{6.59}$$

where $x_1, x_2, \ldots, x_n$ are system states, y is the output; $\sigma(t)$ is the switching signal, which takes its values in a finite set $I_m = \{1, 2, \ldots, m\}$ and $m > 1$ is the number of subsystems; $d(t)$ is an unknown piecewise continuous disturbance belonging to a known compact set $\Omega \in R^s$; $u_k \in R$ is the control input of the k-th subsystem. For $\forall i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$, functions $f_i^k(\bar{x}_i, d(t))$ and $g_i^k(\bar{x}_i)$ are known and smooth with $0 < \underline{g} \le g_i^k(\bar{x}_i) \le \bar{g}$, where $\underline{g}$ and $\bar{g}$ are positive constants. As commonly assumed in the literature, the Zeno behavior for all types of switching signals is not considered. In addition, we assume that the state of system (6.59) is continuous at switching instants.

Remark 6.1 For non-switched nonlinear systems, the structure of (6.59) has been widely investigated (see, e.g., [12, 14, 17–19]). For switched nonlinear systems, the considered system structure of (6.59) was restricted to the design of stabilizing controllers [20–23].

Here, we consider the following output-constrained tracking control problem.

The output-constrained tracking control problem: For the system (6.59) under arbitrary switchings, design state feedback controllers to ensure the output of system (6.59) to track a given constant reference signal y_c such that:

(1) Asymptotic tracking is achieved; i.e.,

$$\lim_{t\to\infty}(y(t) - y_c) = 0. \tag{6.60}$$

(2) The output tracking error is confined to be a pre-specified limit range; i.e.,

$$-\underline{L} \le y(t) - y_c \le \overline{L} \tag{6.61}$$

for all $t \geq t_0 \geq 0$, where $\underline{L}$ and $\overline{L}$ are strictly positive constants. If $\underline{L} = \overline{L}$, the constraint (6.61) is referred to as a symmetric constraint. If $\underline{L} \neq \overline{L}$, the constraint (6.61) is referred to as an asymmetric constraint.

(3) All signals of the closed-loop system (6.59) are bounded.

The following assumptions are needed to develop the main results.

Assumption 6.5 For $i = 2, 3, \ldots, n$,

$$\left| f_i^k(\bar{x}_i, d(t)) \right| \leq (|x_2| + \cdots + |x_i|)\mu_i^k(\bar{x}_i), \forall k \in I_m, \tag{6.62}$$

where $\mu_{i,k}(\bar{x}_i)$ is a set of known non-negative smooth functions.

Assumption 6.6 At t_0, there exist strictly positive constants $\underline{L}_1 < \underline{L}$ and $\overline{L}_1 < \overline{L}$ such that

$$-\underline{L}_1 \leq x_{1d}(t_0) \leq \overline{L}_1, \tag{6.63}$$

where $x_{1d}(t_0) = x_1(t_0) - y_c$ is the initial output tracking error.

Two definitions and two relevant lemmas are addressed in the following for later use.

Definition 6.2 ([2]) A scalar function $h(x, a, b)$ is said to be a p-times differentiable step function if it satisfies the following properties.

(1) $h(x, a, b) = 0, \quad \forall -\infty < x \leq a,$

(2) $h(x, a, b) = 1, \quad \forall b \leq x < +\infty,$

(3) $0 < h(x, a, b) < 1, \quad \forall x \in (a, b),$

(4) $h(x, a, b)$ is p times differentiable with respect to x,

(5) $h'(x, a, b) > 0, \quad \forall x \in (a, b),$

(6) $h'(x, a, b) \geq \delta_1(\rho_1) > 0, \quad \forall x \in (a + \rho_1, b - \rho_1)$ with $0 < \rho_1 < \frac{b-a}{2}$,

where p is a positive integer, $x \in R$, a and b are constants such that $a < 0 < b$, $h'(x, a, b) = \frac{\partial h'(x,a,b)}{\partial x}$, and $\delta_1(\rho_1)$ is a positive constant depending on the positive constant ρ_1. Moreover, if the function $h(x, a, b)$ is infinite times differentiable with respect to x, then it is said to be a smooth step function.

Lemma 6.6 ([14]) *Let the scalar function $h(x, a, b)$ be defined as*

$$h(x, a, b) = \frac{\int_a^x f(\tau - a) f(b - \tau) d\tau}{\int_a^b f(\tau - a) f(b - \tau) d\tau} \tag{6.64}$$

where a and b are constants such that $a < 0 < b$, and the function $f(y)$ is defined below:

$$\begin{aligned} f(y) &= 0, \quad if \;\; y \leq 0, \\ f(y) &= g(y), \quad if \;\; y > 0, \end{aligned} \tag{6.65}$$

where $g(y)$ is a single-valued function satisfying the following properties,

(a) $g(\tau-a)f(b-\tau)>0, \quad \forall\tau\in(a,b)$,

(b) $g(\tau-a)f(b-\tau)\geq\delta_2(\rho_2)>0, \quad \forall\tau\in(a+\rho_2,b-\rho_2)$, *with* $0<\rho_2<\frac{b-a}{2}$,

(c) $g(y)$ *is p times differentiable with respect to y, and* $\lim_{y\to0^+}\frac{\partial^k g(y)}{\partial y^k}=0, \quad k=1,2,\ldots,p-1$, *with p being a positive integer, and* $\delta_2(\rho_2)$ *is a positive constant depending on the positive constant* ρ_2*. Then, the function* $h(x,a,b)$ *is a p-times differentiable step function. Furthermore, if* $g(y)$ *in (6.65) is replaced by* $g(y)=\mathrm{e}^{-1/y}$*, then property (4) in Definition 6.2 is replaced by* $(4)'$*; i.e.,* $h(x,a,b)$ *is a smooth step function.*

Definition 6.3 ([2]) A function $\Psi(x,a,b)$ is said to be a p-times differentiable unbounded function if it holds the following properties.

(1) $x=0\Leftrightarrow\Psi(x,a,b)=0$,

(2) $\lim_{x\to a^-}\Psi(x,a,b)=-\infty$, $\lim_{x\to b^+}\Psi(x,a,b)=\infty$,

(3) $\Psi(x,a,b)$ is p times differentiable with respect to x, for all $x\in(a,b)$,

(4) $\Psi^{'}(x,a,b)>0, \quad \forall x\in(a,b)$,

(5) $\Psi^{'}(x,a,b)\geq\delta_3(\rho_3)>0, \quad \forall x\in(a+\rho_3,b-\rho_3)$, with $0<\rho_3<\frac{b-a}{2}$,

where p is a positive integer, a and b are constants such that $a<0<b$, $\Psi^{'}(x,a,b)=\frac{\partial\Psi(x,a,b)}{\partial x}$, and $\delta_3(\rho_3)$ is a positive constant depending on the positive constant ρ_3. Furthermore, if $p=\infty$, then the function $\Psi(x,a,b)$ is said to be a smooth unbounded function.

Lemma 6.7 ([2]) *Let the scalar function* $\Psi(x,a,b)$ *be defined as*

$$\Psi(x,a,b)=\bar{\Psi}(\varphi(x,a,b))-\bar{\Psi}(\varphi(0,a,b)), \tag{6.66}$$

where the function $\varphi(x,a,b)$ *is defined as follows.*

$$\varphi(x,a,b)=\varepsilon(2h(x,a,b)-1) \tag{6.67}$$

with ε *being a positive constant, and* $h(x,a,b)$ *being the p-times differentiable step functions in Definition 6.2. The function* $\bar{\Psi}(\xi)$ *is such that*

(1) $\xi=0\Leftrightarrow\bar{\Psi}(\xi)=0$,

(2) $\lim_{\xi\to-\varepsilon^-}\bar{\Psi}(\xi)=-\infty$, $\lim_{\xi\to\varepsilon^+}\bar{\Psi}(\xi)=\infty$,

(3) $\bar{\Psi}(\xi)$ *is p times differentiable with respect to* ξ*, for all* $\xi\in(-\varepsilon,\varepsilon)$,

(4) $\bar{\Psi}^{'}(\xi)>0, \quad \forall\xi\in(-\varepsilon,\varepsilon)$,

(5) $\bar{\Psi}^{'}(\xi)>\delta_4(\rho_4)>0, \quad \forall\xi\in(a+\rho_4,b-\rho_4)$, *with* $0<\rho_4<\frac{b-a}{2}$,

where $\bar{\Psi}^{'}(\xi)=\frac{\partial\bar{\Psi}(\xi)}{\partial\xi}>0, \forall\xi\in(-\varepsilon,\varepsilon)$*, and* $\delta_4(\rho_4)$ *is a positive constant depending on the positive constant* ρ_4*. Then the function* $\Psi(x,a,b)$ *is a p-times differentiable unbounded function. Moreover, if* $h(x,a,b)$ *is a smooth step function, then the function* $\Psi(x,a,b)$ *is a smooth unbounded function.*

Remark 6.2 For Lemma 6.6, it can be seen that several functions satisfy properties (a)–(c) of the function $g(y)$, such as $g(y)=\mathrm{y}^p$, $g(y)=\tanh(\mathrm{y})^p$, $g(y)=\arctan(\mathrm{y}^p)$, etc.

Remark 6.3 In Definition 6.3, if $a = -b$, then many p-times differentiable unbounded functions can be obtained. An example is the function $\tan(-\frac{\pi}{2a}x)$. If $a \neq -b$, it is difficult to give a p-times differentiable unbounded function. However, we can construct a p-times differentiable unbounded function by using the p-times differentiable step function in Definition 6.2 with Lemma 6.7. For example, $\Psi(x, a, b) = \tan[\frac{\pi}{2}(2h(x, a, b) - 1)] - \tan[\frac{\pi}{2}(2h(0, a, b) - 1)]$.

Lemma 6.8 ([18]) *For any positive real numbers c, d and any real-valued function $\rho(x, y) > 0$,*

$$|x|^a\,|y|^d \leq \frac{a}{a+d}\rho(x, y)\,|x|^{a+d} + \frac{d}{a+d}\rho^{-a/d}(x, y)\,|y|^{a+d}. \tag{6.68}$$

Lemma 6.9 ([24]) (Barbalat's Lemma) *Consider a differentiable function $h(t)$. If* $\lim_{t\to\infty} h(t)$ *is finite and $\dot{h}(t)$ is uniformly continuous, then* $\lim_{t\to\infty} \dot{h}(t) = 0$.

6.3.2 Main Results

In what follows, a systematic design procedure for output-constrained tracking control of the system (6.59) is presented by using the CLF approach and the p-times differentiable unbounded functions in Definition 6.3.

First, give a coordinate transformation:

$$z_1 = \Psi(x_{1d}, a, b), \tag{6.69}$$

where $x_{1d} = x_1 - y_c = y - y_c$ is the output tracking error, $\Psi(x_{1d}, a, b)$ is a p-times differentiable unbounded function with $p \geq n - 1$, and the constants a and b are chosen such that

$$-\underline{L} \leq a < -\underline{L}_1, \overline{L}_1 < b \leq \bar{L}. \tag{6.70}$$

On the basis of the properties of $\Psi(x_{1d}, a, b)$ presented in Definition 6.3, it is clear that if we design a control input u ensuring $\lim_{t\to\infty} z_1(t) = 0$ and keeping all signals of the corresponding closed-loop system bounded for a bounded $z_1(t_0)$, then the output-constrained tracking control problem of system (6.59) is solved. Note that $z_1(t_0)$ is bounded under the constants a and b in (6.70), the assumption on the initial output tracking error in (6.63), and the properties of the function $\Psi(x_{1d}, a, b)$ listed in Definition 6.3.

Differentiating both sides of (6.69) in conjunction with system (6.59), we can rewrite them as

$$\begin{aligned}
\dot{z}_1 &= \Psi'(x_{1d}, a, b)g_1^{\sigma(t)}(x_1)\,x_2, \\
\dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i, d(t)) + g_i^{\sigma(t)}(\bar{x}_i)x_{i+1}, i = 2, 3, \ldots, n-1,
\end{aligned}$$

$$\dot{x}_n = f_n^{\sigma(t)}(\bar{x}_n, d(t)) + g_n^{\sigma(t)}(\bar{x}_n)u,$$
$$y = x_1, \tag{6.71}$$

Next, the steps of designing controllers are given below.

Step 1. Choose $V_1(z_1) = \frac{1}{2}z_1^2$ and let $z_2 = x_2 - \phi_1(z_1)$, where $\phi_1(z_1)$ is the common stabilizing function to be designed.

The derivative of $V_1(z_1)$ is

$$\dot{V}_1(z_1) = z_1\Psi'(x_{1d}, a, b)g_1^k(x_1)(z_2 + \phi_1(z_1)). \tag{6.72}$$

Choose the common stabilizing function as

$$\phi_1(z_1) = z_1\left[-\frac{1}{\underline{g}}(((n-2)/\Psi'(x_{1d}, a, b) + 1)\overline{g} + n/\Psi'(x_{1d}, a, b))\right]. \tag{6.73}$$

Substituting (6.73) into (6.72) yields that

$$\begin{aligned}\dot{V}_1(z_1) =& -z_1^2\Psi'(x_{1d}, a, b)\frac{g_1^k(x_1)}{\underline{g}}\frac{n}{\Psi'(x_{1d}, a, b)}\\ & -z_1^2\Psi'(x_{1d}, a, b)\frac{g_1^k(x_1)}{\underline{g}}\left(\frac{(n-2)}{\Psi'(x_{1d}, a, b)} + 1\right)\overline{g}\\ & + \Psi'(x_{1d}, a, b)g_1^k(x_1)z_1z_2\\ \le & -nz_1^2 - (\Psi'(x_{1d}, a, b) + n - 2)\overline{g}z_1^2\\ & + \Psi'(x_{1d}, a, b)g_1^k(x_1)z_1z_2,\end{aligned} \tag{6.74}$$

where the coupling term $\Psi'(x_{1d}, a, b)g_1^k(x_1)z_1z_2, \forall k \in I_m$ can be canceled by following the steps below.

Step 2. Let $z_3 = x_3 - \phi_2(\bar{z}_2)$, where $\phi_2(\bar{z}_2)$ is the common stabilizing function to be designed.

Choose $\overline{V}_2(\bar{z}_2) = V_1(z_1) + \frac{1}{2}z_2^2$, and then the time derivative of $\overline{V}_2(\bar{z}_2)$ can be given by

$$\begin{aligned}\dot{\overline{V}}_2(\bar{z}_2) =& -nz_1^2 - (\Psi'(x_{1d}, a, b) + n - 2)\bar{g}z_1^2\\ & + z_2\left(f_2^k(\bar{x}_2, d(t)) - \frac{\partial\phi_1(z_1)}{\partial z_1}\dot{z}_1 + g_2^k(\bar{x}_2)x_3\right)\\ & + \Psi'(x_{1d}, a, b)g_1^k(x_1)z_1z_2\\ \le & -nz_1^2 - (\Psi'(x_{1d}, a, b) + n - 2)\bar{g}z_1^2\\ & + \bar{g}\Psi'(x_{1d}, a, b)|z_1z_2| + \left|z_2\Phi_2^k(\bar{z}_2, d(t))\right|\\ & + g_2^k(\bar{x}_2)(z_3 + \phi_2(\bar{z}_2)),\end{aligned} \tag{6.75}$$

where $\Phi_2^k(\bar{z}_2, d(t)) = f_2^k(\bar{x}_2, d(t)) - \frac{\partial \phi_1(z_1)}{\partial z_1}\Gamma_1^k(\bar{z}_2)$, $\Gamma_1^k(\bar{z}_2) = g_1^k(x_1)(z_2 + \phi_1(z_1))$, $\forall k \in I_m$.

Furthermore, one has $\left| f_2^k(\bar{x}_2, d(t)) \right| \leq |x_2| \mu_2^k(\bar{x}_2) \leq (|z_1| + |z_2|)\hat{\mu}_2^k(\bar{z}_2)$, $\forall k \in I_m$, where $\hat{\mu}_2^k(\bar{z}_2)$ are a set of smooth non-negative functions. It means that

$$\left| \Phi_2^k(\bar{z}_2, d(t)) \right| \leq (|z_1| + |z_2|)\widetilde{\mu}_2^k(\bar{z}_2), \tag{6.76}$$

where $\widetilde{\mu}_2^k(\bar{z}_2)$ are a set of smooth non-negative functions, $\forall k \in I_m$.

According to Lemma 6.8 and (6.76), it holds that $|z_1 z_2| \leq z_1^2 + z_2^2 \tilde{\varphi}_2(\bar{z}_2)$, $|z_2 \Phi_2^k(\bar{z}_2, d(t))| \leq z_1^2 + z_2^2 \tilde{\varphi}_2^k(\bar{z}_2)$, $\forall k \in I_m$, where $\tilde{\varphi}_2(\bar{z}_2) \geq 1$, $\tilde{\varphi}_2^k(\bar{z}_2) \geq 1$ are some smooth functions. Thus, we get that

$$\begin{aligned}
\dot{V}_2(\bar{z}_2) = & -n z_1^2 - (n-2)\bar{g} z_1^2 - \bar{g}\Psi'(x_{1d}, a, b) z_1^2 + z_1^2 \\
& + \bar{g}\Psi'(x_{1d}, a, b) z_1^2 + \bar{g}\Psi'(x_{1d}, a, b) z_2^2 \tilde{\varphi}_2(\bar{z}_2) \\
& + z_2^2 \tilde{\varphi}_2^k(\bar{z}_2) + g_2^k(\bar{x}_2) z_2 \phi_2(\bar{z}_2) + g_2^k(\bar{x}_2) z_2 z_3 \\
\leq & -(n-1) z_1^2 - (n-2)\bar{g} z_1^2 + z_2^2 \varphi_2^k(\bar{z}_2) \\
& + \bar{g}\Psi'(x_{1d}, a, b) z_1^2 + g_2^k(\bar{x}_2) z_2 \phi_2(\bar{z}_2) \\
& + g_2^k(\bar{x}_2) z_2 z_3 \\
\leq & -(n-1) z_1^2 - (n-2)\bar{g} z_1^2 + z_2^2 \varphi_2^{\max}(\bar{z}_2) \\
& + g_2^k(\bar{x}_2) z_2 \phi_2(\bar{z}_2) + g_2^k(\bar{x}_2) z_2 z_3,
\end{aligned} \tag{6.77}$$

where $\varphi_2^{\max}(\bar{z}_2) \geq \varphi_2^k(\bar{z}_2) = \bar{g}\Psi'(x_{1d}, a, b)\tilde{\varphi}_2(\bar{z}_2) + \tilde{\varphi}_2^k(\bar{z}_2)$, $\forall k \in I_m$ is a smooth function.

Design the common stabilizing function as

$$\phi_2(\bar{z}_2) = z_2 \left[-\frac{1}{\underline{g}}(\varphi_2^{\max}(\bar{z}_2) + (n-2)\bar{g} + (n-1)) \right]. \tag{6.78}$$

Substituting (6.78) into (6.77) yields that

$$\begin{aligned}
\dot{V}_2(\bar{z}_2) \leq & -(n-1) z_1^2 - (n-2)\bar{g} z_1^2 + z_2^2 \varphi_2^{\max}(\bar{z}_2) \\
& - \frac{g_2^k(\bar{x}_2)}{\underline{g}} z_2^2 \varphi_2^{\max}(\bar{z}_2) - \frac{g_2^k(\bar{x}_2)}{\underline{g}}(n-2)\bar{g} z_2^2 \\
& - \frac{g_2^k(\bar{x}_2)}{\underline{g}}(n-1) z_2^2 + g_2^k(\bar{x}_2) z_2 z_3 \\
\leq & -(n-1)(z_1^2 + z_2^2) - (n-2)\bar{g}(z_1^2 + z_2^2) \\
& + g_2^k(\bar{x}_2) z_2 z_3,
\end{aligned} \tag{6.79}$$

where the coupling term $g_2^k(\bar{x}_2) z_2 z_3$ can be canceled by the following steps.

Step i. Let $z_{i+1} = x_{i+1} - \phi_i(\bar{z}_i)$, where $\phi_i(\bar{z}_i)$ is a common stabilizing function to be designed.

Assume that the first $i-1$ $(2 \le i \le n)$ steps are finished, that is, for the following auxiliary $(z_1, \ldots, z_{i-1})$-equations:

$$\dot{z}_j = \Phi_j^k(\bar{z}_j, d(t)) + g_j^k\left(\bar{x}_j\right) x_{j+1},\ j = 1, \ldots, i-1, \tag{6.80}$$

where $\Phi_j^k(\bar{z}_j, d(t)) = f_j^k(\bar{z}_j, d(t)) - \sum_{l=1}^{j-1} \frac{\partial \phi_{j-1}(\bar{z}_{j-1})}{\partial z_l} \Gamma_l^k(\bar{z}_{l-1})$, we have a set of common stabilizing functions (6.73), (6.78) and

$$\phi_j(\bar{z}_j) = z_j\left[-\frac{1}{\underline{g}}(\varphi_j^{\max}\left(\bar{z}_j\right) + (n-j)\bar{g} + (n-j+1))\right], \tag{6.81}$$

where $3 \le j \le i-1$, such that there exists a CLF for system (6.80),

$$\overline{V}_{i-1}(\bar{z}_{i-1}) = V_1(z_1) + \frac{1}{2}\sum_{l=2}^{i-1} z_l^2, \tag{6.82}$$

and the time derivative of $\overline{V}_{i-1}(\bar{z}_{i-1})$ fulfills $\dot{\overline{V}}_{i-1}\left(\bar{z}_{i-1}\right) \le -(n-i+2)(z_1^2 + \cdots + z_{i-1}^2) - (n-i+1)\bar{g}(z_1^2 + \cdots + z_{i-1}^2) + g_{i-1}^k\left(\bar{x}_{i-1}\right) z_{i-1} z_i$.

Choosing $\overline{V}_i(\bar{z}_i) = \overline{V}_{i-1}(\bar{z}_{i-1}) + \frac{1}{2}z_i^2$, then we can derive that

$$\begin{aligned}
\dot{\overline{V}}_i\left(\bar{z}_i\right) \le\ & -(n-i+2)(z_1^2 + \cdots + z_{i-1}^2) \\
& -(n-i+1)\bar{g}(z_1^2 + \cdots + z_{i-1}^2) \\
& + z_i(\Phi_i^k\left(\bar{z}_i, d(t)\right) + g_i^k\left(\bar{x}_i\right) x_{i+1}) \\
& + g_{i-1}^k\left(\bar{z}_{i-1}\right) z_{i-1} z_i \\
\le\ & -(n-i+2)(z_1^2 + \cdots + z_{i-1}^2) \\
& -(n-i+1)\bar{g}(z_1^2 + \cdots + z_{i-1}^2) \\
& + \bar{g}\left|z_{i-1} z_i\right| + \left|z_i(\Phi_i^k\left(\bar{z}_i, d(t)\right)\right| \\
& + g_i^k\left(\bar{z}_i\right) z_i \phi_i\left(\bar{z}_i\right) + g_i^k\left(\bar{z}_i\right) z_i z_{i+1},
\end{aligned} \tag{6.83}$$

where $\Phi_i^k\left(\bar{z}_i, d(t)\right) = f_i^k\left(\bar{z}_i, d(t)\right) - \sum_{l=1}^{i-1} \frac{\partial \phi_{l-1}(\bar{z}_{l-1})}{\partial z_i} \Gamma_l^k\left(\bar{z}_{l-1}\right),\ \forall k \in I_m$.

Similar to Step 2, one has $\left|z_{i-1} z_i\right| \le z_1^2 + \cdots + z_{i-1}^2 + z_i^2 \tilde{\varphi}_i(\bar{z}_i)$, $\left|z_i \Phi_i^k\left(\bar{z}_i, d(t)\right)\right| \le z_1^2 + \cdots + z_{i-1}^2 + z_i^k \tilde{\varphi}_i^k(\bar{z}_i), \forall k \in I_m$, where $\tilde{\varphi}_i\left(\bar{z}_i\right) \ge 1, \tilde{\varphi}_i^k\left(\bar{z}_i\right) \ge 1$ are some smooth functions. Therefore, we can arrive at

$$\begin{aligned}\dot{\overline{V}}_i(\bar{z}_i) \le & -(n-i+2)(z_1^2+\cdots+z_{i-1}^2) \\ & -(n-i+1)\bar{g}(z_1^2+\cdots+z_{i-1}^2) \\ & +\bar{g}(z_1^2+\cdots+z_{i-1}^2)+z_1^2+\cdots+z_{i-1}^2 \\ & +z_i^2\tilde{\varphi}_i^k(\bar{z}_i)+\bar{g}z_i^2\tilde{\varphi}_i(\bar{z}_i)+g_i^k(\bar{x}_i)z_i\phi_i(\bar{z}_i) \\ & +g_i^k(\bar{x}_i)z_iz_{i+1} \\ \le & -(n-i+1)(z_1^2+\cdots+z_{i-1}^2) \\ & -(n-i)\bar{g}(z_1^2+\cdots+z_{i-1}^2) \\ & +z_i^2\varphi_i^{\max}(\bar{z}_i)+g_i^k(\bar{x}_i)z_i\phi_i(\bar{z}_i) \\ & +g_i^k(\bar{x}_i)z_iz_{i+1},\end{aligned} \tag{6.84}$$

where $\varphi_i^{\max}(\bar{z}_i) \ge \varphi_i^k(\bar{z}_i) = \bar{g}\tilde{\varphi}_i(\bar{z}_i)+\tilde{\varphi}_i^k(\bar{z}_i), \forall k \in I_m$ are some smooth functions.

Design the common stabilizing function as

$$\phi_i(\bar{z}_i) = z_i\left[-\frac{1}{\underline{g}}(\varphi_i^{\max}(\bar{z}_i)+(n-i)\bar{g}+(n-i+1))\right]. \tag{6.85}$$

Then, substituting (6.85) into (6.84) yields that

$$\begin{aligned}\dot{\overline{V}}_i(\bar{z}_i) \le & -(n-i+1)(z_1^2+\cdots+z_{i-1}^2) \\ & -(n-i)\bar{g}(z_1^2+\cdots+z_{i-1}^2)+z_i^2\varphi_i^{\max}(\bar{z}_i) \\ & -\frac{g_i^k(\bar{x}_i)}{\underline{g}}z_i^2\varphi_i^{\max}(\bar{z}_i)-\frac{g_i^k(\bar{x}_i)}{\underline{g}}(n-i)\bar{g}z_i^2 \\ & -\frac{g_i^k(\bar{x}_i)}{\underline{g}}(n-i+1)z_i^2+g_i^k(\bar{x}_i)z_iz_{i+1} \\ \le & -(n-i+1)(z_1^2+\cdots+z_i^2) \\ & -(n-i)\bar{g}(z_1^2+\cdots+z_i^2)+g_i^k(\bar{x}_i)z_iz_{i+1},\end{aligned} \tag{6.86}$$

where the coupling term $g_i^k(\bar{x}_i)z_iz_{i+1}$ can be canceled by the following steps.

Step n. Repeating the procedures above, it is straightforward to see that there exists a CLF of system (6.59)

$$\overline{V}_n(\bar{z}_n) = V_1(z_1)+\frac{1}{2}\sum_{l=2}^{n} z_l^2. \tag{6.87}$$

Then, we can explicitly design an individual controller for each subsystem

$$u_k(\bar{z}_n) = z_n\left[-\frac{1}{g_{n,k}}(\varphi_{n,k}(\bar{z}_n)+1)\right], \forall k \in I_m \tag{6.88}$$

such that

$$\dot{\overline{V}}_n(\bar{z}_n) \leq -(z_1^2 + \cdots + z_n^2). \tag{6.89}$$

Remark 6.4 In fact, we can also design a common state feedback controller for the system (6.59) as

$$u(\bar{z}_n) = z_n \left[-\frac{1}{\underline{g}} (\varphi_n^{max}(\bar{z}_n) + 1) \right], \tag{6.90}$$

where $\varphi_n^{max}(\bar{z}_n) \geq \varphi_n^k(\bar{z}_n) = \bar{g}\tilde{\varphi}_n(\bar{z}_n) + \tilde{\varphi}_n^k(\bar{z}_n)$ is a smooth function. It can be seen that (6.90) can be extended from (6.88), which illustrates the less conservativeness of the controller to be developed.

Based on the above discussions, we now provide the main result.

Theorem 6.3 *Suppose that Assumption 6.5 holds. The output-constrained tracking controller for system (6.59) under arbitrary switching can be designed as (6.88), and the output tracking error $x_{1d}(t)$ locally exponentially converges to zero.*

Proof (i) Forward completeness. From (6.89) and $\Psi'(x_{1d}, a, b) > 0$ for all $x_{1d}(t) \in (a, b)$, noticing Property (6.58) of the function $\Psi(x_{1d}, a, b)$ in Definition 6.3, one obtains that

$$\dot{\overline{V}}_n \leq 0 \Rightarrow \overline{V}_n(t) \leq \overline{V}(t_0), \forall t \geq t_0 \geq 0. \tag{6.91}$$

This means that

$$\sum_{i=1}^{n} z_i(t) \leq \sum_{i=1}^{n} z_i(t_0) \tag{6.92}$$

for all $t \geq t_0 \geq 0$. Under the initial condition specified in (6.61), and the choice of the constants a and b in (6.70), the right-hand side of (6.92) is bounded. This means that the left-hand side of (6.92) must be bounded. Boundedness of the left-hand side of (6.92) implies that all z_i, $i = 1, 2, \ldots, n$ are bounded. Because $|z_1(t)|$ is bounded for all $t \geq t_0 \geq 0$, the output tracking error $x_{1d}(t)$ never reaches its boundary values $\underline{a}$ and b; i.e., $x_{1d}(t) \in (a, b)$ for all $t \geq t_0 \geq 0$. This together with (6.70), $\underline{L}_1 < \underline{L}$ and $\overline{L}_1 < \overline{L}$ (Assumption 6.2) implies that $x_{1d}(t)$ is always in its constraint range, i.e. $\underline{L} < x_{1d}(t) < \overline{L}$ for all $t \geq t_0 \geq 0$. Boundedness of all x_i, $i = 1, 2, \ldots, n$ follows from the boundedness of all z_i, and smooth property of all functions $f_i^k(\bar{x}_i, d(t))$, $g_i^k(\bar{x}_i)$ and $\Psi(x_{1d}, a, b)$. Boundedness of all x_i, $i = 1, 2, \ldots, n$ also denotes that the closed-loop system (6.55) is forward complete.

(ii) Asymptotic convergence. Noting that $x_i(t)$, $z_i(t)$, $i = 1, 2, \ldots, n$ are bounded, it is not hard to deduce that $\ddot{\overline{V}}_n(\bar{z}_n)$ is bounded, which gives that $\dot{\overline{V}}_n(\bar{z}_n)$ is uniformly continuous. Then, we get from Lemma 6.9 that $\lim_{t\to\infty} z_i(t) = 0$, $i =$

$1, 2, \ldots, n$. Therefore, it follows from Property (6.55) of function $\Psi(x_{1d}, a, b)$. that $\lim_{t\to\infty} x_{1d}(t) = 0$.

(iii) Local exponential convergence of the output tracking error $x_{1d}(t)$. It follows from (6.89) that

$$V_n(t) \leq V_n(t_0)e^{-(t-t_0)}, \forall t \geq t_0. \tag{6.93}$$

One can get from (6.93) that

$$|z_1(t)| \leq \sqrt{2V_n(t_0)}e^{-\frac{1}{2}(t-t_0)}, \forall t \geq t_0, \tag{6.94}$$

which implies that $z_1(t)$ locally exponentially converges to 0. Now, with the help of Taylor expansion of function $\Psi(x_{1d}, a, b)$ around $x_{1d} = 0$ and noticing Property (6.59) of the function $h(x_{1d}, a, b)$, Property (6.60) of the function $\Psi(x_{1d}, a, b)$, and the construction of the function $\Psi(x_{1d}, a, b)$ (see Lemma 6.7), it can be shown that there exists a strictly positive constant $\delta_5(\rho_5)$ depending on the positive constant ρ_5 with $0 < \rho_5 < \frac{b-a}{2}$ such that

$$|\Psi(x_{1d}(t), a, b)| \geq \delta_5(\rho_5)\,|x_{1d}(t)|\,, \forall t \geq t_1, \tag{6.95}$$

where the time instance $t_1 > t_0$. By definition $z_1(t) = \Psi(x_{1d}(t), a, b)$, a combination of (6.93) and (6.95) gives

$$|x_{1d}(t)| \leq \frac{\sqrt{2V_n(t_0)}e^{-\frac{1}{2}(t-t_0)}}{\delta_5(\rho_5)}, \forall t \geq t_1, \tag{6.96}$$

which shows the local exponential convergence of $x_{1d}(t)$ to 0. □

Remark 6.5 When the output-constrained tracking control problem is considered, it is required that $|x_1|$ be absent in Assumption 6.5, and thus $f_1^k(x_1)$ cannot appear in x_1-equation of system (6.59), $k = 1, 2, \ldots, m$. It seems that Assumption 6.1 appears to be conservative. However, if the stabilization problem is considered, $|x_1|$ can be presented in Assumption 6.5, which leads to: $f_1^k(x_1)$ exists in the x_1-equation of (1), $k = 1, 2, \ldots, m$. We give the design procedures for the stabilization problem in the next section.

In what follows, we consider the robust state-constrained stabilization problem for the following uncertain switched nonlinear system,

$$\begin{aligned}\dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i, d(t)) + g_i^{\sigma(t)}(\bar{x}_i)\,x_{i+1}, i = 1, 2, \ldots, n-1,\\ \dot{x}_n &= f_n^{\sigma(t)}(\bar{x}_n, d(t)) + g_n^{\sigma(t)}(\bar{x}_n)\,u_{\sigma(t)},\end{aligned} \tag{6.97}$$

where all functions are smooth with $f_{i,k}(0, d(t)) = 0$ for all $d(t) \in \Omega$ and $0 < \underline{g} < g_{i,k}(\bar{x}_i) \leq \bar{g}$, $\underline{g}, \bar{g}$ are positive constants, respectively, $i = 1, 2, \ldots, n$, $\forall k \in I_m$.

The robust state-constrained stabilization problem: For system (6.97) under arbitrary switching, design state feedback controllers for all subsystems to ensure that:

(1) System (6.97) is asymptotically stabilizable.

(2) The state x_1 is within a pre-specified limit range; i.e.,

$$-\underline{L} \leq x_1 \leq \overline{L} \tag{6.98}$$

for all $t \geq t_0 \geq 0$, where $\underline{L}$ and $\overline{L}$ are strictly positive constants.

(3) All signals of the closed-loop system (6.97) are bounded.

In addition, it is assumed that the following conditions hold.

Assumption 6.7 For $i = 1, 2, \ldots, n$,

$$\left| f_i^k(\bar{x}_i, d(t)) \right| \leq (|x_1|+|x_2|+\cdots+|x_i|)\mu_i^k(\bar{x}_i), \forall k \in I_m, \tag{6.99}$$

where $\mu_i^k(\bar{x}_i)$ are a set of known non-negative smooth functions.

Assumption 6.8 The p-times differentiable unbounded function in Definition 6.2 satisfies

$$\Psi(x, a, b) = x[1 + \chi(x)], \tag{6.100}$$

where $\chi(x)$ is a non-negative smooth function.

Similar to Theorem 6.3, we apply a coordinate transformation:

$$z_1 = \Psi(x_1, a, b), \tag{6.101}$$

where $\Psi(x_1, a, b)$ is a p-times differentiable unbounded function with $p \geq n - 1$.

Differentiating both sides of (6.101) in conjunction with system (6.97), one can rewrite them in the form:

$$\begin{aligned} \dot{z}_1 &= \Psi'(x_1, a, b)(f_1^{\sigma(t)}(x_1, d(t)) + g_1^{\sigma(t)}(x_1)x_2), \\ \dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i, d(t)) + g_i^{\sigma(t)}(\bar{x}_i)x_{i+1}, i = 2, 3, \ldots, n-1, \\ \dot{x}_n &= f_n^{\sigma(t)}(\bar{x}_n, d(t)) + g_n^{\sigma(t)}(\bar{x}_n)u_{\sigma(t)}, \end{aligned} \tag{6.102}$$

Step 1. Choose $V_1(z_1) = \frac{1}{2}z_1^2$ and let $z_2 = x_2 - \phi_1(z_1)$, where $\phi_1(z_1)$ is the common stabilizing function to be designed.

The derivative of $V_1(z_1)$ is given by

$$\begin{aligned} \dot{V}_1(z_1) &= \Psi'(x_1, a, b)z_1[f_1^k(x_1, d(t)) + g_1^k(x_1)(z_2 + \phi_1(z_1))] \\ &= \Psi'(x_1, a, b)z_1 f_1^k(x_1, d(t)) \\ &\quad + \Psi'(x_{1d}, a, b)z_1 g_1^k(x_1)(z_2 + \phi_1(z_1)) \\ &\leq \Psi'(x_1, a, b)|z_1\Phi_1^k(x_1)| \\ &\quad + \Psi'(x_{1d}, a, b)z_1 g_1^k(x_1)(z_2 + \phi_1(z_1)), \end{aligned} \tag{6.103}$$

where $\Phi_1^k(x_1) = f_1^k(x_1, d(t)), \forall k \in I_m$.

Under Assumptions 6.5 and 6.8, one can find that

$$\left| f_1^k(x_1, d(t)) \right| \le |x_1| \mu_1^k(x_1) \le |z_1| \hat{\mu}_1^k(z_1), \forall k \in I_m, \tag{6.104}$$

where $\hat{\mu}_1^k(z_1)$ are a set of smooth non-negative functions.

Then, we can get that

$$\left| z_1 \Phi_1^k(z_1) \right| \le z_1^2 \tilde{\varphi}_1^k(z_1), \forall k \in I_m. \tag{6.105}$$

where $\tilde{\varphi}_1^k(z_1) \ge 1, \forall k \in I_m$ is a smooth function.

Then, one can see that

$$\begin{aligned} \dot{V}_1(z_1) &\le z_1^2 \Psi'(x_1, a, b) \tilde{\varphi}_1^k(z_1) \\ &\quad + z_1 \Psi'(x_1, a, b) g_1^k(x_1)(z_2 + \phi_1(z_1)) \\ &\le z_1^2 \Psi'(x_1, a, b) \tilde{\varphi}_1^{max}(z_1) \\ &\quad + z_1 \Psi'(x_1, a, b) g_1^k(x_1)(z_2 + \phi_1(z_1)), \end{aligned} \tag{6.106}$$

where $\tilde{\varphi}_1^{max}(z_1) \ge \tilde{\varphi}_1^k(z_1) \ge 1, \forall k \in I_m$ is a smooth function.

The common stabilizing function is designed as

$$\begin{aligned} \phi_1(z_1) = z_1 \Bigg[&- \frac{1}{\underline{g}} (\tilde{\varphi}_1^{max}(z_1) + ((n-2)/\Psi'(x_1, a, b) + 1)\overline{g} \\ &+ n/\Psi'(x_1, a, b)) \Bigg]. \end{aligned} \tag{6.107}$$

Substituting (6.107) into (6.106), one can get that

$$\begin{aligned} \dot{V}_1(z_1) = &\ z_1^2 \Psi'(x_1, a, b) \tilde{\varphi}_1^{max}(z_1) \\ &- z_1^2 \Psi'(x_1, a, b) \frac{g_1^k(x_1)}{\underline{g}} \tilde{\varphi}_1^{max}(z_1) \\ &- z_1^2 \Psi'(x_1, a, b) \frac{g_1^k(x_1)}{\underline{g}} \frac{n}{\Psi'(x_1, a, b)} \\ &- z_1^2 \Psi'(x_1, a, b) \frac{g_1^k(x_1)}{\underline{g}} \left(\frac{(n-2)}{\Psi'(x_1, a, b)} + 1 \right) \overline{g} \\ &+ \Psi'(x_1, a, b) g_1^k(x_1) z_1 z_2 \\ \le &- n z_1^2 - (\Psi'(x_1, a, b) + n - 2) \overline{g} z_1^2 \\ &+ \Psi'(x_1, a, b) g_1^k(x_1) z_1 z_2, \end{aligned} \tag{6.108}$$

where the coupling term $\Psi'(x_1, a, b) g_1^k(x_1) z_1 z_2, \forall k \in I_m$ can be canceled by using the steps below.

Similar to the procedures in the above section, we design the individual controllers for the subsystems as

$$u_k(\bar{z}_n) = z_n \left[-\frac{1}{g_{n,k}} (\varphi_{n,k}(\bar{z}_n) + 1) \right], k \in I_m \tag{6.109}$$

such that

$$\dot{\bar{V}}_n(\bar{z}_n) \le -(z_1^2 + \cdots + z_n^2). \tag{6.110}$$

Now, we give the following result focusing on robust state-constrained stabilization problem of system (6.97).

Theorem 6.4 *Suppose that Assumptions 6.6–6.8 are satisfied; then the robust state-constrained stabilization problem of system (6.97) under arbitrary switchings can be solved by the controller in (6.109).*

Proof The proof is similar to the one of Theorem 6.3. □

6.3.3 Simulation Results

In this section, the following example is provided to illustrate the effectiveness of the proposed results.

Consider the switched nonlinear system:

$$\begin{cases} \dot{x}_1 = f_1^{\sigma(t)}(x_1) + x_2, \\ \dot{x}_2 = f_2^{\sigma(t)}(\bar{x}_2) + g^{\sigma(t)}(\bar{x}_2)\, u, \\ \; y \; = x_1, \quad \sigma(t) : [0, \infty) \to \{1, 2\}, \end{cases} \tag{6.111}$$

where $f_1^1(x_1) = 0$, $f_2^1(\bar{x}_2) = 3x_1^2 x_2^3$, $f_1^2(x_1) = 2x_1 - 0.4$, $f_2^2(\bar{x}_2) = x_1 x_2 \left(1 + x_1^2\right)$, $g^1(\bar{x}_2) = \left[- \sin^2\left(x_1^3 + 2x_2\right), 1.4 - \cos(x_1 x_2)\right]$, $g^2(\bar{x}_2) = \left[- 4x_1^4 x_2^2, 1.2\right]$. The objective is enable $y(t)$ to track the desired trajectory $y_d = 0.2$ subject to asymmetric time-varying output constraints $-(0.2 + 0.1\cos(t)) < y(t) < 0.7 + 0.1\cos(t)$.

According to Assumption 6.4, we choose $B_1(t) = 0.1 + 0.1\cos(t)$ and $B_2(t) = 0.3 + 0.1\cos(t)$. Based on (6.38), we can get an asymmetric barrier Lyapunov function:

$$V_1 = (1 - q(z_1)) \log \frac{0.09}{\left(0.09 - z_1^2\right)} + q(z_1) \log \frac{0.16}{\left(0.16 - z_1^2\right)}. \tag{6.112}$$

Defining $z_1 = x_1 - 0.2$, one can see that $\phi_1 = (1 - q(z_1))(-2z_1 - z_1(0.09 - z_1^2)) + q(z_1)(-2z_1 - z_1(0.16 - z_1^2))$ is a dominating feedback law for the auxiliary first-order subsystems. In that scenario

$$d_1^1 = (1 - q(z_1))(-z_1^2) + q(z_1)(-z_1^2),$$
$$d_1^2 = (1 - q(z_1))\left(-z_1^2 - \frac{2z_1^2}{(0.09 - z_1^2)}\right) + q(z_1)\left(-z_1^2 - \frac{2z_1^2}{(0.16 - z_1^2)}\right). \tag{6.113}$$

Define $z_2 = x_2 - \phi_1$. Then, $\overline{V}_2 = V_1 + \frac{1}{2}z_2^2$ is continuously differentiable and positive definite when $-0.3 < z_1(t) < 0.4$. Furthermore, $\overline{V}_2$ is a common Lyapunov function for the two subsystems in (6.111). For $k = 1, 2,$ let

$$\begin{aligned} a_k =& (1 - q(z_1))(d_1^k + z_2\left(\frac{z_1}{0.09 - z_1^2} + f_2^k(\bar{x}_2) - \frac{\partial \phi_1}{\partial x_1}(x_2 + f_1^k(\bar{x}_1))\right) \\ &+ q(z_1)(d_1^k + z_2\left(\frac{z_1}{0.16 - z_1^2} + f_2^k(\bar{x}_2) - \frac{\partial \phi_1}{\partial x_1}\left(x_2 + f_1^k(\bar{x}_1)\right)\right), \\ b_k =& z_2 g^k(\bar{x}_2). \end{aligned} \tag{6.114}$$

It is clear that $M = \{2\}$, $F = \{1\}$. From (6.49), we can obtain the controller:

$$u = [u_1, u_2]^T \tag{6.115}$$

with

$$u_1 = \begin{cases} \max_{i\in\{1,2\}}\left\{p_{i,1}\right\}, & if\ z_2 > 0 \\ \min_{i\in\{1,2\}}\left\{p_{i,1}\right\}, & if\ z_2 < 0 \\ z_2 = 0, \quad if\ z_2 = 0 \end{cases} \tag{6.116}$$

and

$$u_2 = \begin{cases} \min_{i\in\{1,2\}}\left\{p_{i,2}\right\}, & if\ \ z_2 > 0 \\ \max_{i\in\{1,2\}}\left\{p_{i,2}\right\}, & if\ \ z_2 < 0 \\ z_2 = 0, \quad if\ z_2 = 0 \end{cases} \tag{6.117}$$

where

$$p_k = \left[p_{k,1}, p_{k,2}\right] = \begin{cases} -b_k^T \frac{\max\left\{a_k + b_k b_k^T, 0\right\}}{b_k b_k^T}, & if\ \ z_2 \neq 0, \\ 0, \quad if\ z_2 \neq 0. \end{cases} \tag{6.118}$$

Given $x_1(0) = -0.05$ and $x_2(0) = -2.2$, Fig. 6.5 shows that asymptotic output tracking performance is achieved and the output stays within the set $(-0.2 - 0.1cos(t), 0.7 + 0.1cos\,(t))$ when the Lyapunov function obtained in (6.112) is used. The switching signal for switched system (6.111) is shown in Fig. 6.6. Moreover, given different initial values of z_1, Fig. 6.7 indicates that the error z_1 converges to 0 while remaining in the set $(-0.4 - 0.1cos(t), 0.5 + 0.1cos(t)),\ \forall t \geq 0$. The phase portraits of z_1 and z_2 are depicted in Fig. 6.8, from which we can see that the error $z_1(t)$ does not transgress its barriers as long as its initial value satisfies $-0.3 < z_1(0) < 0.4$.

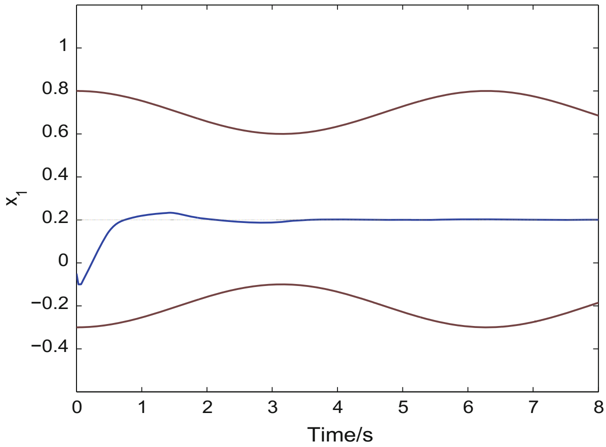

Fig. 6.5 Output tracking for the desired signal $y_d = 0.2$

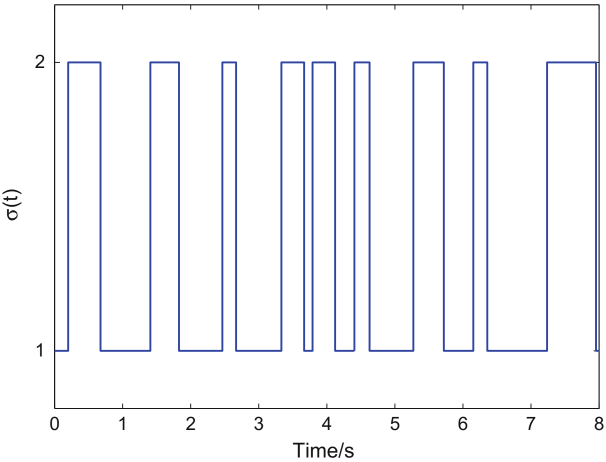

Fig. 6.6 The switching signals for the switched system (6.111)

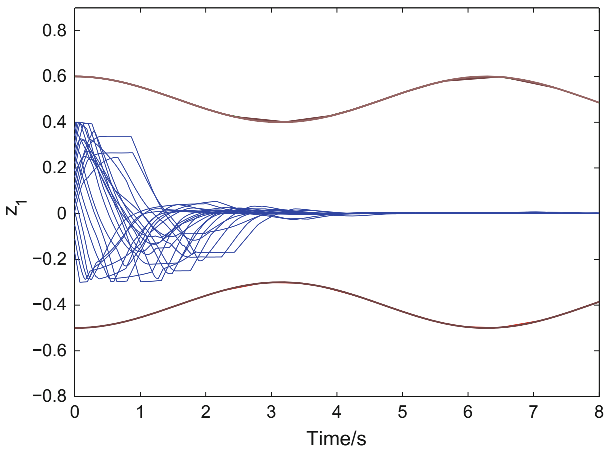

Fig. 6.7 Tracking error z_1 for various initial values satisfying $-0.3 < z_1(0) < 0.4$

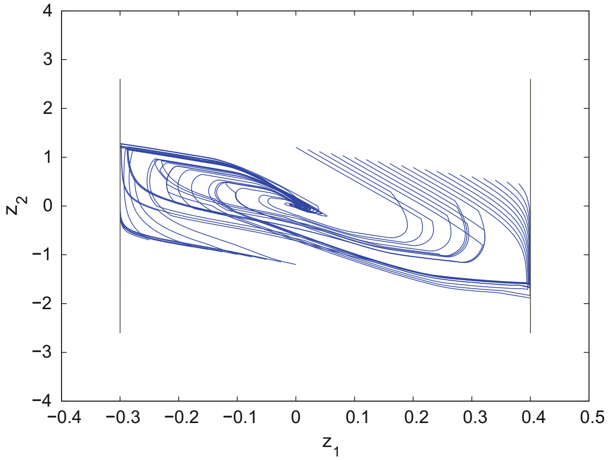

Fig. 6.8 Phase portraits of z_1, z_2 for the closed-loop system (6.111)

6.3.4 Conclusions

The problems of robust output-constrained tracking control and state-constrained stabilization for uncertain switched nonlinear systems in lower triangular form have been respectively studied. In the proposed approach, the p-times differentiable unbounded functions are introduced and incorporated in output tracking error transformations to convert the problem of controlling the switched systems with output tracking error constraints to a new problem of regulating the converted systems without a constraint. The backstepping technique is resorted to design controllers for the transformed systems.

References

1. Alamir M, Murilo A (2008) Swing-up and stabilization of a twin-pendulum under state and control constraints by a fast nmpc scheme. Automatica 44(5):1319–1324
2. Cinquemani E, Agarwal M, Chatterjee D, Lygeros J (2011) Convexity and convex approximations of discrete-time stochastic control problems with constraints. Automatica 47(9):2082–2087
3. Sun YJ, Lien CH, Hsieh JG (1998) Global exponential stabilization for a class of uncertain nonlinear systems with control constraint. IEEE Trans Autom Control 43(5):674–677
4. Derong L, Michel A (1993) Dynamical systems with saturation nonlinearities. Lecture Notes in Control and Information Sciences 195
5. Hu T, Lin Z (2003) Composite quadratic lyapunov functions for constrained control systems. IEEE Trans Autom Control 48(3):440–450
6. Saberi A, Han J, Stoorvogel AA (2002) Constrained stabilization problems for linear plants. Automatica 38(4):639–654
7. Blanchini F (1999) Survey paper: set invariance in control. Autom(J IFAC) 35(11):1747–1767
8. Gilbert EG, Tan KT (1991) Linear systems with state and control constraints: The theory and application of maximal output admissible sets. IEEE Trans Autom Control 36(9):1008–1020
9. Bemporad A, Borrelli F, Morari M et al (2002) Model predictive control based on linear programming~ the explicit solution. IEEE Trans Autom Control 47(12):1974–1985
10. Mayne DQ, Rawlings JB, Rao CV, Scokaert PO (2000) Constrained model predictive control: stability and optimality. Automatica 36(6):789–814
11. Kogiso K, Hirata K (2009) Reference governor for constrained systems with time-varying references. Robot Auton Syst 57(3):289–295
12. Ngo KB, Mahony R, Jiang ZP (2005) Integrator backstepping using barrier functions for systems with multiple state constraints. In: 2005 44th IEEE Conference on Decision and Control and 2005 European Control Conference, CDC-ECC'05, pp 8306–8312. IEEE
13. Tee KP, Ge SS, Tay EH (2009) Barrier lyapunov functions for the control of output-constrained nonlinear systems. Automatica 45(4):918–927
14. Do K (2010) Control of nonlinear systems with output tracking error constraints and its application to magnetic bearings. Int J Control 83(6):1199–1216
15. Xu X, Antsaklis PJ (2005) On time optimal control of integrator switched systems with state constraints. Nonlinear Anal: Theory Methods Appl 62(8):1453–1465
16. Mhaskar P, El-Farra NH, Christofides PD (2005) Predictive control of switched nonlinear systems with scheduled mode transitions. IEEE Trans Autom Control 50(11):1670–1680
17. Fu Y, Tian Z, Shi S (2003) State feedback stabilization for a class of stochastic time-delay nonlinear systems. IEEE Trans Autom Control 48(2):282–286
18. Qian C, Lin W (2001) Non-lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization. Syst Control Lett 42(3):185–200
19. Tee KP, Ren B, Ge SS (2011) Control of nonlinear systems with time-varying output constraints. Automatica 47(11):2511–2516
20. Long L, Zhao J (2012) Control of switched nonlinear systems in-normal form using multiple lyapunov functions. IEEE Trans Autom Control 57(5):1285–1291
21. Long L, Zhao J (2013) Output-feedback stabilisation for a class of switched nonlinear systems with unknown control coefficients. Int J Control 86(3):386–395
22. Ma R, Zhao J (2010) Backstepping design for global stabilization of switched nonlinear systems in lower triangular form under arbitrary switchings. Automatica 46(11):1819–1823
23. Ma R, Zhao J, Dimirovski GM (2013) Backstepping design for global robust stabilisation of switched nonlinear systems in lower triangular form. Int J Syst Sci 44(4):615–624
24. Slotine JJE, Li W (1998) Applied nonlinear control. 1991. NJ: Prantice-Hall, Englewood Cliffs