Chapter 5
Adaptive Control of Switched Stochastic Nonlinear Systems

5.1 Background and Motivation

The last chapter discussed adaptive control design methods for switched nonlinear systems with uncertainties. However, the system structures considered in the last chapter are somewhat simple, which greatly limits the applications of the results in practice.

It is well known that stochastic disturbance is inevitably encountered in practical systems. Therefore, control of stochastic systems with or without switching has become an active research field and received much attention recently, see, e.g., [1–5] and the references therein. The authors in [6] considered global stabilization for high-order stochastic nonlinear systems with stochastic integral input-to-state stability inverse dynamics. The moment stability and sample path stability of switched stochastic linear systems were investigated in [7]. In [8] dissipativity-based sliding mode control for switched stochastic linear systems was adopted. Stabilization problems for stochastic nonlinear systems with Markovian switching were studied in [9]. The p^{th} moment exponential stability and global asymptotic stability in probability for a class of switched stochastic nonlinear retarded systems with asynchronous switching were solved in [10].

Moreover, dead-zone characteristics are encountered in many physical components of control systems. They are particularly common in actuators, such as hydraulic servovalves and electric servomotors. They also appear in biomedical systems. The system model is more realistic and reliable when the dead-zone nonlinearities are taken into consideration.

On the other hand, since the input-to-state stability (ISS) property was proposed in [11], it has rapidly become an important tool to investigate the stability problem of nonlinear systems. In view of the crucial importance of ISS, it is natural to introduce this concept to switched nonlinear systems. In this chapter, we consider some control problems of switched high-order nonlinear systems. Some complex dynamics such as stochastic disturbances, uncertainties, dead-zone nonlinearities and input-to-state stability inverse dynamics are considered in the systems under investigations. The

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considered mathematical models can provide a good description of a large number of practical switched nonlinear systems.

Notation $\mathbb{R}$ denotes the n-dimensional space, $\mathbb{R}^n$ is the set of all nonnegative real numbers. $\mathscr{C}^i$ stands for a set of functions with continuous i^{th} partial derivatives. For a given matrix A (or vector v), A^T (or v^T) denotes its transpose, and $Tr\{A\}$ denotes its trace when A is a square. $\mathscr{K}$ represents the set of functions: $\mathbb{R}^+ \to \mathbb{R}^+$, which are continuous, strictly increasing and vanishing at zero; $\mathscr{K}_\infty$ denotes a set of functions that is of class $\mathscr{K}$ and unbounded. In addition, $\|\cdot\|$ refers to the Euclidean vector norm. $\mathbb{R}$ denotes the n-dimensional space, $\mathbb{R}^+$ denotes the set of all nonnegative real numbers, and $\mathbb{R}^* = \{q \in \mathbb{R}^+ : q \geq 1 \text{ is an odd integer}\}$. $\mathscr{C}^i$ denotes a set of all functions with continuous i^{th} partial derivatives. For a given matrix A (or vector v), A^T (or v^T) denotes its transpose, and $Tr\{A\}$ denotes its trace when A is a square. $\mathscr{K}$ denotes the set of all functions: $\mathbb{R}^+ \to \mathbb{R}^+$, which are continuous, strictly increasing and vanishing at zero; $\mathscr{K}_\infty$ denotes a set of functions that are of class $\mathscr{K}$ and unbounded. In addition, $\|\cdot\|$ refers to the Euclidean vector norm.

5.2 Adaptive Tracking Control for Switched Stochastic Nonlinear Systems with Unknown Actuator Dead-Zone

5.2.1 Problem Formulation and Preliminaries

Consider the following switched stochastic nonlinear system in nonstrict-feedback form.

$$\begin{aligned}
dx_i &= (g_{i,\sigma(t)}x_{i+1} + f_{i,\sigma(t)}(x))dt + \psi_{i,\sigma(t)}^T(x)dw, \\
&\quad 1 \leq i \leq n-1, \\
dx_n &= (g_{n,\sigma(t)}v_{\sigma(t)} + f_{n,\sigma(t)}(x))dt + \psi_{n,\sigma(t)}^T(x)dw, \\
v_{\sigma(t)} &= D_{\sigma(t)}(u_{\sigma(t)}), \\
y &= x_1,
\end{aligned} \tag{5.1}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is the system state, w is an r-dimensional independent standard Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ with Ω being a sample space, $\mathscr{F}$ being a σ-field, $\{\mathscr{F}_t\}_{t\geq 0}$ being a filtration, and P being a probability measure, and y is the system output; $\sigma(t) : [0, \infty) \to M = \{1, 2, \ldots, m\}$ represents the switching signal; $v_{\sigma(t)}, u_{\sigma(t)} \in \mathbb{R}$ are the actuator output and input. For any $i = 1, 2, \ldots, n$ and $k \in M$, $f_{i,k}(x) : \mathbb{R}^n \to \mathbb{R}$, $\psi_{i,k} : \mathbb{R}^n \to \mathbb{R}^r$ are locally Lipschitz unknown nonlinear functions and $g_{i,k}$ are positive known constants.

The nonsymmetric dead-zone nonlinearity is considered in the chapter, which is defined as the form in [12]:

$$v_k = D_k(u_k) = \begin{cases} m_{r_k}(u_k - b_{r_k}), & u_k \geq b_{r_k} \\ 0, & -b_{l_k} < u_k < b_{r_k} \\ m_{l_k}(u_k + b_{l_k}), & u_k \leq -b_{l_k} \end{cases} \tag{5.2}$$

Here, $m_{r_k} > 0$ and $m_{l_k} > 0$ represent the right and the left slopes of the dead-zone characteristic. $b_{r_k} > 0$ and $b_{l_k} > 0$ stand for the breakpoints of the input nonlinearity.

It is assumed that the nonsymmetric dead-zone nonlinearity can be reformulated as:

$$v_k = D_k^{'}(u_k) + \iota_k, \tag{5.3}$$

where $D_k^{'}(u_k)$ is a smooth function, ι_k is the error between $D_k(u_k)$ and $D_k^{'}(u_k)$ with $|\iota_k| \leq \bar{\iota}_k$.

Moreover, we have

$$\begin{aligned} v_k &= u_k + (D_k^{'}(u_k) - u_k + \iota_k) \\ &= u_k + \eta_k^{'}(u_k) + \iota_k, \end{aligned} \tag{5.4}$$

where $\eta_k^{'}(u_k) = D_k^{'}(u_k) - u_k$ is an unknown function.

The controller can be designed as

$$u_k = u_{c_k} - u_{\phi_k}. \tag{5.5}$$

Then (5.4) can be rewritten as

$$v_k = u_{c_k} + \eta_k^{'}(u_k) - u_{\phi_k} + \iota_k. \tag{5.6}$$

where u_{ϕ_k} is the compensator of dead-zone nonlinearity and u_{c_k} is a main controller of system (5.1).

Our control objective is to design a state-feedback controller such that the output of system (5.1) can track a given time-varying signal $y_d(t)$, and the problem of the actuator dead-zone can be solved. The following assumptions are supposed to be true.

Assumption 5.1 The tracking target $y_d(t)$ and its time derivatives up to n^{th} order $y_d^{(n)}(t)$ are continuous and bounded; it is further assumed that $|y_d(t)| \leq d$.

Assumption 5.2 There exist strictly increasing smooth functions $\phi_{i,k}(\cdot)$, $\rho_{i,k}(\cdot)$: $R^+ \to R^+$ with $\phi_{i,k}(0) = \rho_{i,k}(0) = 0$ such that for $i = 1, 2, \ldots, n$ and $k \in M$,

$$|f_{i,k}(x)| \leq \phi_{i,k}(\|x\|). \tag{5.7}$$

$$\left|\psi_{i,k}(x)\right| \leq \rho_{i,k}(\|x\|). \tag{5.8}$$

Remark 5.1 The increasing properties of $\phi_{i,k}(\cdot)$, $\rho_{i,k}(\cdot)$ imply that if $a_i, b_i \geq 0$, for $i = 1, 2, \ldots, n$, then $\phi_{i,k}(\sum_{i=1}^n a_i) \leq \sum_{i=1}^n \phi_{i,k}(na_i)$, $\rho_{i,k}(\sum_{i=1}^n b_i) \leq \sum_{i=1}^n \rho_{i,k}(nb_i)$. Notice that $\phi_{i,k}(s)$, $\rho_{i,k}(s)$ are smooth functions, and $\phi_{i,k}(0) =$

$\rho_{i,k}(0) = 0$. Therefore, there exist smooth functions $h_{i,k}(s)$, $\eta_{i,k}(s)$ such that $\phi_{i,k}(s) = sh_{i,k}(s)$, $\rho_{i,k}(s) = s\eta_{i,k}(s)$ which results in

$$\phi_{i,k}\left(\sum_{j=1}^{n} a_j\right) \le \sum_{j=1}^{n} na_j h_{i,k}(na_j). \tag{5.9}$$

$$\rho_{i,k}\left(\sum_{j=1}^{n} b_j\right) \le \sum_{j=1}^{n} nb_j \eta_{i,k}(nb_j). \tag{5.10}$$

We use the radial basis function (RBF) neural networks to approximate any a real function $f(Z)$ over a compact set $\Omega_Z \subset \mathbb{R}^q$. For arbitrary $\bar{\varepsilon} > 0$, there exists a neural network $W^T S(Z)$ such that

$$f(Z) = W^T S(Z) + \varepsilon(Z),\ \varepsilon(Z) \le \bar{\varepsilon}, \tag{5.11}$$

where $Z \in \Omega_Z \subset R^q$, $W = [w_1, w_2, \ldots, w_l]^T$ is the ideal constant weight vector, and $S(Z) = [s_1(Z), s_2(Z), \ldots, s_l(Z)]^T$ is the basis function vector, with $l > 1$ being the number of the neural network nodes and $s_i(Z)$ being chosen as Gaussian functions; i.e., for $i = 1, 2, \ldots, l$,

$$s_i(Z) = \exp\left[\frac{-(Z - \mu_i)^T (Z - \mu_i)}{\zeta_i^2}\right], \tag{5.12}$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^T$ is the center vector, and ζ_i is the width of the Gaussian function.

Definition 5.1 For any given $V(x_i, t) \in \mathscr{C}^{2,1}$ associated with system (5.1), define the differential operator $\mathscr{L}$ as follows;

$$\mathscr{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i} F_{i,k} + \frac{1}{2} Tr\left\{\psi_{i,k}^T \frac{\partial^2 V}{\partial x_i^2} \psi_{i,k}\right\}, \tag{5.13}$$

where $F_{i,k} = g_{i,k} x_{i+1} + f_{i,k}(x)$.

Definition 5.2 The trajectory $\{x(t), t \ge 0\}$ of switched stochastic system (5.1) is said to be semi-globally uniformly ultimately bounded (SGUUB) in the p^{th} moment, if for some compact set $\Omega \in \mathbb{R}^n$ and any initial state $x_0 = x(t_0)$, there exist a constant $\varepsilon > 0$, and a time constant $T = T(\varepsilon, x_0)$ such that $E(|x(t)|^p) < \varepsilon$, for all $t > t_0 + T$. Especially, when $p = 2$, it is usually called SGUUB in mean square.

Lemma 5.1 ([13]) *Suppose that there exist a $C^{2,1}$ function $V(x,t):\mathbb{R}^n\times\mathbb{R}^+\to\mathbb{R}^+$, two constants $c_1>0$ and $c_2>0$, class $\mathscr{K}_\infty$ functions $\bar{\alpha}_1$ and $\bar{\alpha}_2$ such that*

$$\begin{cases}\bar{\alpha}_1(|x|)\le V(x,t)\le\bar{\alpha}_2(|x|)\\ \mathscr{L}V\le -c_1V(x,t)+c_2\end{cases}$$

for all $x\in\mathbb{R}^n$ and $t>t_0$. Then, there is an unique strong solution of system (5.1) for each $x_0\in\mathbb{R}^n$, that satisfies

$$E[V(x,t)]\le V(x_0)e^{-c_1t}+\frac{c_2}{c_1},\ \forall t>t_0$$

Lemma 5.2 ([14]) *For any $\xi\in\mathbb{R}$ and $\varpi>0$, the following inequality holds:*

$$0\le|\xi|-\xi\tanh\left(\frac{\xi}{\varpi}\right)\le\delta\varpi, \tag{5.14}$$

with $\delta=0.2785$.

5.2.2 Main Results

Based on the backstepping technique, a control design and stability analysis procedure is presented in this section. For $i=1,2,\ldots,n-1$, define a common virtual control function α_i as

$$\alpha_i=\frac{1}{g_{i,\min}}\left[-\left(\lambda_i+\frac{3}{4}\right)z_i-\frac{1}{2a_i^2}z_i^3\hat{\theta}S_i^TS_i\right], \tag{5.15}$$

where $\lambda_i,\ a_i>0$ are design parameters, $g_{i,\min}=\min\{g_{i,k}:k\in M\}$, z_i is the new state variable after the coordinate transformation: $z_i=x_i-\alpha_{i-1}$, $\alpha_0=y_d$. $\hat{\theta}$ is an unknown constant that is specified later. $S_i=S_i(X_i)$ is the basis function vector. $X_i=[\bar{x}_i^T,\bar{\hat{\theta}}_i,\bar{y}_d^{(i)}]^T$ with $\bar{x}_i=[x_1,x_2,\ldots,x_i]^T$, $\bar{\hat{\theta}}_i=[\hat{\theta}_1,\hat{\theta}_2,\ldots,\hat{\theta}_i]^T$, $\bar{y}_d^{(i)}=[y_d,\dot{y}_d,\ldots,y_d^{(i)}]^T$. The z-system is obtained as

$$\begin{aligned}dz_i&=(g_{i,k}x_{i+1}+f_{i,k}-\mathscr{L}\alpha_{i-1})dt+\left(\psi_{i,k}-\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_{j,k}\right)^Tdw,\ 1\le i\le n-1\\ dz_n&=(g_{n,k}v_k+f_{n,k}-\mathscr{L}\alpha_{n-1})dt+\left(\psi_{n,k}-\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\psi_{j,k}\right)^Tdw,\end{aligned} \tag{5.16}$$

where the differential operator $\mathscr{L}$ is defined in Definition 5.1; $\mathscr{L}\alpha_{i-1}$ is given by:

$$\mathscr{L}\alpha_{i-1} = \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s} (f_{s,k} + g_{s,k} x_{s+1}) + \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} + \frac{1}{2} \sum_{p,q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q} \psi_{p,k}^T \psi_{q,k}. \tag{5.17}$$

Consider the following common stochastic Lyapunov function candidate

$$V = \sum_{i=1}^{n} \frac{1}{4} z_i^4 + \frac{1}{2r_1} \tilde{\theta}^2 + \frac{1}{2r_2} \tilde{\vartheta}^2, \tag{5.18}$$

where $r_1, \ r_2 > 0$ are design parameters; θ and ϑ are specified later. $\hat{\theta}$ and $\hat{\vartheta}$ stand for the estimations of θ and ϑ, respectively; $\tilde{\theta} = \theta - \hat{\theta}$, $\tilde{\vartheta} = \vartheta - \hat{\vartheta}$.

Lemma 5.3 *From the coordinate transformations* $z_i = x_i - \alpha_{i-1}$, $i = 1, 2, \ldots, n$, $\alpha_0 = y_d$, *the following results hold,*

$$\|x\| \le \sum_{i=1}^{n} |z_i| \varphi_i(z_i, \hat{\theta}) + d, \tag{5.19}$$

where $\varphi_i(z_i, \hat{\theta}) = \frac{1}{g_{i,\min}}[(\lambda_i + \frac{3}{4}) + \frac{1}{2a_i^2} z_i^2 \hat{\theta} S_i^T S_i] + 1$, *for* $i = 1, 2, \ldots, n-1$, *and* $\varphi_n = 1$.

Proof From Assumption 5.1 and (5.15), one can get that

$$\begin{aligned}
\|x\| &\le \sum_{i=1}^{n} |x_i| \\
&\le \sum_{i=1}^{n} (|z_i| + |\alpha_{i-1}|) \\
&\le \sum_{i=1}^{n} |z_i| + y_d + \sum_{i=1}^{n-1} \left(\frac{1}{g_{i,\min}} [(\lambda_i + \frac{3}{4}) + \frac{1}{2a_i^2} z_i^2 \hat{\theta} S_i^T S_i] \right) |z_i| \\
&\le \sum_{i=1}^{n} |z_i| \varphi_i(z_i, \hat{\theta}) + d.
\end{aligned}$$

The proof of Lemma 5.3 is completed here. □

The $\mathscr{L}V$ can be given by

$$\begin{aligned}\mathscr{L}V =& \sum_{i=1}^{n-1}\Bigg\{z_i^3\Bigg(f_{i,k}+g_{i,k}x_{i+1}-\sum_{s=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(s)}}y_d^{(s+1)}-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\\&-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}(f_{s,k}+g_{s,k}x_{s+1})-\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\psi_{p,k}^T\psi_{q,k}\Bigg)\\&+\frac{3}{2}z_i^2\left\|\psi_{i,k}-\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_{j,k}\right\|^2\Bigg\}+z_n^3\Bigg(f_{n,k}+g_{n,k}v_k-\sum_{s=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_d^{(s)}}y_d^{(s+1)}\\&-\sum_{s=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_s}(f_{s,k}+g_{s,k}x_{s+1})-\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}-\frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\psi_{p,k}^T\psi_{q,k}\Bigg)\\&-\frac{1}{r_1}\tilde{\theta}\dot{\hat{\theta}}+\frac{3}{2}z_n^2\left\|\psi_{n,k}-\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\psi_{n,k}\right\|^2-\frac{1}{r_2}\tilde{\vartheta}\dot{\hat{\vartheta}}\end{aligned}$$

$$\begin{aligned}=&\sum_{i=1}^{n}\Bigg\{z_i^3\Bigg(f_{i,k}-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}f_{s,k}-\sum_{s=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(s)}}y_d^{(s+1)}-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}g_{s,k}x_{s+1}\\&-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}-\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\psi_{p,k}^T\psi_{q,k}\Bigg)+\frac{3}{2}z_i^2\left\|\psi_{i,k}-\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_{j,k}\right\|^2\Bigg\}\\&-\frac{1}{r_1}\tilde{\theta}\dot{\hat{\theta}}-\frac{1}{r_2}\tilde{\vartheta}\dot{\hat{\vartheta}}+\sum_{i=1}^{n}z_i^3g_{i,k}x_{i+1}+z_n^3g_{n,k}v_k. \qquad (5.20)\end{aligned}$$

By resorting to Assumption 5.2 and Lemma 5.3, one has that

$$\begin{aligned}&z_i^3(f_{i,k}-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}f_{s,k}(x))\\=&-z_i^3\sum_{s=1}^{i}\frac{\partial\alpha_{i-1}}{\partial x_s}f_{s,k}(x)\\\le&\frac{3}{4}nz_i^4\sum_{s=1}^{i}\left(\frac{\partial\alpha_{i-1}}{\partial x_s}\right)^{\frac{4}{3}}+\sum_{s=1}^{i}\sum_{l=1}^{n}z_l^4\bar{\phi}_{s,k}^4(z_l,\hat{\theta})+\left|z_i^3\right|\sum_{s=1}^{i}\left|\frac{\partial\alpha_{i-1}}{\partial x_s}\right|\phi_{s,k}((n+1)d),\end{aligned} \qquad (5.21)$$

where $\bar{\phi}_{s,k}^4(z_l,\hat{\theta})=\frac{1}{4}(n+1)^4\varphi_l^4(z_l,\hat{\theta})h_{s,k}^4((n+1)|z_l|\varphi_l(z_l,\hat{\theta}))$, $\frac{\partial\alpha_0}{\partial x_i}=0$ and $\frac{\partial\alpha_{i-1}}{\partial x_i}=-1$.

Then, the following inequality can be obtained,

$$\begin{aligned}
&\frac{3}{2}z_i^2\left\|\psi_{i,k}-\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_{j,k}\right\|^2\\
&\quad\le\frac{9}{8}i^2(n+1)^2nz_i^4+\sum_{l=1}^{n}z_l^4\bar{\rho}_{i,k}^4(z_l,\hat{\theta})+\sum_{j=1}^{i-1}\sum_{l=1}^{n}z_l^4\bar{\rho}_{j,k}^4(z_l,\hat{\theta})\\
&\qquad+\frac{9}{8}i^2(n+1)^2nz_i^4\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j})^4+\frac{9}{8}i^2(n+1)^2z_i^4l_{ii}^{-2}\rho_{i,k}^4((n+1)d)\\
&\qquad+\sum_{j=1}^{i}l_{ij}^2+\frac{9}{8}i^2(n+1)^2z_i^4\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j})^4l_{ij}^{-2}\rho_{j,k}^4((n+1)d),
\end{aligned}\tag{5.22}$$

where l_{ij} is a positive constant, and $\frac{\partial\alpha_0}{\partial x_j}=0$ because $\alpha_0=y_d$, and

$$\begin{aligned}
&-\frac{1}{2}z_i^3\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\psi_{p,k}^T\psi_{q,k}\\
&\le(i-1)\sum_{s=1}^{i-1}\sum_{l=1}^{n}z_l^4\bar{\rho}_{s,k}^4(z_l,\hat{\theta})\\
&\quad+\frac{1}{8}(n+1)^2nz_i^6\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\left(\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right)^2\\
&\quad+\frac{1}{2}(n+1)\left|z_i^3\right|\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\left|\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right|\rho_{s,k}^2((n+1)d),
\end{aligned}\tag{5.23}$$

where $\bar{\rho}_{s,k}^4(z_l,\hat{\theta})=\frac{1}{2}(n+1)^4\varphi_l^4(z_l,\hat{\theta})\eta_{s,k}^4((n+1)|z_l|\varphi_l(z_l,\hat{\theta}))$, $s=1,2,\ldots,i-1$.

Substituting (5.21), (5.22) and (5.23) into (5.20) gives that

$$\begin{aligned}
\mathscr{L}V&\le\sum_{i=1}^{n}\frac{3}{4}nz_i^4\sum_{s=1}^{i}\left(\frac{\partial\alpha_{i-1}}{\partial x_s}\right)^{\frac{4}{3}}+\sum_{i=1}^{n}\sum_{s=1}^{i}\sum_{l=1}^{n}z_l^4\bar{\phi}_{s,k}^4(z_l,\hat{\theta})\\
&+\sum_{i=1}^{n}\left|z_i^3\right|\sum_{s=1}^{i}\left|\frac{\partial\alpha_{i-1}}{\partial x_s}\right|\phi_{s,k}((n+1)d)+\sum_{i=1}^{n}\sum_{s=1}^{i-1}\sum_{l=1}^{n}(i-1)z_l^4\bar{\rho}_{s,k}^4(z_l,\hat{\theta})\\
&+\sum_{i=1}^{n}\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\frac{1}{8}(n+1)^2nz_i^6\left(\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right)^2\\
&+\sum_{i=1}^{n}\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\frac{1}{2}(n+1)\left|z_i^3\right|\rho_{s,k}^2((n+1)d)\left|\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right|
\end{aligned}$$

$$+\sum_{i=1}^{n}\Bigg\{\frac{9}{8}i^2(n+1)^2nz_i^4+\sum_{l=1}^{n}z_l^4\bar{\rho}_{i,k}^4(z_l,\hat{\theta})+\sum_{j=1}^{i-1}\sum_{l=1}^{n}z_l^4\bar{\rho}_{j,k}^4(z_l,\hat{\theta})$$
$$+\frac{9}{8}i^2(n+1)^2nz_i^4\sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\right)^4+\frac{9}{8}i^2(n+1)^2z_i^4l_{ii}^{-2}\rho_{i,k}^4((n+1)d)+\sum_{j=1}^{i}l_{ij}^2$$
$$+\frac{9}{8}i^2(n+1)^2z_i^4\sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\right)^4l_{ij}^{-2}\rho_{j,k}^4((n+1)d)\Bigg\}$$
$$+\sum_{i=1}^{n}z_i^3\left(-\sum_{s=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(s)}}y_d^{(s+1)}-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}g_{s,k}x_{s+1}\right)$$
$$+\sum_{i=1}^{n-1}z_i^3g_{i,k}x_{i+1}+z_n^3g_{n,k}v_k-\frac{1}{r_1}\tilde{\theta}\dot{\hat{\theta}}-\frac{1}{r_2}\tilde{\vartheta}\dot{\hat{\vartheta}}. \tag{5.24}$$

Define $U_{i,k}$ as

$$U_{i,k}=\sum_{s=1}^{i}|\frac{\partial\alpha_{i-1}}{\partial x_s}|\phi_{s,k}((n+1)d)$$
$$+\frac{1}{2}(n+1)\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\left|\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right|\rho_{s,k}^2((n+1)d). \tag{5.25}$$

By using Lemma 5.2 one has

$$\left|z_i^3\right|U_{i,k}\le z_i^3U_{i,k}\tanh(\frac{z_i^3U_{i,k}}{\varpi_{i,k}})+\delta\varpi_{i,k}. \tag{5.26}$$

Note that

$$\sum_{i=1}^{n-1}z_i^3g_{i,k}x_{i+1}=\sum_{i=1}^{n-1}z_i^3g_{i,k}z_{i+1}+\sum_{i=1}^{n-1}g_{i,k}z_i^3\alpha_i, \tag{5.27}$$

Therefore, one has

$$\sum_{i=1}^{n}\sum_{s=1}^{i}\sum_{l=1}^{n}z_l^4\bar{\phi}_{s,k}^4(z_l,\hat{\theta})=\sum_{i=1}^{n}z_i^4\sum_{s=1}^{n}(n-s+1)\bar{\phi}_{s,k}^4(z_i,\hat{\theta}),$$
$$\sum_{i=1}^{n}(i-1)\sum_{s=1}^{i-1}\sum_{l=1}^{n}z_l^4\bar{\rho}_{s,k}^4(z_l,\hat{\theta})=\sum_{i=1}^{n}z_i^4\sum_{s=1}^{n-1}(n-s)(i-1)\bar{\rho}_{s,k}^4(z_i,\hat{\theta}),$$
$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{l=1}^{n}z_l^4\bar{\rho}_{j,k}^4(z_l,\hat{\theta})=\sum_{i=1}^{n}z_i^4\sum_{j=1}^{n}(n-j+1)\bar{\rho}_{j,k}^4(z_i,\hat{\theta}).$$

For any $i = 1, 2, \ldots, n$ and $k \in M$, define $\bar{f}_{i,k}$ as

$$
\begin{aligned}
\bar{f}_{i,k} &= \frac{3}{4} n z_i \sum_{s=1}^{i} \left(\frac{\partial \alpha_{i-1}}{\partial x_s} \right)^{\frac{4}{3}} + z_i \sum_{s=1}^{n} (n - s + 1) \bar{\phi}_{s,k}^4 (z_i, \hat{\theta}) \\
&\quad + z_i \sum_{s=1}^{n-1} (n - s)(i - 1) \bar{\rho}_{s,k}^4 (z_i, \hat{\theta}) + \sum_{s=1}^{i-1} \sum_{j=1}^{i-1} \frac{1}{8} (n + 1)^2 n z_i^3 \left(\frac{\partial^2 \alpha_{i-1}}{\partial x_s \partial x_j} \right)^2 \\
&\quad + \frac{9}{8} i^2 (n + 1)^2 z_i \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_j} \right)^4 l_{ij}^{-2} \rho_{j,k}^4 ((n + 1)d) \\
&\quad + z_i \sum_{j=1}^{n} (n - j + 1) \bar{\rho}_{j,k}^4 (z_i, \hat{\theta}) + \frac{9}{8} i^2 (n + 1)^2 n z_i \\
&\quad + \frac{9}{8} i^2 (n + 1)^2 z_i l_{ii}^{-2} \rho_{i,k}^4 ((n + 1)d) + \frac{9}{8} i^2 (n + 1)^2 n z_i \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_j} \right)^4 \\
&\quad - \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s} g_{s,k} x_{s+1} \\
&\quad + U_{i,k} \tanh\left(\frac{z_i^3 U_{i,k}}{\varpi_{i,k}}\right) + g_{i,k} z_{i+1},
\end{aligned}
\tag{5.28}
$$

with $z_{n+1} = 0$.
Substituting (5.6) and (5.26)–(5.28) into (5.24) yields that

$$
\begin{aligned}
\mathscr{L}V \leq & \sum_{i=1}^{n-1} z_i^3 (\bar{f}_{i,k} + g_{i,k} \alpha_i) + z_n^3 \bar{f}_{n,k} + z_n^3 g_{n,k} (u_{c_k} + \eta_k^{'} - u_{\phi_k} + \iota_k) \\
& - \frac{1}{r_1} \tilde{\theta} \dot{\hat{\theta}} - \frac{1}{r_2} \tilde{\vartheta} \dot{\hat{\vartheta}} + \sum_{i=1}^{n} \left(\delta \varpi_{i,k} + \sum_{j=1}^{i} l_{ij}^2 \right).
\end{aligned}
\tag{5.29}
$$

By exploring the neural networks' approximation capability and Young's inequality, one can get the following inequalities.

$$
\begin{aligned}
z_i^3 \bar{f}_{i,k} &= z_i^3 W_{i,k}^T S_{i,k} + z_i^3 \varepsilon_{i,k} \\
&\le \frac{1}{2a_i^2} z_i^6 \left\| W_{i,k} \right\|^2 S_{i,k}^T S_{i,k} + \frac{a_i^2}{2} + \frac{3}{4} z_n^4 + \frac{\bar{\varepsilon}_{i,k}^4}{4}, \\
&\le \frac{1}{2a_i^2} z_i^6 \theta_i S_i^T S_i + \frac{a_i^2}{2} + \frac{3}{4} z_i^4 + \frac{\bar{\varepsilon}_i^4}{4},
\end{aligned} \tag{5.30}
$$

$$
\begin{aligned}
z_n^3(\eta_k^{'} + \iota_k) &= z_n^3 W_{\eta,k}^T S_{\eta,k} + z_n^3(\varepsilon_{\eta,k} + \iota_k) \\
&\le \frac{1}{2a_\eta^2} z_n^6 \vartheta_\eta S_\eta^T S_\eta + \frac{a_\eta^2}{2} + \frac{3z_n^4 + \bar{\varepsilon}_\eta^4}{4},
\end{aligned} \tag{5.31}
$$

where $\theta_{i,k} = \left\| W_{i,k} \right\|^2$, $\vartheta_{\eta,k} = \left\| W_{\eta,k} \right\|^2$, $\theta_i = \max\{\theta_{i,k} : k \in M\}$, $\vartheta_\eta = \max\{\vartheta_{\eta,k} : k \in M\}$, $\left|\varepsilon_{i,k}\right| \le \bar{\varepsilon}_i$, $\left|\varepsilon_{\eta,k} + \iota_k\right| \le \bar{\varepsilon}_\eta$.

Substituting (5.30) and (5.31) into (5.29) gives

$$
\begin{aligned}
\mathscr{L}V \le& \sum_{i=1}^{n-1} z_i^3 \left(\frac{z_i^3 \theta_i}{2a_i^2} S_i^T S_i + g_{i,k} \alpha_i \right) + z_n^3 \left(\frac{z_n^3 \theta_n}{2a_n^2} S_n^T S_n + g_{n,k} u_{c_k} \right) \\
&+ z_n^3 g_{n,k} \left(\frac{1}{2a_\eta^2} z_n^3 \theta_\eta S_\eta^T S_\eta - u_{\phi_k} \right) + g_{n,k} \left(\frac{a_\eta^2}{2} + \frac{3}{4} z_n^4 + \frac{\bar{\varepsilon}_\eta^4}{4} \right) \\
&+ \sum_{i=1}^{n} \left(\frac{2a_i^2 + 3z_i^4 + \bar{\varepsilon}_i^4}{4} \right) - \frac{1}{r_1} \tilde{\theta} \dot{\hat{\theta}} - \frac{1}{r_2} \tilde{\vartheta} \dot{\hat{\vartheta}} + \sum_{i=1}^{n} \left(\delta \varpi_i + \sum_{j=1}^{i} l_{ij}^2 \right),
\end{aligned} \tag{5.32}
$$

where $\varpi_i := \max\{\varpi_{i,k},\ k \in M\}$.

Design the virtual control function as

$$
\alpha_i = \frac{1}{g_{i,\min}} \left[-\left(\lambda_i + \frac{3}{4} \right) z_i - \frac{1}{2a_i^2} z_i^3 \hat{\theta} S_i^T S_i \right], \tag{5.33}
$$

where $\hat{\theta} = \sum_{i=1}^{n} \hat{\theta}_i$ is the estimation of θ; $\lambda_i > 0$ is a design parameter.

The actual actuator input is given as

$$
u_k = u_{c_k} - u_{\phi_k}, \tag{5.34}
$$

where

$$
u_{c_k} = \frac{1}{g_{n,k}} \left[-\left(\lambda_n + \frac{3}{4} \right) z_n - \frac{1}{2a_n^2} z_n^3 \hat{\theta} S_n^T S_n \right], \tag{5.35}
$$

$$
u_{\phi_k} = \left(\lambda_\eta + \frac{3}{4} \right) z_n + \frac{g_{n,\max}}{2a_\eta^2 g_{n,k}} z_n^3 \hat{\vartheta} S_\eta^T S_\eta, \tag{5.36}
$$

λ_n, λ_η, a_n, $a_\eta > 0$ are design parameters, $g_{n,\max} = \max\{g_{n,k}, k \in M\}$, $g_{n,\min} = \min\{g_{n,k}, k \in M\}$, $\hat{\vartheta}$ is the estimation of ϑ.

The adaptive laws can be designed as

$$\dot{\hat{\theta}} = \sum_{i=1}^{n} \frac{r_1}{2a_{i,\min}^2} z_i^6 S_i^T S_i - \beta_1 \hat{\theta}, \tag{5.37}$$

$$\dot{\hat{\vartheta}} = \frac{g_{n,\max} r_2}{2a_{\eta,\min}^2} z_n^6 S_\eta^T S_\eta - \beta_2 \hat{\vartheta}. \tag{5.38}$$

Then, one can get from (5.32)–(5.38) that

$$\begin{aligned}\mathscr{L}V \leq & -\sum_{i=1}^{n} \lambda_i z_i^4 - \lambda_\eta z_n^4 + g_{n,k}\left(\frac{a_\eta^2}{2} + \frac{\bar{\varepsilon}_\eta^4}{4}\right) + \sum_{i=1}^{n}\left(\frac{a_i^2}{2} + \frac{\bar{\varepsilon}_i^4}{4}\right) + \frac{\beta_1}{r_1}\tilde{\theta}\hat{\theta} \\ & + \frac{\beta_2}{r_2}\tilde{\vartheta}\hat{\vartheta} + \sum_{i=1}^{n}\left(\delta\varpi_i + \sum_{j=1}^{i} l_{ij}^2\right).\end{aligned} \tag{5.39}$$

It is clear that

$$\tilde{\theta}\hat{\theta} = \tilde{\theta}(\theta - \tilde{\theta}) \leq -\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\theta^2, \tag{5.40}$$

$$\tilde{\vartheta}\hat{\vartheta} = \tilde{\vartheta}(\vartheta - \tilde{\vartheta}) \leq -\frac{1}{2}\tilde{\vartheta}^2 + \frac{1}{2}\vartheta^2. \tag{5.41}$$

Combining (5.39) with (5.40) and (5.41) yields that

$$\begin{aligned}\mathscr{L}V \leq & -\sum_{i=1}^{n} \lambda_i z_i^4 - \frac{\beta_1}{2r_1}\tilde{\theta}^2 - \frac{\beta_2}{2r_2}\tilde{\vartheta}^2 + g_{n,k}\left(\frac{a_\eta^2}{2} + \frac{\bar{\varepsilon}_\eta^4}{4}\right) + \sum_{i=1}^{n}\left(\frac{a_i^2}{2} + \frac{\bar{\varepsilon}_i^4}{4}\right) \\ & + \sum_{i=1}^{n}\left(\delta\varpi_i + \sum_{j=1}^{i} l_{ij}^2\right) + \frac{\beta_1\theta^2}{2r_1} + \frac{\beta_2\vartheta^2}{2r_2} \\ \leq & - p_0 V + q_0,\end{aligned} \tag{5.42}$$

where $\lambda_n := \lambda_n + \lambda_\eta,\ p_0 = \min\{4\lambda_i, \beta_1, \beta_2 : 1 \leq i \leq n\},\ q_0 = \sum_{i=1}^{n}(\frac{a_i^2}{2} + \frac{\bar{\varepsilon}_i^4}{4}) + \sum_{i=1}^{n}\left(\delta\varpi_i + \sum_{j=1}^{i} l_{ij}^2\right) + \frac{\beta_1\theta^2}{2r_1} + \frac{\beta_2\vartheta^2}{2r_2} + g_{n,k}(\frac{a_\eta^2}{2} + \frac{\bar{\varepsilon}_\eta^4}{4})$.

By using Lemma 5.1, we have

$$\frac{dE[V(t)]}{dt} \leq -p_0 E[V(t)] + q_0; \tag{5.43}$$

Then, the following inequality holds

$$0 \leq E[V(t)] \leq V(0)e^{-p_0 t} + \frac{q_0}{p_0}, \tag{5.44}$$

where $V(0) = \sum_{j=1}^{n} \frac{z_j^2(0)}{4} + \frac{1}{2r_1}\tilde{\theta}(0)^2 + \frac{1}{2r_2}\tilde{\vartheta}(0)^2$. Equation (5.44) implies that all the signals in the closed-loop system are bounded in probability. In particular, we have

$$E[|z_i|^4] \leq \frac{4q_0}{p_0}, \quad t \to \infty. \tag{5.45}$$

Now, we are ready to provide our main result in the following theorem.

Theorem 5.1 *Consider the closed-loop system (5.1) with unknown nonsymmetric actuator dead-zone (5.2). Suppose that for* $1 \leq i \leq n$, $k \in M$, *the packaged unknown functions* $\bar{f}_{i,k}$ *can be approximated by neural networks in the sense that the approximation error* $\varepsilon_{i,k}$ *are bounded. Under the state feedback controller (5.34) and the adaptive laws (5.37), (5.38), the following statements hold,*

(i) All the signals of the closed-loop z-system (5.17) are SGUUB in the fourth moment and

$$P\left\{\lim_{t\to\infty}\sum_{i=1}^{n} E[|z_i|^4] \leq \frac{4q_0}{p_0}\right\} = 1.$$

(ii) The output y of the closed-loop system (5.1) can be almost surely regulated to a small neighborhood of the target signal.

Proof It is not difficult to complete the proof by using the above developments. □

5.2.3 Simulation Results

In this section, an example about the control of a ship manoeuvring system are used to illustrate the effectiveness of the obtained results.

The ship maneuvering system can be described by the following Norrbin nonlinear model [15].

$$T_{\sigma(v_s)}\dot{h} + h + \alpha_{\sigma(v_s)}h^3 = K_{\sigma(v_s)}\delta + \phi_{\sigma(v_s)}^T(\psi, h, \delta)w,$$

where $T_{\sigma(v_s)}$ is the time constant, $h = \dot{\psi}$ denotes the yaw rate, ψ stands for the heading angle, $\alpha_{\sigma(v_s)}$ is the Norrbin coefficient, $K_{\sigma(v_s)}$ represents the rudder gain, δ is the rudder angle and w stands for an r-dimensional independent standard Brownian motion, $\phi_{\sigma(v_s)}(\psi, h, \delta) : \mathbb{R}^3 \to \mathbb{R}^{3\times r}$ is an unknown function, and $\sigma(v_s)$ is the switching signal that satisfies:

$$\sigma(v_s) = \begin{cases} 1, \ 0 < v_s \leq v_L \\ 2, \ v_L < v_s \leq v_M \\ 3, \ v_M < v_s \leq v_T \end{cases}$$

v_L, v_M, v_T represent the value of low speed, middle speed and top speed, respectively.

A simplified mathematical model of the rudder system can be described as follows,

$$T_{E,\sigma(v_s)}\dot{\delta} + \delta = K_{E,\sigma(v_s)}\delta_{E,\sigma(v_s)},$$

where $T_{E,\sigma(v_s)}$ represents the rudder time constant, δ stands for the actual rudder angle, $K_{E,\sigma(v_s)}$ denotes the rudder control gain and $\delta_{E,\sigma(v_s)}$ is the rudder order.

Let $x_1 = \psi$, $x_2 = h$, $x_3 = \delta$, $v_{\sigma(v_s)} = \delta_{E,\sigma(v_s)}$; we can get the following switched nonlinear system model with actuator dead-zone to describe the dynamic behavior of the ship with low speed, middle speed and high speed, respectively.

$$\begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= (f_{\sigma(v_s)} + b_{\sigma(v_s)} x_3)dt + \phi^T_{\sigma(v_s)} d\omega, \\ dx_3 &= \left(-\frac{1}{T_{E,\sigma(v_s)}} x_3 + \frac{K_{E,\sigma(t)}}{T_{E,\sigma(v_s)}} v_{\sigma(v_s)}\right) dt, \\ v_{\sigma(v_s)} &= D(u_{\sigma(v_s)}) \end{aligned}$$

where $f_{\sigma(v_s)} = -\frac{1}{T_{\sigma(v_s)}} x_2 - \frac{\tau_{\sigma(v_s)}}{T_{\sigma(v_s)}} x_2^3$, $b_{\sigma(v_s)} = \frac{K_{\sigma(v_s)}}{T_{\sigma(v_s)}}$.

The vessel data comes from a ship that has a length overall of 160.9 m. $v_L = 3.7$ m/s, $K_1 = 32$ s^{-1}, $T_1 = 30$ s, $\tau_1 = 40$ s^2, $T_{E,1} = 4$ s, $K_{E,1} = 2$; $v_M = 7.5$ m/s, $K_2 = 11.4$ s^{-1}, $T_2 = 63.69$ s, $\tau_2 = 30$ s^2, $T_{E,2} = 2.5$ s, $K_{E,2} = 1$; $v_T = 15.3$ m/s, $K_3 = 5.1$ s^{-1}, $T_3 = 80.47$ s, $\tau_3 = 25$ s^2, $T_{E,3} = 1$ s, $K_{E,3} = 0.72$. The initial conditions are $x_1(0) = 2, x_2(0) = -0.05, x_3(0) = 0.03, \hat{\theta}(0) = 10, \hat{\vartheta}(0) = 1$. We construct the basis function vectors S_1, S_2, S_3 and S_η using 11, 15, 21 and 48 nodes, the centers μ_1, μ_2, μ_3, μ_η evenly spaced on $[-1.5,\ 4.5] \times [-3,\ 4] \times [-10,\ 8]$, $[-5,\ 4] \times [-30,\ 20] \times [-0.5,\ 5.5]$, $[-5.5,\ 8] \times [-12,\ 25] \times [-0.1,\ 2]$ and $[-10,\ 2] \times [-60,\ 2] \times [-0.2,\ 10.5]$, and the widths $\zeta_1 = 1.2$, $\zeta_2 = 2.2$, $\zeta_3 = 2$, $\zeta_\eta = 1.8$. The design parameters are $a_1 = a_2 = a_3 = a_\eta = 10$, $r_1 = 2$, $r_2 = 10$, $\beta_1 = 0.5$, $\beta_2 = 0.1$, $\lambda_1 = \lambda_2 = \lambda_3 = 5$, and $\lambda_\eta = 3$. The desired trajectory is $y_d = 10 \sin 0.08t$.

According to Theorem 5.1, the adaptive laws $\dot{\hat{\theta}}$, $\dot{\hat{\vartheta}}$ and the control laws u_{c_k}, u_{ϕ_k} are chosen, respectively, as

$$\begin{aligned} \dot{\hat{\theta}} &= \sum_{i=1}^{3} 0.01 z_i^6 S_i^T S_i - 0.5\hat{\theta}, \\ \dot{\hat{\vartheta}} &= 0.036 z_3^6 S_\eta^T S_\eta - 0.1\hat{\vartheta}, \\ u_{c_k} &= \frac{1}{g_{3,k}}[-5.75 z_3 - 0.005 z_3^3 \hat{\theta} S_3^T S_3], \\ u_{\phi_k} &= 3.75 z_3 + \frac{0.00057}{g_{3,k}} z_3^3 \hat{\vartheta} S_\eta^T S_\eta, \end{aligned}$$

where $u_k = u_{c_k} - u_{\phi_k}$, $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$, $z_3 = x_3 - \alpha_2$ and α_1, α_2 are given by

$$\alpha_1 = -5.75z_1 - 0.005z_1^3\hat{\theta}S_1^T S_1,$$
$$\alpha_2 = -92z_2 - 0.08z_2^3\hat{\theta}S_2^T S_2.$$

In order to give the simulation results, we assume that

$$v_k = D(u_k) = \begin{cases} 10(u_k - 50), & u_k \geq 50 \\ 0, & -60 < u_k < 50 \\ 20(u_k + 60), & u_k \leq -60 \end{cases}$$

and $\phi_1 = 0.5x_1 \sin x_2 x_3$, $\phi_2 = 0.25x_1^2 x_2 \cos x_2$, $\phi_3 = 0.1x_1x_3$. The simulation results are shown in Figs. 5.1–5.4. Figure 5.1 depicts the responses of system output ψ and target signal y_d. Figure 5.2 shows the trajectories of adaptive laws. Figure 5.3 demonstrates the responses of $D(u_{c_k})$ (without dead-zone compensation controller) and $D(u_{c_k} - u_{\phi_k})$ (with dead-zone compensation controller) and Fig. 5.4 illustrates the evolution of the switching signal. From Fig. 5.1, it can be seen that the output ψ can track the target signal y_d within a small bounded error. On the other hand, Fig. 5.3 proves that the dead-zone nonlinearity can be compensated by u_{ϕ_k}.

5.2.4 Conclusions

The tracking control problem for a class of stochastic switched nonlinear systems under arbitrary switchings has been investigated, where the unknown nonsymmetric actuator dead-zone is taken into account. A state feedback controller is designed for the systems under consideration. It is shown that the target signal can be almost

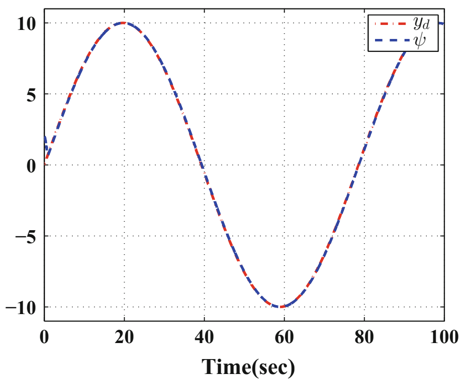


Fig. 5.1 Tracking performance

Fig. 5.2 The responses of adaptive laws

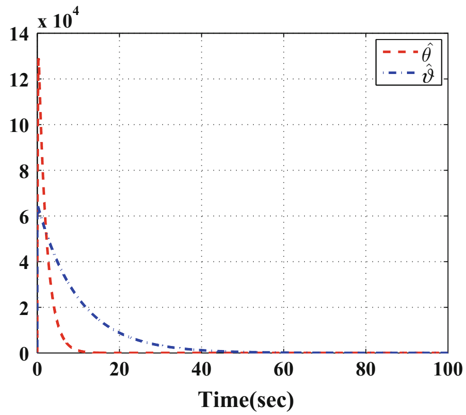


Fig. 5.3 The responses of $D(u_{c_k} - u_{\phi_k})$ and $D(u_{c_k})$

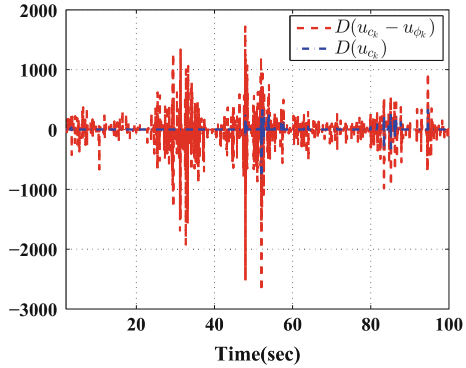
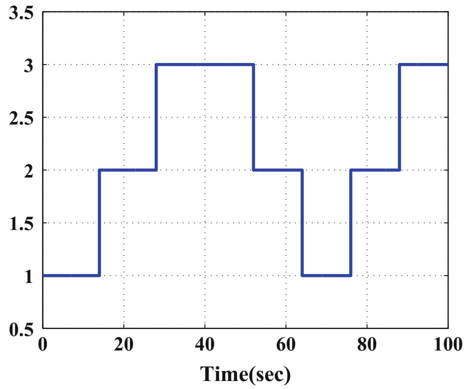


Fig. 5.4 The response of switching signal



surely tracked by the system output within a small bounded error, and the tracking error is SGUUB in 4^{th} moment.

5.3 Adaptive Neural Control for Switched Stochastic High-Order Uncertain Nonlinear Systems with SISS Inverse Dynamic

5.3.1 Problem Formulation and Preliminaries

Here, we consider the following stochastic switched high-order nonlinear systems with SISS inverse dynamic,

$$\begin{aligned}
d\zeta &= f_{0,\sigma(t)}(\zeta, x_1)\,dt + \psi^T_{0,\sigma(t)}(\zeta, x_1)\,d\omega,\\
dx_i &= \left(g_{i,\sigma(t)}(\zeta, x)x_{i+1}^{p_i} + f_{i,\sigma(t)}(\zeta, x)\right)dt + \psi^T_{i,\sigma(t)}(\zeta, x)\,d\omega,\ i = 1, 2, \ldots, n-1,\\
dx_n &= \left(g_{n,\sigma(t)}(\zeta, x)u^{p_n}_{\sigma(t)} + f_{n,\sigma(t)}(\zeta, x)\right)dt + \psi^T_{n,\sigma(t)}(\zeta, x)\,d\omega,\\
y &= x_1,
\end{aligned} \tag{5.46}$$

where $\zeta \in \mathbb{R}^r$ are immeasurable stochastic inverse dynamics; $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and $y \in R$ are the system state and output, respectively; p_i is a positive odd integer and ω is an m-dimensional standard Wiener process defined on the complete probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P\right)$ with Ω being a sample space, $\mathscr{F}$ being a σ-field, $\{\mathscr{F}_t\}_{t\geq 0}$ being a filtration, and P being a probability measure; $\sigma(t) : [0, +\infty) \to M = \{1, 2, \ldots, m\}$ is the switching signal; $u_k \in R$ is the control input of the k-th subsystem; $f_{0,k} : \mathbb{R}^r \times R \to \mathbb{R}^r$, $\psi_{0,k} : \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^{m\times r}$; For any $i = 1, 2 \ldots, n$ and $k = 1, 2, \ldots, m$, $f_{i,k} : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$, $\psi_{i,k} : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^m$ are unknown nonlinear functions assumed to be locally Lipschitz with $f_{i,k}(0) = 0$, $\psi_{i,k}(0) = 0$, and $g_{i,k} : R^r \times \mathbb{R}^n \to R$ is a strictly either positive or negative known function.

Remark 5.2 System (5.46) reduces to the well-known normal form when $p_i = 1$, $\zeta = 0$ and $m = 1$. In the case that $p_i > 1$, $\zeta = 0$ and $m = 1$, the Jacobian linearization of the system is neither controllable nor feedback linearizable. This makes the control design very challenging. To solve this problem, Lin and Qian [16] proposed a fruitful deterministic technique: adding a power integrator. Subsequently, many excellent results are proposed based on the adding a power integrator technique, see, e.g., [17–19] and the references therein.

Definition 5.3 For any given $V(x_i, t) \in \mathscr{C}^{2,1}$ associated with system (5.46), define the differential operator $\mathscr{L}$ as follows,

$$\mathscr{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i} F_{i,k} + \frac{1}{2} Tr \left\{ \psi_{i,k}^T \frac{\partial^2 V}{\partial x_i^2} \psi_{i,k} \right\}, \tag{5.47}$$

where $F_{i,k} = g_{i,\sigma(t)}(\zeta, x)x_{i+1}^{p_i} + f_{i,\sigma(t)}(\zeta, x)$.

Assumption 5.3 The sign and the upper bound of function $g_{i,k}$ for $1 \le i \le n$ and $k \in M$, are known, and without loss of generality, it is assumed that

$$0 < \underline{d}_i \le g_{i,k}(\zeta, x) \le \bar{d}_i,$$

where $\underline{d}_i$ and $\bar{d}_i$ stand for the lower and upper bound values of $g_{i,k}(\zeta, x)$, respectively.

Assumption 5.4 For $1 \le i \le n$ and $k \in M$, there exists a $\mathscr{C}^2$ function $V_0(\zeta)$, which is positive definite and proper, such that $\mathscr{L}V_0 \le -\lambda_0\zeta^4 + \bar{\lambda}_0 x_1^{p+3}$, where λ_0 and $\bar{\lambda}_0$ are positive constants.

Lemma 5.4 *Let $p \in \mathbb{R}^*$ and x, y be real-valued functions. There exists a constant $c > 0$ such that*

$$\left|x^p - y^p\right| \le c\,|x-y|\,\left|(x-y)^{p-1} + y^{p-1}\right|.$$

Lemma 5.5 *Suppose that there exists a $\mathscr{C}^{2,1}$ function $V(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$, two constants $c_1 > 0$, $c_2 > 0$, and $\mathscr{K}_\infty$ functions $\bar{c}_1$, $\bar{c}_2$ such that*

$$\begin{cases} \bar{c}_1(|x|) \le V(x,t) \le \bar{c}_2(|x|) \\ \mathscr{L}V(x,t) \le -c_1 V(x,t) + c_2 \end{cases}$$

for all $x \in R^n$ and $t > t_0$. Then, there is an unique strong solution for each $x_0 \in \mathbb{R}^n$ and it satisfies:

$$E[V(x,t)] \le V(x_0)e^{-c_1 t} + \frac{c_2}{c_1},\ \forall t > t_0.$$

In the following control design procedure, radial basis function (RBF) neural networks are used to approximate a continuous real function $f(X)$. For arbitrary $\varepsilon > 0$, there exists a neural network $W^T S(X)$ such that

$$f(X) = W^T S(X) + \delta(X),\ \delta(X) \le \varepsilon,$$

where $X \in \Omega_X \subset \mathbb{R}^q$ is the input vector with q dimension, $S(X) = [s_1(X), s_2(X), \ldots, s_l(X)]^T$ is the basis function vector, and $W = [w_1, w_2, \ldots, w_l]^T$ is the ideal constant weight vector with $l > 1$ being the number of the neural network nodes, and $s_i(X)$ are chosen as Gaussian functions; i.e., for $i = 1, 2, \ldots, l$,

$$s_i(X) = \exp\left(-\frac{(X-\mu_i)^T(X-\mu_i)}{\zeta_i^2}\right),$$

where ζ_i is the width of the Gaussian function, and $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^T$ is the center vector.

Lemma 5.6 *Consider the Gaussian RBF networks. Let* $\rho := \frac{1}{2}\min_{i\neq j}||\mu_i - \mu_j||$*; then an upper bound of* $||S(X)||$ *is taken as*

$$||S(X)|| \leq \sum_{k=0}^{\infty} 3q(k+2)^{q-1}e^{-2\rho^2 k^2/\zeta^2} := D.$$

It has been proven in [20] that the constant D in Lemma 5.6 is a limited value and is independent of the variable X.

5.3.2 Main Results

In the following, the adaptive tracking control design is carried out by using a standard backstepping procedure. Firstly, define $p = \max_{i=1,\ldots,n}\{p_i\}$. The following lemma is also given.

Lemma 5.7 *Suppose that the Lyapunov function*

$$V(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} \frac{\xi_i^{p-p_i+4}}{p-p_i+4}$$

is positive-definite and proper, satisfying

$$\mathscr{L}V \leq -\sum_{i=1}^{n} \xi_i^{p+3} + \phi. \tag{5.48}$$

Then, the following inequality holds

$$\mathscr{L}V \leq -a_0 V + b_0,$$

where

$$a_0 = \min\left(\phi^{(p_i-1)/(p+3)}\right),\ b_0 = (n+1)\phi.$$

Proof Let $a = \phi^{1/(p+3)}$ and $b = \xi_i$. Then, by using Young's inequality

$$\begin{aligned} a^{p_i-1}b^{p-p_i+4} &\leq \frac{p_i-1}{p+3}a^{p+3} + \frac{p-p_i+4}{p+3}b^{p+3} \\ &\leq a^{p+3} + b^{p+3}, \end{aligned}$$

which implies that

$$-\xi_i^{p+3} \leq -\phi^{(p_i-1)/(p+3)}\xi_i^{p-p_i+4} + \phi. \tag{5.49}$$

Substituting (5.49) into (5.48) yields that

$$\mathscr{L}V \le -\sum_{i=1}^{n} \phi^{(p_i-1)/(p+1)} \xi_i^{p-p_i+4} + (n+1)\phi.$$

The proof of Lemma 5.7 is completed here. □

Step 1: Define the variable $z_1 = x_1$. Then, consider the following Lyapunov function candidate

$$V_1 = \frac{\zeta^4}{4} + \frac{z_1^{p-p_1+4}}{p-p_1+4}.$$

It follows from (5.47) and Assumption 5.4 that

$$\mathscr{L}V_1 = -\lambda_0\zeta^4 + \bar{\lambda}_0 z_1^{p+3} + z_1^{p-p_1+3}\left(g_{1,k}x_2^{p_1} + f_{1,k}\right) + \frac{p-p_1+3}{2}\left\|\psi_{1,k}\right\|^2 z_1^{p-p_1+2}, \tag{5.50}$$

where $f_{1,k}$ and $\left\|\psi_{1,k}\right\|^2$ are unknown. Then, two neural networks $W_{1,k}S_1$ and $\Phi_{1,k}P_{1,k}$ are used to approximate the unknown function $f_{1,k}$ and the norm $\left\|\psi_{1,k}\right\|$ such that for any given $\varepsilon_{1,k} > 0$ and $\tau_{1,k} > 0$,

$$\begin{aligned} f_{1,k} &= W_{1,k}^T S_{1,k}(X_1) + \delta_{1,k}(X_1), \\ \left\|\psi_{1,k}\right\|^2 &= \Phi_{1,k}^T P_{1,k}(X_1) + \bar{\delta}_{1,k}(X_1), \end{aligned}$$

where $X_1 := [\zeta^T, x^T]^T \in R^{r+n}$, $\left|\delta_{1,k}(X)\right| \le \varepsilon_{1,k}$, $\bar{\delta}_{1,k}(X_1) \le \tau_{1,k}$.

One can get from the Young's inequality and Lemma 5.6 that

$$\begin{aligned} &z_1^{p-p_1+3} f_{1,k} \\ &= z_1^{p-p_1+3}(W_{1,k}^T S_{1,k}(X_1) + \delta_{1,k}(X_1)) \\ &\le \frac{p-p_1+3}{p+3} l_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} \left\|W_{1,k}\right\|^{\frac{p+3}{p-p_1+3}} \left\|S_{1,k}\right\|^{\frac{p+3}{p-p_1+3}} + \frac{p_1}{p+3} l_1^{-\frac{p+3}{p_1}} \\ &+ \frac{p-p_1+3}{p+3}\eta_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} + \frac{p_1}{p+3}\eta_1^{-\frac{p+3}{p_1}} \varepsilon_{1,k}^{\frac{p+3}{p_1}} \\ &\le l_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} \left\|W_{1,k}\right\|^{\frac{p+3}{p-p_1+3}} \left\|S_{1,k}\right\|^{\frac{p+3}{p-p_1+3}} + \eta_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} + l_1^{-\frac{p+3}{p_1}} + \eta_1^{-\frac{p+3}{p_1}} \varepsilon_{1,k}^{\frac{p+3}{p_1}} \\ &\le z_1^{p+3}\left(l_1^{\frac{p+3}{p-p_1+3}}\theta_1 D_1^{\frac{p+3}{p-p_1+3}} + \eta_1^{\frac{p+3}{p-p_1+3}}\right) + b_1, \end{aligned} \tag{5.51}$$

where $l_1, \eta_1 > 0$ are design parameters; $\|S_1\| \le D_1$; $\theta_1 := \max\{\left\|W_{1,k}\right\|^{(p+3)/(p-p_1+3)} : k \in M\}$; $b_1 = l_1^{-(p+3)/p_1} + \eta_1^{-(p+3)/p_1}\varepsilon_1^{(p+3)/p_1}$.

Moreover, one has that

$$\begin{aligned}
&\left\|\psi_{1,k}\right\|^2 z_1^{p-p_1+2} \\
&= z_1^{p-p_1+2}\left(\Phi_{1,k}^T P_{1,k}(X_1)+\bar{\delta}_{1,k}(X_1)\right) \\
&\le z_1^{p-p_1+2}\Phi_{1,k}^T P_{1,k} + z_1^{p-p_1+2}\bar{\delta}_{1,k} \\
&\le \frac{p-p_1+2}{p+3}\xi_1^{\frac{p+3}{p-p_1+2}} z_1^{p+3}\left\|\Phi_{1,k}\right\|^{\frac{p+3}{p-p_1+2}}\left\|P_{1,k}\right\|^{\frac{p+3}{p-p_1+2}} + \frac{p_1+1}{p+3}\xi_1^{-\frac{p+3}{p_1+1}} \\
&\quad + \frac{p-p_1+2}{p+3} m_1^{\frac{p+3}{p-p_1+2}} z_1^{p+3} + \frac{p_1+1}{p+3} m_1^{-\frac{p+3}{p_1+1}} \tau_{1,k}^{\frac{p+3}{p_1+1}} \\
&\le z_1^{p+3}\left(\xi_1^{\frac{p+3}{p-p_1+2}}\varphi_1 Q_1^{\frac{p+3}{p-p_1+2}} + m_1^{\frac{p+3}{p-p_1+2}}\right) + \bar{b}_1, \qquad (5.52)
\end{aligned}$$

where $\xi_1,\ m_1 > 0$ are design parameters; $\|P_1\| \le Q_1$; $\bar{b}_1 = \xi_1^{-(p+3)/(p_1+1)} + m_1^{-(p+3)/(p_1+1)}\tau_1^{(p+3)/(p_1+1)}$; $\varphi_1 = \max\{\left\|\Phi_{1,k}\right\|^{(p+3)/(p-p_1+2)} : k \in M\}$.

Substituting (5.51) and (5.52) into (5.50), yields that

$$\begin{aligned}
\mathscr{L}V_1 \le & -\lambda_0\zeta^4 + \bar{\lambda}_0 z_1^{p+3} + g_{1,k} z_1^{p-p_1+3} x_2^{p_1} + z_1^{p+3}\left(l_1^{\frac{p+3}{p-p_1+3}}\theta_1 D_1^{\frac{p+3}{p-p_1+3}} + \eta_1^{\frac{p+3}{p-p_1+3}}\right) \\
& + z_1^{p+3}\left(\frac{1}{2}(p-p_1+3)\xi_1^{\frac{p+3}{p-p_1+2}}\varphi_1 Q_1^{\frac{p+3}{p-p_1+2}} + \frac{1}{2}(p-p_1+3) m_1^{\frac{p+3}{p-p_1+2}}\right) + \tilde{b}_1, \qquad (5.53)
\end{aligned}$$

where $\tilde{b}_1 := b_1 + 0.5(p-p_1+3)\bar{b}_1$.

Then, the common virtual control function can be designed as

$$\begin{aligned}
\alpha_1 = & -z_1\left\{\frac{1}{\underline{d}_1}\left(\bar{\lambda}_1 + l_1^{\frac{p+3}{p-p_1+3}}\hat{\theta}_1 D_1^{\frac{p+3}{p-p_1+3}} + \frac{1}{2}(p-p_1+3)\xi_1^{\frac{p+3}{p-p_1+2}}\hat{\varphi}_1 Q_1^{\frac{p+3}{p-p_1+2}}\right.\right. \\
& \left.\left. + \eta_1^{\frac{p+3}{p-p_1+3}} + \frac{1}{2}(p-p_1+3) m_1^{\frac{p+3}{p-p_1+2}}\right)\right\}^{\frac{1}{p_1}} \\
= & -z_1\beta_1, \qquad (5.54)
\end{aligned}$$

where $\hat{\theta}_1$, $\hat{\varphi}_1$ are the estimations of θ_1, φ_1 respectively; $\bar{\lambda}_1 > 1 + \bar{\lambda}_0$ is a positive design parameter; $\underline{d}_1$ is defined in Assumption 5.3.

It follows from (5.53) and (5.54) that

$$\begin{aligned}
\mathscr{L}V_1 \le & -\lambda_0\zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0) z_1^{p+3} + g_{1,k} z_1^{p-p_1+3}\left(x_2^{p_1} - \alpha_1^{p_1}\right) \\
& + l_1^{\frac{p+3}{p-p_1+3}}\tilde{\theta}_1 D_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} + 0.5(p-p_1+3)\xi_1^{\frac{p+3}{p-p_1+2}}\tilde{\varphi}_1 Q_1^{\frac{p+3}{p-p_1+2}} z_1^{p+3} + \tilde{b}_1, \qquad (5.55)
\end{aligned}$$

where $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$, $\tilde{\varphi}_1 = \varphi_1 - \hat{\varphi}_1$.

Step 2: Denote $z_2 = x_2 - \alpha_1$, and define $d\alpha_1$ as

$$d\alpha_1 = \left(\sum_{j=1}^{n} \frac{\partial \alpha_1}{\partial x_j} \left(g_{j,k} x_{j+1}^{p_2} + f_{1,k} \right) + \frac{\partial \alpha_1}{\partial \hat{\theta}_1} \dot{\hat{\theta}}_1 + \frac{\partial \alpha_1}{\partial \hat{\varphi}_1} \dot{\hat{\varphi}}_1 \right) dt + \frac{\partial \alpha_1}{\partial x_1} \psi_{1,k}^T d\omega$$
$$= \bar{a}_1 dt + \frac{\partial \alpha_1}{\partial x_1} \psi_1^T d\omega, \tag{5.56}$$

where $\dot{\hat{\theta}}_1$ and $\dot{\hat{\varphi}}_1$ will be specified later, $x_{n+1} := u$ will given at final step.

Choose the Lyapunov function as

$$V_2 = V_1 + \frac{z_2^{p-p_2+4}}{p-p_2+4}.$$

Then, $\mathscr{L}V_2$ is given by

$$\begin{aligned}\mathscr{L}V_2 \leq & -\lambda_0 \zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0) z_1^{p+3} + g_{1,k} z_1^{p-p_1+3} \left(x_2^{p_1} - \alpha_1^{p_1} \right) \\ & + l_1^{\frac{p+3}{p-p_1+3}} \tilde{\theta}_1 D_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} + 0.5(p-p_1+3) \xi_1^{\frac{p+3}{p-p_1+2}} \tilde{\varphi}_1 Q_1^{\frac{p+3}{p-p_1+2}} z_1^{p+3} + \tilde{b}_1 \\ & + z_2^{p-p_2+3} \left(g_{2,k} x_3^{p_2} + f_{2,k} - \bar{a}_1 \right) + \frac{p-p_2+3}{2} \left\| \psi_{2,k} - \frac{\partial \alpha_1}{\partial x_1} \psi_{1,k} \right\|^2 z_2^{p-p_2+2}.\end{aligned} \tag{5.57}$$

By using Lemma 5.4 and Young's inequality, one can obtain that

$$\begin{aligned}& \left| g_{1,k} z_1^{p-p_1+3} (x_2^{p_1} - \alpha_1^{p_1}) \right| \\ & \leq c_1 \bar{d}_1 \left| z_1^{p-p_1+3} \right| |z_2| \left| z_2^{p_1-1} + (z_1 \beta_1)^{p_1-1} \right| \\ & \leq c_1 \bar{d}_1 \frac{p-p_1+3}{p+3} z_1^{p+3} + c_1 \bar{d}_1 \frac{p_1}{p+3} z_2^{p+3} + c_1 \bar{d}_1 \frac{p+2}{p+3} z_1^{p+3} \\ & \quad + c_1 \bar{d}_1 \frac{1}{p+3} z_2^{p+3} \beta_1^{(p_1-1)(p+3)} \\ & \leq z_1^{p+3} + z_2^{p+3} \left(1 + \beta_1^{(p_1-1)(p+3)} \right) \\ & = z_1^{p+3} + z_2^{p+3} \bar{\beta}_1,\end{aligned} \tag{5.58}$$

where $\bar{\beta}_1 = 1 + \beta_1^{(p_1-1)(p+1)}$; c_1 is chosen as $1/2\bar{d}_1$; $\bar{d}_1$ is defined in Assumption 5.3.

Substituting (5.58) into (5.57), gives that

$$\begin{aligned}\mathscr{L}V_2 \leq & -\lambda_0 \zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0 - 1) z_1^{p+3} + z_2^{p+3} \bar{\beta}_1 + l_1^{\frac{p+3}{p-p_1+3}} \tilde{\theta}_1 D_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} \\ & + \frac{1}{2}(p-p_1+3) \xi_1^{\frac{p+3}{p-p_1+2}} \tilde{\varphi}_1 Q_1^{\frac{2p+6}{p-p_1+2}} z_1^{p+3} + \tilde{b}_1 + z_2^{p-p_2+3} \left(g_{2,k} x_3^{p_2} + \bar{f}_{2,k} \right) \\ & + \frac{1}{2} (p-p_2+3) \bar{\psi}_{2,k} z_2^{p-p_2+2},\end{aligned} \tag{5.59}$$

where $\bar{f}_{2,k} = f_{2,k} - \bar{\alpha}_1$, $\bar{\psi}_{2,k} := \left\| \psi_{2,k} - \frac{\partial \alpha_1}{\partial x_1} \psi_{1,k} \right\|^2$. Then, neural networks $W_{2,k}^T S_{2,k}(X_2)$ and $\Phi_{2,k}^T P_{2,k}(X_2)$ are used to approximate the unknown functions $\bar{f}_{2,k}$ and $\bar{\psi}_{2,k}$ such that for any given $\varepsilon_{2,k} > 0$ and $\tau_{2,k} > 0$,

$$\bar{f}_{2,k} = W_{2,k}^T S_{2,k}(X_2) + \delta_{2,k}(X_1),$$
$$\bar{\psi}_{2,k} = \Phi_{2,k}^T P_{2,k}(X_2) + \bar{\delta}_{2,k}(X_1),$$

where $X_2 := [\zeta^T, x^T, \hat{\theta}_1, \hat{\varphi}_1]^T \in R^{r+n+2}$, $\left|\delta_{2,k}(X_2)\right| \le \varepsilon_{2,k}, \bar{\delta}_{2,k}(X_2) \le \tau_{2,k}$.

Similar to the procedure in (5.51), one can obtain that

$$\begin{aligned}
& z_2^{p-p_2+3} f_{2,k} \\
&= z_2^{p-p_2+3}(W_{2,k}^T S_{2,k}(X_2) + \delta_{2,k}(X_2)) \\
&\le \frac{p-p_2+3}{p+3} l_2^{\frac{p+3}{p-p_2+3}} z_2^{p+3} \left\| W_{2,k} \right\|^{\frac{p+3}{p-p_2+3}} \left\| S_{2,k} \right\|^{\frac{p+3}{p-p_2+3}} + \frac{p_2}{p+3} l_2^{-\frac{p+3}{p_2}} \\
&\quad + \frac{p-p_2+3}{p+3} \eta_2^{\frac{p+3}{p-p_2+3}} z_2^{p+3} + \frac{p_2}{p+3} \eta_2^{-\frac{p+3}{p_2}} \varepsilon_{2,k}^{\frac{p+3}{p_2}} \\
&\le l_2^{\frac{p+3}{p-p_2+3}} z_2^{p+3} \left\| W_{2,k} \right\|^{\frac{p+3}{p-p_2+3}} \left\| S_{2,k} \right\|^{\frac{p+3}{p-p_2+3}} + \eta_2^{\frac{p+3}{p-p_2+3}} z_2^{p+3} + l_2^{-\frac{p+3}{p_2}} + \eta_2^{-\frac{p+3}{p_2}} \varepsilon_{2,k}^{\frac{p+3}{p_2}} \\
&\le z_2^{p+3} \left(l_2^{\frac{p+3}{p-p_2+3}} \theta_2 D_2^{\frac{p+3}{p-p_2+3}} + \eta_2^{\frac{p+3}{p-p_2+3}} \right) + b_2,
\end{aligned} \tag{5.60}$$

where $l_2, \ \eta_2 > 0$ are design parameters, $\theta_2 := \max\{\left\| W_{2,k} \right\|^{(p+3)/(p-p_2+3)} : k \in M\}$, $b_2 = l_2^{-(p+3)/p_2} + \eta_2^{-(p+3)/p_2} \varepsilon_2^{(p+3)/p_2}$.

Using a similar way to (5.52), one gets that

$$\begin{aligned}
& \bar{\psi}_{2,k} z_2^{p-p_2+2} \\
&= z_2^{p-p_2+2} \left(\Phi_{2,k}^T P_{2,k}(X_2) + \bar{\delta}_{2,k}(X_2) \right) \\
&\le z_2^{p-p_2+2} \Phi_{2,k}^T P_{2,k} + z_2^{p-p_2+2} \bar{\delta}_{2,k} \\
&\le \frac{p-p_2+2}{p+3} \xi_2^{\frac{p+3}{p-p_2+2}} z_2^{p+3} \left\| \Phi_{2,k} \right\|^{\frac{p+3}{p-p_2+2}} \left\| P_{2,k} \right\|^{\frac{p+3}{p-p_2+2}} + \frac{p_2+1}{p+3} \xi_2^{-\frac{p+3}{p_2+1}} \\
&\quad + \frac{p-p_2+2}{p+3} m_2^{\frac{p+3}{p-p_2+2}} z_2^{p+3} + \frac{p_2+1}{p+3} m_2^{-\frac{p+3}{p_2+1}} \tau_{2,k}^{\frac{p+3}{p_2+1}} \\
&\le z_2^{p+3} \left(\xi_2^{\frac{p+3}{p-p_2+2}} \varphi_2 Q_2^{\frac{p+3}{p-p_2+2}} + m_2^{\frac{p+3}{p-p_2+2}} \right) + \bar{b}_2,
\end{aligned} \tag{5.61}$$

where $\xi_2, \ m_2 > 0$ are design parameters; $\bar{b}_2 = \xi_2^{-\frac{p+3}{p_2+1}} + m_2^{-\frac{p+3}{p_2+1}} \tau_2^{\frac{p+3}{p_2+1}}$; $\varphi_2 = \max\{\left\| \Phi_{2,k} \right\|^{\frac{p+3}{p-p_2+2}} : k \in M\}$; $P_2(X_2)$ and $\tau_2(X_2)$ represent the basis function vector and the estimation error of φ_2.

Design the common virtual control function as

$$\alpha_2 = -z_2 \left\{ \frac{1}{\underline{d}_2} \left(\bar{\beta}_1 + \lambda_2 + l_2^{\frac{p+3}{p-p_2+3}} \hat{\theta}_2 D_2^{\frac{p+3}{p-p_2+3}} + \frac{1}{2}(p - p_2 + 3)\xi_2^{\frac{p+3}{p-p_2+2}} \hat{\varphi}_2 Q_2^{\frac{p+3}{p-p_2+2}} \right.\right.$$
$$\left.\left. + \eta_2^{\frac{p+3}{p-p_2+3}} + \frac{1}{2}(p - p_2 + 3) m_2^{\frac{p+3}{p-p_2+2}} \right) \right\}^{\frac{1}{p_2}}$$
$$= -z_2 \beta_2, \tag{5.62}$$

where $\hat{\theta}_2, \hat{\varphi}_2$ are the estimation of θ_2 and φ_2 respectively; $\lambda_2 > 1$ is a design parameter; $\underline{d}_2$ is defined in Assumption 5.3.

By substituting (5.60)–(5.62) into (5.59), one has

$$\begin{aligned} &\mathscr{L}V_2 \\ &\le -\lambda_0 \zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0 - 1) z_1^{p+3} - \lambda_2 z_2^{p+3} + g_{2,k} z_2^{p-p_2+3}(x_3^{p_2} - \alpha_2^{p_2}) \\ &\quad + \sum_{j=1}^{2} \left(l_j^{\frac{p+3}{p-p_j+3}} \tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p - p_j + 3)\xi_j^{\frac{p+3}{p-p_j+2}} \tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} + \tilde{b}_j \right), \end{aligned}$$

where $\tilde{b}_j := b_j + 0.5(p - p_j + 3)\bar{b}_j, \tilde{\theta}_j = \theta_j - \hat{\theta}_j, \tilde{\varphi}_j = \varphi_j - \hat{\varphi}_j, j = 1, 2.$

Step i: Suppose at step i $(3 \le i \le n - 1)$ that, there is a set of virtual control functions $\alpha_3, \ldots, \alpha_{n-1}$, defined by

$$\alpha_i = z_i \beta_i,\ z_i = x_{i+1} - \alpha_i \tag{5.63}$$

and assume that a set of unknown nonlinear functions $\bar{f}_{i,k}$ and $\bar{\psi}_{i,k}$ can be approximated by neural networks $W_{i,k}^T S_{i,k}(X_i)$ and $\Phi_{i,k}^T P_{i,k}(X_i)$ for any given $\varepsilon_{i,k} > 0$, $\tau_{i,k} > 0$.

$$\begin{aligned} \bar{f}_{i,k} &= W_{i,k}^T S_{i,k}(X_i) + \delta_{i,k},\ \left|\delta_{i,k}(X_i)\right| \le \varepsilon_{i,k}, \\ \bar{\psi}_{i,k} &= \Phi_{i,k}^T P_{i,k}(X_i) + \bar{\delta}_{i,k},\ \left|\bar{\delta}_{i,k}(X_i)\right| \le \tau_{i,k}, \end{aligned}$$

where $X_i := [\zeta^T, x^T, \hat{\theta}_1, \ldots, \hat{\theta}_i, \hat{\varphi}_1, \ldots, \hat{\varphi}_i]^T \in R^{r+n+2i}$.

A straightforward calculation gives that

$$\begin{aligned} &\mathscr{L}V_i \\ &\le -\lambda_0 \zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0 - 1) z_1^{p+3} - \sum_{j=2}^{i-1} (\lambda_j - 1) z_j^{p+3} \\ &- \lambda_i z_i^{p+3} + g_{i,k} z_i^{p-p_i+3}(x_{i+1}^{p_i} - \alpha_i^{p_i}) \\ &+ \sum_{j=1}^{i} \left(l_j^{\frac{p+3}{p-p_j+3}} \tilde{\theta}_j \left\|S_j\right\|^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p - p_j + 3)\xi_j^{\frac{p+3}{p-p_j+2}} \tilde{\varphi}_j \left\|P_j\right\|^{\frac{p+3}{p-p_j+2}} z_j^{p+3} + \tilde{b}_j \right), \end{aligned} \tag{5.64}$$

where $\tilde{\theta}_j = \theta_j - \hat{\theta}_j$, $\tilde{\varphi}_j = \varphi_j - \hat{\varphi}_j$; $[\hat{\theta}_j, \hat{\varphi}_j]$ is the estimation of

$$[\theta_j,\ \varphi_j] := \max\{\left\| W_{j,k} \right\|^{\frac{p+3}{p-p_j+3}},\ \left\| \Phi_{j,k} \right\|^{\frac{p+3}{p-p_j+2}} : k \in M\};$$
$$\tilde{b}_j = b_j + 0.5(p - p_j + 3)\bar{b}_j,$$
$$b_j = l_j^{-(p+3)/p_j} + \eta_j^{-(p+3)/p_j} \varepsilon_j^{(p+3)/p_j}$$

and $\bar{b}_j = \xi_j^{-(p+3)/(p_j+1)} + m_j^{-(p+3)/(p_j+1)} \tau_j^{(p+3)/(p_j+1)}$.

Step n: Let $z_n = x_n - \alpha_{n-1}$, define $d\alpha_{n-1}$ as

$$\begin{aligned} d\alpha_{n-1} = & \left(\sum_{j=1}^{n} \frac{\partial \alpha_{n-1}}{\partial x_j} \left(g_{j,k} x_{j+1}^{p_2} + f_{j,k} \right) + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_j} \dot{\hat{\theta}}_j + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\varphi}_j} \dot{\hat{\varphi}}_j \right) dt \\ & + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_{j,k}^T d\omega \\ & = \bar{a}_{n-1} dt + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_{j,k}^T d\omega. \end{aligned}$$

where $x_{n+1} := u$ is provided later.

We construct the Lyapunov function as

$$V_n = V_{n-1} + \frac{z_n^{p-p_n+4}}{p - p_n + 4}.$$

By using (5.64), $\mathscr{L} V_n$ is given by

$$\mathscr{L} V_n \tag{5.65}$$

$$\leq -\lambda_0 \zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0 - 1) z_1^{p+3} - \sum_{j=2}^{n-2} (\lambda_j - 1) z_j^{p+3} \tag{5.66}$$

$$\begin{aligned} & - \lambda_{n-1} z_{n-1}^{p+3} + g_{n-1,k} z_{n-1}^{p-p_n+3} (x_n^{p_{n-1}} - \alpha_{n-1}^{p_{n-1}}) \\ & + \sum_{j=1}^{n-1} \left(l_j^{\frac{p+3}{p-p_j+3}} \tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p - p_j + 3) \xi_j^{\frac{p+3}{p-p_j+2}} \tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} + \tilde{b}_j \right) \\ & + z_n^{p-p_n+3} \left(g_{n,k} u^{p_n} + \bar{f}_{n,k} \right) + \frac{p - p_n + 3}{2} \bar{\psi}_{n,k} z_n^{p-p_n+2}, \end{aligned} \tag{5.67}$$

where $\bar{f}_{n,k} = f_{n,k} - \bar{\alpha}_{n-1}$, $\bar{\psi}_{n,k} = \left\| \psi_{n,k} - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} \psi_{i,k} \right\|^2$. Then, neural networks $W_{n,k}^T S_{n,k}(X_n)$ and $\Phi_{n,k}^T P_{n,k}(X_n)$ are used to approximate unknown functions $\bar{f}_{n,k}$ and $\bar{\psi}_{n,k}$ such that for any given $\varepsilon_{n,k} > 0$ and $\tau_{n,k} > 0$,

$$\bar{f}_{n,k} = W_{n,k}^T S_{n,k}(X_n) + \delta_{n,k}(X_n),$$
$$\bar{\psi}_{n,k} = \Phi_{n,k}^T P_{n,k}(X_n) + \bar{\delta}_{n,k}(X_n),$$

where $X_n := [\zeta^T, x^T, \hat{\theta}_1, \ldots, \hat{\theta}_n, \hat{\varphi}_1, \ldots, \hat{\varphi}_n]^T \in R^{r+3n}$, $\left|\delta_{n,k}(X_n)\right| \leq \varepsilon_{n,k}$, $\bar{\delta}_{n,k}(X_n) \leq \tau_{n,k}$.

Similar to (5.60) and (5.61), one has

$$\begin{aligned} z_n^{p-p_n+3}\bar{f}_{n,k} &\leq z_n^{p+3}\left(l_n^{\frac{p+3}{p-p_n+3}}\theta_n D_n^{\frac{p+3}{p-p_n+3}} + \eta_n^{\frac{p+3}{p-p_n+3}}\right) + b_n, \\ z_n^{p-p_n+3}\bar{\psi}_{n,k} &\leq z_n^{p+3}\left(\xi_n^{\frac{p+3}{p-p_n+2}}\varphi_n Q_n^{\frac{p+3}{p-p_n+2}} + m_n^{\frac{p+3}{p-p_n+2}}\right) + \bar{b}_n, \end{aligned} \tag{5.68}$$

where $l_n,\ \eta_n,\ \xi_n,\ m_n > 0$ are design parameters; $\theta_n := \max\{\left\|W_{n,k}\right\|^{(p+3)/(p-p_n+3)} : k \in M\}$; $b_n = l_n^{-\frac{p+3}{p_n}} + \eta_n^{-\frac{p+3}{p_n}}\varepsilon_n^{\frac{p+3}{p_n}}$, $\varphi_n := \max\{\left\|\Phi_{n,k}\right\|^{(p+3)/(p-p_n+2)} : k \in M\}$; $\bar{b}_n = \xi_n^{-(p+3)/(p_n+1)} + m_n^{-(p+3)/(p_n+1)}\tau_n^{(p+3)/(p_n+1)}$.

Furthermore, it is not hard to get that

$$\begin{aligned} &\left|g_{n-1,k}z_{n-1}^{p-p_n+3}(x_n^{p_{n-1}} - \alpha_{n-1}^{p_{n-1}})\right| \\ &\leq c_{n-1}\bar{d}_{n-1}\left|z_{n-1}^{p-p_{n-1}+3}\right||z_n|\left|z_n^{p_{n-1}-1} + (z_{n-1}\beta_{n-1})^{p_{n-1}-1}\right| \\ &\leq c_{n-1}\bar{d}_{n-1}\frac{p-p_{n-1}+3}{p+3}z_{n-1}^{p+3} + c_{n-1}\bar{d}_{n-1}\frac{p_{n-1}}{p+3}z_n^{p+3} + c_{n-1}\bar{d}_{n-1}\frac{p+2}{p+3}z_{n-1}^{p+3} \\ &\quad + c_{n-1}\bar{d}_{n-1}\frac{1}{p+3}z_n^{p+3}\beta_{n-1}^{(p_{n-1}-1)(p+3)} \\ &\leq z_{n-1}^{p+3} + z_n^{p+3}\left(1 + \beta_{n-1}^{(p_{n-1}-1)(p+3)}\right) \\ &= z_{n-1}^{p+3} + z_n^{p+3}\bar{\beta}_{n-1}, \end{aligned} \tag{5.69}$$

where $\bar{\beta}_{n-1} = 1 + \beta_{n-1}^{(p_{n-1}-1)(p+1)}$, c_{n-1} is chosen as $1/2\bar{d}_{n-1}$, and $\bar{d}_{n-1}$ is defined in Assumption 5.3.

Substituting (5.68) and (5.69) into (5.67), the following inequality can be obtained.

$$\begin{aligned} &\mathscr{L}V_n \qquad (5.70) \\ &\leq -\lambda_0\zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0 - 1)z_1^{p+3} - \sum_{j=2}^{n-1}(\lambda_j - 1)z_j^{p+3} + z_n^{p+3}\bar{\beta}_{n-1} \\ &+ \sum_{j=1}^{n-1}\left(l_j^{\frac{p+3}{p-p_j+3}}\tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p-p_j+3)\xi_j^{\frac{p+3}{p-p_j+2}}\tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3}\right) + \sum_{j=1}^{n}\tilde{b}_j \\ &+ z_n^{p+3}\left(l_n^{\frac{p+3}{p-p_n+3}}\theta_n D_n^{\frac{p+3}{p-p_n+3}} + \eta_n^{\frac{p+3}{p-p_n+3}} + 0.5(p-p_n+3)\xi_n^{\frac{p+3}{p-p_n+2}}\varphi_n Q_n^{\frac{p+3}{p-p_n+2}}\right. \end{aligned}$$

$$+ 0.5(p - p_n + 3)m_n^{\frac{p+3}{p-p_n+2}}\Bigg) + z_n^{p-p_n+3} g_{n,k} u^{p_n}. \tag{5.71}$$

Design the controller u as

$$\begin{aligned} u =& - z_n \left\{ \frac{1}{\underline{d}_n} \Bigg(\lambda_n + \bar{\beta}_{n-1} + l_n^{\frac{p+3}{p-p_n+3}} \hat{\theta}_n D_n^{\frac{p+3}{p-p_n+3}} + \eta_n^{\frac{p+3}{p-p_n+3}} \right. \\ &\left. + \ 0.5(p - p_n + 3)\xi_n^{\frac{p+3}{p-p_n+2}} \hat{\varphi}_n Q_n^{\frac{p+3}{p-p_n+2}} + 0.5(p - p_n + 3)m_n^{\frac{p+3}{p-p_n+2}} \Bigg) \right\}^{\frac{1}{p_n}} \\ =& - z_n \beta_n, \end{aligned} \tag{5.72}$$

where $\hat{\theta}_n$ is the estimation of θ_n; $\lambda_n > 1$ is a positive design parameter; d_n is defined in Assumption 5.3.

It follows from (5.71) and (5.72) that

$$\mathscr{L}V_n \le \tag{5.73}$$

$$- \lambda_0 \zeta^4 - \sum_{j=1}^{n} (\lambda_j - 1) z_j^{p+3}$$

$$+ \sum_{j=1}^{n} \left(l_j^{\frac{p+3}{p-p_j+3}} \tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p - p_j + 3)\xi_j^{\frac{p+3}{p-p_j+2}} \tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} + \tilde{b}_j \right) \tag{5.74}$$

where $\lambda_1 := \bar{\lambda}_1 - \bar{\lambda}_0$.

Last Step: Choose the final Lyapunov function as

$$V = V_n + \sum_{j=1}^{n} \left(\frac{1}{2r_j} \tilde{\theta}_j^2 + \frac{1}{2\bar{r}_j} \tilde{\varphi}_j^2 \right) \tag{5.75}$$

where r_j is a positive design parameter.

$\mathscr{L}V$ is given by

$$\mathscr{L}V \le \tag{5.76}$$

$$- \lambda_0 \zeta^4 - \sum_{j=1}^{n} (\lambda_j - 1) z_j^{p+3} - \sum_{j=1}^{n} \left(\frac{1}{r_j} \tilde{\theta}_j \dot{\hat{\theta}}_j + \frac{1}{\bar{r}_j} \tilde{\varphi}_j \dot{\hat{\varphi}}_j \right)$$

$$+ \sum_{j=1}^{n} \left(l_j^{\frac{p+3}{p-p_j+3}} \tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p - p_j + 3)\xi_j^{\frac{p+3}{p-p_j+2}} \tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} + \tilde{b}_j \right). \tag{5.77}$$

The adaptive laws are defined as the solutions to the following differential equations

$$\dot{\hat{\theta}}_j = r_j l_j^{\frac{p+3}{p-p_j+3}} D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} - B_j\hat{\theta}_j,$$
$$\dot{\hat{\varphi}}_j = \frac{1}{2}(p - p_j + 3)\bar{r}_j \xi_j^{\frac{p+3}{p-p_j+2}} Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} - \bar{B}_j\hat{\varphi}_j, \tag{5.78}$$

where $j = 1, 2, \ldots, n,\ B_j,\ \bar{B}_j > 0$ are design parameters.

This, together with (5.77), means that

$$\mathscr{L}V \leq -\lambda_0\zeta^4 - \sum_{j=1}^{n}(\lambda_j - 1)z_j^{p+3} + \sum_{j=1}^{n}\frac{B_j\hat{\theta}_j\tilde{\theta}_j}{r_j} + \sum_{j=1}^{n}\frac{\bar{B}_j\hat{\varphi}_j\tilde{\varphi}_j}{\bar{r}_j} + \sum_{j=1}^{n}\tilde{b}_j. \tag{5.79}$$

Notice that

$$\tilde{\theta}_j\hat{\theta}_j = \tilde{\theta}_j(\theta_j - \tilde{\theta}_j) \leq -\frac{1}{2}\tilde{\theta}_j^2 + \frac{1}{2}\theta_j^2,$$
$$\tilde{\varphi}_j\hat{\varphi}_j = \tilde{\varphi}_j(\varphi_j - \tilde{\varphi}_j) \leq -\frac{1}{2}\tilde{\varphi}_j^2 + \frac{1}{2}\varphi_j^2, \tag{5.80}$$

By using (5.79), (5.80) and Lemma 5.7, one has

$$\begin{aligned}\mathscr{L}V \leq& -\lambda_0\zeta^4 - \sum_{j=1}^{n}\left((\lambda_j - 1)z_j^{p+3} + \frac{B_j}{2r_j}\tilde{\theta}_j^2 + \frac{\bar{B}_j}{2\bar{r}_j}\tilde{\varphi}_j^2\right)\\ &+ \sum_{j=1}^{n}\left(\frac{B_j}{2r_j}\theta_j^2 + \frac{\bar{B}_j}{2\bar{r}_j}\varphi_j^2 + \tilde{b}_j\right)\\ \leq& -q_0V + q_1,\end{aligned}$$

where $q_0 = \min\{(p - p_i + 4)(\lambda_j - 1)\phi^{(p_i-1)/(p+3)},\ B_j,\ \bar{B}_j,\ 2\lambda_0 : 1 \leq j \leq n\}$, $\phi = \sum_{j=1}^{n}(\frac{B_j}{2r_j}\theta_j^2 + \frac{\bar{B}_j}{2\bar{r}_j}\varphi_j^2 + \tilde{b}_j)$, $q_1 = (\lambda_j - 1)(n + 1)\phi$.

According to Lemma 5.5, we have that

$$E[V(x,t)] \leq V(x_0)e^{-q_0t} + \frac{q_1}{q_0},\ \forall t \geq 0, \tag{5.81}$$

which indicates that all the signals in the closed-loop system are bounded. The design is completed here. Next, we address our main result.

Theorem 5.2 *For $1 \leq i \leq n, k \in M$ assume that all the unknown nonlinear functions $\bar{f}_{i,k}$ and $\bar{\psi}_{i,k}$ can be approximated by neural networks in the sense that the approximation errors are bounded, and all the initial values of $\hat{\theta}_i$ and $\hat{\varphi}_i$ satisfy $\hat{\theta}_i(0) \geq 0$ and $\hat{\varphi}_i(0) \geq 0$, respectively. Then, under the state feedback controller*

(5.72) and the adaptive laws (5.78), the equilibrium at the origin of the closed-loop system is boundedly stable in probability and

$$P\left\{\lim_{t\to\infty}\left(\frac{|\zeta|^4}{4}+\sum_{i=1}^{n}\frac{|z_i|^{p-p_i+4}}{p-p_i+4}\right)\le\frac{q_1}{q_0}\right\}=1.$$

Proof It is not difficult to complete the proof by the above discussions, and thus we omit the proof here.

In the following, a corollary is given by using only two adaptive laws.

Corollary 5.1 *For $1\le i\le n, k\in M$ assume that all the unknown nonlinear functions $\bar{f}_{i,k}$ and $\bar{\psi}_{i,k}$ can be approximated by neural networks in the sense that the approximation errors are bounded, and all the initial values of $\hat{\theta}_i$ and $\hat{\varphi}_i$ satisfy $\hat{\theta}_i(0)\ge 0$ and $\hat{\varphi}_i(0)\ge 0$, respectively. Consider the following controller and adaptive laws:*

$$\begin{aligned}u=&-z_n\left\{\frac{1}{\underline{d}_n}\left(\lambda_n+\bar{\beta}_{n-1}+l_n^{\frac{p+3}{p-p_n+3}}\hat{\theta}D_n^{\frac{p+3}{p-p_n+3}}+\eta_n^{\frac{p+3}{p-p_n+3}}\right.\right.\\&\left.\left.+\frac{1}{2}(p-p_n+3)\xi_n^{\frac{p+3}{p-p_n+2}}\hat{\varphi}Q_n^{\frac{p+3}{p-p_n+2}}+0.5(p-p_n+3)m_n^{\frac{p+3}{p-p_n+2}}\right)\right\}^{\frac{1}{p_n}},\\\dot{\hat{\theta}}=&\sum_{j=1}^{n}rl_j^{\frac{p+3}{p-p_j+3}}D_j^{\frac{p+3}{p-p_j+3}}z_j^{p+3}-B\hat{\theta},\\\dot{\hat{\varphi}}=&\frac{1}{2}\sum_{j=1}^{n}(p-p_j+3)\bar{r}\xi_j^{\frac{p+3}{p-p_j+2}}Q_j^{\frac{p+3}{p-p_j+2}}z_j^{p+3}-\bar{B}\hat{\varphi},\end{aligned}$$

where $\lambda_n>1$, l_j, ξ_j, m_n, η_n, r, B, $\bar{r}$, $\bar{B}>0$ are positive design parameters, $\hat{\theta}=\sum_{j=1}^{n}\hat{\theta}_j$, $\hat{\varphi}=\sum_{j=1}^{n}\hat{\varphi}_j$. Then, the equilibrium at the origin of the closed-loop system is boundedly stable in probability and

$$P\left\{\lim_{t\to\infty}\left(\frac{|\zeta|^4}{4}+\sum_{i=1}^{n}\frac{|z_i|^{p-p_i+4}}{p-p_i+4}\right)\le\frac{q_1}{q_0}\right\}=1.$$

Proof It should be pointed out that $\hat{\theta}\ge\hat{\theta}_j\ge 0$, $\hat{\varphi}\ge\hat{\varphi}_j\ge 0$, $j=1,\ldots,n$. Therefore, we can use $\hat{\theta}$ and $\hat{\varphi}$ instead of $\hat{\theta}_j$ and $\hat{\varphi}_j$ in (5.54), (5.62), (5.63) and (5.72). In (5.75), the parameters $\tilde{\theta}_j$ and $\tilde{\varphi}_j$ in Lyapunov function V should be rewritten as $\tilde{\theta}$ and $\tilde{\varphi}$. The detailed proof is omitted here because it is similar to the one of Theorem 5.2. □

5.3.3 Simulation Results

An example with two controllers (multiple adaptive laws and two adaptive laws respectively) is presented in the following to demonstrate the effectiveness of our main results.

Consider the following switched stochastic high-order nonlinear systems with SISS inverse dynamics:

$$\sum\nolimits_1 = \begin{cases} d\zeta = f_{0,1}(\zeta, x_1)\,dt + \psi_{0,1}^T(\zeta, x_1)\,d\omega, \\ dx_1 = \left[g_{1,1}(\zeta, x_1, x_2)x_2^{p_1} + f_{1,1}(\zeta, x_1, x_2)\right]dt + \psi_{1,1}^T(\zeta, x_1, x_2)\,d\omega, \\ dx_2 = \left[g_{2,1}(\zeta, x_1, x_2)u^{p_2} + f_{2,1}(\zeta, x_1, x_2)\right]dt + \psi_{2,1}^T(\zeta, x_1, x_2)\,d\omega, \end{cases}$$

$$\sum\nolimits_2 = \begin{cases} d\zeta = f_{0,2}(\zeta, x_1)\,dt + \psi_{0,2}^T(\zeta, x_1)\,d\omega, \\ dx_1 = \left[g_{1,2}(\zeta, x_1, x_2)x_2^{p_1} + f_{1,2}(\zeta, x_1, x_2)\right]dt + \psi_{1,2}^T(\zeta, x_1, x_2)\,d\omega, \\ dx_2 = \left[g_{2,2}(\zeta, x_1, x_2)u^{p_2} + f_{2,2}(\zeta, x_1, x_2)\right]dt + \psi_{2,2}^T(\zeta, x_1, x_2)\,d\omega, \end{cases}$$

where $g_{1,1}$, $f_{1,1}$, $\psi_{1,1}$, $g_{2,1}$, $f_{2,1}$, $\psi_{2,1}$, $g_{1,2}$, $f_{1,2}$, $\psi_{1,2}$, $g_{2,2}$, $f_{2,2}$, and $\psi_{2,2}$ are all completely unknown functions; $p_1 = 3$, $p_2 = 5$. First, a controller under multiple adaptive laws is designed by Theorem 5.2. The initial conditions are $\zeta(0) = 1$ $x_1(0) = 0.5$, $x_2(0) = -0.5$ and $\hat{\theta}_1(0) = 2$, $\hat{\theta}_2(0) = 3.5$, $\hat{\varphi}_1(0) = 3$, $\hat{\varphi}_2(0) = 4$. The controller parameters are chosen as: $\lambda_1 = \lambda_2 = 5$, $l_2 = l_2 = \eta_1 = \eta_2 = \xi_1 = \xi_2 = m_1 = m_2 = 4$, $r_1 = r_2 = \bar{r}_1 = \bar{r}_2 = 1$, $B_1 = B_2 = \bar{B}_1 = \bar{B}_2 = 0.1$. We apply three nodes for each input dimension of $W_1^T S_1$, $W_2^T S_2$, $\Phi_1^T P_1$ and $\Phi_2^T P_2$. Therefore, each of them contains 81 nodes with centers spaced evenly in the interval $[-0.5, 0.5] \times [-0.5, 0.5] \times [-0.5, 0.5] \times [-0.5, 0.5]$, and the widths still being equal to 2.5. Second, a controller under two adaptive laws is designed by Corollary 5.1 with same conditions except $\hat{\theta}(0) = 3$, $\hat{\varphi}(0) = 4$, $r = 1$, $B = 0.1$.

In order to give the simulation results, it is assumed that $f_{0,1} = -15\zeta + 0.1x_1^2$, $\psi_{0,1} = \left(\zeta^2 + 0.3x_1^4\right)^{1/2}$, $g_{1,1} = \sin(x_1x_2+\zeta)+2$, $f_{1,1} = x_1x_2+\zeta$, $\psi_{1,1} = \sin(x_1x_2+\zeta)$, $g_{2,1} = \cos(x_1 + x_2^2 + \zeta) + 2$, $f_{2,1} = x_1x_2^2 + \zeta\sin\zeta$, $\psi_{2,1} = x_1\cos x_2 + \zeta^2$;

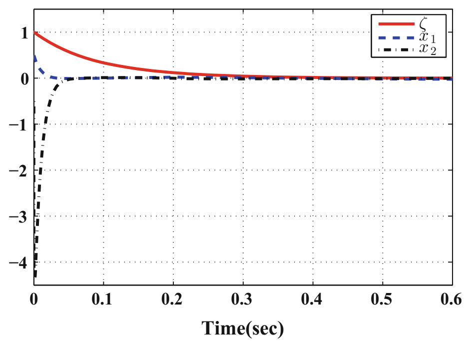


Fig. 5.5 Responses of system states by using multiple adaptive laws

Fig. 5.6 Responses of the multiple adaptive laws

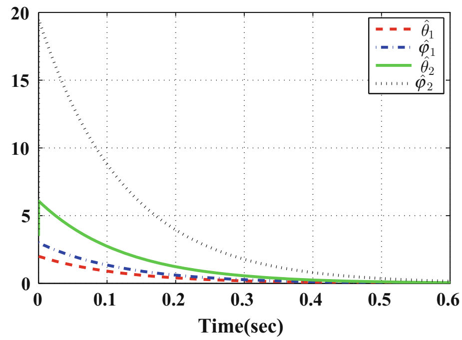


Fig. 5.7 Response of switching signal

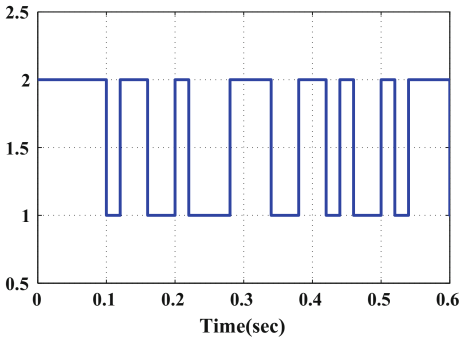
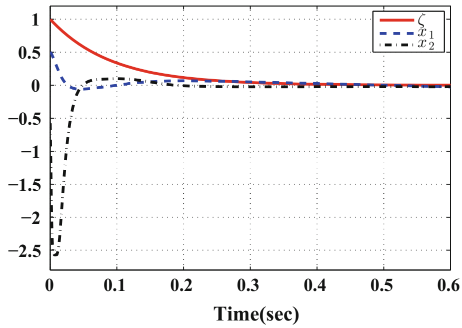


Fig. 5.8 Responses of system states by using two adaptive laws



$f_{0,2} = -13\zeta + 0.3x_1^2$, $\psi_{0,2} = \left(0.17\zeta^2 + 0.13x_1^4\right)^{1/2}$, $g_{1,2} = \sin(x_1^2 + x_2 + \zeta) + 2$, $f_{1,2} = x_1^2x_2 + \zeta^2$, $\psi_{1,2} = \sin(x_1 + x_2) + \zeta^3$, $g_{2,2} = \cos(x_1x_2^2 + \zeta) + 2$, $f_{2,2} = x_1x_2 + \zeta\cos\zeta$, $\psi_{2,2} = x_1\sin(x_1x_2) + \zeta^2$.

The simulation results based on Theorem 5.2 are shown in Figs. 5.5, 5.6 and 5.7, respectively. Figure 5.5 depicts the responses of system states. The trajectories of adaptive laws are shown in Figs. 5.6, and 5.7 describes the switching signal. From Fig. 5.5, it can be seen that all the system states eventually converge to a small neighborhood of the origin. The simulation results based on Corollary 5.1 are shown in

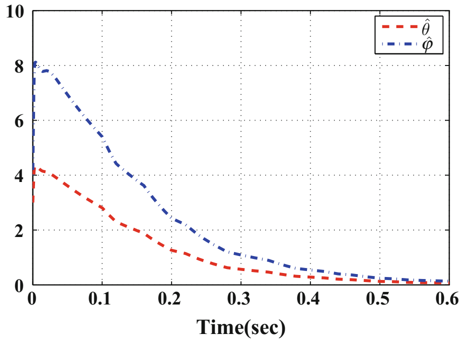


Fig. 5.9 Responses of two adaptive laws

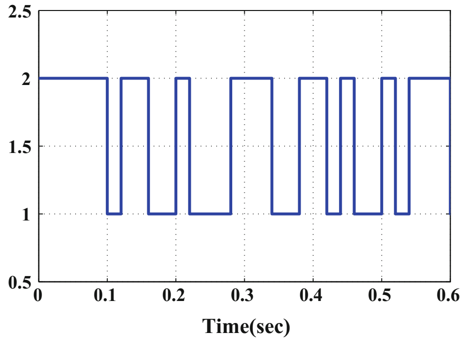


Fig. 5.10 Response of switching signal

Figs. 5.8, 5.9 and 5.10, respectively. It can be seen that all the system states eventually converge to a small neighborhood of the origin by using only two adaptive laws.

5.3.4 Conclusions

The adaptive neural control for a class of stochastic high-order switched nonlinear systems with SISS inverse dynamic is studied. An adaptive neural control algorithm is proposed. It can be shown that the equilibrium at the origin of the closed-loop system is BIBO stable in probability.

References

1. Hou M, Fu F, Duan G (2013) Global stabilization of switched stochastic nonlinear systems in strict-feedback form under arbitrary switchings. Automatica 49(8):2571–2575
2. Liu L, Xie X (2013) State feedback stabilization for stochastic feedforward nonlinear systems with time-varying delay. Automatica 49(4):936–942

3. Wu Z, Xia Y, Xie X (2012) Stochastic Barbalat's lemma and its applications. IEEE Trans Autom Control 57(6):1537–1543
4. Xie X, Duan N (2010) Output tracking of high-order stochastic nonlinear systems with application to benchmark mechanical system. IEEE Trans Autom Control 55(5):1197–1202
5. Zhang H, Wu Z, Xia Y (2014) Exponential stability of stochastic systems with hysteresis switching. Automatica 50(2):599–606
6. Xie X, Duan N, Yu X (2011) State-feedback control of high-order stochastic nonlinear systems with SiISS inverse dynamics. IEEE Trans Autom Control 56(8):1921–1926
7. Feng W, Tian J, Zhao P (2011) Stability analysis of switched stochastic systems. Automatica 47(1):148–157
8. Wu L, Zheng WX, Gao H (2013) Dissipativity-based sliding mode control of switched stochastic systems. IEEE Trans Autom Control 58(3):785–791
9. Wu Z, Yang J, Shi P (2010) Adaptive tracking for stochastic nonlinear systems with Markovian switching. IEEE Trans Autom Control 55(9):2135–2141
10. Zhai D, Kang Y, Zhao P, Zhao Y (2012) Stability of a class of switched stochastic nonlinear systems under asynchronous switching. Int. J. Control, Autom. Syst. 10(6):1182–1192
11. Sontag E (1989) Smooth stabilization implies coprime factorization. IEEE Trans Autom Control 34(4):435–443
12. Tao G, Kokotović P (1994) Adaptive control of plants with unknown dead-zones. IEEE Trans Autom Control 39(1):59–68
13. Krstić M, Hua D (1998) Stabilization of nonlinear uncertain systems. Springer, London
14. Polycarpou M, Ioannou P (1996) A robust adaptive nonlinear control design. Automatica 32(3):423–427
15. Lim C, Forsythe W (1983) Autopilot for ship control, part 1: theoretical design. IEE Proc D Control Theory Appl 130(6):281–287
16. Lin W, Pongvuthithum R (2003) Nonsmooth adaptive stabilization of cascade systems with nonlinear parameterization via partial-state feedback. IEEE Trans Autom Control 48(10):1809–1816
17. Li W, Xie X (2009) Inverse optimal stabilization for stochastic nonlinear systems whose linearizations are not stabilizable. Automatica 45(2):498–503
18. liu L, Duan N (2010) State-feedback stabilization for stochastic high-order nonlinear systems with a ratio of odd integers power. Nonlinear Anal Model Control 15(1):39–53
19. Xie X, Tian J (2007) State-feedback stabilization for high-order stochastic nonlinear systems with stochastic inverse dynamics. Int. J. Robust Nonlinear Control 17(14):1343–1362
20. Wang C, Hill DJ, Ge SS, Chen G (2006) An iss-modular approach for adaptive neural control of pure-feedback systems. Automatica 42(5):723–731