Chapter 3
Switching Stabilization of Switched Systems Composed of Unstable Subsystems

3.1 Background and Motivation

As mentioned in Chap. 2, for a switched system, even if all its subsystems are stable, it may fail to preserve stability under arbitrary switching, but may be stable under restricted switching signals. Therefore, it is of significance to study the controlled-switching stabilization problems of switched systems. The controlled switching may result from the physical constraints of a system or the designers' intervention [1] which is actually related to the controlled-switching stabilization problem [2]. Generally, the controlled switching in systems could be classified into state-dependent and time-constrained ones.

During the past few years, the problems of state-dependent switching stabilization problems have been widely studied for switched systems with or without unstable subsystems [3, 4]. In the state-dependent case, the whole state space is usually divided into pieces so as to facilitate the search for corresponding Lyapunov-like functions. Then, the state-dependent switching can be designed to ensure the non-increasing conditions when switching occurs. Note that, state-dependent switching is applicable only for the systems whose states are measurable or observable, which also suffers from the problems of high cost, reliability and real-time ability.

However, the time-constrained switching is more applicable in practice, and has been used for controlled-switching stabilization of switched systems in recent years [5–7]. It is noticed that the results on time-constrained switching stabilization of switched systems mainly focus on systems with stable subsystems (or at least one stable subsystem). The basic idea of the existing works is to activate the stable subsystem for a sufficiently large time that we could call slow switching, to compensate the state divergence [8]. In [9], the stability analysis of continuous-time linear switched systems comprising both Hurwitz stable and unstable subsystems is studied by exploring a new type of Lyaounov-like function whose energy can rise with a bounded rate for each active mode. After the bounded increment, the minimal average dwell time should be designed sufficiently large to compensate the energy increment produced during the unstable time. Recently, the mode-dependent dwell-

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time switching is used in [10] for stabilization of switched linear systems with both stable and unstable modes. It is very worth pointing out that there are few efforts put on time-constrained switching stabilization of switched linear systems with all unstable subsystems, which is both theoretically challenging and of fundamental importance to numerous applications.

On the other hand, as many applications of switched systems, such as mobile robots, automotive, DC converters etc., appear to be described by nonlinear models, it is natural to extend the time-constrained switching stabilization theory of switched linear systems to switched nonlinear systems [11–13]. When a switched system is composed of unstable nonlinear subsystems, some promising ideas are not effective any more. Therefore, it will be very meaningful and challenging to carry out the studies on time-constrained switching stabilization of switched systems with possibly all unstable nonlinear subsystems.

Based on the above observations, in this chapter, the problems of time-constrained switching stabilization for switched systems composed of unstable subsystems are investigated in both linear and nonlinear cases.

Notations:

$\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and n-dimensional Euclidean space respectively; $\mathbb{I}_n = \{1, 2, \ldots, n\}$. For a given vector $\mathbf{x}$, the notation $\|\mathbf{x}\|$ refers to the Euclidean vector norm. For a given subspace $S \subseteq \mathbb{R}^n$, $\|A\|$ and $\|A\|_S$ represent the spectral norm of A and the spectral norm of A with restriction in S, respectively, and $\mathscr{C}(S)$ stands for the complement subspace. $\oplus$ denotes the direct sum. In addition, $\lambda(A)$ and $\delta(A)$ refer to the eigenvalues and singular values of A, and $Re\{\lambda(A)\}$ is the real part of $\lambda(A)$. $\mathscr{C}^1$ denotes the space of continuously differentiable functions, and a function α: $[0, \infty) \to [0, \infty)$ is said to be of class $\mathscr{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. Class $\mathscr{K}_\infty$ denotes the subset of $\mathscr{K}$ consisting of all those functions that are unbounded. A function β: $[0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of class $\mathscr{K}\mathscr{L}$ if $\beta(\cdot, t)$ is of class $\mathscr{K}$ for each fixed $t > 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$. The notation $P > 0 (\geq 0)$ means that P is a real symmetric and positive definite (semi-positive definite) matrix.

3.2 Switching Stabilization of Switched Linear Systems

3.2.1 Problem Formulation and Preliminaries

Consider the following switched linear systems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) \tag{3.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\sigma(t)$ is the switching signal to be designed, which takes its values in the finite set $\mathscr{I} = \{1, \ldots, \Bbbk\}$; $\Bbbk$ is the number of subsystems. Also, for a switching sequence $0 < t_1 < \cdots < t_i < t_{i+1} < \cdots$, $\sigma(t)$ is continuous from the right everywhere. Moreover, when $t \in [t_i, t_{i+1})$, $\sigma(t) = \sigma(t_i) = p \in \mathscr{I}$, and we say the p^{th} subsystem A_p of (3.1) is activated. In this chapter, we suppose that all the subsystems of (3.1) are unstable.

We first introduce the following definition and lemmas for later development.

Definition 3.1 ([14]) Suppose $A \in \mathbb{C}^{n\times n}$, and $S \subseteq \mathbb{C}^n$ is a subspace. S is A-invariant if $AS \subseteq S$, that is, $\forall v \in S \Rightarrow Av \in S$.

Lemma 3.1 ([14]) *For any subspaces S_1, S_2, $S_1 + S_2$ is also a subspace.*

Lemma 3.2 ([14]) *For any subspaces S_1, S_2, $S_1 \cap S_2$ is also a subspace.*

Next, the following exponential stability definition of system (3.1) is also recalled.

Definition 3.2 ([9]) The equilibrium $\mathbf{x} = \mathbf{0}$ of system (3.1) is globally uniformly exponentially stable (GUES) under certain switching signal $\sigma(t)$ if for initial conditions $\mathbf{x}(t_0)$, there exist constants $\eta_1 > 0$, $\eta_2 > 0$ such that the solution of the system satisfies $\|\mathbf{x}(t)\| \leq \rho_1 \mathrm{e}^{-\rho_2(t-t_0)} \|\mathbf{x}(t_0)\|$, $\forall t \geq t_0$.

In this chapter, we aim at designing a set of switching signals $\sigma(t)$ with the mode-dependent average dwell time (MDADT) property, such that the system (3.1) is GUES. For this purpose, let us now recall the definition of MDADT switching.

Definition 3.3 For a switching signal $\sigma(t)$ and any $T \geq t \geq 0$, let $N_{\sigma p}(T, t)$ be the switching numbers that the p^{th} subsystem is activated over the interval $[t, T]$ and $\mathscr{T}_p(T, t)$ denotes the total running time of the p^{th} subsystem over the interval $[t, T]$, $p \in S$. We say that $\sigma(t)$ has a mode-dependent average dwell time τ_{ap} if there exist positive numbers N_{0p} (we call N_{0p} the mode-dependent chatter bounds here) and τ_{ap} such that

$$N_{\sigma p}(T, t) \leq N_{0p} + \frac{\mathscr{T}_p(T, t)}{\tau_{ap}}, \quad \forall T \geq t \geq 0 \tag{3.2}$$

Remark 3.1 For simplicity, we mark $\sigma(t) \in \digamma_{MDADT}[N_{0p}, \tau_{ap}]$ in this chapter if $\sigma(t)$ is a class of the switching signals defined in Definition 3.2.

3.2.2 Main Results

In correspondence with each subsystem A_p, $p \in \mathscr{I}$, the whole state space can be divided into the two subspaces S_p^s and S_p^u which are defined below.

Definition 3.4 The stable subspace S_p^s, $p \in \mathscr{I}$, is spanned by the eigenvectors corresponding to the eigenvalues $\lambda_k(A_p)$, $k \in \mathbb{k}_p^s = \{m \in \mathbb{I}_n \mid Re(\lambda_m(A_p)) < 0, p \in \mathscr{I}\}$,

Definition 3.5 The unstable subspace S_p^u, $p \in \mathscr{I}$, is spanned by the eigenvectors corresponding to the eigenvalues $\lambda_k(A_p)$, $k \in \mathbb{k}_p^u = \{m \in \mathbb{I}_n \mid Re(\lambda_m(A_p)) \geq 0, p \in \mathscr{I}\}$,

Before providing our main results, the following lemmas are first developed for later use.

Lemma 3.3 *Consider the switched linear system (3.1). If S is A_p-invariant, $\forall p \in \mathscr{I}$, then, S is $e^{A_p t}$-invariant, $p \in \mathscr{I}$, $\forall t \geq 0$.*

Proof It is noted that, $\forall p \in \mathscr{I}, t \geq 0$,

$$e^{A_p t} = I + tA_p + \frac{t^2}{2!}A_p^2 + \cdots + \frac{t^n}{n!}A_p^n + \cdots \tag{3.3}$$

On the other hand, because S is A_p-invariant, $\forall p \in \mathscr{I}$, one has, $\forall \mathbf{x} \in S, n \in \mathbb{Z}^+$,

$$\begin{aligned} A_p^n\mathbf{x} &= A_p^{n-1}A_p\mathbf{x} \\ &= A_p^{n-1}\mathbf{x}_1, (\mathbf{x}_1 = A_p\mathbf{x} \in S) \\ &= A_p^{n-2}\mathbf{x}_2, (\mathbf{x}_2 = A_p\mathbf{x}_1 \in S) \\ &\cdots \\ &= A_p\mathbf{x}_n \in S, (\mathbf{x}_n = A_p\mathbf{x}_{n-1} \in S) \end{aligned} \tag{3.4}$$

Therefore, one can get from (3.3) and (3.4) that, $\forall p \in \mathscr{I}, t \geq 0, \mathbf{x} \in S$,

$$\begin{aligned} e^{A_p t}\mathbf{x} &= \mathbf{x} + tA_p\mathbf{x} + \frac{t^2}{2!}A_p^2\mathbf{x} + \cdots + \frac{t^n}{n!}A_p^n\mathbf{x} + \cdots \\ &\in S \end{aligned} \tag{3.5}$$

which completes the proof. □

Remark 3.2 Lemma 3.3 implies that, if the p^{th} subsystem of system (3.1) is activated with initial condition $\mathbf{x}(t_0) \in S$, the state will stay in S during the running time of the p^{th} operation mode; i.e., $\mathbf{x}(t) = e^{A_p(t-t_0)}\mathbf{x}(t_0) \in S$ if $\mathbf{x}(t_0) \in S$.

Lemma 3.4 *Consider the linear system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$. Let $\lambda^m = \{-\max_k\{\lambda_k(A)\} \mid Re(\lambda_k(A)) < 0, k \in \mathbb{I}_n\}$ and $\lambda^M = \{\max_k\{\lambda_k(A)\} \mid Re(\lambda_k(A)) \geq 0, k \in \mathbb{I}_n\}$; then, there exists a constant $\varepsilon > 0$ such that*

$$\|\exp\{At\}\|_{S^s} \leq \exp\{\varepsilon - \lambda^m t\} \tag{3.6}$$

$$\|\exp\{At\}\|_{S^u} \leq \exp\{\varepsilon + \lambda^M t\} \tag{3.7}$$

where S^s and S^u are the stable subspace and unstable subspace of A, respectively.

Proof It is obvious that both S^s and S^u are A-invariant, and thus are e^{At}-invariant. We can choose the following orthogonal matrix

$$T = [a_1, a_2, \ldots, a_r, b_{r+1}, b_{r+2}, \ldots, b_n] \tag{3.8}$$

appropriately, where $\{a_1, a_2, \ldots, a_r\}$ and $\{b_{r+1}, b_{r+2}, \ldots, b_n\}$ are the bases of S^s and S^u. Note that S^s and S^u are also the stable subspace and unstable subspace corresponding to e^A. Then, one has that

$$T^{-1} \exp\{At\}T = \exp\{diag\{A^s t, A^u t\}\} \tag{3.9}$$

where A^s, A^u are appropriate matrices satisfying $\lambda(A^s) < 0$ and $\lambda(A^u) \geq 0$, respectively. Therefore, it follows from (3.9) that

$$\begin{aligned} \|\exp\{At\}\|_{S^s} &\leq \|T\|_{S^s} \left\|T^{-1}\right\|_{S^s} \left\|\exp\{diag\{A^s t, A^u t\}\}\right\|_{S^s} \\ &= \|T\|_{S^s} \left\|T^{-1}\right\|_{S^s} \left\|\exp\{A^s t\}\right\| \\ &\leq \|T\|_{S^s} \left\|T^{-1}\right\|_{S^s} \exp\{-\lambda^m t\} \end{aligned} \tag{3.10}$$

$$\begin{aligned} \|\exp\{At\}\|_{S^s} &\leq \|T\|_{S^u} \left\|T^{-1}\right\|_{S^u} \left\|\exp\{diag\{A^s t, A^u t\}\}\right\|_{S^s} \\ &= \|T\|_{S^u} \left\|T^{-1}\right\|_{S^u} \left\|\exp\{diag\{A^u t\}\right\| \\ &\leq \|T\|_{S^u} \left\|T^{-1}\right\|_{S^u} \exp\{\lambda^M t\} \end{aligned} \tag{3.11}$$

Finally, set $\varepsilon = \ln(\frac{\varepsilon_1}{\varepsilon_2})$, $\varepsilon_1 = \max \delta(T)$ and $\varepsilon_2 = \min \delta(T)$. This together with (3.10) and (3.11) completes the proof. □

Subsequently, we define $\lambda_p^m = \{-\max_k\{\lambda_k(A_p)\} \mid Re(\lambda_k(A_p)) < 0, k \in \mathbb{I}_n, p \in \mathscr{I}\}$, and $\lambda_p^M = \{\max_k\{\lambda_k(A_p)\} \mid Re(\lambda_k(A_p)) \geq 0, k \in \mathbb{I}_n, p \in \mathscr{I}\}$ for switched system (1). Then, Lemma 3.4 can be trivially extended to the following result for switched system (3.1).

Lemma 3.5 *Consider the switched linear system (3.1). There exist some constants $\varepsilon_p > 0$, $p \in \mathscr{I}$, such that*

$$\left\|\exp\{A_p t\}\right\|_{S_p^s} \leq \exp\{\varepsilon_p - \lambda_p^m t\} \tag{3.12}$$

$$\left\|\exp\{A_p t\}\right\|_{S_p^u} \leq \exp\{\varepsilon_p + \lambda_p^M t\} \tag{3.13}$$

Theorem 3.1 *Consider the switched linear system (3.1). For given constants $\alpha_p > \lambda_p^M > 0$, $\lambda_p^m > \beta_p > 0$, $p \in \mathscr{I}$, and η_p, if there exist two sets $\mathscr{I}_1, \mathscr{I}_2 \subset \mathscr{I}$ ($\mathscr{I}_1 \cup \mathscr{I}_2 = \mathscr{I}$) such that $\Omega_1 = \sum_{p \in \mathscr{I}_1} S_p^u$ and $\Omega_2 = \cap_{p \in \mathscr{I}_1} S_p^s$ are A_p-invariant, $p \in \mathscr{I}$, and*

$$\Omega_1 \subseteq \cap_{p\in\mathscr{I}_2} S_p^s \tag{3.14}$$

then, the system ([3.1]) is GUES for any switching signal $\sigma(t) \in F_{MDADT}[N_{0p}, \tau_{ap}]$ *satisfying*

$$\tau_{ap} \geq \frac{\varepsilon_p}{\alpha_p - \lambda_p^M}, \forall p \in \mathscr{I} \tag{3.15}$$

$$\tau_{ap} \geq \frac{\varepsilon_p}{\lambda_p^m - \beta_p}, \forall p \in \mathscr{I} \tag{3.16}$$

$$\sum_{p\in\mathscr{I}_1} (\alpha_p \mathscr{T}_p(T,0) + \eta_p \mathscr{T}_p(T,0)) \leq \sum_{p\in\mathscr{I}_2} (\beta_p \mathscr{T}_p(T,0) - \eta_p \mathscr{T}_p(T,0)) \tag{3.17}$$

$$\sum_{p\in\mathscr{I}_2} (\alpha_p \mathscr{T}_p(T,0) + \eta_p \mathscr{T}_p(T,0)) \leq \sum_{p\in\mathscr{I}_1} (\beta_p \mathscr{T}_p(T,0) - \eta_p \mathscr{T}_p(T,0)) \tag{3.18}$$

Proof By Lemmas 3.1 and 3.2, it is obvious that Ω_1 and Ω_2 are two subspaces in $\mathbb{R}^n$, and it is also clear from the definitions of Ω_1 and Ω_2 that,

$$\Omega_1 \cap \Omega_2 = \varnothing \tag{3.19}$$

It is also true that,

$$\mathscr{C}(\Omega_2) = \mathscr{C}(\cap_{p\in\mathscr{I}_1} S_p^s) = \sum_{p\in\mathscr{I}_1} \mathscr{C}(S_p^s) = \sum_{p\in\mathscr{I}_1} S_p^u = \Omega_1 \tag{3.20}$$

which implies

$$\Omega_1 \oplus \Omega_2 = \mathbb{R}^n \tag{3.21}$$

Next, for any sufficiently large $T > 0$, let $t_0 = 0$ and $t_1, t_2 \ldots t_i, t_{i+1}, \ldots t_{N_\sigma(T,0)}$ denote the switching times on the interval $[0, T]$, where $N_\sigma(T, 0) = \sum_{p=1}^{\Bbbk} N_{\sigma p}(T, 0)$. Then, when the initial condition $\mathbf{x}(0) \in \Omega_1$, it yields from Lemma 3.3 that, $\forall T > 0$,

$$\begin{aligned}\mathbf{x}(T) &= \exp\{A_{\sigma(t_{N_\sigma(T,0)})}(T - t_{N_\sigma(T,\,0)})\} \cdots \exp\{A_{\sigma(t_i)}(t_{i+1} - t_i)\} \cdots \\ &\quad \exp\{A_{\sigma(t_0)}(t_1 - t_0)\}\mathbf{x}(0) \\ &\in \Omega_1\end{aligned} \tag{3.22}$$

Therefore, by (3.14), (3.22), Definition 3.3 and Lemma 3.5, it arrives at, $\forall T > 0$,

$$
\begin{aligned}
\|\mathbf{x}(T)\| &\leq \prod_{s\in\Phi_1} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_1} \prod_{s\in\Phi_2} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_1} \|\mathbf{x}(0)\| \\
&\leq \prod_{s\in\Phi_1} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_1} \prod_{s\in\Phi_2} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{S_p^s} \|x(0)\| \\
&\leq \prod_{p\in\mathscr{I}_1} \exp\{N_{\sigma p}(T,0)\varepsilon_p\}\exp\{\lambda_p^M \mathscr{T}_p(T,0)\} \\
&\quad \prod_{p\in\mathscr{I}_2} \exp\{N_{\sigma p}(T,0)\varepsilon_p\}\exp\{-\lambda_p^m \mathscr{T}_p(T,0)\}\, \|\mathbf{x}(0)\| \\
&= \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{\sigma p}(T,0)\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}_1} \lambda_p^M \mathscr{T}_p(T,0)\right. \\
&\quad \left. -\sum\nolimits_{p\in\mathscr{I}_2} \lambda_p^m \mathscr{T}_p(T,0)\right\} \|x(0)\| \\
&\leq \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}_1} \lambda_p^M \mathscr{T}_p(T,0) - \sum\nolimits_{p\in\mathscr{I}_2} \lambda_p^m \mathscr{T}_p(T,0)\right. \\
&\quad \left. +\sum\nolimits_{p\in\mathscr{I}} \frac{\varepsilon_p \mathscr{T}_p(T,0)}{\tau_{ap}}\right\} \|\mathbf{x}(0)\| \qquad (3.23) \\
&= \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}_1} (\lambda_p^M + \frac{\varepsilon_p}{\tau_{ap}}) \mathscr{T}_p(T,0)\right. \\
&\quad \left. -\sum\nolimits_{p\in\mathscr{I}_2} (\lambda_p^m - \frac{\varepsilon_p}{\tau_{ap}}) \mathscr{T}_p(T,0)\right\} \|\mathbf{x}(0)\| \qquad (3.24)
\end{aligned}
$$

where Φ_1 and Φ_2 denote the sets of s satisfying $\sigma(t_s) \in \mathscr{I}_1$ and $\mathscr{I}_2$, respectively. Therefore, if we specify

$$\tau_{ap} \geq \frac{\varepsilon_p}{\alpha_p - \lambda_p^M},\ p \in \mathscr{I}_1 \qquad (3.25)$$

$$\tau_{ap} \geq \frac{\varepsilon_p}{\lambda_p^m - \beta_p},\ p \in \mathscr{I}_2 \qquad (3.26)$$

then, it is clear from (3.17) and (3.23) that

$$
\begin{aligned}
&\|\mathbf{x}(T)\| \\
&\leq \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}_1} \alpha_p \mathscr{T}_p(T,0) - \sum\nolimits_{p\in\mathscr{I}_2} \beta_p \mathscr{T}_p(T,0)\right\} \|\mathbf{x}(0)\| \\
&\leq \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}} -\eta_p \mathscr{T}_p(T,0)\right\} \|\mathbf{x}(0)\| \\
&\leq \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{-\min_{p\in\mathscr{I}}\{\eta_p\}T\right\} \|\mathbf{x}(0)\| \qquad (3.27)
\end{aligned}
$$

which means that the system is GUES under MDADT satisfying (3.18), (3.24) and (3.25).

On the other hand, when the initial condition $\mathbf{x}(0) \in \Omega_2$, it is true that $\mathbf{x}(T) \in \Omega_2, \forall T > 0$, and

$$
\begin{aligned}
\|\mathbf{x}(T)\| &\leq \prod_{s\in\Phi_1} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_2} \prod_{s\in\Phi_2} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_2} \|\mathbf{x}(0)\| \\
&\leq \prod_{s\in\Phi_1} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{S_p^s} \prod_{s\in\Phi_2} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_2} \|\mathbf{x}(0)\| \\
&\leq \prod_{p\in\mathscr{I}_1} \exp\{N_{\sigma p}(T,0)\varepsilon_p\}\exp\{-\lambda_p^m \mathscr{T}_p(T,0)\} \\
&\quad \prod_{p\in\mathscr{I}_2} \exp\{N_{\sigma p}(T,0)\varepsilon_p\}\exp\{\lambda_p^M \mathscr{T}_p(T,0)\}\,\|\mathbf{x}(0)\| \\
&= \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{\sigma p}(T,0)\varepsilon_p\right\}\exp\left\{-\sum\nolimits_{p\in\mathscr{I}_1}\lambda_p^m\mathscr{T}_p(T,0)\right. \\
&\quad \left. +\sum\nolimits_{p\in\mathscr{I}_2}\lambda_p^M\mathscr{T}_p(T,0)\right\}\|\mathbf{x}(0)\| \\
&\leq \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{-\sum\nolimits_{p\in\mathscr{I}_1}\lambda_p^m\mathscr{T}_p(T,0)\right. \\
&\quad \left. +\sum\nolimits_{p\in\mathscr{I}_2}\lambda_p^M\mathscr{T}_p(T,0)+\sum\nolimits_{p\in\mathscr{I}}\frac{\varepsilon_p\mathscr{T}_p(T,0)}{\tau_{ap}}\right\}\|\mathbf{x}(0)\| \\
&= \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{-\sum\nolimits_{p\in\mathscr{I}_1}\left(\lambda_p^m-\frac{\varepsilon_p}{\tau_{ap}}\right)\mathscr{T}_p(T,0)\right. \\
&\quad \left. +\sum\nolimits_{p\in\mathscr{I}_2}\left(\lambda_p^M+\frac{\varepsilon_p}{\tau_{ap}}\right)\mathscr{T}_p(T,0)\right\}\|\mathbf{x}(0)\|
\end{aligned} \tag{3.28}
$$

Similarly, if we choose

$$\tau_{ap} \geq \frac{\varepsilon_p}{\alpha_p-\lambda_p^M},\ p\in\mathscr{I}_2 \tag{3.29}$$

$$\tau_{ap} \geq \frac{\varepsilon_p}{\lambda_p^m-\beta_p},\ p\in\mathscr{I}_1 \tag{3.30}$$

then, it is immediate from (3.18) and (3.27) that

$$
\begin{aligned}
&\|\mathbf{x}(T)\| \\
&\leq \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{-\sum\nolimits_{p\in\mathscr{I}_1}\beta_p\mathscr{T}_p(T,0)+\sum\nolimits_{p\in\mathscr{I}_2}\alpha_p\mathscr{T}_p(T,0)\right\}\|\mathbf{x}(0)\| \\
&\leq \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{\sum\nolimits_{p\in\mathscr{I}} -\eta_p\mathscr{T}_p(T,0)\right\}\|\mathbf{x}(0)\| \\
&\leq \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{-\min_{p\in\mathscr{I}}\{\eta_p\}T\right\}\|\mathbf{x}(0)\|
\end{aligned} \tag{3.31}
$$

Thus, the system is GUES with MDADT satisfying (3.18), (3.28) and (3.29).

Now, we consider the case that the initial condition $\mathbf{x}(0)\in\Omega_3=\overline{\Omega_1\cup\Omega_2}$. By (3.21), for any $\mathbf{x}(0)\in\Omega_3$, one can always find

$$\bar{\mathbf{x}}(0)\in\Omega_1 \tag{3.32}$$

and

$$\tilde{\mathbf{x}}(0) \in \Omega_2 \tag{3.33}$$

such that

$$\mathbf{x}(0) = \bar{\mathbf{x}}(0) + \tilde{\mathbf{x}}(0) \tag{3.34}$$

It yields from (3.33) that

$$\begin{aligned} \mathbf{x}(T) &= \exp\{A_{\sigma(t_{N_\sigma(T,0)})}(T - t_{N_\sigma(T,\,0)})\} \cdots \exp\{A_{\sigma(t_i)}(t_{i+1} - t_i)\} \cdots \\ &\quad \exp\{A_{\sigma(t_0)}(t_1 - t_0)\}\mathbf{x}(0) \\ &= \exp\{A_{\sigma(t_{N_\sigma(T,0)})}(T - t_{N_\sigma(T,\,0)})\} \cdots \exp\{A_{\sigma(t_0)}(t_1 - t_0)\}\bar{\mathbf{x}}(0) \\ &\quad + \exp\{A_{\sigma(t_{N_\sigma(T,0)})}(T - t_{N_\sigma(T,\,0)})\} \cdots \exp\{A_{\sigma(t_0)}(t_1 - t_0)\}\tilde{\mathbf{x}}(0) \\ &= \bar{\mathbf{x}}(T) + \tilde{\mathbf{x}}(T) \end{aligned} \tag{3.35}$$

where $\bar{\mathbf{x}}(T) \in \Omega_1$ and $\tilde{\mathbf{x}}(T) \in \Omega_2$ are the state responses of initial conditions $\bar{\mathbf{x}}(0)$ and $\tilde{\mathbf{x}}(0)$, respectively. It then follows from (3.26) and (3.30) that the underlying system is stabilized by MDADT satisfying (3.17)–(3.18), (3.24)–(3.25) and (3.28)–(3.29).

Finally, we can conclude from (3.24)–(3.26), (3.27)–(3.29) and (3.34) that if (3.14) holds, the switched system (3.1) is GUES under MDADT meeting (3.15)–(3.18), which completes the proof. □

Remark 3.3 It is noted from the proof of Theorem 3.1 that switched system (3.1) is stabilized via the designed MDADT switching, and the decay rate of the state can be set in advance via a scalar $\eta = \min_{p \in \mathscr{I}}\{\eta_p\}$.

As a special case, if all the subsystems of switched system (3.1) are Hurwitz stable, then the sufficient condition for stabilization via MDADT switching is addressed in the following corollary.

Corollary 3.1 *Consider the switched linear system (3.1) composed of all Hurwitz stable subsystems. The system is GUES for any switching signal* $\sigma(t) \in \digamma_{MDADT}[N_{0p}, \tau_{ap}]$ *satisfying*

$$\tau_{ap} \geq \frac{\varepsilon_p}{\lambda_p^m},\ p \in \mathscr{I} \tag{3.36}$$

Proof Note the fact that all the subsystems are stable. Therefore, in Theorem 3.1, $\mathscr{I}_1 = \varnothing$, $\mathscr{I}_2 = \mathscr{I}$, $\Omega_1 = \varnothing$, and $\Omega_2 = \mathbb{R}^n$. Then, $\forall T > 0$,

$$\begin{aligned}\|\mathbf{x}(T)\| &= \left\|\exp\left\{\sum_{s\in\Phi_1\cup\Phi_2} A_{\sigma(t_s)}(t_{s+1}-t_s)\right\}\right\| \|\mathbf{x}(0)\| \\ &\le \prod_{p\in\mathscr{I}} \exp\{N_{\sigma p}(T,0)\varepsilon_p - \lambda_p^m \mathscr{T}_p(T,0)\} \|\mathbf{x}(0)\| \\ &\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}} \left(\frac{\varepsilon_p}{\tau_{ap}} - \lambda_p^m\right)\mathscr{T}_p(T,0)\right\} \|\mathbf{x}(0)\| \\ &\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\max_{p\in\mathscr{I}} \left(\frac{\varepsilon_p}{\tau_{ap}} - \lambda_p^m\right) T\right\} \|\mathbf{x}(0)\| \end{aligned} \tag{3.37}$$

Thus, we can see from Definition 3.2 and (3.36) that the underlying system is exponentially stabilized via MDADT satisfying (3.35). □

Remark 3.4 The above theorem and corollary provide sufficient conditions of switching stabilization for switched system (3.1) comprising all unstable subsystems and all stable subsystems, respectively. An example in the next section will show the validity of the obtained criteria.

3.2.3 Simulation Results

In this section, a numerical example of switched linear systems with all unstable subsystems is presented to show the effectiveness of the developed approaches.

Example 3.1 Consider the switched linear systems consisting of three subsystems described by:

$$A_1 = \begin{bmatrix} -20 & -12.5 & -12.5 \\ 0 & -7.5 & 12.5 \\ 0 & 12.5 & -7.5 \end{bmatrix}, A_2 = \begin{bmatrix} -7.5 & 15 & -2.5 \\ 17.5 & -5 & 2.5 \\ -17.5 & -15 & -22.5 \end{bmatrix}, A_3 = \begin{bmatrix} -7.5 & -12.5 & 0 \\ -12.5 & -7.5 & 0 \\ 12.5 & 12.5 & 5 \end{bmatrix}.$$

First, the state responses of each subsystem with the same initial condition $\mathbf{x}(0) = [\,5\ -5\ 10\,]^T$ are depicted in Fig. 3.1 from which it is seen that all the three subsystems are unstable. Furthermore, the simulation results with four random switching signals are given in Fig. 3.2 which shows that the above switched system is unstable under these switching signals.

Then, our purpose here is to design a set of mode-dependent average dwell time switching to exponentially stabilize the above switched systems. It is clear that $\lambda(A_1) = \{5, -20, -20\}$, $\lambda(A_2) = \{-20, -25, 10\}$, $\lambda(A_3) = \{5, 5, -20\}$, $\lambda_1^M = 5$, $\lambda_2^M = 10$, $\lambda_3^M = 5$, $\lambda_1^m = 20$, $\lambda_2^m = 20$, $\lambda_3^m = 20$. We choose $\mathscr{I}_1 = \{1, 3\}$, $\mathscr{I}_2 = \{2\}$. Therefore,

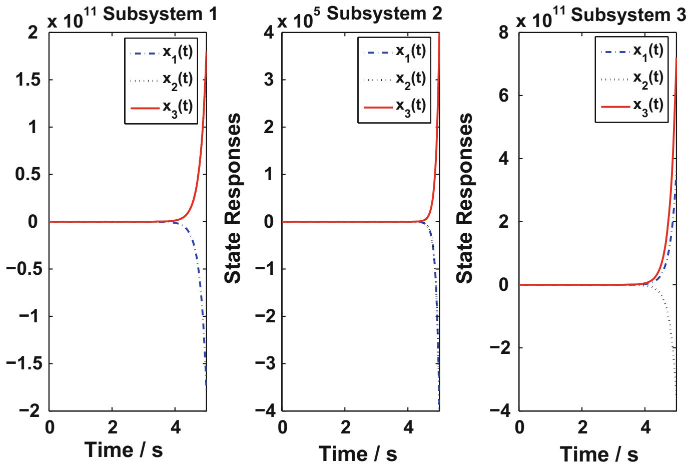

Fig. 3.1 The state responses of each subsystem

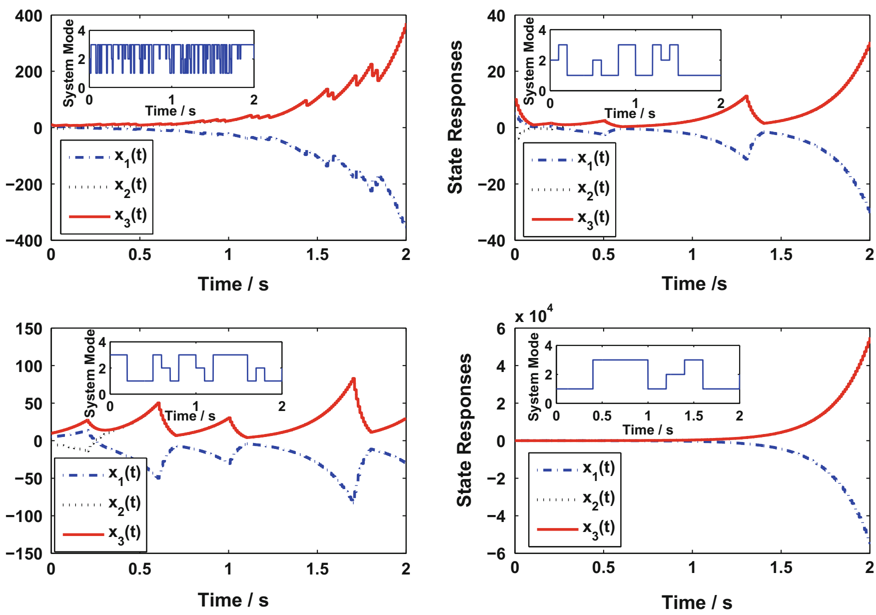

Fig. 3.2 The state responses of the system with different random switching signals

$$\Omega_1 = span\left\{\begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \end{bmatrix}\right\}, \Omega_2 = span\left\{\begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}\right\},$$

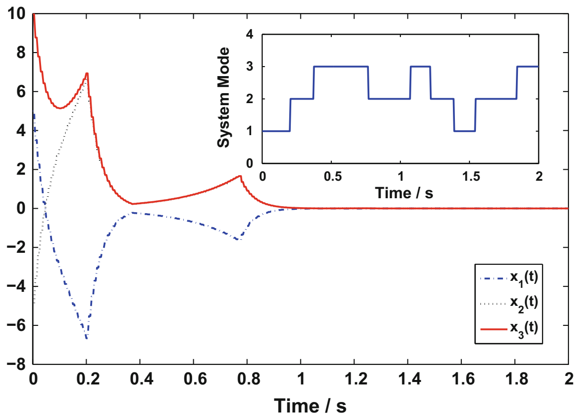

Fig. 3.3 The state responses of the system under the designed MDADT switching signals

On the other hand, it is not hard to get that Ω_1 and Ω_2 are A_p-invariant, $p \in \{1, 2, 3\}$, and satisfy the condition (3.14).

Set $\eta_p = 0.1$, $\varepsilon_p = 0.69$, $p = \{1, 2, 3\}$, $\alpha_1 = \alpha_3 = 10$, $\alpha_2 = 14$, $\beta_1 = \beta_3 = 15$, $\beta_2 = 16$. Based on Theorem 3.1, one can get a MDADT switching signal satisfying (3.17), (3.18) and

$$\tau_{a1} \geq 0.14, \tau_{a2} \geq 0.17, \tau_{a3} \geq 0.14 \tag{3.38}$$

To illustrate the correctness of the theoretical results, we now generate one possible switching sequences with the MDADT property (3.37). Then, one can obtain the corresponding state responses of the system as shown in Fig. 3.3, for the same initial state condition. It can be concluded from the curves that the underlying system is stabilized by the designed MDADT switching signal.

Finally, from the above demonstrations, we obtain that Theorem 3.1 provides an effective stabilization approach via MDADT switching for switched linear systems composed of unstable subsystems.

3.2.4 Conclusions

This section is concerned with switching stabilization for switched linear systems consisting of unstable modes. Based on the invariant subspace theory, the advanced mode-dependent average dwell time (MDADT) switching, is introduced to stabilize the systems under consideration. Then, the corresponding result is extended to switched systems composed of all Hurwitz stable subsystems. Finally, a numerical example is provided to demonstrate the correctness and effectiveness of the obtained results.

3.3 Switching Stabilization of Switched Nonlinear Systems

3.3.1 Problem Formulation and Preliminaries

This section presents some definitions and preliminary results that will be used throughout the remainder of this chapter. Consider the following switched nonlinear systems,

$$\dot{\mathbf{x}}(t) = \sum_{p=1}^{m} \delta_p(\sigma(t)) f_p(\mathbf{x}(t), t), \mathbf{x}(t_0) = \mathbf{x}_0, t \geq t_0, \tag{3.39}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, and $\mathbf{x}_0$ and $t_0 \geq 0$ denote the initial state and initial time, respectively; $\sigma(t)$ is a switching signal which is a piecewise constant function from the right of time and takes its values in the finite set $S = \{1, \ldots, m\}$, where $m > 1$ is the number of subsystems. $f_p : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ are smooth functions for any $\sigma(t) = p \in S$. Moreover, all the subsystems in system (3.39) may be unstable.

For a switching sequence, $0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $\sigma(t)$ may be either autonomous or controlled. When $t \in [t_k, t_{k+1})$, we say $\sigma(t_k)^{th}$ mode is active; i.e., the indication functions $\delta_p(\sigma(t))$ satisfy:

$$\delta_p(\sigma(t)) = \begin{cases} 1, & if\ \sigma(t) = p, \\ 0, & otherwise. \end{cases} \tag{3.40}$$

The switched nonlinear system (3.39) can be described by fuzzy systems, and the p^{th} fuzzy subsystem is represented as follows.

Model rule R_p^i : IF $\theta_1(t)$ is M_{p1}^i and $\cdots$ and $\theta_l(t)$ is M_{pl}^i, THEN

$$\dot{\mathbf{x}}(t) = A_{pi}\mathbf{x}(t),\ t \geq t_0,\ i \in R = \{1, 2, \cdots, r\}, p \in S, \tag{3.41}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector; M_{pj}^i $(j = 1, 2, \ldots, l)$ is the fuzzy set, and r is the number of IF-THEN rules; $\theta_1(t), \theta_2(t) \cdots \theta_p(t)$ are the premise variables; Furthermore, $A_{pi}, i \in R,\ p \in S$ is a real matrix with appropriate dimensions. Thus, through fuzzy blending, the global model of the p^{th} subsystem can be given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(h(t))\mathbf{x}(t) \\ &= \sum_{i=1}^{r} h_{pi}(\theta(t)) A_{pi}\mathbf{x}(t),\ p \in S. \end{aligned} \tag{3.42}$$

$h_{pi}(\theta(t))$ are the normalized membership functions satisfying:

$$h_{pi}(\theta(t)) = \frac{\prod_{j=1}^{l} M_{pj}^i(\theta_j(t))}{\sum_{i=1}^{r} \prod_{j=1}^{l} M_{pj}^i(\theta_j(t))} \geq 0,\ \sum_{i=1}^{r} h_{pi}(\theta(t)) = 1, \tag{3.43}$$

where $M^i_{pj}(\theta_j(t))$ represent the grade of the membership function of premise variable $\theta_j(t)$ in M^i_{pj}. Finally, we can describe switched nonlinear system (3.39) in the following form,

$$\dot{\mathbf{x}}(t) = \sum_{p=1}^{m}\sum_{i=1}^{r}\delta_p(\sigma(t))h_{pi}(\theta(t))A_{pi}\mathbf{x}(t). \tag{3.44}$$

Next, we introduce the following definition for later use.

Definition 3.6 [15] The equilibrium $x = 0$ of switched system (3.39) is globally asymptotically stable (GAS) under a certain switching signal $\sigma(t)$ if there exists a $\mathscr{K}\mathscr{L}$ function β such that the solution of the system satisfies the inequality $\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(t_0)\|, t),\ \forall t \geq t_0$, with any initial conditions $x(t_0)$.

In the following, our goal is to find a set of switching signals with the ADT property, such that the switched system (3.39) is GAS. For this purpose, we first define a new class of ADT switching signals.

Definition 3.7 For a switching signal $\sigma(t)$ and each $T \geq t \geq 0$, let $N_\sigma(T, t)$ denote the number of discontinuities of $\sigma(t)$ in the interval (t, T). We say that $\sigma(t)$ has an average dwell time τ_a if there exist two positive numbers N_0 (we call N_0 the chatter bound here) and τ_a such that

$$N_\sigma(T, t) \geq N_0 + \frac{T - t}{\tau_a}, \quad \forall T \geq t \geq 0. \tag{3.45}$$

3.3.2 Main Results

In this section, we consider the switching stabilization for switched nonlinear systems described in the previous section. Next, we are in a position to provide the first switching stabilization condition for switched nonlinear systems (3.39) in the following theorem by designing ADT switching signals defined in Definition 3.7.

Theorem 3.2 *Consider switched nonlinear system (3.39). Suppose that there exist a switching sequence* $\xi = \{t_0, t_1, \ldots t_k, \ldots t_{N_{\sigma(t)}}\}$ *satisfying (3.45), a set of* C^1 *non-negative functions* $V_p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $p \in S$, *two class* K_∞ *functions* α_1 *and* α_2, *and two positive numbers* $\lambda > 0$ *and* $0 < \mu < 1$ *such that*

$$\alpha_1(\|\mathbf{x}(t)\|) \leq V_p(\mathbf{x}(t), t) \leq \alpha_2(\|\mathbf{x}(t)\|), \forall p \in S, \tag{3.46}$$

$$\dot{V}_p(\mathbf{x}(t), t)) \leq \lambda V_p(\mathbf{x}(t), t), \forall p \in S, \tag{3.47}$$

$$V_q(\mathbf{x}(t_k^+), t_k^+) \leq \mu V_p(\mathbf{x}(t_k^-), t_k^-), \quad \forall p, q \in S \tag{3.48}$$

$$\tau_a \leq \frac{-ln\mu}{\lambda}. \tag{3.49}$$

Then switched system (3.39) is globally asymptotically stable under the switching sequence ξ generated by $\sigma(t)$.

Proof Without loss of generality, we denote $\xi = \{t_0, t_1, \ldots t_k, \ldots t_{N_{\sigma(t)}}\}$ as the switching sequence on time interval $[0, T]$ for any $T > 0,\ t_0 = 0$.

Next, we establish a multiple Lyapunov function (MLF) for switched nonlinear system (3.39) as follows,

$$V(\mathbf{x}(t), t) = \sum_{p=1}^{m} \delta_p(\sigma(t)) V_p(\mathbf{x}(t), t). \tag{3.50}$$

Then we consider the function

$$W(t) = e^{-\lambda t} \sum_{p=1}^{m} \delta_p(\sigma(t)) V_p(\mathbf{x}(t), t). \tag{3.51}$$

It is clear that it is piecewise differentiable along solutions of (3.39). When $t \in [t_k, t_{k+1})$, we get from (3.47) that

$$\begin{aligned} \dot{W}(t) &= -\lambda e^{-\lambda t} V_p(\mathbf{x}(t), t) + e^{-\lambda t} \dot{V}_p(\mathbf{x}(t), t) \\ &\le -\lambda e^{-\lambda t} V_p(\mathbf{x}(t), t) + e^{-\lambda t} \lambda V_p(\mathbf{x}(t), t) \\ &= 0. \end{aligned} \tag{3.52}$$

Thus $W(t)$ is nonincreasing when $t \in [t_k, t_{k+1})$. This together with (3.48) gives that

$$\begin{aligned} W(t_{k+1}^+) &= e^{-\lambda t_{k+1}^+} V_p(\mathbf{x}(t_{k+1}^+), t_{k+1}^+) \\ &\le \mu e^{-\lambda t_{k+1}^-} V_p(x(t_{k+1}^-), t_{k+1}^-) \\ &= \mu W(t_{k+1}^-) \\ &\le \mu W(t_k). \end{aligned} \tag{3.53}$$

By integrating this for $t \in [t_k, t_{k+1})$, it yields that

$$\begin{aligned} W(T^-) &\le W(t_{N_\delta}) \\ &\le \mu W(t_{N_\delta}^-) \\ &\le \mu W(t_{N_\delta - 1}) \\ &\quad \ldots \\ &\le \mu^{N_\delta} W(t_0). \end{aligned} \tag{3.54}$$

One can easily obtain from the definition of $W(t)$ that

$$e^{-\lambda T} V_{\delta(T^-)}(\mathbf{x}(T), T) \le \mu^{N_\delta} V_{\delta(t_0)}(\mathbf{x}(t_0), t_0). \tag{3.55}$$

Moreover, it can be derived from (3.45) and (3.55) that

$$\begin{aligned} V_{\delta(T^-)}(\mathbf{x}(T), T) &\le e^{\lambda T} e^{N_\delta \ln \mu} V_{\delta(t_0)}(\mathbf{x}(t_0), t_0) \\ &\le e^{\lambda T} e^{(N_0 + \frac{T}{\tau_a}) \ln \mu} V_{\delta(t_0)}(\mathbf{x}(t_0), t_0) \\ &= e^{N_0 \ln \mu} e^{(\lambda + \frac{\ln \mu}{\tau_a}) T} V_{\delta(t_0)}(\mathbf{x}(t_0), t_0). \end{aligned} \tag{3.56}$$

Finally, we can conclude from (3.56) that, if τ_a satisfies the condition in (3.49), then $V_{\delta(T^-)}(\mathbf{x}(T), T)$ exponentially converges to zero as $T \to \infty$,

By (3.46), we can get that

$$\|\mathbf{x}(T)\| = \alpha_1^{-1}(\mu^{N_0} e^{\lambda T} \alpha_2(\|\mathbf{x}_0\|),$$

which verifies the global asymptotic stability by Definition 3.6. Therefore, switched nonlinear system (3.39) is asymptotically stabilized by our proposed ADT switching signals (3.45) with (3.49) if the conditions (3.46)–(3.48) hold. This completes the proof.

In the following, we utilize the T-S fuzzy modeling approach to represent nonlinear system (3.39), to develop more applicable results.

Note that the traditional linear multiple quadratic Lyapunov function $V_p(\mathbf{x}(t)) = \mathbf{x}^T(t) P_p \mathbf{x}(t)$, where $P_p > 0$, $\forall p \in S$, will not satisfy the condition $P_q \le \mu P_p \, \forall p, q \in S$ because $0 < \mu < 1$. Hence, we choose a time-variant (TV) positive definite matrix $P_p(t)$ to construct a TV-MQLF for switched T-S fuzzy system (3.44) as follows,

$$V_p(\mathbf{x}(t), t) = \mathbf{x}^T(t) P_p(t) \mathbf{x}(t), \ \forall p \in S. \tag{3.57}$$

Then it is immediately clear that $V_q(\mathbf{x}(t_k^+), t_k^+) \le \mu V_p(\mathbf{x}(t_k^-), t_k^-), \ \forall p, q \in S$ can be expressed by $P_q(t_k^+) \le \mu P_p(t_k^-), \ p \ne q, \ \forall p, q \in S$. Next, we resort to the discretized Lyapunov function technique to numerically check the existence of such a matrix function $P_p(t)$ which is, however, difficult to be checked in the continuous case.

First of all, giving τ_a a sufficient small lower bound $\tau^* > 0$, we divide the interval $[t_k, t_k + \tau^*)$ into K segments. The length of each section is equal to $l = \frac{\tau^*}{K}$, and then the interval $[t_k, t_k + \tau^*)$ can be described as $G_{p,n} = [t_k + H_n, t_k + H_{n+1})$, $H_n = nl, \ n = 1, 2, \ldots, K-1$, Next, we use a linear interpolation formula to describe the continuous-time matrix function $P_p(t)$ which is chosen to be linear within each segment $G_{p,n} = [t_k + H_n, t_k + H_{n+1}), \ n = 1, 2, \ldots, K-1$. When $t \in G_{p,n}, \ n = 1, 2, \ldots, K-1$

$$\begin{aligned}
P_p(t) &= \frac{t - t_k - H_{n+1}}{t_k + H_n - t_k - H_{n+1}} P_{p,n} + \frac{t - t_k - H_n}{t_k + H_{n+1} - t_k - H_n} P_{p,n+1} \\
&= \frac{t - t_k - H_{n+1}}{-l} P_{p,n} + \frac{t - t_k - H_n}{l} P_{p,n+1} \\
&= (1-\gamma) P_{p,n} + \gamma P_{p,n+1} \\
&= P_p^{(n)}(\gamma), \qquad (3.58)
\end{aligned}$$

where $P_{p,n} = P_p(t_k + H_n)$, $P_{p,n+1} = P_p(t_k + H_{n+1})$, $0 < \gamma = \frac{t - t_k - H_n}{l} < 1$. In the interval $[t_k, t_k + \tau^*)$, the continuous-time matrix function $P_p(t),\ p \in S$, is determined by $P_{p,n}\ n = 1, 2, \ldots, K,\ p \in S$. On the other hand, in the interval $[t_k + \tau^*, t_{k+1})$, the matrix function $P_p(t),\ p \in S$ is fixed by a constant matrix $P_p(t) = P_{p,K},\ p \in S$. Thus, the TV-MQLF for switched T-S fuzzy system (3.44) for mode $p \in S$ can be described as

$$V_p(\mathbf{x}(t), t) = \begin{cases} \mathbf{x}^T(t) P_p^{(n)} \mathbf{x}(t), & t \in G_{p,n},\ n = 1, 2, \ldots, K-1 \\ \mathbf{x}^T(t) P_{p,K} \mathbf{x}(t), & t \in [t_k + \tau^*, t_{k+1}). \end{cases} \qquad (3.59)$$

Moreover, it can be derived from (3.59) that for any $t \in G_{p,n},\ n = 1, 2, \ldots, K-1$

$$\begin{aligned}
\dot{V}_p(\mathbf{x}(t), t) &= \dot{\mathbf{x}}^T(t) P_p(t) \mathbf{x}(t) + \mathbf{x}^T(t) \dot{P}_p(t) \mathbf{x}(t) + \mathbf{x}^T(t) P_{pi}(t) \dot{\mathbf{x}}(t) \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t)) [(A_{pi}\mathbf{x}(t))^T P_p(t) \mathbf{x}(t) + \mathbf{x}^T(t) \dot{P}_p(t) \mathbf{x}(t) + \mathbf{x}^T(t) P_p(t) A_{pi} \mathbf{x}(t)] \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t)) \mathbf{x}^T(t) [A_{pi}{}^T P_p(t) + P_p(t) A_{pi} + \dot{P}_p(t)] \mathbf{x}(t). \qquad (3.60)
\end{aligned}$$

When $t \in G_{p,n},\ n = 1, 2, \ldots, K-1$, one can immediately get from (3.58) that

$$\begin{aligned}
\dot{P}_p(t) &= -\dot{\gamma} P_{p,n} + \dot{\gamma} P_{p,n+1} \\
&= (P_{p,n+1} - P_{p,n}) \frac{K}{\tau^*} \\
&= \Pi_p^n. \qquad (3.61)
\end{aligned}$$

In the sequel, we can obtain from (3.58), (3.60) and (3.61) that for any $t \in G_{p,n},\ n = 1, 2, \ldots, K-1$,

$$\begin{aligned}
\dot{V}_p(\mathbf{x}(t), t) &= \sum_{i=1}^{r} h_{pi}(\theta(t)) \mathbf{x}^T(t) [A_{pi}^T P_p^{(n)} + P_p^{(n)} A_{pi} + \Pi_{pi}^n] \mathbf{x}(t) \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t)) \mathbf{x}^T(t) [(1-\gamma)(A_{pi}^T P_{p,n} + P_{p,n} A_{pi} + \Pi_{pi}^n) \\
&\quad + \gamma (A_{pi}^T P_{p,n+1} + P_{p,n+1} A_{pi} + \Pi_{pi}^n)] \mathbf{x}(t)
\end{aligned}$$

$$= \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)\Phi_{pi,1}^{(n)} + \gamma\Phi_{pi,2}^{(n)}]\mathbf{x}(t), \tag{3.62}$$

where $\Phi_{pi,1}^{(n)} = A_{pi}^T P_{p,n} + P_{p,n}A_{pi} + \Pi_p^n$ and $\Phi_{pi,2}^{(n)} = A_{pi}^T P_{p,n+1} + P_{p,n+1}A_{pi} + \Pi_p^n$.

Thus, a switching stabilization condition for switched T-S fuzzy system (3.44) can be obtained on the basis of the above developments.

Theorem 3.3 *Consider switched T-S fuzzy system (3.44), and let* $\lambda > 0$, $0 < \mu < 1$, *and* $\tau^* > 0$ *be given constants. If there exists a set of matrices* $P_{p,n} > 0$, $n = 0, 1, 2, \ldots, K$, $p \in S$, *such that* $\forall n = 0, 1, 2, \ldots, K, \forall i \in R$, $p \neq q$, $\forall (p \times q) \in S \times S$,

$$\Phi_{pi,1}^{(n)} - \lambda P_{p,n} < 0, \tag{3.63}$$

$$\Phi_{pi,2}^{(n)} - \lambda P_{p,n+1} < 0, \tag{3.64}$$

$$A_{pi}^T P_{p,K} + P_{p,K}A_{pi} - \lambda P_{p,K} < 0, \tag{3.65}$$

$$P_{q,0} - \mu P_{p,K} \leq 0, \tag{3.66}$$

where $\Phi_{pi,1}^{(n)}$ *and* $\Phi_{pi,2}^{(n)}$ *are defined in (3.62), then, the system is GAS for any switching signal with ADT satisfying*

$$\tau^* \leq \tau_a \leq \frac{-ln\mu}{\lambda}. \tag{3.67}$$

Proof. When $t \in G_{p,n}$, $n = 1, 2, \ldots, K-1$, by the discussions in (3.62), it can be seen that if (3.63) and (3.64) hold, then,

$$\begin{aligned}
&\dot{V}_p(\mathbf{x}(t), t) - \lambda V_p(\mathbf{x}(t), t) \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)\Phi_{pi,1}^{(n)} + \gamma\Phi_{pi,2}^{(n)} - \lambda P_p^{(n)}(\gamma)]\mathbf{x}(t) \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)(A_{pi}^T P_{pi,n} + P_{pi,n}A_{pi} + \Pi_p^n - \lambda P_{p,n}) \\
&\quad + \gamma(A_{pi}^T P_{p,n+1} + P_{p,n+1}A_{pi} + \Pi_p^n - \lambda P_{p,n+1})]\mathbf{x}(t) \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)(\Phi_{p,1}^{(n)} - \lambda P_{p,n}) + \gamma(\Phi_{pi,2}^{(n)} - \lambda P_{p,n+1})]\mathbf{x}(t) \\
&< 0.
\end{aligned} \tag{3.68}$$

Moreover, when $t \in [t_k + \tau^*, t_{k+1})$, we have from (3.59), (3.65) and (3.68) that

$$\dot{V}_p(\mathbf{x}(t), t) - \lambda V_p(\mathbf{x}(t), t) = \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)(A_{pi}^T P_{p,K} + P_{p,K}A_{pi} - \lambda P_{pi,K})\mathbf{x}(t) < 0. \tag{3.69}$$

Thus, we can get that (3.68) and (3.69) hold, which means that

$$\dot{V}_p(\mathbf{x}(t), t) \leq \lambda V_p(\mathbf{x}(t), t).$$

Then, according to (3.59) and (3.65), it can be obtained that

$$V_q(t_k^+, t^+) \leq \mu V_p(t_k^-, t^-).$$

Finally, one can readily conclude from Theorem 3.2 that switched T-S fuzzy system (3.44) is GAS for any switching signal with our proposed ADT (3.45).

Remark 3.5 Compared with Theorem 3.2, the advantage of Theorem 3.3 lies in that the obtained stability condition is formulated in terms of linear matrix inequalities that can be efficiently solved by the LMI toolbox.

3.3.3 Simulation Results

We provide the following example to verify the main results developed in this Sect. 3.2. By using a T-S fuzzy model to represent a given switched nonlinear system composed of all unstable subsystems, a switching signal with our proposed ADT property is designed to asymptotically stabilize the system.

Example 3.2 Consider the switched nonlinear system composed of the following two subsystems,

$$\Sigma_1 = \begin{cases} \dot{x}_1(t) = -7.64x_1(t) + 5.03\sin^2(x_1(t))x_2(t) + 5.84x_2(t) - 6.66\sin^2(x_1(t))x_1(t) \\ \dot{x}_2(t) = -6.44x_1(t) + 4.94x_2(t) - 5.58\sin^2(x_1(t))x_1(t) + 4.21\sin^2(x_1(t))x_2(t), \end{cases}$$

$$\Sigma_2 = \begin{cases} \dot{x}_1(t) = 7.23x_1(t) + 5.031.9\sin^2(x_1(t))x_2(t) - 8.58x_2(t) + 2.96\sin^2(x_1(t))x_1(t) \\ \dot{x}_2(t) = 9.48x_1(t) - 11.28x_2(t) + 3.82\sin^2(x_1(t))x_1(t) - 4.52\sin^2(x_1(t))x_2(t). \end{cases}$$

The state trajectories shown in Figs. 3.4 and 3.5 demonstrate that both subsystems $\sum_1$ and $\sum_2$ are unstable.

Next, we are interested in designing a class of switching signal $\sigma(t)$ with property (3.45) to asymptotically stabilize the above switched system. First, we formulate the T-S fuzzy model of the switched nonlinear system in the following.

When $p = 1$, the Σ_1 can be written as

$$\dot{x}(t) = \begin{bmatrix} -7.64 - 6.66\sin^2(x_1(t)) & 5.84 + 5.03\sin^2(x_1(t)) \\ -6.44 - 5.58\sin^2(x_1(t)) & 4.94 + 4.21\sin^2(x_1(t)) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

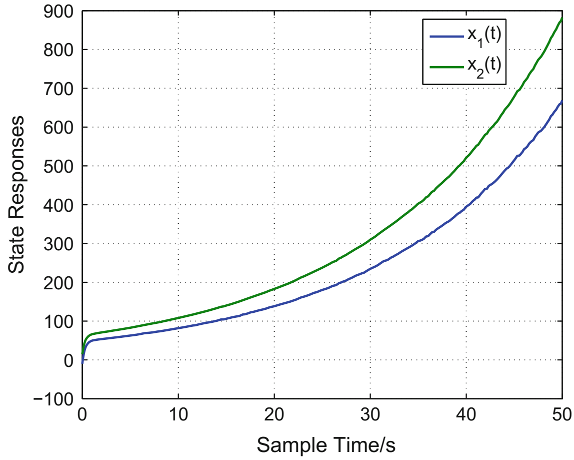

Fig. 3.4 State response of the subsystem $\sum_1$

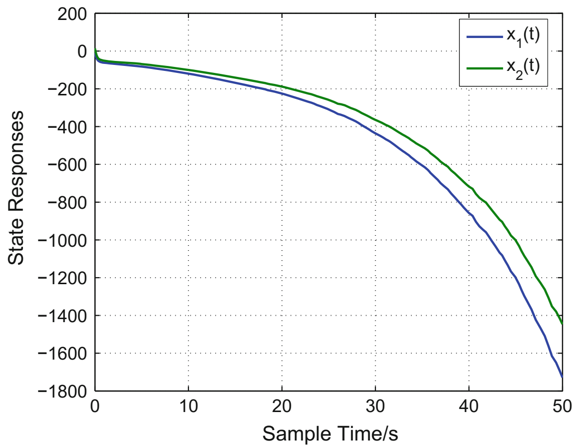

Fig. 3.5 State response of the subsystem $\sum_2$

For the nonlinear term $\sin^2(x_1(t))$, define $\theta(t) = \sin^2(x_1(t))$. Then we have

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -7.64 - 6.66\theta(t) & 5.84 + 5.03\theta(t) \\ 0.6 + 0.4\theta(t) & -0.1 + 3.1\theta(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Next, calculate the minimum and maximum values of $\theta(t)$. The minimum and maximum values of $\theta(t)$ are 0 and 1, respectively. From the minimum and maximum values, $\theta(t)$ can be represented by

$$\theta(t) = \sin^2(x_1(t)) = M_{11}(\theta(t)) \times 0 + M_{12}(\theta(t)) \times 1,$$

where

$$M_{11}(\theta(t)) + M_{12}(\theta(t)) = 1.$$

Therefore the membership functions can be secleted as

$$M_{11}(\theta(t)) = 1 - \sin^2(x_1(t)),\, M_{12}(\theta(t)) = \sin^2(x_1(t)).$$

Then, the first nonlinear subsystem Σ_1 is represented by the following fuzzy model.

Model rule R_1^1 : If $\theta(t)$ is 0, THEN

$$\dot{\mathbf{x}}(t) = A_{11}\mathbf{x}(t),$$

Model rule R_1^2 : If $\theta(t)$ is 1, THEN

$$\dot{\mathbf{x}}(t) = A_{12}\mathbf{x}(t).$$

Its normalized membership functions are $h_1(\theta(t)) = 1 - \sin^2(x_1(t))$, $h_2(\theta(t)) = \sin^2(x_1(t))$, and here,

$$A_{11} = \begin{pmatrix} -7.64 & 5.84 \\ -6.44 & 4.94 \end{pmatrix},\, A_{12} = \begin{pmatrix} -14.3 & 10.87 \\ -12.02 & 9.15 \end{pmatrix}.$$

Thus, through the use of fuzzy blending, the global mode of the 1^{st} fuzzy subsystem can be given by

$$\begin{aligned} \dot{x}(t) &= A(h(t))\mathbf{x}(t) \\ &= \sum_{i=1}^{2} h_{1i}(\theta(t))A_{1i}\mathbf{x}(t), \end{aligned}$$

where

$$\begin{aligned} h_{11}(\theta(t)) &= \frac{M_{11}(\theta(t))}{M_{11}(\theta(t)) + M_{12}(\theta(t))} = 1 - \sin^2(x_1(t)), \\ h_{12}(\theta(t)) &= \frac{M_{12}(\theta(t))}{M_{11}(\theta(t)) + M_{12}(\theta(t))} = \sin^2(x_1(t)). \end{aligned}$$

Similarly, the second nonlinear subsystem Σ_2 can be represented by the following fuzzy model.

Model rule R_2^1 : If $\theta(t)$ is 0, THEN

$$\dot{\mathbf{x}}(t) = A_{21}\mathbf{x}(t),$$

Model rule R_2^2 : If $\theta(t)$ is 1, THEN

$$\dot{\mathbf{x}}(t) = A_{22}\mathbf{x}(t),$$

where

$$A_{21} = \begin{pmatrix} 7.23 & -8.58 \\ 9.48 & -11.28 \end{pmatrix}, A_{22} = \begin{pmatrix} 10.18 & -12.05 \\ 13.30 & -15.80 \end{pmatrix}.$$

Therefore, we can describe switched nonlinear system (3.70) in the following form

$$\dot{x}(t) = \sum_{p=1}^{2}\sum_{i=1}^{2} \delta_p(\sigma(t))h_{pi}(\theta(t))A_{pi}\mathbf{x}(t)\ , i \in R = \{1, 2\},\ \ p = \{1, 2\},$$

where

$$\delta_p(\sigma(t)) = \begin{cases} 1, & if\ \sigma(t) = p, \\ 0, & otherwise. \end{cases}$$

Next, by using Theorem 3.3 and choosing $K = 1$, $\mu = 0.6$, $\eta = 0.7$, $\tau^* = 0.3$, the feasible solutions are obtained as below:

$$P_{1,0} = \begin{pmatrix} 0.5355 & -0.5210 \\ -0.5210 & 0.5436 \end{pmatrix}, P_{1,1} = \begin{pmatrix} 1.0411 & -0.8787 \\ -0.8787 & 0.7933 \end{pmatrix},$$

$$P_{2,0} = \begin{pmatrix} 0.6034 & -0.5065 \\ -0.5065 & 0.4539 \end{pmatrix}, P_{2,1} = \begin{pmatrix} 0.9275 & -0.9049 \\ -0.9049 & 0.9477 \end{pmatrix}.$$

Finally, generating one possible switching sequence by our proposed ADT switching ($\tau_a = 0.5 < -\frac{ln\lambda}{\lambda} = 0.59$), the corresponding state responses of the system under initial state condition $\mathbf{x}(0) = [-10\ 15]^T$, are shown in Fig. 3.6, from which one can see that the switched nonlinear system is stabilized by the designed ADT switching.

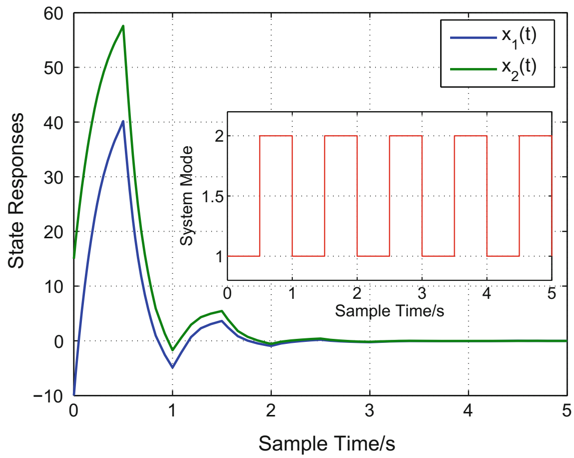

Fig. 3.6 State responses of switched nonlinear system (32) under switching signal $\sigma(t)$ with $\tau_a = 0.5$

3.3.4 Conclusions

The problem of stabilization for switched nonlinear systems composed of unstable subsystems is investigated in the above section by using ADT switching with new property. The stabilization result for the system under consideration is first derived on the basis of our proposed switching signals. After that, the T-S fuzzy modeling method together with a new type of Lyapunov function approach is also used to establish an improved stabilization condition. Finally, a numerical example is provided to verify the correctness and effectiveness of the proposed approach.

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