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Control Synthesis of Switched Systems

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Preface

Switched systems are hybrid systems with both continuous dynamics and discrete events. During the past decades, considerable attention has been devoted to the investigation of such systems due to the fact that switched systems provide a unified framework for mathematical modeling of many practical systems such as networked control systems, near space vehicle control systems and circuit and power systems. As the most important issues in the study of switched linear or nonlinear systems, stability analysis and control synthesis are discussed extensively by many researchers.

Switched linear systems have been investigated for a long time, and many excellent results have been obtained for the systems under arbitrary switching or constraint switching. As far as the stability with arbitrary switching is concerned, it is necessary to require that all the subsystems be asymptotically stable. However, even when all the subsystems of a switched linear system are exponentially stable, such a system may fail to preserve stability under arbitrary switching, but may be stable under constraint switching signals. The constraint switching may result from the physical constraints of the system or the designers' intervention that is actually related to the switching stabilization problem. As an important class of controlled switching signals, time-constraint switching has been widely used for switching stabilization, and a number of effective concepts and powerful tools have been developed. Despite of the rapid progress, some fundamental problems are still either unsolved or less well understood. In particular, the existing time-constraint switching signals are somewhat too strict to be applied in some circumstances, and the switching stabilization among unstable linear subsystems has not been successfully solved. These issues are considered in the current monograph.

On the other hand, the switched systems considered in the literature mostly consist of linear subsystems or first-order nonlinear subsystems, and various types of complicated dynamics such as stochastic noises, unknown uncertainties and actuator dead-zone are not taken into account. However, many industrial systems or physical systems cannot be described by simple switched system models, and thus those traditional control synthesis methods are not applicable for such systems. Considering these, we will focus on the problem of control synthesis for more general switched nonlinear systems containing complicated dynamics, and some intelligent control design methods are also proposed for our considered systems by introducing novel design approaches.

This monograph addresses theoretical explorations on stabilization and intelligent control for both switched linear systems and switched nonlinear systems. A systematic design method of control synthesis is given by establishing new concepts and state-of-the-art results. The book can be used for researchers to carry out studies on switched systems, and is suitable for graduate students of control theory and engineering. It may also be a valuable reference for control design of switched systems by engineers.

The contents of the book are divided into six chapters which contain several independent yet related topics, and they are organized as follows. Chapter 1 introduces some basic background knowledge on switched systems, and also describes the main work of the book. Chapter 2 considers the problem of stabilization of switched linear systems. Chapter 3 addresses the problem of switching stabilization for switched systems composed of unstable subsystems in both linear and nonlinear cases. Chapters 4 and 5 give theoretical developments in detail for adaptive intelligent control for some classes of switched nonlinear systems with uncertainties. Some control problems for constrained switched nonlinear systems are discussed in Chap. 6. Finally, Chap. 7 concludes the book and highlights some future study directions relating to the contents of the book.

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Contents

Chapter 1
Introduction

1.1 Switched Systems

Switched systems provide a unified framework for mathematical modeling of many physical or man-made systems displaying switching features such as power electronics, flight control systems, and network control systems. The systems consists of a collection of indexed differential or difference equations and a switching signal governing the switching among them. The various switching signals differentiate switched systems from the general time-varying systems, because the solutions of the former are dependent on not only the system's initial conditions but also the switching signals.

In general, a switched system can be mathematically described by

$$\begin{aligned} \delta \mathbf{x}(t) &= f_{\sigma(t)}(\mathbf{x}(t), \mathbf{u}(t), \omega(t)) \\ \mathbf{y}(t) &= g_{\sigma(t)}(\mathbf{x}(t), \omega(t)) \\ \mathbf{x}(t_0) &= x_0 \end{aligned}$$

where $\mathbf{x}(t)$, $\mathbf{u}(t)$ and $\mathbf{y}(t)$ are the system state, control input and measurement output, respectively; $\omega(t)$ represents the external disturbance signals; the symbol stands for the derivative operator in the continuous-time context ($\delta \mathbf{x}(t) = \frac{d}{dt}\mathbf{x}(t)$) and the shift forward operator in the discrete-time case ($\delta \mathbf{x}(t) = \mathbf{x}(t+1)$); $\sigma(t)$ is a piecewise constant function of time, called a switching signal, which takes its values in the finite set $S = \{1, 2, \ldots, M\}$ with M being the number of subsystems. In addition, for a series of switching instances $0 < t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots$, $\sigma(t)$ is continuous from the right everywhere. When $t \in [t_i, t_{i+1})$, we say the $\sigma(t_i)^{th}$ subsystem is active. In addition, f_k, $k \in S$ are vector fields, and g_k, $k \in S$ are vector functions.

The configuration of a general switched system is shown in Fig. 1.1. For such systems, the subsystems represent the low-level "local" dynamics governed by conventional differential and/or difference equations, whereas the supervisor is the high-

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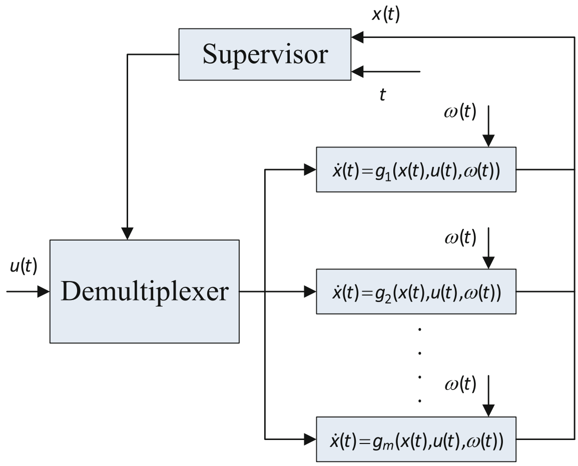


Fig. 1.1 Digram of switched system

level coordinator yielding the switchings among the subsystems [1]. The dynamics of the system is determined by both the switching signal and the subsystems.

Switching is the most important factor in a switched system, which gives the control problems of the switched systems some features and difficulties. The switching of switched systems can be classified into two categories: autonomous switching and active switching. The former is the switching law of switched systems without the influences of external switching logic, which only displays the characteristics of the system itself. Autonomous switching may be arbitrary switching, stochastic switching, time-dependent switching, and state-dependent switching, etc. The latter stands for the switching rules produced by the designers according to some control purposes. Active switching mainly comprises state-driven switching, time-driven switching, and event-driven switching, etc. In addition to the traditional control methods such as feedforward control and feedback control, the active switching design provides us another efficient control strategy for switched systems to achieve the desired state or performances.

1.2 Background and Examples

Switching among different system modes make a switched systems display very complicated dynamic behaviors such as the phenomena of chaos, Zeno, and multiple limit cycles, etc. Also, as far as the stability of a switched system is concerned, it is interesting to see that the stability cannot be ensured for a system composed of all stable subsystems, and switching among unstable subsystems may lead to stability of the whole switched system. For example,

Example 1.1 Consider the switched linear system composed of two subsystems with the following system matrices,

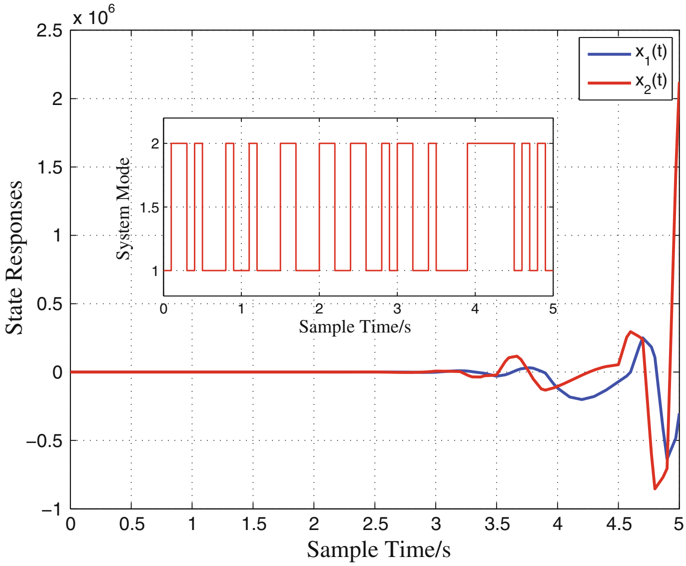


Fig. 1.2 State responses for Example 1.1

$$A_1 = \begin{bmatrix} -1.49 & 3.2 \\ -49.1 & 2.1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1.3 & 9.9 \\ -1.9 & -1.2 \end{bmatrix}$$

It is clear that both subsystems are stable. However, it can be seen in Fig. 1.2 that the system is not stable under the switching shown in the figure.

Example 1.2 Consider the switched linear system composed of two subsystems with the following system matrices:

$$A_1 = \begin{bmatrix} -1.8930 & 0.5846 \\ 0.6124 & -0.0992 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.1024 & -0.8879 \\ 0.0959 & -1.3974 \end{bmatrix}$$

It is clear that none of the subsystems is stable. However, it can be seen in Fig. 1.3 that the system is stable under the switching shown in the figure.

Switched systems clearly have attracted much attention for their wide practical applications in many areas. A few examples are listed in the following to illustrate their potential applications.

Example 1.3 Consider a simplified Pulse Width Modulation (PWM)-driven boost converter shown in Fig. 1.4.

There are two storage elements in the circuit: inductor L and capacitor C. In addition, the source voltage and load are, respectively, represented by E and R.

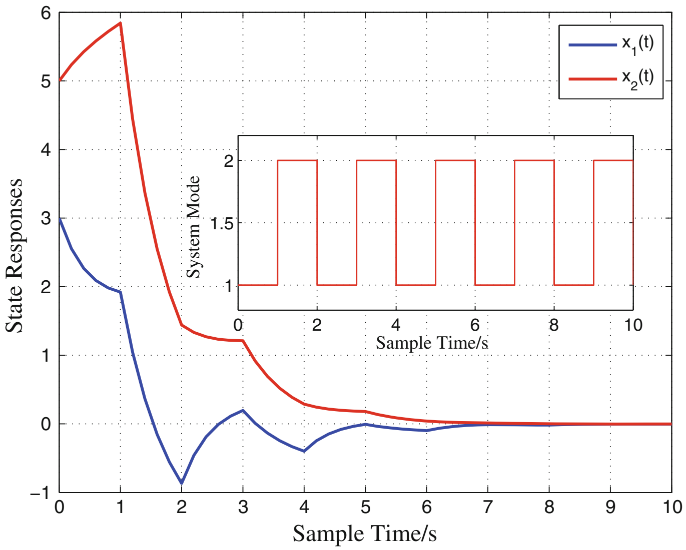
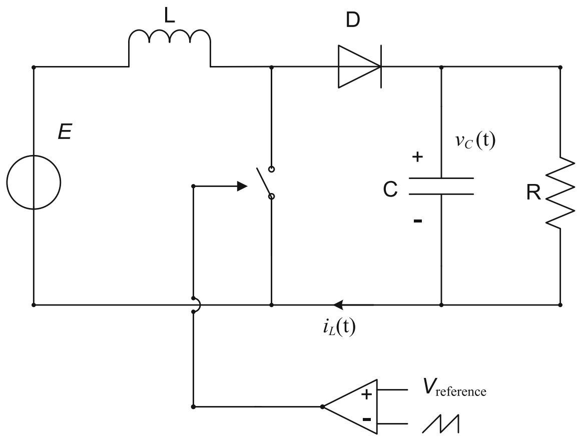


Fig. 1.3 State responses for Example 1.2

Fig. 1.4 A PWM-driven boost converter



The PWM-driven switching signal $s(t)$ that controls the on (1) and off (0) state of the switch is generated by comparing a reference signal V_{ref} and a repetitive triangular waveform. That is, $s(t) \in \{0, 1\}$. Then, the differential equations for the boost converter are given as follows.

$$\dot{v}_C(t) = -\frac{1}{RC}v_C(t) + (1 - s(t))\frac{1}{C}i_L(t)$$
$$\dot{i}_L(t) = -(1 - s(t))\frac{1}{L}v_C(t) + s(t)\frac{1}{L}E$$

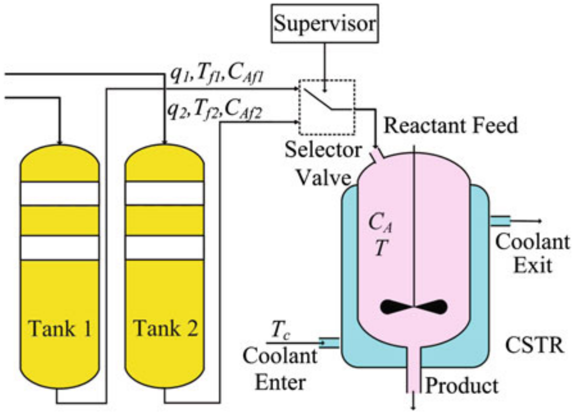


Fig. 1.5 Schematic diagram of the process

Define $\mathbf{x}_1(t) = v_C(t),\ \mathbf{x}_2(t) = i_L(t), \mathbf{u}(t) = E, \sigma(t) = s(t) + 1$, and

$$A_1 = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ L \end{bmatrix}.$$

Then the boost converter can be described by the following state-space model

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t),\ \sigma(t) \in \{1, 2\},$$

which is exactly the switched linear system with two subsystems.

Example 1.4 Consider the continuous stirred tank reactor (CSTR) with two modes feed stream in Fig. 1.5.

In the cases of constant liquid volume, negligible heat losses, perfectly mixing and a first-order reaction in reactant A, the continuous stirred tank reactor at each operating mode can be described by the following differential equations.

$$\dot{C}_A = \frac{q_\sigma}{V}(C_{Af_\sigma} - C_A) - a_0 e^{-\frac{E}{RT}} C_A,$$
$$\dot{T} = \frac{q_\sigma}{V}(T_{f_\sigma} - T) - a_1 e^{-\frac{E}{RT}} C_A + \frac{UA}{V_\rho C_p}(T_c - T).$$

where the C_A is the reactant A concentration, T is the reactor temperature, T_c is the coolant temperature, q is the feed flow rate, V is the volume of the reactor, E is the activation energy, R is the gas constant, and $a0$, $a1$ and $a2$ are constant coefficients. Denote the nominal operating conditions corresponding to an unstable equilibrium point as T^*, T_c^* and C_A^* for both modes.

Define the states as $x_1 = C_A - C_A^*$, $x_2 = T - T^*$ and $x_2 = T_c - T_c^*$, and the control input $u = T_C - T_c^*$. Then, it is clear that the system can be represented by a switched nonlinear system model:

$$\begin{aligned}\dot{x}_1 &= f_1^i(x_1, x_2) + g_1^i(x_1, x_2)u \\ \dot{x}_2 &= f_2^i(x_1, x_2) + g_2^i(x_1, x_2)u\end{aligned}$$

where $i \in \{1, 2\}$, and

$$\begin{aligned}f_1^i &= \tfrac{q_i}{V}(C_{Afi} - C_A^* - x_1) - a_0(x_1 + C_A^*)\exp\left(-\tfrac{E/R}{x_2+T^*}\right)(x_1 + C_A^*) \\ f_2^i &= \tfrac{q_i}{V}(T_{fi} - T^* - x_2) - a_1\exp\left(-\tfrac{E/R}{x_2+T^*}\right)(x_1 + C_A^*) + a_2(T_C^* - x_2 - T^*)\end{aligned}$$

$$\begin{aligned}g_1^i &= 0 \\ g_2^i &= a_2\end{aligned}$$

Example 1.5 Consider the problem of parking the wheeled mobile robot of the unicycle type as shown in Fig. 1.6, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the coordinates of the point in the middle of the rear axle, and θ stands for the angle between the vertical axis of the vehicle and x_1 -axis. The kinematics of the robot can be modelled as below

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{u}_1\cos\theta \\ \dot{\mathbf{x}}_2 &= \mathbf{u}_1\sin\theta \\ \dot{\theta} &= \mathbf{u}_2\end{aligned}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the control inputs (the forward and the angular velocity, respectively) such that $\mathbf{x}_1, \mathbf{x}_2$ and θ tend to zero. It is interesting to see that the corresponding system is nonholonomic and thus cannot be asymptotically stabilized by any time-invariant continuous state feedback law. However, the hybrid control scheme can tackle this problem. Introduce

$$\begin{aligned}y_1 &= \theta \\ y_2 &= \mathbf{x}_1\cos\theta + \mathbf{x}_2\sin\theta \\ y_3 &= \mathbf{x}_1\sin\theta - \mathbf{x}_2\cos\theta \\ D_1 &= \left\{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}_3| > \frac{\|\mathbf{x}\|}{2}\right\} \\ D_2 &= \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \notin D_1\}\end{aligned}$$

Then, a feasible set of candidate controllers can be designed as

$$\begin{aligned}\mathbf{u}^1 &= \begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} = \begin{bmatrix} -4y_2 - 6\frac{y_3}{y_1} - y_3 y_1 \\ -y_1 \end{bmatrix} \\ \mathbf{u}^2 &= \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} -y_2 - sgn(y_2 y_3) y_3 \\ -sgn(y_2 y_3) \end{bmatrix}\end{aligned}$$

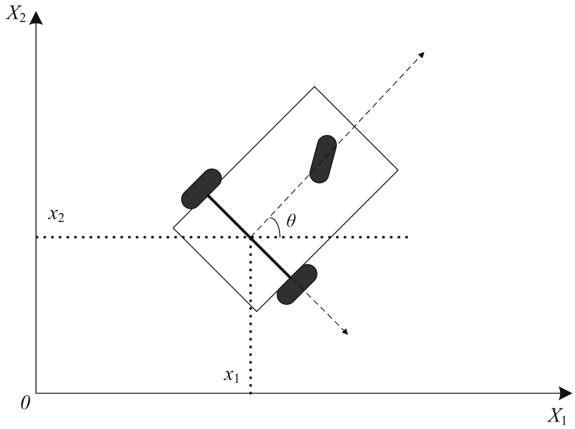


Fig. 1.6 Wheeled mobile robot of unicycle type

where function $sgn(\alpha)$ is defined as

$$sgn(\alpha) = \begin{cases} 1, \text{ if } \alpha \geq 0 \\ -1, \text{ otherwise} \end{cases}$$

Under these controllers, the system can be rewritten by an unforced switched nonlinear system

$$\dot{\mathbf{x}}(t) = f_{\sigma(t)}(\mathbf{x}(t))$$

where $\mathbf{x}(t) = [x_1, x_2, \theta]^T$, $\sigma(t) \in \{1, 2\}$, and

$$f_i(\mathbf{x}(t)) = \begin{bmatrix} u_1^i \cos\theta \\ u_1^i \sin\theta \\ u_2^i \end{bmatrix}, i = 1, 2.$$

To achieve stabilization, the switching law $\sigma(t)$ is chosen as

$$\sigma(t) = \begin{cases} 1, \text{ if } \mathbf{x}(t) \in D_1 \\ 2, \text{ if } \mathbf{x}(t) \in D_2 \end{cases}$$

Therefore, it is clear that the problem of parking the wheeled mobile robot of the unicycle type is described by a switching design problem of a switched nonlinear system.

In addition, a switched system system also finds its numerous applications in multi-controller-switching control systems, robot control systems, asynchronous switching control systems, etc. On the other hand, study on switched systems is also of great theoretical importance because it can provide additional insights and ideas to some long-standing and complicated problems, such as reset control, robust control, intelligent control, control of multi-agent systems and time-delay systems, only to list a few. In summary,a switched system deserves investigation because it is of both theoretical and practical importance.

1.3 Motivations

In recent years, research on control issues of switched systems has received great interest from both academic and engineering experts, and obtained successful achievements. In a certain sense, research on control of switched systems includes three basic issues: control problems of switched systems under arbitrary switching signals,and control problems of switched systems under certain specific switching signals,and control problems of designing certain switching signals to achieve certain performances. A brief review and discussions on the developments of these three basic problems are given in the following.

(1) Control problems of switched systems under arbitrary switching signals

A great number of works have been carried out for such problems in as much as the corresponding results are of general sense. One often resorts to the common Lyapunov function approaches to investigate control problems of switched systems under arbitrary switching. That is, a switched system is stable if there exists a common Lyapunov function for all the subsystems. To list a few Representative works, it was proved by Mosca that stable subsystems must share a common Lyapunov function if the system state matrices are exchangeable [2]. Mehmet [3] probed the existence condition of common Lyapunov functions for second- order switched systems, and proposed concrete methods for obtaining a common Lyapunov function. Stability conditions were established by Shorten for some special switched systems based on the common quadratic Lyapunov function [4]. In [5], Daafouz developed a method for constructing switched quadratic Lyapunov functions in correspondence with discrete-time switched systems, upon which, less conservative stability conditions were given. Based on such a type of Lyapunov function approach, Xie [6] proposed L_2-gain conditions in LMI formulation and controller design method for uncertain discrete-time switched systems, and Wang [7] investigated the problem of fault detection for switched systems with state delay. Liu established stability criteria in [8] for a class of delay switched positive systems with arbitrary switching, and also indicated that the stability of such a type of systems is independent of time delay. On the basis of the approximation of state transition matrices and Gronwall inequality, Sun designed state-feedback controllers for switched nonlinear systems with impulsive effects [9]. By applying the Green formula and Poincaré inequalities, Dong gave a design method of fault-tolerant controller for a class of switched delay systems with distributed parameters [10]. For switched nonlinear impulsive systems with completely unknown uncertainties, Long designed adaptive impulsive tracking controllers in [11], and found that the tracking performance can be improved by using disturbance compensation. On the other hand, the investigations on both necessary and sufficient stability conditions for switched systems under arbitrary switching have also attracted much attention by researchers [12].

It should be pointed out that although the results in the arbitrary switching case are of general sense, the conservatism to require all subsystems be stable and achieve desired control performance under arbitrary switching cannot be ignored. In reality, many switched systems own their specific switching logic, such as liquid level control system, vehicle shift system, etc., and thus there is no need to achieve the control objective with respect to arbitrary switching. It is particularly necessary to study switched systems with some specific switching rules to develop less conservative and more efficient control methods and conclusions compared with the ones in the arbitrary switching case.

(2) Control problems of switched systems under specific switching signals

For some practical switched systems, we can obtain some knowledge of the switching rules among their modes in advance, and these rules are generally described by three classes of switching signals: stochastic switching signals possessing statistical properties, state-dependent switching signals and time-dependent switching signals. Switched systems with these three types of switching signals have been widely studied in recent years. Due to successful applications in network control systems, related control theory of stochastic Markovian switching systems have received considerable attention, and developed well [13]. Meanwhile, systems with state-dependent switching and time-dependent switching have also been paid much attention for their remarkable application backgrounds. The authors in [8] established mathematical models for a Mars exploration unmanned aerial vehicle with umbrella and without umbrella, respectively, and gave the simplified switched system model for the re-entry process where the switching between the models with umbrella and without umbrella was determined by the aircraft speed. Then, an integrated control system of the Mars exploration unmanned aerial vehicle was designed via gain pre-fabricated method on the basis of the proposed switched system model. In [14], seven characteristic points were selected for the whole flight process of a BTT missile, around which, constant subsystem models were established to obtain a switched system model for the flight process. Then, the authors designed subsystem controllers and autopilot switch points under the cases that the switching instances were dependent on the system state and the switching sequence was known, such that the missile could rapidly and accurately track the guidance command, and the switching chatter was effectively suppressed. The bifurcation characteristic and chaos switching oscillation behavior were systematically investigated in [12] for the Rőssler oscillator and Chua’s circuit under state-dependent switching, respectively, and the complex dynamic behavior caused by periodic switching between two Lorenz oscillators was also analyzed. In [15], the system mutation dynamics of an electro hydraulic servo actuator under different voltage supply was modelled by several subsystems of a switched system whose switching law represented the voltage supply variation, and then the authors designed a control system for an aero electro hydraulic servo actuator according to the system actual voltage supply variation to achieve good performance.

The aforementioned literatures mainly focuses on analysis and synthesis of switched systems with certain specific switching laws. On the other hand, one can also actively design switching signals to achieve some required control performances of a switched system.

(3) Control problems of switched systems via switching signal design

For switched systems, switching itself provides us a very efficient control strategy in addition to those classical control methods widely used in control theory. We can properly design switching rules to enable a switched system to achieve desired performances. It should be noted that switching control can complete some control tasks that cannot be accomplished by traditional control methods. Active switching strategies generally comprise state-driven switching control, time-driven switching control and event-driven switching control, etc. Control issues of switched systems based on these three active switching strategies have been noticed by many researchers.

In the state-driven switching control aspect, by resorting to the minimum projection strategy, the problems of quadratic stabilization and state-driven switching signal design were addressed by Pettersson in [16]. The state-driven switching design method for uncertain switched linear systems with polytopic uncertainties was proposed in [17] by the LMI technique. Allerhand discussed several control problems of switched systems with polytopic uncertainties in [18], and gave the design approach of state-driven switching signals. Sangswang systematically investigated the problems of performance analysis and state-driven switching control for power electronic converters with a pulse width modulation circuit driver [19]. Corona proposed a class of state-driven switching law via the LQ performance optimization method such that the considered systems without stable subsystems were exponentially stable [20]. For switched systems with partially unstable or all unstable subsystems, the authors in [21] developed a novel concept of multiple generalized Lyapunov-like function to solve the problems of stability, L_2-gain analysis and H_∞ control for switched nonlinear systems under state-driven switching signals. It can be seen that the investigations on state-driven switching control of switched systems have been extended from switched linear systems, systems with stable subsystems and simple systems to switched nonlinear systems, systems with unstable subsystems and complex systems, and gradually form a relatively complete theory framework. But it is noted that there are some constraints in applying state-driven switching to switched systems, such as state measurability, observability, estimation cost, and real-time ability, etc.

Due to great advantages in the aspects of applicability, reliability, real-time ability and application cost, etc., time-driven switching control of switched systems has been widely noticed by many researchers. The concepts of dwell time and average dwell time have been successively proposed and applied to the time-driven switching control of switched systems. Through a practical example, Liberzon [22] revealed the divergence phenomenon of the state trajectory of systems switched between two stable subsystems, and pointed out that the essential reason behind this phenomenon is the energy increment caused by switching was not compensated by stable subsystems. In addition, the work also indicated that for systems comprising unstable subsystems, dwelling on unstable modes or frequently switching to unstable modes would lead to the instability of a switched system. Therefore, an effective guarantee of the stability of a switched system is to activate stable modes for a long time and reduce the switching frequency (that is, slow switching). Based on this idea, the concept of dwell time was proposed and extensively used for the control of switched systems. Geromel discussed the problem of minimal dwell time stabilization for continuous-time switched systems [23]. Then, in [24], Briat extended the results in [23], and established convex stability conditions via the "Lifting Setting" technique. The obtained conditions are convenient for robust analysis and synthesis of the systems. Dwell time switching requires the dwell time on each subsystem be larger than a sufficient big constant, which greatly restricts its applications. Considering this point, Hespanha [25] creatively gave the concept of average dwell time that relaxed some restrictions on switching rules and owned more flexibilities in switching design of switched systems.Recently, on the basis of average dwell time switching, studies on control problems related to switched systems have made considerable progress. The authors in were [26] concerned with the weighted L_2-gain analysis for switched systems with time varying delay under average dwell time switching. For switched systems with polytopic uncertain parameters,a time-driven switching exponential H_∞ filter was designed in [27] by resorting to the parameter-dependent idea. Then, Zong [28] was devoted to the exponential l_2-l_∞ filtering design for discrete-time uncertain switched systems under average dwell time switching. The authors in [29] provided a time-driven switching observer scheme for delayed switched recurrent neural networks by exploring the free weighting matrix technique. In the meantime, switching control of switched nonlinear systems in the framework of average dwell time has also obtained synchronous development.

The above-mentioned literature related to time-driven switching control only considers stable open-loop or closed-loop subsystems. However, in practice, many controlled plants are unstable, and designing feedback controllers are often impractical due to an unmeasurable or unobservable state, high cost, low real-time capability, etc. On the other hand, uncontrollable subsystems, controller faults and asynchronous switching are sometimes encountered in practical switched systems. In addition, some control problems of many systems can be transformed into control problems of switching among unstable subsystems of switched systems. Causally, there have been a few reports on time-driven switching control problems of switched systems with unstable subsystems in the last decade, which are of both theoretical and practical significance. The authors in [30] studied the average dwell time switching stabilization of switched systems comprising both stable and unstable subsystems by proposing a novel class of Lyapunov-like functions, and extended the corresponding results to asynchronously switched control of switched systems. Through constructing a Lyapunov looped-function, Briat [31] solved the mode-dependent dwell time switching control and computation of the minimal dwell time for switched systems composed of stable and unstable subsystems. The problems of mode-dependent average dwell time switching control and asynchronous L_1 control of delayed switched positive systems with stable and unstable subsystems were considered in [32] based on a copositive Lyapunov–Krasovskii function approach. In [33], finite-time stability was investigated for impulsive switched systems with unstable subsystems.

As can be seen in the above illustrations, many control issues of switched systems have been noticed and developed in the past few years, some of which, however, have not been successfully solved so far. For example, time-driven switching design for switched systems composed of unstable subsystems is still an open problem in both linear and nonlinear contexts. Also, it is urgent to carry out investigations on more complicated switched system models for practical applications, such as high-order switched systems, stochastic switched systems, switched systems with completely unknown uncertainties, etc.

1.4 Structure of the Book

Structure of the book is summarized as follows.

This chapter has introduced the system description and some background knowledge, and also addressed the motivations of the book.

Chapter 2 investigates the stability and stabilization problems for a class of switched systems with mode-dependent average dwell time (MDADT) in both continuous-time and discrete-time contexts. The proposed MDADT switching law is more applicable in practice than the ADT switching. Some improved stability criteria for switched systems with our proposed switching in nonlinear settings are first derived, by which the conditions for stability and stabilization for linear systems are also presented. Finally, the results are extended to the ones for switched systems with unstable subsystems.

Chapter 3 studies the problems of switching stabilization for both switched linear systems and switched nonlinear systems with time-driven switching signals. In particular, the considered systems can be composed of all unstable subsystems. In the linear case, the switching signal is designed to exponentially stabilize the underlying system based on the invariant subspace theory. Then, some sufficient conditions are also established in the nonlinear case. Furthermore, the T-S fuzzy modeling approach is applied to represent the underlying switched nonlinear system to make the obtained conditions easily verified.

Chapter 4 is concerned with the adaptive control design for a class of switched nonlinear systems in lower triangular form with unknown functions and arbitrary switchings. First, switched strict-feedback nonlinear systems are considered. Two classes of state feedback controllers are constructed by adopting an adaptive backstepping technique, and both of them are designed by using the common Lyapunov function (CLF) method. The first controller is designed under multiple adaptive laws. Then, the second one is designed based on constructing a maximum common adaptive parameter, which can overcome the problem of over-parameterization of the first controllers. Then, controller design methods for switched nonstrict-feedback nonlinear systems are also carried out. It is shown that the designed state-feedback controllers can ensure that all the signals remain bounded and the tracking error converges to a small neighborhood of the origin.

Chapter 5 considers the problem of adaptive control for switched stochastic nonlinear systems. First, controller design approaches for stochastic switched nonstrict-feedback nonlinear systems with unknown nonsymmetric actuator dead-zone are proposed. By combining radial basis function neural networks universal approximation capability, adaptive backstepping technique with common stochastic Lyapunov function method, adaptive control algorithms are given for the considered systems.

Furthermore, under the framework of adding a power integrator technique, adaptive controllers of switched stochastic high-order uncertain nonlinear systems with SISS inverse dynamic are also designed.

Chapter 6 is focused on the output tracking control problem of constrained nonlinear switched systems in lower triangular form. First, when all the states are subjected to constraints, a Barrier Lyapunov Function (BLF) is explored, which grows to infinity whenever its arguments approach some finite limits, to prevent the states from violating the constraints. Based on the simultaneous domination assumption, we design a continuous feedback controller for the switched system, which guarantees that asymptotic output tracking is achieved without transgression of the constraints and all closed-loop signals remain bounded, provided that the initial states are feasible. Then, we further consider the case of asymmetric time-varying output constraints by constructing an appropriate BLF. In addition, we also resort to p-times differentiable unbounded functions to deal with asymmetric output constraints, which avoids the defect caused by the discontinuity of the constructed asymmetric BLF.

Chapter 7 concludes the monograph by briefly summarizing the main theoretical findings presented in our book, and proposing unsolved problems for further investigations.

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Chapter 2
Stabilization of Switched Linear Systems with Stable Subsystems

2.1 Background and Motivation

In a certain sense, switching signals in systems can be classified into autonomous (uncontrolled) or controlled ones [1, 2], that respectively, result from the system itself and the designers' intervention [3]. The stabilization problems of switched systems with both classes of switching signals, have always been the hottest topic in the studies of switched systems. Relatively, plenty of theoretical results have been available for systems under the uncontrolled switching signals, in both the continuous-time domain [4], and discrete-time domain [5]. However, for the switched systems with controlled switching signals, the corresponding stabilization problem is complicated in finding suitable switching signals to ensure system stability and improve system performances.

In practice, the time-constrained switching signals [6] with restrictions on switching instants are frequently encountered, and have drawn considerable attention. A minimum time interval called dwell time (DT) is first introduced for switched systems. By using multiple Lyapunov functions, it has been proved in [7] that the switched linear systems with stable subsystems are exponentially stable if the dwell time τ is sufficiently large. However, in many practical switched systems, specifying a fixed dwell time may be restrictive. The concept of average dwell time (ADT) extending the concept of DT allows the possibility of dwell time being less than a fixed constant. The ADT switching signal has been found important in not only theory but also in practice, and many sound and pioneered results have been obtained for analysis and synthesis of switched systems by using ADT switching signal [8–12].

However, the property in the ADT switching that the average time interval between any two consecutive switchings is not smaller than a constant independent of the system modes, is probably still not anticipated. In addition, it has been well shown in the literature that, the minimum of admissible ADT is computed by two mode-independent parameters. It is straightforward that such a setup of the two *common* parameters for all subsystems in a mode-independent manner will give rise to a certain conservativeness.

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Furthermore, controller failures, uncontrollable/unobservable modes, and sensor faults are often encountered in real plants, which may lead to switched system models with unstable modes. Therefore, it is of fundamental importance to numerous applications but theoretically challenging to carry out studies of switched systems with unstable subsystems [13–15].

A new class of switching signals called mode-dependent average dwell time (MDADT) switching is proposed in this chapter. Then, the stabilization problems of switched systems composed of stable subsystems are discussed via MDADT switching. Furthermore, the results are extended to the systems comprising unstable subsystems.

Notations: In this chapter, the used notations are standard. $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of the real numbers and n-dimensional Euclidean space, respectively; $\mathbb{Z}^+$ represents the set of positive integers; the notation $\|\cdot\|$ refers to the Euclidean norm. $\mathscr{C}^1$ denotes the set of continuously differentiable functions, and a function α: $[0, \infty) \to [0, \infty)$ is said to be of class $\mathscr{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. Class $\mathscr{K}_\infty$ denotes the subset of $\mathscr{K}$ consisting of all those functions that are unbounded. In addition, the notation $P > 0 (\geq 0)$ means that P is a real symmetric and positive definite (semi-positive definite) matrix.

2.2 Stabilization for Switched Systems Composed of Stable Subsystems

2.2.1 Problem Formulation and Preliminaries

Consider a class of switched linear systems given by

$$\delta\mathbf{x}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t) \tag{2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, the symbol δ denotes the derivative operator in the continuous-time context ($\delta\mathbf{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)$) and the shift forward operator in the discrete-time case ($\delta\mathbf{x}(t) = \mathbf{x}(t+1)$). $\sigma(t)$ is a piecewise constant function of time, called a switching signal, which takes its values in the finite set $S = \{1, \ldots, M\}$; M is the number of subsystems. Also, for a switching sequence $0 < t_1 < \cdots < t_i < t_{i+1} < \cdots$, $\sigma(t)$ is continuous from the right everywhere and may be either autonomous or controlled. When $t \in [t_i, t_{i+1})$, we say the $\sigma(t_i)^{th}$ subsystem is active. The two-matrix pair (A_p, B_p), $\forall\sigma(t) = p \in S$, represents the p^{th} subsystem or p^{th} mode of (2.1).

The following stability definition of system (2.1) is first introduced for later developments, and we denote time by k in the discrete-time case.

Definition 2.1 ([2]) The equilibrium $x = 0$ of system (2.1) is globally uniformly exponentially stable (GUES) under a certain switching signal $\sigma(t)$ if for $\mathbf{u}(t) = 0$ (or

$\mathbf{u}(k) = 0$) and initial conditions $\mathbf{x}(t_0)$ (or $\mathbf{x}(k_0)$), there exist constants $\alpha > 0$, $\delta > 0$ (respectively, $0 < \varsigma < 1$) such that the solution of the system satisfies $\|\mathbf{x}(t)\| \leq \alpha e^{-\delta(t-t_0)} \|\mathbf{x}(t_0)\|$, $\forall t \geq t_0$ (respectively, $\|\mathbf{x}(k)\| \leq \alpha \varsigma^{(k-k_0)} \|\mathbf{x}(k_0)\|$, $\forall k \geq k_0$).

The control input $\mathbf{u}(t)$ (or $\mathbf{u}(k)$) in (2.1) is used to achieve system stability or certain performances for certain switching signals. The state feedback is considered with $\mathbf{u}(t) = K_{\sigma(t)}\mathbf{x}(t)$ (or $\mathbf{u}(k) = K_{\sigma(k)}\mathbf{x}(k)$), where K_p, $\forall \sigma(t) = p \in S$, is the controller gain to be determined. Then, the resulting closed-loop system is given by

$$\delta \mathbf{x}(t) = \bar{A}_p \mathbf{x}(t) \tag{2.2}$$

where,

$$\bar{A}_p = A_p + B_p K_p \tag{2.3}$$

Next, we aim at finding a more general set of admissible switching signals and the corresponding state-feedback controllers, such that the resulting closed-loop system (2.2) is GUES. For this purpose, let us first revisit the definition of the ADT property and the stability results for switched nonlinear systems with ADT.

Definition 2.2 ([16]) For a switching signal $\sigma(t)$ and each $t_2 \geq t_1 \geq 0$, let $N_\sigma(t_2, t_1)$ denote the number of discontinuities of $\sigma(t)$ in the open interval (t_1, t_2). We say that $\sigma(t)$ has an average dwell time τ_a if there exist two positive numbers N_0 (we call N_0 the chatter bound here) and τ_a such that

$$N_\sigma(t_2, t_1) \leq N_0 + \frac{t_2 - t_1}{\tau_a}, \quad \forall t_2 \geq t_1 \geq 0$$

Lemma 2.1 ([16]) *Consider the continuous-time switched system* $\dot{\mathbf{x}}(t) = \mathbf{f}_{\sigma(t)}(\mathbf{x}(t))$, $\sigma(t) \in S$ *and let* $\lambda > 0$, $\mu > 1$ *be given constants. Suppose that there exist* $\mathscr{C}^1$ *functions* $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, *and two class* $\mathscr{K}_\infty$ *functions* κ_1, κ_2 *such that,* $\forall p \in S$

$$\kappa_1(\|\mathbf{x}(t)\|) \leq V_p(\mathbf{x}(t)) \leq \kappa_2(\|\mathbf{x}(t)\|) \tag{2.4}$$

$$\dot{V}_p(\mathbf{x}(t)) \leq -\lambda V_p(\mathbf{x}(t)) \tag{2.5}$$

and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in S \times S$, $p \neq q$,

$$V_p(\mathbf{x}(t_i)) \leq \mu V_q(\mathbf{x}(t_i)) \tag{2.6}$$

then the system is globally uniformly asymptotically stable (GUAS) for any switching signal with ADT

$$\tau_a \geq \tau_a^* = \frac{\ln \mu}{\lambda} \tag{2.7}$$

Lemma 2.2 ([10]) *Consider the discrete-time switched system* $\mathbf{x}(k+1) = f_{\sigma(k)}(\mathbf{x}(k))$, $\sigma(k) \in S$ *and let* $0 < \lambda < 1$ *and* $\mu > 0$, $\forall p \in S$ *be given constants.*

Suppose that there exists positive definite $\mathscr{C}^1$ *functions* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in S$ *and two class* $\mathscr{K}_\infty$ *functions* κ_1, κ_2 *such that,*

$$\kappa_1(\|\mathbf{x}(k)\|) \leq V_p(\mathbf{x}_k) \leq \kappa_2(\|\mathbf{x}(k)\|) \tag{2.8}$$

$$\Delta V_p(\mathbf{x}(k)) \leq -\lambda V_p(\mathbf{x}(k)) \tag{2.9}$$

and $\forall(\sigma(k_i) = p, \sigma(k_{i-1}) = q) \in S \times S,\ p \neq q,$

$$V_p(\mathbf{x}(k_i)) \leq \mu V_q(\mathbf{x}(k_i)) \tag{2.10}$$

then the system is GUAS for any switching signal with ADT

$$\tau_a > \tau_a^* = -\frac{\ln \mu}{\ln(1-\lambda)}. \tag{2.11}$$

2.2.2 Main Results

The definition of the MDADT property used to restrict a new class of switching signals is first given in the following.

Definition 2.3 For a switching signal $\sigma(t)$ and any $T \geq t \geq 0$, let $N_{\sigma p}(T, t)$ be the switching numbers that the p^{th} subsystem is activated over the interval $[t, T]$ and $T_p(T, t)$ denote the total running time of the p^{th} subsystem over the interval $[t, T]$, $p \in S$. We say that $\sigma(t)$ has a mode-dependent average dwell time τ_{ap} if there exist positive numbers N_{0p} (we call N_{0p} the mode-dependent chatter bounds here) and τ_{ap} such that

$$N_{\sigma p}(T, t) \leq N_{0p} + \frac{T_p(T, t)}{\tau_{ap}}, \quad \forall T \geq t \geq 0$$

Remark 2.1 Definition 2.3 constructs a new set of switching signals with a MDADT property. If there exist positive scalars τ_{ap}, $p \in S$ such that a switching signal has the MDADT property, it only requires the average time among the intervals associated with the p^{th} subsystem is larger than τ_{ap}.

The following lemmas present the stability results for the switched nonlinear systems with MDADT.

Lemma 2.3 (Continuous-Time Version) *Consider the continuous-time switched system*

$$\dot{\mathbf{x}}(t) = \mathbf{f}_{\sigma(t)}(\mathbf{x}(t)), \sigma(t) \in S \tag{2.12}$$

and let $\lambda_p > 0$, $\mu_p > 1$, $p \in S$ *be given constants. Suppose that there exist* $\mathscr{C}^1$ *functions* $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, *and class* $\mathscr{K}_\infty$ *functions* κ_{1p}, κ_{2p}, $p \in S$ *such that,* $\forall p \in S$,

$$\kappa_{1p}(\|\mathbf{x}(t)\|) \leq V_p(\mathbf{x}(t)) \leq \kappa_{2p}(\|\mathbf{x}(t)\|) \tag{2.13}$$

$$\dot{V}_p(\mathbf{x}(t)) \leq -\lambda_p V_p(\mathbf{x}(t)) \tag{2.14}$$

and $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in S \times S,\ p \neq q,$

$$V_p(\mathbf{x}(t_i)) \leq \mu_p V_q(\mathbf{x}(t_i)) \tag{2.15}$$

then the system is GUAS for any switching signal with MDADT

$$\tau_{ap} \geq \tau_{ap}^* = \frac{\ln \mu_p}{\lambda_p} \tag{2.16}$$

Proof For any $T > 0$, let $t_0 = 0$ and denote $t_1, t_2 \cdots t_i, t_{i+1}, \ldots t_{N_\sigma(T,0)}$ the switching times on the interval $[0, T]$, where $N_\sigma(T, 0) = \sum_{p=1}^{M} N_{\sigma p}(T, 0)$.

Then, we set

$$\phi(t) := \mathrm{e}^{\lambda_{\sigma(t)} t} V_{\sigma(t)}(x(t)) \tag{2.17}$$

Function (2.17) is piecewise differentiable along solution (2.12). For any $t \in [t_i, t_{i+1}]$, we have:

$$\dot{\phi}(t) = \lambda_{\sigma(t_i)}\phi(t) + \mathrm{e}^{\lambda_{\sigma(t_i)} t} \dot{V}_{\sigma(t_i)}(x(t))$$

By (2.14), we obtain that $\dot{\phi}(t) \leq 0$. This, together with (2.15) and (2.17), implies that

$$\begin{aligned}
\phi(t_{i+1}) &= \mathrm{e}^{\lambda_{\sigma(t_{i+1})} t_{i+1}} V_{\sigma(t_{i+1})}(x(t_{i+1})) \\
&\leq \mu_{\sigma(t_{i+1})} \mathrm{e}^{\lambda_{\sigma(t_{i+1})} t_{i+1}} V_{\sigma(t_i)}(x(t_{i+1})) \\
&= \mu_{\sigma(t_{i+1})} \mathrm{e}^{\lambda_{\sigma(t_{i+1})} t_{i+1} - \lambda_{\sigma(t_i)} t_{i+1}} \phi(t_{i+1}^-) \\
&\leq \mu_{\sigma(t_{i+1})} \mathrm{e}^{(\lambda_{\sigma(t_{i+1})} - \lambda_{\sigma(t_i)}) t_{i+1}} \phi(t_i) \\
&\leq \mu_{\sigma(t_i)} \mu_{\sigma(t_{i+1})} \mathrm{e}^{(\lambda_{\sigma(t_{i+1})} - \lambda_{\sigma(t_i)}) t_{i+1} + (\lambda_{\sigma(t_i)} - \lambda_{\sigma(t_{i-1})}) t_i} \phi(t_{i-1}) \\
&\leq \prod_{j=0}^{i} \mu_{\sigma(t_{j+1})} \mathrm{e}^{\sum_{j=0}^{i} (\lambda_{\sigma(t_{j+1})} - \lambda_{\sigma(t_j)}) t_{j+1}} \phi(t_0)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi(T^-) &\leq \phi(t_{N_\sigma}) \\
&\leq \prod_{j=0}^{N_\sigma - 1} \mu_{\sigma(t_{j+1})} e^{\sum_{j=0}^{N_\sigma - 1} (\lambda_{\sigma(t_{j+1})} - \lambda_{\sigma(t_j)}) t_{j+1}} \phi(0)
\end{aligned}$$

Then, it follows from (2.17) that:

$$\exp(\lambda_{\sigma(T^-)}T)V_{\sigma(T^-)}(\mathbf{x}(T)) \le \prod_{j=0}^{N_\sigma-1}\mu_{\sigma(t_{j+1})}\mathrm{e}^{\sum_{j=0}^{N_\sigma-1}(\lambda_{\sigma(t_{j+1})}-\lambda_{\sigma(t_j)})t_{j+1}}V_{\sigma(0)}(\mathbf{x}(0))$$

This implies that

$$\begin{aligned}
V_{\sigma(T^-)}(\mathbf{x}(T)) &\le \prod_{j=0}^{N_\sigma-1}\mu_{\sigma(t_{j+1})}\exp\left\{\sum_{j=0}^{N_\sigma-1}(\lambda_{\sigma(t_{j+1})}-\lambda_{\sigma(t_j)})t_{j+1}\right.\\
&\quad\left.-\lambda_{\sigma(t_{N_\sigma})}T+\lambda_{\sigma(t_0)}t_0\right\}V_{\sigma(0)}(\mathbf{x}(0))\\
&\le \prod_{p=1}^{M}\mu_p^{N_{\sigma p}}\exp\left\{-\sum_{p=1}^{M}\left[\lambda_p\sum_{s\in\psi(p)}(t_{s+1}-t_s)\right]\right.\\
&\quad\left.-\lambda_{\sigma(t_{N_\sigma})}(T-t_{N_\sigma})\right\}V_{\sigma(0)}(\mathbf{x}(0))\\
&\le \exp\left\{\sum_{p=1}^{M}N_{0p}\ln\mu_p\right\}\exp\left\{\sum_{p=1}^{M}\frac{T_p}{\tau_{ap}}\ln\mu_p-\sum_{p=1}^{M}\lambda_pT_p\right\}V_{\sigma(0)}(x(0))\\
&= \exp\left\{\sum_{p=1}^{M}N_{0p}\ln\mu_p\right\}\exp\left\{\sum_{p=1}^{M}\left(\frac{\ln\mu_p}{\tau_{ap}}-\lambda_p\right)T_p\right\}V_{\sigma(0)}(x(0))
\end{aligned}$$

where $\psi(p)$ denotes the set of s satisfying $\sigma(t_s)=p$, $t_s\in\{t_0,t_1\cdots t_i,t_{i+1},\ldots t_{N_\sigma-1}\}$. Therefore, if there exist constants $\tau_{ap},\ p\in S$ satisfying (2.16), one has:

$$V_{\sigma(T^-)}(x(T)) \le \exp\left\{\sum_{p=1}^{M}N_{0p}\ln\mu_p\right\}\exp\left\{\max_{p\in S}\left(\frac{\ln\mu_p}{\tau_{ap}}-\lambda_p\right)T\right\}V_{\sigma(0)}(\mathbf{x}(0))$$

Thus, one can conclude that $V_{\sigma(T^-)}(\mathbf{x}(T))$ convergences to zero as $T\to\infty$ if the MDADT satisfies (2.16). Then, the asymptotic stability can be deduced with the aid of (2.13). □

Lemma 2.4 (Discrete-Time Version) *Consider the discrete-time switched system*

$$\mathbf{x}(k+1)=\mathbf{f}_{\sigma(k)}(\mathbf{x}(k)),\sigma(k)\in S \tag{2.18}$$

and let $0<\lambda_p<1$ *and* $\mu_p\ge 1,\ p\in S$ *be given constants. Suppose that there exist* $\mathscr{C}^1$ *functions* $V_{\sigma(k)}:\mathbb{R}^n\to\mathbb{R}$, $\sigma(k)\in S$, *and class* $\mathscr{K}_\infty$ *functions* κ_{1p} *and* κ_{2p}, $p\in S$, *such that* $\forall\sigma(k)=p\in S$

$$\kappa_{1p}(\|\mathbf{x}(k)\|)\le V_p(\mathbf{x}_k)\le\kappa_{2p}(\|\mathbf{x}(k)\|) \tag{2.19}$$

$$\varDelta V_p(\mathbf{x}(k)) \le -\lambda_p V_p(\mathbf{x}(k)) \tag{2.20}$$

and $\forall(\sigma(k_i) = p, \sigma(k_{i-1}) = q) \in S \times S,\ p \ne q,$

$$V_p(\mathbf{x}(k_i)) \le \mu_p V_q(\mathbf{x}(k_i)) \tag{2.21}$$

then the system is GUAS for any switching signal with MDADT

$$\tau_{ap} > \tau_{ap}^* = -\frac{\ln \mu_p}{\ln(1-\lambda_p)} \tag{2.22}$$

Proof For any $K > 0$, let $k_0 = 0$ and denote $k_1, k_2, \ldots k_i, k_{i+1}, \ldots k_{N_\sigma(K,0)}$ the switching times on interval $[0, K]$, where $N_\sigma(K, 0) = \sum\limits_{p=1}^{M} N_{\sigma p}(K, 0)$.

One can get from (2.20) that, $\forall p \in S$:

$$V_p(\mathbf{x}(k+1)) - V_p(\mathbf{x}(k)) < 0 \tag{2.23}$$

$$V_p(\mathbf{x}(k+1)) \le (1-\lambda_p) V_p(\mathbf{x}(k)) \tag{2.24}$$

This together with (2.21) means that

$$\begin{aligned}
V_{\sigma(k_{i+1})}(\mathbf{x}(k_{i+1})) &\le \mu_{\sigma(k_{i+1})} V_{\sigma(k_{i+1}-1)}(\mathbf{x}(k_{i+1})) \\
&\le \mu_{\sigma(k_{i+1})} V_{\sigma(k_{i+1}-1)}(\mathbf{x}(k_{i+1}-1))(1-\lambda_{\sigma(k_{i+1}-1)}) \\
&= \mu_{\sigma(k_{i+1})} (1-\lambda_{\sigma(k_i)}) V_{\sigma(k_i)}(\mathbf{x}(k_{i+1}-1)) \\
&\le \mu_{\sigma(k_{i+1})} (1-\lambda_{\sigma(k_i)})^{k_{i+1}-k_i} V_{\sigma(k_i)}(\mathbf{x}(k_i)) \\
&\quad \ldots \\
&\le \prod_{j=0}^{i} \mu_{\sigma(k_{j+1})} \prod_{j=0}^{i} (1-\lambda_{\sigma(k_j)})^{k_{j+1}-k_j} V_{\sigma(k_0)}(\mathbf{x}(k_0))
\end{aligned}$$

Then, by (2.24), one gets that

$$\begin{aligned}
V_{\sigma(K)}(\mathbf{x}(K)) &\le (1-\lambda_{\sigma(k_{N_\sigma})})^{K-k_{N_\sigma}} V_{\sigma(k_{N_\sigma})}(\mathbf{x}(k_{N_\sigma})) \\
&\le (1-\lambda_{\sigma(k_{N_\sigma})})^{K-k_{N_\sigma}} \prod_{j=0}^{N_\sigma-1} \mu_{\sigma(k_{j+1})} \prod_{j=0}^{N_\sigma-1} (1-\lambda_{\sigma(k_j)})^{k_{j+1}-k_j} V_{\sigma(0)}(\mathbf{x}(0)) \\
&= \prod_{p=1}^{M} \mu_p^{N_{\sigma p}} \prod_{p=1}^{M} (1-\lambda_p)^{T_p} V_{\sigma(0)}(\mathbf{x}(0)) \\
&= \prod_{p=1}^{M} \mu_p^{N_{\sigma p}} \exp\left\{ \sum_{p=1}^{M} \left[T_p \ln(1-\lambda_p) \right] \right\} V_{\sigma(0)}(\mathbf{x}(0))
\end{aligned}$$

$$\leq \exp\left\{\sum_{p=1}^{M} N_{0p} \ln \mu_p\right\} \exp\left\{\sum_{p=1}^{M} \frac{T_p}{\tau_{ap}} \ln \mu_p + \sum_{p=1}^{M} \ln(1-\lambda_p) T_p\right\} V_{\sigma(0)}(\mathbf{x}(0))$$

Thus, if there exist constants $\tau_{ap},\ p \in S$ satisfying (2.22), one has:

$$V_{\sigma(K)}(\mathbf{x}(K)) \leq \exp\left\{\sum_{p=1}^{M} N_{0p} \ln \mu_p\right\} \exp\left\{\max_{p\in S}\left[\frac{\ln \mu_p}{\tau_{ap}} + \ln(1-\lambda_p)\right] K\right\} V_{\sigma(0)}(\mathbf{x}(0))$$

Then, one can conclude that $V_{\sigma(K)}(\mathbf{x}(K))$ converges to zero as $K \to \infty$ if the MDADT satisfies (2.22). Subsequently, the asymptotic stability can be obtained by resorting to (2.19). □

Remark 2.2 It can be seen from Lemmas 2.1 and 2.2 that the parameters λ and μ are mode-independent for all subsystems. However, the parameters λ_p, μ_p prescribed in Lemmas 2.3 and 2.4 are mode-dependent, therefore, we can conclude that $\tau_{ap}^* \leq \tau_a^*$, $\forall p \in S$ from (2.5)–(2.7) and (2.14)–(2.16), and the mode-dependent features would reduce the conservativeness existing in Lemmas 2.1 and 2.2.

Remark 2.3 It is clear that Lemma 2.3 (or Lemma 2.4 in the discrete-time case) presents a more general switching signal than Lemma 2.1 (respectively, Lemma 2.2) which corresponds to the special case of $\lambda = \lambda_p$, $\mu = \mu_p$, $\tau_a = \tau_{ap}$, $\forall p \in S$. In fact, we note that if $\tau_a = \tau_{ap}$, $\forall p \in S$, one readily knows from Definition 2.3 that

$$\sum_{p\in S} N_{\sigma p}(T,t) \leq \sum_{p\in S} N_{0p} + \sum_{p\in S} \frac{T_p}{\tau_a}, \quad \forall T \geq t \geq 0$$

Thus, there exist positive numbers $N_0 = \sum\limits_{p\in S} N_{0p}$ and $\tau_a = \tau_{ap}$ such that

$$N_\sigma(T,t) \leq N_0 + \frac{T-t}{\tau_a}, \quad \forall T \geq t \geq 0$$

Based on the results obtained above, we present the stability conditions for system (2.1) with MDADT.

Theorem 2.1 (Continuous-Time Case) *Consider the switched linear system (2.1) when $\mathbf{u}(t) \equiv 0$ and let $\lambda_p > 0,\ \mu_p > 1,\ p \in S$ be given constants. If there exist matrices $P_p > 0$, $\forall p \in S$, such that, $\forall (p,q) \in S \times S,\ p \neq q$,*

$$A_p^T P_p + P_p A_p + \lambda_p P_p \leq 0 \tag{2.25}$$

$$P_p - \mu_p P_q \leq 0 \tag{2.26}$$

then, the switched linear system (2.1) is GUES with MDADT satisfying (2.16).

Proof Here, we choose the Lyapunov function candidate as follows,

$$V_p(\mathbf{x}(t)) = \mathbf{x}^T(t)P_p\mathbf{x}(t), \quad \forall\sigma(t) = p \in S \tag{2.27}$$

where P_p, $\forall p \in S$ is a positive definite matrix satisfying (2.25) and (2.26). Then, from (2.1), (2.14), (2.15) and (2.27), we have, $\forall(p, q) \in S \times S,\ p \neq q$,

$$\dot{V}_p(\mathbf{x}(t)) + \lambda_p V_p(\mathbf{x}(t)) = \lambda_p \mathbf{x}^T(t)P_p\mathbf{x}(t) + \mathbf{x}^T(t)P_pA_p\mathbf{x}(t) + \mathbf{x}^T(t)A_p^TP_p\mathbf{x}(t)$$

$$V_p(\mathbf{x}(t_i)) - \mu_p V_q(\mathbf{x}(t_i)) = \mathbf{x}^T(t_i)P_p\mathbf{x}(t_i) - \mu_p\mathbf{x}^T(t_i)P_q\mathbf{x}(t_i)$$

Thus, if (2.25) and (2.26) hold, system (2.1) is GUAS for any switching signal with MDADT (2.16). In addition, by denoting $\delta = -\frac{1}{2}[\max\limits_{p\in S}(\frac{\ln\mu_p}{\tau_{ap}} - \lambda_p)]$, we can obtain from (2.13) and (2.27) that the system state satisfies $\|\mathbf{x}(t)\| \leq \alpha e^{-\delta(t-t_0)}\|\mathbf{x}(t_0)\|$, $\forall t \geq t_0$ for a certain $\alpha > 0$,; that is the underlying system is GUES. □

Theorem 2.2 (Discrete-Time Case) *Consider the switched linear system (2.1) when* $\mathbf{u}(t) \equiv 0$ *and let* $0 < \lambda_p < 1$ *and* $\mu_p \geq 1$, $p \in S$ *be given constants. If there exist matrices* $P_p > 0$, $\forall p \in S$, *such that,* $\forall(p, q) \in S \times S,\ p \neq q$,

$$A_p^TP_pA_p + \lambda_pP_p - P_p \leq 0 \tag{2.28}$$

$$P_p - \mu_pP_q \leq 0 \tag{2.29}$$

then, the switched linear systems (2.1) is GUES with MDADT satisfying (2.22).

Proof We establish the Lyapunov function

$$V_p(\mathbf{x}(k)) = \mathbf{x}^T(k)P_p\mathbf{x}(k), \quad \forall\sigma(k) = p \in S \tag{2.30}$$

where P_p, $\forall p \in S$ is a positive definite matrix satisfying (2.28) and (2.29). Then, together with (2.1), (2.20), (2.21) and (2.30), we can get, $\forall(p, q) \in S \times S,\ p \neq q$,

$$\Delta V_p(\mathbf{x}(k)) + \lambda_pV_p(\mathbf{x}(k)) = \lambda_p\mathbf{x}^T(k)P_p\mathbf{x}(k) - \mathbf{x}^T(k)P_p\mathbf{x}(k) + \mathbf{x}^T(k)A_p^TP_pA_p\mathbf{x}(k)$$

$$V_p(\mathbf{x}(k_i)) - \mu_pV_q(\mathbf{x}(k_i)) = \mathbf{x}^T(k_i)P_p\mathbf{x}(k_i) - \mu_p\mathbf{x}^T(k_i)P_q\mathbf{x}(k_i)$$

Therefore, if (2.28) and (2.29) hold, system (2.1) is GUAS for any switching signal with MDADT (2.22) in the light of Lemma 2.4. Subsequently, by denoting $\varsigma = \sqrt{\exp\{\max\limits_{p\in S}[\frac{\ln\mu_p}{\tau_{ap}} + \ln(1-\lambda_p)]\}}$, we can obtain from (2.19) and (2.30) that $\|\mathbf{x}(k)\| \leq \alpha\varsigma^{(k-k_0)}\|\mathbf{x}(k_0)\|$, $\forall k \geq k_0$ for a certain $\alpha > 0$, that is, the underlying system is GUES. □

Now, we give a stabilizing controller design approach for system (2.1) with the MDADT switching.

Theorem 2.3 (Continuous-Time Case) *Consider the switched linear systems (2.2) and let $\lambda_p > 0$, $\mu_p > 1$, $p \in S$ be given constants. If there exist matrices $U_p > 0$, and T_p, $\forall p \in S$, such that, $\forall (p, q) \in S \times S$, $p \neq q$,*

$$A_p U_p + B_p T_p + U_p A_p^T + T_p^T B_p^T + \lambda_p U_p \leq 0 \tag{2.31}$$

$$U_q \leq \mu_p U_p \tag{2.32}$$

then there exists a set of stabilizing controllers such that system (2.2) is GUES for any switching signal with MDADT satisfying (2.16). Moreover, if (2.31) and (2.32) are feasible, the controller gains can be provided by

$$K_p = T_p U_p^{-1} \tag{2.33}$$

Proof Theorem 2.1 implies that if

$$\bar{A}_p^T P_p + P_p \bar{A}_p + \lambda_p P_p \leq 0 \tag{2.34}$$

$$P_p - \mu_p P_q \leq 0 \tag{2.35}$$

system (2.2) is GUES for any switching signal with MDADT (2.16). Replacing $\bar{A}_p$ in (2.34) by (2.3), setting $U_p = P_p^{-1}$ and $T_p = K_p P_p^{-1}$, we can see that, if (2.31) holds, (2.34) is satisfied. Moreover, if (2.32) holds, we can obtain that $U_q - \mu_p U_p \leq 0$. By Schur complement, we note that $U_q - \mu_p U_p \leq 0$ is equivalent to

$$\Lambda = \begin{bmatrix} -\mu_p U_p & I \\ I & -U_q^{-1} \end{bmatrix} \leq 0.$$

Furthermore, by Schur complement, one has that $\Lambda \leq 0$ is equivalent to $-U_q^{-1} - I^T (\mu_p U_p)^{-1} I \leq 0$; that is, (2.35) holds. In addition, if the inequalities (2.31) and (2.32) have feasible solutions, the admissible controller gains can be given by (2.33) because $T_p = K_p P_p^{-1}$, which ends the proof. □

Theorem 2.4 (Discrete-Time Case) *Consider the switched linear systems (2.2) and let $0 < \lambda_p < 1$ and $\mu_p \geq 1$, $p \in S$ be given constants. If there exist matrices $U_p > 0$, and T_p, $\forall p \in S$, such that, $\forall (p, q) \in S \times S$, $p \neq q$,*

$$\begin{bmatrix} -U_p & A_p U_p + B_p T_p \\ * & -(1 - \lambda_p) U_p \end{bmatrix} \leq 0 \tag{2.36}$$

$$U_q \leq \mu_p U_p \tag{2.37}$$

then there exists a set of controllers such that system (2.2) is GUES for any switching signal with MDADT satisfying (2.22). Moreover, if (2.36) and (2.4) have a solution, the admissible controllers can be given by (2.33).

Proof By Theorem 2.2 we have that if

$$\bar{A}_p^T P_p \bar{A}_p + \lambda_p P_p - P_p \le 0 \tag{2.38}$$

$$P_p - \mu_p P_q \le 0 \tag{2.39}$$

system (2.2) is GUES for any switching signal with MDADT (2.22). Substituting $\bar{A}_p$ in (2.38) and by Schur complement, we have

$$\begin{bmatrix} -P_p & P_p B_p K_p + P_p A_p \\ * & -(1-\lambda_p)P_p \end{bmatrix} \le 0 \tag{2.40}$$

setting $U_p = P_p^{-1}$ and $T_p = K_p P_p^{-1}$ and performing a congruence transformation to (40) via $diag\{U_p, U_p\}$, we can obtain (2.36). Therefore, (2.36) and (2.4) ensure (2.38) and (2.39). In addition, if the inequalities (2.36) and (2.4) have feasible solutions, the admissible controller gains can be given by (2.33), which ends the proof. □

2.2.3 Simulation Results

An example in the continuous-time domain is presented to demonstrate the potential and validity of the results obtained above.

Example 2.1 Consider the switched linear systems consisting of three subsystems described by:

$$A_1 = \begin{bmatrix} 3.9 & 1.5 \\ 2.5 & 2.3 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1.4 & 0.3 \\ 1 & -2.7 \end{bmatrix}, \ A_3 = \begin{bmatrix} -2.2 & 0.1 \\ -2 & -0.4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

Here, we aim to design a set of mode-dependent stabilizing controllers and find corresponding switching signals with MDADT property such that the resulting closed-loop system is stable.

To illustrate the advantages of the proposed MDADT switching, we shall also present the design results of both controllers and switching signals for the systems with ADT switching. By different approaches and setting the relevant parameters appropriately, the computation results for the system with two different switching schemes are listed in Table 2.1.

Table 2.1 Computation results for the system under two different switching schemes

Switching schemes	ADT switching	MDADT switching
Criteria for controller design	Corollary 2.1 in [12]	Theorem 2.3 in the chapter
Controller gains	Γ_1 : $K_1 = \begin{bmatrix} 73.66 & 66.14 \end{bmatrix}$ $K_2 = \begin{bmatrix} -19.94 & -2.75 \end{bmatrix}$ $K_3 = \begin{bmatrix} 3.25 & -15.24 \end{bmatrix}$	Γ_2 : $K_1 = \begin{bmatrix} 93.79 & 69.75 \end{bmatrix}$ $K_2 = \begin{bmatrix} -59.81 & -34.25 \end{bmatrix}$ $K_3 = \begin{bmatrix} -53.91 & -63.58 \end{bmatrix}$
Switching signals	$\tau_{\mathrm{a}}^* = 0.99$ $(\mu = 2, \lambda \leq 0.7)$	$\tau_{\mathrm{a}1}^* = 0.22, \tau_{\mathrm{a}2}^* = 0.49, \tau_{\mathrm{a}3}^* = 0.99$ $(\mu_1 = \mu_2 = \mu_3 = 2,$ $\lambda_1 \leq 3.1, \lambda_2 \leq 1.4, \lambda_3 \leq 0.7)$

It can be seen from Table 2.1 that the minimal MDADT are reduced to $\tau_{a1}^* = 0.22$, $\tau_{a2}^* = 0.49$, $\tau_{a3}^* = 0.99$, for given $\mu = \mu_1 = \mu_2 = \mu_3 = 2$, and one special case of MDADT switching is $\tau_{\mathrm{a}}^* = \tau_{\mathrm{a}1}^* = \tau_{\mathrm{a}2}^* = \tau_{\mathrm{a}3}^* = 0.99$ by setting $\lambda = \lambda_1 = \lambda_2 = \lambda_3 = 0.7$, which corresponds to minimal ADT.

To further show the merit of MDADT switching, let us now consider the resulting closed-loop system performances. Applying the obtained controller, under the scheme of ADT switching and MDADT switching, respectively, we can obtain the state responses for each closed-loop subsystem as shown in Fig. 2.1. It is seen that there are some fluctuations with larger amplitude in the state response of closed-loop subsystem 1.

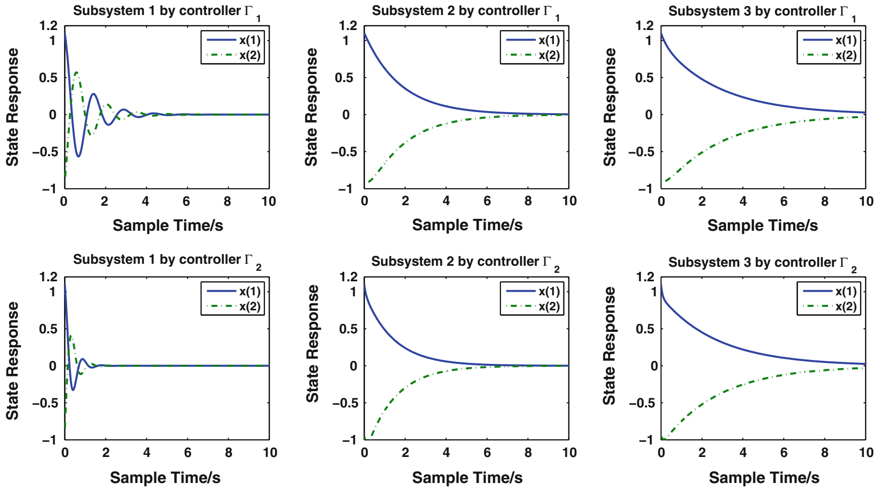


Fig. 2.1 The state response comparisons of the closed-loop subsystems by controllers Γ_1 and Γ_2

Now, generating one possible switching sequences with the ADT property and the MDADT property, one can obtain the corresponding state responses of the closed-loop system as shown in Figs. 2.2 and 2.3, respectively, for the same initial state condition. It can be seen from the curves that the state response of the closed-loop system fluctuates under the ADT switching scheme, but is smooth under the MDADT switching scheme.

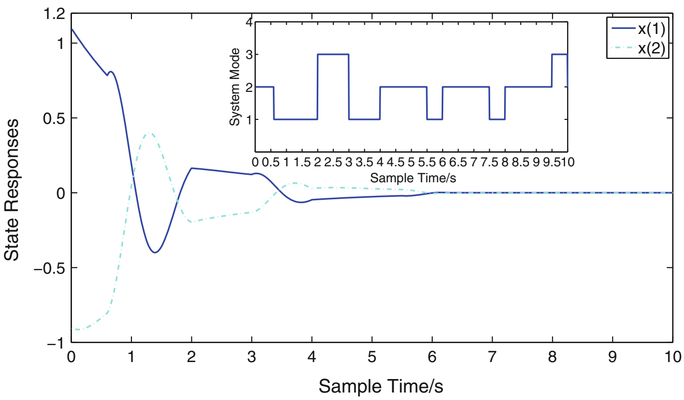


Fig. 2.2 State response of the closed-loop systems by controllers Γ_1 under switching signal σ with $\tau_a = 1.0$

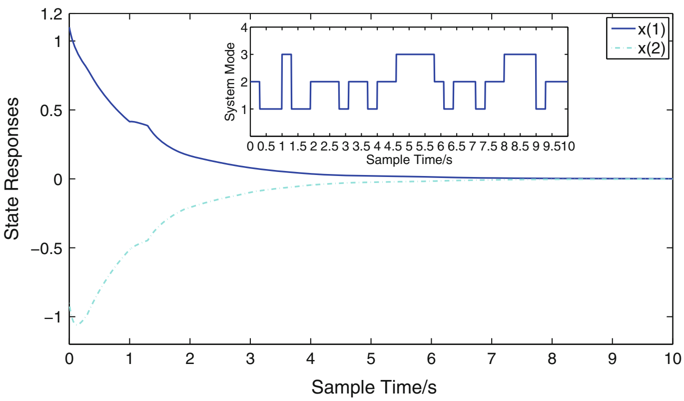


Fig. 2.3 State response of the closed-loop systems by controllers Γ_2 under switching signal σ with $\tau_{a1} = 0.3,\ \tau_{a2} = 0.6,\ \tau_{a3} = 1.0$

2.2.4 Conclusions

The MDADT switching stabilization problems for switched linear systems with stable subsystems are investigated. First, the stability results for a class of switched systems with MDADT are derived in both linear and nonlinear contexts. The minimal MDADT for admissible switching signals and the corresponding state feedback controller are designed for switched linear systems in both continuous-time and discrete-time cases. Finally, a numerical example is given to demonstrate the validity and effectiveness of the developed results.

2.3 Stabilization of Switched Linear Systems with Unstable Subsystems

2.3.1 Problem Formulation and Preliminaries

Consider the following switched linear systems,

$$\delta \mathbf{x}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t), \mathbf{x}(t_0) = x_0, t \geq t_0, \tag{2.41}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, x_0 and $t_0 \geq 0$ denote the state vector, control input, initial state and initial time, respectively; the symbol δ denotes the derivative operator in the continuous-time case ($\delta\mathbf{x}(t) = \dot{\mathbf{x}}(t)$) and the shift forward operator in the discrete-time case ($\delta\mathbf{x}(t) = \mathbf{x}(t+1)$); $\sigma(t)$ represents a switching signal which is a piecewise constant function from the right of time and takes its values in the finite set $\mathfrak{L} = \{1, 2, \ldots, m\}$, where $m > 1$ is the number of subsystems. Moreover, the A_r, $\forall r \in \mathfrak{L}$ is either a Hurwitz stable or unstable subsystem matrix. Without lose of generality, we assume that $\mathfrak{L} = \mathfrak{S} \bigcup \mathfrak{U}$, where $\mathfrak{S} = \{1, 2, \ldots, s\}$ and $\mathfrak{U} = \{s+1, \ldots, m\}$; that is, there are s stable subsystems and $m - s$ unstable subsystems. When $t \in [t_k, t_{k+1})$, $\forall k \in \mathbb{Z}^+$, the $\sigma(t_k)^{th}$ mode is activated. Let $\{A_r \in \mathbb{R}^{n\times n}, B_r \in \mathbb{R}^{n\times m}, r \in \mathfrak{L}\}$ be a family of constant matrices describing subsystems.

Next, some definitions are introduced for later developments of the main results in this chapter.

Definition 2.4 ([17]) The equilibrium $x = 0$ of switched system (2.41) is globally uniformly exponentially stable (GUES) under a certain switching signal $\sigma(t)$, if for $\mathbf{u}(t) = 0$ there exists positive numbers $\lambda > 0, \alpha > 0$, (resp., $0 < \nu < 1$) such that $\|\mathbf{x}(t)\| \leq \lambda e^{-\alpha(t-t_0)}\|\mathbf{x}(t_0)\|$, (resp., $\|\mathbf{x}(t)\| \leq \lambda\nu^{-(t-t_0)}\|\mathbf{x}(t_0)\|$), $\forall t \geq t_0$ with any initial conditions $\mathbf{x}(t_0)$.

Definition 2.5 For any time interval $[t_1, t_2]$, denote $N_{\sigma p}(t_2, t_1)$ as the numbers of the p^{th} subsystem being activated, and $T_p(t_2, t_1)$ as the overall running time of the p^{th} subsystem, $p \in \mathfrak{S}$. We can find two constants N_{0p} and τ_{ap} satisfying

$$N_{\sigma p}(t_2, t_1) \leq N_{0p} + \frac{T_p(t_2, t_1)}{\tau_{ap}}, \quad \forall t_2 \geq t_1 \geq 0. \tag{2.42}$$

where τ_{ap} is called the mode-dependent average dwell time of the switching signal $\sigma(t)$.

In this chapter, we also define another type of MDADT called fast MDADT in the following.

Definition 2.6 For any time interval $[t_1, t_2]$, denote $N_{\sigma q}(t_2, t_1)$ as the numbers of the q^{th} subsystem being activated, and $T_q(t_2, t_1)$ as the overall running time of the q^{th} subsystem, $q \in \mathfrak{U}$. We can find two constants N_{0q} and τ_{aq} satisfying

$$N_{\sigma q}(t_2, t_1) \geq N_{0q} + \frac{T_q(t_2, t_1)}{\tau_{aq}}, \quad \forall t_2 \geq t_1 \geq 0. \tag{2.43}$$

where τ_{aq} is called the mode-dependent average dwell time of the switching signal $\sigma(t)$.

Remark 2.4 The MDADT in Definition 2.5 requiring $N_{\sigma p}(t_2, t_1) \leq N_{0p} + \frac{t_2 - t_1}{\tau_a} \Longleftrightarrow \frac{T_p(t_2,t_1)}{N_{\sigma p(t_2,t_1)} - N_{0p}} \geq \tau_{ap}$, $\forall t_2 \geq t_1 \geq 0$ can be called slow switching (in average sense), which means that average time among the intervals associated with the p^{th} subsystem is larger than τ_{ap}. By resorting to this MDADT to achieve stabilization, the basic idea is to allow the transient effect to dissipate after each switching. In this framework, the energy decrement of the Lyapunov function during dwelling on stable subsystems can compensate possible energy at the switching instance and/or during dwelling at unstable subsystems. However, Definition 2.6 requires $N_{\sigma q}(t_2, t_1) \geq N_{0q} + \frac{t_2 - t_1}{\tau_a} \Longleftrightarrow \frac{T_q(t_2,t_1)}{N_{\sigma q(t_2,t_1)} - N_{0q}} \leq \tau_{aq}$, $\forall t_2 \geq t_1 \geq 0$. It is called fast switching (in average sense), because the average time among the intervals associated with the q^{th} subsystem is no more than τ_{aq}. The basic idea of using the fast MDADT is to compensate the state divergence via dwelling at appropriate unstable subsystems, but obviously the dwell time cannot be too big. Therefore, in order to achieve stabilization, we apply the slow MDADT to stable subsystems and fast MDADT to unstable subsystems in the following.

2.3.2 Main Results

In this section, we consider the problems of stability and stabilization for switched linear systems described in the previous section.

2.3.2.1 Stability Analysis

We first introduce a class of quasi-alternative switching signals.

Definition 2.7 Suppose that a switching law $\sigma(t)$ satisfies the following conditions.
(1) If $\sigma(t_k) \in \mathfrak{S}$, then $\sigma(t_{k+1}) \in \mathfrak{L}$,
(2) If $\sigma(t_k) \in \mathfrak{U}$, then $\sigma(t_{k+1}) \in \mathfrak{S}$,
The switching signal $\sigma(t)$ satisfying the above conditions is called a quasi-alternative switching signal.

Remark 2.5 Definition 2.7 implies that a switched system cannot directly switches from an unstable mode to another unstable mode. If condition (1) is changed as: "If $\sigma(t_k) \in \mathfrak{S}$, then $\sigma(t_{k+1}) \in \mathfrak{U}$;", Definition 2.7 implies that $\sigma(t)$ is a alternative switching signal, that is, stable subsystems and unstable subsystems alternately switch to each other.

Next, stability conditions for switched nonlinear system

$$\delta \mathbf{x}(t) = f_{\sigma(t)}(\mathbf{x}(t)). \tag{2.44}$$

are first presented in the following lemmas by designing quasi-alternative switching signals with MDADT property.

Lemma 2.5 *Consider continuous-time switched nonlinear system (2.44), $\sigma(t) \in \mathfrak{L}$, and let $\eta_p < 0$, $\mu_p > 1$, $p \in \mathfrak{S}$ and $\eta_q > 0$, $0 < \mu_q < 1, q \in \mathfrak{U}$. Suppose that there exist two sets of $\mathscr{C}^1$ non-negative functions $V_p(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathfrak{S}$ and $V_q(x(t)) : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathfrak{U}$, two class $\mathscr{K}_\infty$ functions α_1 and α_2, such that*

$$\alpha_1(\|\mathbf{x}(t)\|) \leq V_p(\mathbf{x}(t)) \leq \alpha_2(\|\mathbf{x}(t)\|), \forall p \in \mathfrak{S}, \tag{2.45}$$
$$\alpha_1(\|\mathbf{x}(t)\|) \leq V_q(\mathbf{x}(t)) \leq \alpha_2(\|\mathbf{x}(t)\|), \forall q \in \mathfrak{U}, \tag{2.46}$$
$$\dot{V}_p(\mathbf{x}(t)) \leq \eta_p V_p(\mathbf{x}(t)), \quad \forall p \in \mathfrak{S}, \tag{2.47}$$
$$\dot{V}_q(\mathbf{x}(t)) \leq \eta_q V_q(\mathbf{x}(t)), \quad \forall q \in \mathfrak{U}, \tag{2.48}$$
$$V_p(\mathbf{x}(t_k)) \leq \mu_p V_r(\mathbf{x}(t_k^-)), \quad \forall p \in \mathfrak{S}, \forall r \in \mathfrak{L}, \quad p \neq r, \tag{2.49}$$
$$V_q(\mathbf{x}(t_k)) \leq \mu_q V_p(\mathbf{x}(t_k^-)), \quad \forall p \in \mathfrak{S}, \forall q \in \mathfrak{U}. \tag{2.50}$$

Then switched system (2.44) is GUES for any quasi-alternative switching signals with MDADT

$$\begin{cases} \tau_{ap} \geq \frac{-ln\mu_p}{\eta_p}, \forall p \in \mathfrak{S}, \\ \tau_{aq} \leq \frac{-ln\mu_q}{\eta_q}, \forall q \in \mathfrak{U}. \end{cases} \tag{2.51}$$

Proof Without loss of generality, we denote $t_1, t_2 \ldots t_k, t_{k+1} \ldots t_{N_{N_{\sigma(T,0)}}}$ as the switching times on time interval $[0, T]$. Then we consider the function

$$W(t) = e^{-\eta_{\sigma(t)} t} V_{\sigma(t)}(\mathbf{x}(t)). \tag{2.52}$$

It is clear that this function is piecewise differentiable along solutions of (2.44). When $t \in [t_k, t_{k+1})$, we get from (2.47), (2.48), (2.52) that

$$\begin{aligned}\dot{W}(t) &= -\eta_{\sigma(t_k)}e^{-\eta_{\sigma(t_k)}t}V_{\sigma(t_k)}(\mathbf{x}(t)) + e^{-\eta_{\sigma(t_k)}t}\dot{V}_{\sigma(t_k)}(\mathbf{x}(t))\\ &\le -\eta_{\sigma(t_k)}e^{-\eta_{\sigma(t_k)}t}V_{\sigma(t_k)}(\mathbf{x}(t)) + e^{-\eta_{\sigma(t_k)}t}\eta_{\sigma(t_k)}V_{\sigma(t_k)}(\mathbf{x}(t))\\ &= 0. \end{aligned} \tag{2.53}$$

Thus $W(t)$ is non-increasing when $t \in [t_k, t_{k+1})$. This together with (2.49), (2.50), (2.52) gives that

$$\begin{aligned}W(t_{k+1}) &= e^{-\eta_{\sigma(t_{k+1})}t_{k+1}}V_{\sigma(t_{k+1})}(\mathbf{x}(t_{k+1}))\\ &\le \mu_{\sigma(t_{k+1})}e^{-\eta_{\sigma(t_{k+1})}t_{k+1}}V_{\sigma(t_k)}(\mathbf{x}(t_{k+1}))\\ &= \mu_{\sigma(t_{k+1})}e^{-\eta_{\sigma(t_{k+1})}t_{k+1}+\eta_{\sigma(t_k)}t_{k+1}}W(\mathbf{x}(t_{k+1}^-))\\ &\le \mu_{\sigma(t_{k+1})}e^{-(\eta_{\sigma(t_{k+1})}-\eta_{\sigma(t_k)})t_{k+1}}W(\mathbf{x}(t_k))\\ &\le \mu_{\sigma(t_{k+1})}\mu_{\sigma(t_k)}e^{-[(\eta_{\sigma(t_{k+1})}-\eta_{\sigma(t_k)})t_{k+1}+(\eta_{\sigma(t_k)}-\eta_{\sigma(t_{k-1})})t_k]}W(\mathbf{x}(t_{k-1}))\\ &\quad \cdots\\ &\le \prod_{i=0}^{k}\mu_{\sigma(t_{i+1})}\exp\{-[(\eta_{\sigma(t_{k+1})}-\eta_{\sigma(t_k)})t_{k+1}+(\eta_{\sigma(t_k)}-\eta_{\sigma(t_{k-1})})t_k\\ &\quad +\cdots+(\eta_{\sigma(t_1)}-\eta_{\sigma(t_0)})t_1]\}W(\mathbf{x}(t_0)). \end{aligned} \tag{2.54}$$

Then, from (2.52) and (2.54), one can obtain that

$$e^{-\eta_{\sigma(T^-)}T}W(\mathbf{x}(T^-)) \le \prod_{i=0}^{N_\sigma-1}\mu_{\sigma(t_{i+1})}e^{\sum_{i=0}^{N_\sigma-1}-(\eta_{\sigma(t_{i+1})}-\eta_{\sigma(t_i)})t_i}V_{\sigma(t_0)}(\mathbf{x}(t_0)). \tag{2.55}$$

Moreover, it can be derived from (2.42), (2.43) and (2.55) that

$$\begin{aligned}V_{\delta(T^-)}(\mathbf{x}(T)) &\le \prod_{p=1}^{s}\mu_p^{N_{\sigma p}}\prod_{q=s+1}^{m}\mu_q^{N_{\sigma q}}e^{(\sum_{p=1}^{s}\eta_pT_p(T,0)+\sum_{q=s+1}^{m}\eta_qT_q(T,0))}V_{\sigma(0)}(\mathbf{x}(0))\\ &\le \prod_{p=1}^{s}\mu_p^{(N_{0p}+\frac{T_p(T,0)}{\tau_{ap}})}\prod_{q=s+1}^{m}\mu_q^{(N_{0q}+\frac{T_q(T,0)}{\tau_{aq}})}\\ &\quad \times e^{(\sum_{p=1}^{s}\eta_pT_p(T,0)+\sum_{q=s+1}^{m}\eta_qT_q(T,0))}V_{\sigma(0)}(\mathbf{x}(0))\\ &= e^{(\sum_{p=1}^{s}(N_{0p}+\frac{T_p(T,0)}{\tau_{ap}})\ln\mu_p+\sum_{q=s+1}^{m}(N_{0q}+\frac{T_q(T,0)}{\tau_{aq}})\ln\mu_q)}\\ &\quad \times e^{(\sum_{p=1}^{s}\eta_pT_p(T,0)+\sum_{q=s+1}^{m}\eta_qT_q(T,0))}\times V_{\sigma(0)}(\mathbf{x}(0))\\ &\le e^{(\sum_{p=1}^{s}N_{0p}\ln\mu_p+\sum_{q=s+1}^{m}N_{0q}\ln\mu_q)} \end{aligned}$$

$$\times e^{(\sum_{p=1}^{s}(\eta_p+\frac{\ln\mu_p}{\tau_{ap}})T_p(T,0)+\sum_{q=s+1}^{m}(\eta_q+\frac{\ln\mu_q}{\tau_{aq}})T_q(T,0))} V_{\sigma(0)}(\mathbf{x}(0)). \tag{2.56}$$

By (2.56), it can be got that, if τ_{ap}, $p \in \mathfrak{S}$ and τ_{aq}, $q \in \mathfrak{U}$ satisfy the conditions in (2.51), then

$$V_{\delta(T^-)}(\mathbf{x}(T)) \leq \lambda e^{-\alpha(T-t_0)} V_{\sigma(0)}(\mathbf{x}(0)),$$

where $\lambda = e^{(\sum_{p=1}^{s} N_{0p}\ln\mu_p + \sum_{q=s+1}^{m} N_{0q}\ln\mu_q)}$, $-\alpha = \max_{(p,q)\in(S\times U)}\left\{\left(\eta_p + \frac{\ln\mu_p}{\tau_{ap}}\right), \left(\eta_q + \frac{\ln\mu_q}{\tau_{aq}}\right)\right\}$, which associated with Definition 2.4 verifies that $V_{\delta(T^-)}(\mathbf{x}(T))$ exponentially converges to zero as $T \to \infty$.

Finally, we conclude that switched nonlinear system (2.44) is GUES under quasi-alternative switching signals satisfying (2.51) if the conditions (2.45)–(2.50) hold. This completes the proof. □

Lemma 2.6 *Consider discrete-time switched nonlinear system (2.44), $\sigma(t) \in \mathfrak{L}$, and let $-1 < \eta_p < 0$, $\mu_p > 1$, $p \in \mathfrak{S}$ and $\eta_q > 0$, $0 < \mu_q < 1$, $q \in \mathfrak{U}$. Suppose that there exist two sets of $\mathscr{C}^1$ non-negative functions $V_p(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathfrak{S}$ and $V_q(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathfrak{U}$, two class $\mathscr{K}_\infty$ functions α_1 and α_2, such that*

$$\alpha_1(\|\mathbf{x}(t)\|) \leq V_p(\mathbf{x}(t)) \leq \alpha_2(\|\mathbf{x}(t)\|), \forall p \in \mathfrak{S}, \tag{2.57}$$
$$\alpha_1(\|\mathbf{x}(t)\|) \leq V_q(\mathbf{x}(t)) \leq \alpha_2(\|\mathbf{x}(t)\|), \forall q \in \mathfrak{U}, \tag{2.58}$$
$$\triangle V_p(\mathbf{x}(t)) \leq \eta_p V_p(\mathbf{x}(t)), \quad \forall p \in \mathfrak{S}, \tag{2.59}$$
$$\triangle V_q(\mathbf{x}(t)) \leq \eta_q V_q(\mathbf{x}(t)), \quad \forall p \in \mathfrak{U}, \tag{2.60}$$
$$V_p(\mathbf{x}(t_k)) \leq \mu_p V_r(\mathbf{x}(t_k^-)), \quad \forall q \in \mathfrak{S}, \forall r \in \mathfrak{L}, \quad p \neq r, \tag{2.61}$$
$$V_q(\mathbf{x}(t_k)) \leq \mu_q V_p(\mathbf{x}(t_k^-)), \quad \forall p \in \mathfrak{S}, \forall q \in \mathfrak{U}. \tag{2.62}$$

Then switched system (2.44) is GUES for any quasi-alternative switching signals with MDADT

$$\begin{cases} \tau_{ap} \geq \frac{-ln\mu_p}{1+\eta_p}, \forall p \in \mathfrak{S}, \\ \tau_{aq} \leq \frac{-ln\mu_q}{1+\eta_q}, \forall q \in \mathfrak{U}. \end{cases} \tag{2.63}$$

Proof The proof of Lemma 2.6 is similar to that of Lemma 2.5. We omit it here. □

Remark 2.6 Different from Lemma 2.3 (or Lemma 2.4 in the discrete-time case), unstable subsystems are considered in Lemma 2.5 (resp., Lemma 2.6). For stable subsystems, it also follows the slow switching scheme (Definition 2.5). But for unstable subsystems, it adopts the fast switching scheme (Definition 2.6). Such a switching strategy can guarantee to dwell on stable subsystems long enough to compensate possible energy increments at the switching instance and during dwelling on unstable subsystems, and avoid dwelling on unstable subsystems too long. Anyway, it should be pointed out that the dwell time on stable subsystems is not required to be bigger than that on unstable subsystems. In fact, if a switched system is composed of stable subsystems, Lemma 2.5 (Lemma 2.6 in the discrete-time case) will reduce to Lemma 2.3 (resp., Lemma 2.4).

Next, the following two theorems for switched linear system (2.41) can be given on the basis of the Lemmas 2.5 and 2.6. Theorem 2.6 corresponds to the continuous-time version and Theorem 2.7 corresponds to the discrete-time version.

Theorem 2.5 *Consider switched linear system (2.41) when $u(t) = 0$, and let $\eta_p < 0$, $\mu_p > 1$, $p \in \mathfrak{S}$ and $\eta_q > 0$, $0 < \mu_q < 1$, $q \in \mathfrak{U}$ be given constants. If there exists a set of matrices $P_p > 0$, $P_q > 0$, $p \in \mathfrak{S}, q \in \mathfrak{U}$, such that*

$$A_p^T P_p + P_p A_p \leq \eta_p P_p, \quad \forall p \in \mathfrak{S}, \tag{2.64}$$

$$A_q^T P_q + P_q A_q \leq \eta_q P_q, \quad \forall q \in \mathfrak{U}, \tag{2.65}$$

$$P_p \leq \mu_p P_r, \quad \forall q \in \mathfrak{S}, \ \forall r \in \mathfrak{L}, \quad p \neq r, \tag{2.66}$$

$$P_q \leq \mu_q P_p, \quad \forall p \in \mathfrak{S}, \ \forall q \in \mathfrak{U}. \tag{2.67}$$

Then, the system is GUES for any quasi-alternative switching signals with MDADT satisfying (2.51).

Proof Construct a multiple Lyapunov function for continuous-time switched system (2.41) in the form of

$$V_{\sigma(t)}(\mathbf{x}(t)) = \begin{cases} \mathbf{x}(t)^T P_p \mathbf{x}(t), & \sigma(t) = p \in \mathfrak{S} \\ \mathbf{x}(t)^T P_q \mathbf{x}(t) & \sigma(t) = q \in \mathfrak{U}, \end{cases} \tag{2.68}$$

where $P_p > 0, P_q > 0, \quad p \in \mathfrak{S}, q \in \mathfrak{U}$ are positive definite matrices satisfying (2.64)–(2.67).

In the sequel, one can obtain from (2.64)–(2.67) that $\forall (p, q) \in \mathfrak{S} \times \mathfrak{U}$,

$$\begin{aligned}
\dot{V}_p(\mathbf{x}(t)) - \eta_p V_p(\mathbf{x}(t)) &= \mathbf{x}^T(t)(A_p^T P_p + P_p A_p - \eta_p P_p)\mathbf{x}(t), \\
&\leq 0, \ p \in \mathfrak{S}, \\
\dot{V}_q(\mathbf{x}(t)) - \eta_q V_p(\mathbf{x}(t)) &= \mathbf{x}^T(t)(A_q^T P_q + P_q A_q - \eta_q P_q)\mathbf{x}(t), \\
&\leq 0, \ q \in \mathfrak{U}. \\
V_p(\mathbf{x}(t)) - \mu_p V_r(\mathbf{x}(t)) &= \mathbf{x}^T(t)(P_p - \mu_p P_r)\mathbf{x}(t), \\
&\leq 0, \ p \in \mathfrak{S}, \ r \in \mathfrak{L}, \ p \neq r. \\
V_q(\mathbf{x}(t)) - \mu_q V_p(\mathbf{x}(t)) &= \mathbf{x}^T(t)(P_q - \mu_q P_p)\mathbf{x}(t), \\
&\leq 0, \ p \in \mathfrak{S}, \ q \in \mathfrak{U}.
\end{aligned}$$

Finally, one can readily conclude by Lemma 2.5 that switched system (2.41) is GUES for any quasi-alternative switching signals with MDADT satisfying (2.51). □

Theorem 2.6 *Consider switched linear system (2.41) when* $\mathbf{u}(t) = 0$*, and let* $-1 < \eta_p < 0$*,* $\mu_p > 1$*,* $p \in \mathfrak{S}$ *and* $\eta_q > 0$*,* $0 < \mu_q < 1$*,* $q \in \mathfrak{U}$ *be given constants. If there exists a set of matrices* $P_p > 0$*,* $P_q > 0$*,* $p \in \mathfrak{S}, q \in \mathfrak{U}$*, such that*

$$A_p^T P_p A_p - P_p \leq \eta_p P_p, \quad \forall p \in \mathfrak{S}, \tag{2.69}$$

$$A_q^T P_q A_q - P_q \leq \eta_q P_q, \quad \forall q \in \mathfrak{U}, \tag{2.70}$$

$$P_p \leq \mu_p P_r, \quad \forall p \in \mathfrak{S}, \ \forall r \in \mathfrak{L}, \quad p \neq r, \tag{2.71}$$

$$P_q \leq \mu_q P_p, \quad \forall p \in \mathfrak{S}, \ \forall q \in \mathfrak{U}. \tag{2.72}$$

then, the system is GUES for any quasi-alternative switching signals with MDADT satisfying (2.63).

Proof The proof of Theorem 2.6 is similar to that of Theorem 2.5. We omit it here. □

2.3.2.2 Controller Design

In this subsection, the problem of controller design for switched system (2.41) with MDADT switching is presented. Unlike some control methods requiring all subsystems be controllable, we only require the existence of at least one controllable subsystem. Without loss of generality, we assume that $\{A_p \in \mathbb{R}^{n\times n}, B_p \in \mathbb{R}^{n\times m}, p \in \mathfrak{C}\}$ are controllable subsystems, where $\mathfrak{C} = \{1, 2, \ldots, s\}$, and $\{A_q \in \mathbb{R}^{n\times n}, q \in \mathfrak{B}\}$ are subsystems that can not be stabilized, where $\mathfrak{B} = \{s+1, s+2, \ldots, m\}$. Our objective is to design p controllers to ensure switched system (2.41) to be GUES with MDADT switching. In this subsection, the state feedback is considered with $u(t) = K_p x(t)$, $p \in \mathfrak{C}$, where K_p is the controller gain to be determined. Then the closed-loop system (3.1) can be obtained as follows,

$$\delta\mathbf{x}(t) = \begin{cases} A_p\mathbf{x}(t) + B_p K_p \mathbf{x}(t), & \forall p \in \mathfrak{C}, \\ A_q \mathbf{x}(t), & \forall q \in \mathfrak{B}. \end{cases} \tag{2.73}$$

However, it should be pointed out that if the A_p, $\forall p \in \mathfrak{C}$ itself is a Hurwitz matrix, the controller gain K_p is chosen as 0.

Theorem 2.7 *Consider switched linear system (2.73), and let* $\eta_p < 0$*,* $\mu_p > 1$*,* $p \in \mathfrak{C}$ *and* $\eta_q > 0$*,* $0 < \mu_q < 1$*,* $q \in \mathfrak{P}$ *be given constants. If there exists a set of matrices* $Q_r > 0$*,* $r \in \mathfrak{L}$*, and* R_p*,* $p \in \mathfrak{C}$ *such that*

$$Q_p A_p^T + A_p Q_p + R_p^T B_p^T + B_p R_p \leq \eta_p Q_p, \quad \forall p \in \mathfrak{C}, \tag{2.74}$$

$$Q_q A_q^T + A_q Q_q \leq \eta_q Q_q, \quad \forall q \in \mathfrak{B}, \tag{2.75}$$

$$Q_r \leq \mu_p Q_p, \quad \forall p \in \mathfrak{C}, \ \forall r \in \mathfrak{L}, \quad p \neq r, \tag{2.76}$$

$$Q_p \leq \mu_q Q_q, \quad \forall p \in \mathfrak{C}, \ \forall q \in \mathfrak{B}. \tag{2.77}$$

then, there is a set of stabilizing controllers such that the system is GUES for any quasi-alternative switching signals with MDADT satisfying (2.51). Moreover, if a feasible solution of (2.74)–(2.77) exists, the controller gains are given by

$$K_p = R_p Q_p^{-1}. \tag{2.78}$$

Proof When $\sigma(t) \in \mathfrak{S}$, perform a congruence transformation to (2.74) via Q_p^{-1}. Then by (2.78), one can obtain that

$$A_p^T Q_p^{-1} + Q_p^{-1} A_p + K_p^T B_p^T Q_p^{-1} + Q_p^{-1} B_p K_p \leq \eta_p Q_p^{-1}, \quad \forall p \in \mathfrak{C}, \tag{2.79}$$

which is equivalent to

$$(A_p + B_p K_p)^T Q_p^{-1} + Q_p^{-1}(A_p + B_p K_p) \leq \eta_p Q_p^{-1}, \quad \forall p \in \mathfrak{C}. \tag{2.80}$$

Then, by the Schur complement theorem, we can get that (2.76) is equivalent to

$$Q_p{}^{-1} \leq \mu_p Q_r{}^{-1}, \quad \forall p \in \mathfrak{C}, \ \forall r \in \mathfrak{L}, \ p \neq r. \tag{2.81}$$

Similarly, when $\sigma(t) \in \mathfrak{U}$, it can be derived that (2.76) and (2.78) are also equivalent to the following inequalities, respectively,

$$A_q^T Q_q^{-1} + Q_q^{-1} A_q \leq \eta_q Q_q{}^{-1}, \quad \forall p \in \mathfrak{B}, \tag{2.82}$$

$$Q_q{}^{-1} \leq \mu_q Q_p{}^{-1}, \quad \forall p \in \mathfrak{C}, \ \forall q \in \mathfrak{B}. \tag{2.83}$$

Finally, by Theorem 2.5 and letting $P_p = Q_p^{-1}$, we can conclude that, if (2.80)–(2.83) hold, switched system (2.73) is GUES for any quasi-alternative switching signal with MDADT satisfying (2.51). This completes the proof. □

Theorem 2.8 *Consider switched linear system (2.73), and let* $-1 < \eta_p < 0$, $\mu_p > 1$, $p \in \mathfrak{C}$ *and* $\eta_q > 0$, $0 < \mu_q < 1$, $q \in \mathfrak{B}$ *be given constants. If there exists a set of matrices* $Q_r > 0$, $r \in \mathfrak{L}$, *and* R_p, $p \in \mathfrak{C}$ *such that*

$$\begin{bmatrix} -Q_p & A_p Q_p + B_p R_p \\ * & -(1+\eta_p) Q_p \end{bmatrix} \leq 0, \quad \forall p \in \mathfrak{C}, \tag{2.84}$$

$$\begin{bmatrix} -Q_q & A_q Q_q \\ * & -(1+\eta_q) Q_q \end{bmatrix} \leq 0, \quad \forall q \in \mathfrak{B}, \tag{2.85}$$

$$Q_r \leq \mu_p Q_p, \quad \forall p \in \mathfrak{C}, \ \forall r \in \mathfrak{L}, \quad p \neq r, \tag{2.86}$$

$$Q_p \leq \mu_q Q_q, \quad \forall p \in \mathfrak{C}, \ \forall q \in \mathfrak{B}. \tag{2.87}$$

Then, there is a set of stabilizing controllers such that the system is GUES for any quasi-alternative switching signals with MDADT satisfying (2.63). Moreover, if a feasible solution of (2.84)–(2.87) exists, the controller gains are given by

$$K_p = R_p Q_p^{-1}. \tag{2.88}$$

Proof The proof of Theorem 2.8 is similar to that of Theorem 2.7. We omit it here. □

2.3.3 Simulation Results

The following numerical example is given in this section to verify our main results developed above.

Example 2.2 Consider the continuous-time switched linear system (2.41) consisting of four subsystems and assume that the third and fourth are uncontrollable subsystems. The corresponding subsystem matrices are

$$A_1 = \begin{bmatrix} -10.11 & 10.32 \\ -8.60 & 8.81 \end{bmatrix}, B_1 = \begin{bmatrix} -2.2 \\ 0.8 \end{bmatrix}, A_2 = \begin{bmatrix} 11.12 & -13.32 \\ 11.10 & -13.30 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 3.4 \\ -1.2 \end{bmatrix}, A_3 = \begin{bmatrix} 9.72 & -9.69 \\ 12.92 & -12.89 \end{bmatrix}, A_4 = \begin{bmatrix} 10.24 & -10.23 \\ 13.64 & -13.63 \end{bmatrix}.$$

The eigenvalues of A_1 are $\lambda_{11} = -1.51$ and $\lambda_{12} = 0.21$, eigenvalues of A_2 are $\lambda_{21} = 0.02$ and $\lambda_{22} = -2.2$, eigenvalues of A_3 are $\lambda_{31} = 0.03$ and $\lambda_{32} = -3.2$ and eigenvalues of A_4 are $\lambda_{41} = 0.01$ and $\lambda_{42} = -3.4$, It can be seen that none of these matrices is Hurwitz stable. In addition, one can easily check that $\{A_p \in \mathbb{R}^{2\times 2}, B_p \in \mathbb{R}^{2\times 1}, p = 1, 2\}$ are controllable.

Next, we are interested in designing a set of controllers and a kind of quasi-alternative switching signal $\sigma(t)$ with properties (2.42) and (2.43) to asymptotically

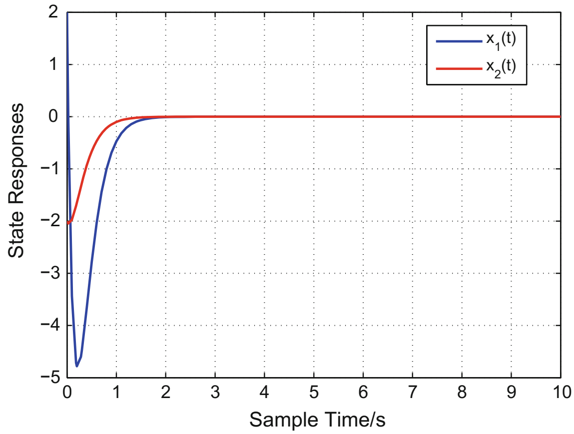


Fig. 2.4 State responses of the first subsystem

stabilize the system. By using Theorem 2.7, if we choose $\mu_1 = 2.9,\ \eta_1 = -1.0, \mu_2 = 2.3,\ \eta_2 = -3.1, \mu_3 = 0.44,\ \eta_3 = 3.0, \mu_4 = 0.51,\ \eta_4 = 1.3$, the feasible solutions are obtained as follows,

$$Q_1 = \begin{bmatrix} 77.0146 & 69.8370 \\ 69.8370 & 65.1947 \end{bmatrix}, Q_2 = \begin{bmatrix} 83.2764 & 77.1246 \\ 77.1246 & 73.6036 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 190.6688 & 176.4114 \\ 176.4114 & 168.1891 \end{bmatrix}, Q_4 = \begin{bmatrix} 180.3970 & 169.1126 \\ 169.1126 & 163.0650 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 6.5871 & -13.5623 \end{bmatrix}, R_2 = \begin{bmatrix} -18.2149 & -3.7398 \end{bmatrix},$$

$$K_1 = R_1 Q_1^{-1} = \begin{bmatrix} 9.5768 & -10.4667 \end{bmatrix}, K_2 = R_2 Q_2^{-1} = \begin{bmatrix} -5.8056 & 6.0325 \end{bmatrix}.$$

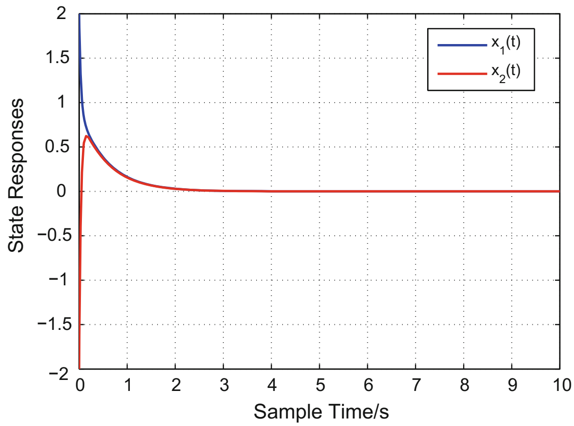
Applying the obtained controllers to the first and second subsystems, respectively, the corresponding state responses of the subsystems under initial state condition $\mathbf{x}(0) = [2\ \ -2]^T$ are shown in Figs. 2.4 and 2.5, in which we can see that the closed-loop subsystems are asymptotically stable. Then, one can obtain that the requirements of MDADT for subsystem $A_i, i = 1, 2, 3, 4$ are:

$$\tau_{a1} \geq \frac{\ln \mu_1}{\eta_1} = \frac{-\ln 2.9}{-1.0} = 1.065,$$

$$\tau_{a2} \geq \frac{\ln \mu_2}{\eta_2} = \frac{-\ln 2.3}{-3.1} = 0.269,$$

$$\tau_{a3} \leq \frac{\ln \mu_3}{\eta_3} = \frac{-\ln 0.44}{3.0} = 0.274,$$

Fig. 2.5 State responses of the second subsystem



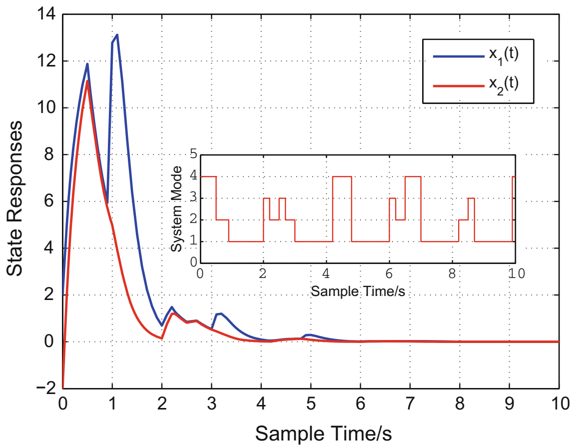


Fig. 2.6 State responses of switched linear system under quasi-alternative switching signal with MDADT

$$\tau_{a4} \leq \frac{\ln \mu_4}{\eta_4} = \frac{-\ln 0.51}{1.3} = 0.518.$$

Furthermore, we generate one possible quasi-alternative switching sequence (4, 2, 1, 3, 2, 3, 2, 1, 4, 1, 3, 2, 4…) with the MDADT property ($\tau_{a1} = 1.2 > 1.065$, $\tau_{a2} = 0.3 > 0.269$, $\tau_{a3} = 0.2 < 0.274$, $\tau_{a4} = 0.5 < 0.518$). The corresponding state responses of the system under initial state condition $x(0) = [2 \ -2]^T$, are shown in Fig. 2.6, from which we can see that the switched linear system is stable under MDADT switching.

2.3.4 Conclusions

In the above, the problems of stability and stabilization for switched systems comprising unstable subsystems are studied in both continuous-time and discrete-time contexts by using a new defined class of switching signal. The proposed switching signal is very efficient for analysis and design for switched systems comprising unstable subsystems. The stability results for switched systems comprising unstable subsystems are first derived on the basis of our proposed switching signals. Moreover, based on the obtained results, improved stabilization conditions are also established, which are concerned with uncontrollable subsystems. Finally, a numerical example is provided to verify the advantages of the proposed approach.

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Chapter 3
Switching Stabilization of Switched Systems Composed of Unstable Subsystems

3.1 Background and Motivation

As mentioned in Chap. 2, for a switched system, even if all its subsystems are stable, it may fail to preserve stability under arbitrary switching, but may be stable under restricted switching signals. Therefore, it is of significance to study the controlled-switching stabilization problems of switched systems. The controlled switching may result from the physical constraints of a system or the designers' intervention [1] which is actually related to the controlled-switching stabilization problem [2]. Generally, the controlled switching in systems could be classified into state-dependent and time-constrained ones.

During the past few years, the problems of state-dependent switching stabilization problems have been widely studied for switched systems with or without unstable subsystems [3, 4]. In the state-dependent case, the whole state space is usually divided into pieces so as to facilitate the search for corresponding Lyapunov-like functions. Then, the state-dependent switching can be designed to ensure the non-increasing conditions when switching occurs. Note that, state-dependent switching is applicable only for the systems whose states are measurable or observable, which also suffers from the problems of high cost, reliability and real-time ability.

However, the time-constrained switching is more applicable in practice, and has been used for controlled-switching stabilization of switched systems in recent years [5–7]. It is noticed that the results on time-constrained switching stabilization of switched systems mainly focus on systems with stable subsystems (or at least one stable subsystem). The basic idea of the existing works is to activate the stable subsystem for a sufficiently large time that we could call slow switching, to compensate the state divergence [8]. In [9], the stability analysis of continuous-time linear switched systems comprising both Hurwitz stable and unstable subsystems is studied by exploring a new type of Lyaounov-like function whose energy can rise with a bounded rate for each active mode. After the bounded increment, the minimal average dwell time should be designed sufficiently large to compensate the energy increment produced during the unstable time. Recently, the mode-dependent dwell-

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time switching is used in [10] for stabilization of switched linear systems with both stable and unstable modes. It is very worth pointing out that there are few efforts put on time-constrained switching stabilization of switched linear systems with all unstable subsystems, which is both theoretically challenging and of fundamental importance to numerous applications.

On the other hand, as many applications of switched systems, such as mobile robots, automotive, DC converters etc., appear to be described by nonlinear models, it is natural to extend the time-constrained switching stabilization theory of switched linear systems to switched nonlinear systems [11–13]. When a switched system is composed of unstable nonlinear subsystems, some promising ideas are not effective any more. Therefore, it will be very meaningful and challenging to carry out the studies on time-constrained switching stabilization of switched systems with possibly all unstable nonlinear subsystems.

Based on the above observations, in this chapter, the problems of time-constrained switching stabilization for switched systems composed of unstable subsystems are investigated in both linear and nonlinear cases.

Notations:

$\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and n-dimensional Euclidean space respectively; $\mathbb{I}_n = \{1, 2, \ldots, n\}$. For a given vector $\mathbf{x}$, the notation $\|\mathbf{x}\|$ refers to the Euclidean vector norm. For a given subspace $S \subseteq \mathbb{R}^n$, $\|A\|$ and $\|A\|_S$ represent the spectral norm of A and the spectral norm of A with restriction in S, respectively, and $\mathscr{C}(S)$ stands for the complement subspace. $\oplus$ denotes the direct sum. In addition, $\lambda(A)$ and $\delta(A)$ refer to the eigenvalues and singular values of A, and $Re\{\lambda(A)\}$ is the real part of $\lambda(A)$. $\mathscr{C}^1$ denotes the space of continuously differentiable functions, and a function α: $[0, \infty) \to [0, \infty)$ is said to be of class $\mathscr{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. Class $\mathscr{K}_\infty$ denotes the subset of $\mathscr{K}$ consisting of all those functions that are unbounded. A function β: $[0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of class $\mathscr{KL}$ if $\beta(\cdot, t)$ is of class $\mathscr{K}$ for each fixed $t > 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$. The notation $P > 0 (\geq 0)$ means that P is a real symmetric and positive definite (semi-positive definite) matrix.

3.2 Switching Stabilization of Switched Linear Systems

3.2.1 Problem Formulation and Preliminaries

Consider the following switched linear systems

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) \tag{3.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\sigma(t)$ is the switching signal to be designed, which takes its values in the finite set $\mathscr{I} = \{1, \ldots, \mathbb{k}\}$; $\mathbb{k}$ is the number of subsystems. Also, for a switching sequence $0 < t_1 < \cdots < t_i < t_{i+1} < \cdots$, $\sigma(t)$ is continuous from the right everywhere. Moreover, when $t \in [t_i, t_{i+1})$, $\sigma(t) = \sigma(t_i) = p \in \mathscr{I}$, and we say the p^{th} subsystem A_p of (3.1) is activated. In this chapter, we suppose that all the subsystems of (3.1) are unstable.

We first introduce the following definition and lemmas for later development.

Definition 3.1 ([14]) Suppose $A \in \mathbb{C}^{n\times n}$, and $S \subseteq \mathbb{C}^n$ is a subspace. S is A-invariant if $AS \subseteq S$, that is, $\forall v \in S \Rightarrow Av \in S$.

Lemma 3.1 ([14]) *For any subspaces S_1, S_2, $S_1 + S_2$ is also a subspace.*

Lemma 3.2 ([14]) *For any subspaces S_1, S_2, $S_1 \cap S_2$ is also a subspace.*

Next, the following exponential stability definition of system (3.1) is also recalled.

Definition 3.2 ([9]) The equilibrium $\mathbf{x} = \mathbf{0}$ of system (3.1) is globally uniformly exponentially stable (GUES) under certain switching signal $\sigma(t)$ if for initial conditions $\mathbf{x}(t_0)$, there exist constants $\eta_1 > 0$, $\eta_2 > 0$ such that the solution of the system satisfies $\|\mathbf{x}(t)\| \leq \rho_1 \mathrm{e}^{-\rho_2(t-t_0)} \|\mathbf{x}(t_0)\|$, $\forall t \geq t_0$.

In this chapter, we aim at designing a set of switching signals $\sigma(t)$ with the mode-dependent average dwell time (MDADT) property, such that the system (3.1) is GUES. For this purpose, let us now recall the definition of MDADT switching.

Definition 3.3 For a switching signal $\sigma(t)$ and any $T \geq t \geq 0$, let $N_{\sigma p}(T, t)$ be the switching numbers that the p^{th} subsystem is activated over the interval $[t, T]$ and $\mathscr{T}_p(T, t)$ denotes the total running time of the p^{th} subsystem over the interval $[t, T]$, $p \in S$. We say that $\sigma(t)$ has a mode-dependent average dwell time τ_{ap} if there exist positive numbers N_{0p} (we call N_{0p} the mode-dependent chatter bounds here) and τ_{ap} such that

$$N_{\sigma p}(T, t) \leq N_{0p} + \frac{\mathscr{T}_p(T, t)}{\tau_{ap}}, \quad \forall T \geq t \geq 0 \tag{3.2}$$

Remark 3.1 For simplicity, we mark $\sigma(t) \in F_{MDADT}[N_{0p}, \tau_{ap}]$ in this chapter if $\sigma(t)$ is a class of the switching signals defined in Definition 3.2.

3.2.2 Main Results

In correspondence with each subsystem A_p, $p \in \mathscr{I}$, the whole state space can be divided into the two subspaces S_p^s and S_p^u which are defined below.

Definition 3.4 The stable subspace S_p^s, $p \in \mathscr{I}$, is spanned by the eigenvectors corresponding to the eigenvalues $\lambda_k(A_p)$, $k \in \Bbbk_p^s = \{m \in \mathbb{I}_n \mid Re(\lambda_m(A_p)) < 0, p \in \mathscr{I}\}$,

Definition 3.5 The unstable subspace S_p^u, $p \in \mathscr{I}$, is spanned by the eigenvectors corresponding to the eigenvalues $\lambda_k(A_p)$, $k \in \mathbb{k}_p^u = \{m \in \mathbb{I}_n \mid Re(\lambda_m(A_p)) \geq 0, p \in \mathscr{I}\}$,

Before providing our main results, the following lemmas are first developed for later use.

Lemma 3.3 *Consider the switched linear system (3.1). If S is A_p-invariant, $\forall p \in \mathscr{I}$, then, S is $e^{A_p t}$-invariant, $p \in \mathscr{I}$, $\forall t \geq 0$.*

Proof It is noted that, $\forall p \in \mathscr{I}, t \geq 0$,

$$e^{A_p t} = I + tA_p + \frac{t^2}{2!}A_p^2 + \cdots + \frac{t^n}{n!}A_p^n + \cdots \tag{3.3}$$

On the other hand, because S is A_p-invariant, $\forall p \in \mathscr{I}$, one has, $\forall \mathbf{x} \in S$, $n \in \mathbb{Z}^+$,

$$\begin{aligned} A_p^n \mathbf{x} &= A_p^{n-1} A_p \mathbf{x} \\ &= A_p^{n-1}\mathbf{x}_1, (\mathbf{x}_1 = A_p \mathbf{x} \in S) \\ &= A_p^{n-2}\mathbf{x}_2, (\mathbf{x}_2 = A_p \mathbf{x}_1 \in S) \\ &\cdots \\ &= A_p \mathbf{x}_n \in S, (\mathbf{x}_n = A_p \mathbf{x}_{n-1} \in S) \end{aligned} \tag{3.4}$$

Therefore, one can get from (3.3) and (3.4) that, $\forall p \in \mathscr{I}, t \geq 0, \mathbf{x} \in S$,

$$\begin{aligned} e^{A_p t}\mathbf{x} &= \mathbf{x} + tA_p\mathbf{x} + \frac{t^2}{2!}A_p^2\mathbf{x} + \cdots + \frac{t^n}{n!}A_p^n\mathbf{x} + \cdots \\ &\in S \end{aligned} \tag{3.5}$$

which completes the proof. □

Remark 3.2 Lemma 3.3 implies that, if the p^{th} subsystem of system (3.1) is activated with initial condition $\mathbf{x}(t_0) \in S$, the state will stay in S during the running time of the p^{th} operation mode; i.e., $\mathbf{x}(t) = e^{A_p(t-t_0)}\mathbf{x}(t_0) \in S$ if $\mathbf{x}(t_0) \in S$.

Lemma 3.4 *Consider the linear system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$. Let $\lambda^m = \{-\max_k\{\lambda_k(A)\} \mid Re(\lambda_k(A)) < 0, k \in \mathbb{I}_n\}$ and $\lambda^M = \{\max_k\{\lambda_k(A)\} \mid Re(\lambda_k(A)) \geq 0, k \in \mathbb{I}_n\}$; then, there exists a constant $\varepsilon > 0$ such that*

$$\|\exp\{At\}\|_{S^s} \leq \exp\{\varepsilon - \lambda^m t\} \tag{3.6}$$

$$\|\exp\{At\}\|_{S^u} \leq \exp\{\varepsilon + \lambda^M t\} \tag{3.7}$$

where S^s and S^u are the stable subspace and unstable subspace of A, respectively.

Proof It is obvious that both S^s and S^u are A-invariant, and thus are e^{At}-invariant. We can choose the following orthogonal matrix

$$T = [a_1, a_2, \ldots, a_r, b_{r+1}, b_{r+2}, \ldots, b_n] \tag{3.8}$$

appropriately, where $\{a_1, a_2, \ldots, a_r\}$ and $\{b_{r+1}, b_{r+2}, \ldots, b_n\}$ are the bases of S^s and S^u. Note that S^s and S^u are also the stable subspace and unstable subspace corresponding to e^A. Then, one has that

$$T^{-1}\exp\{At\}T = \exp\{diag\{A^s t, A^u t\}\} \tag{3.9}$$

where A^s, A^u are appropriate matrices satisfying $\lambda(A^s) < 0$ and $\lambda(A^u) \geq 0$, respectively. Therefore, it follows from (3.9) that

$$\begin{aligned}\|\exp\{At\}\|_{S^s} &\leq \|T\|_{S^s}\left\|T^{-1}\right\|_{S^s}\left\|\exp\{diag\{A^s t, A^u t\}\}\right\|_{S^s}\\ &= \|T\|_{S^s}\left\|T^{-1}\right\|_{S^s}\left\|\exp\{A^s t\}\right\|\\ &\leq \|T\|_{S^s}\left\|T^{-1}\right\|_{S^s}\exp\{-\lambda^m t\}\end{aligned} \tag{3.10}$$

$$\begin{aligned}\|\exp\{At\}\|_{S^s} &\leq \|T\|_{S^u}\left\|T^{-1}\right\|_{S^u}\left\|\exp\{diag\{A^s t, A^u t\}\}\right\|_{S^s}\\ &= \|T\|_{S^u}\left\|T^{-1}\right\|_{S^u}\left\|\exp\{diag\{A^u t\}\right\|\\ &\leq \|T\|_{S^u}\left\|T^{-1}\right\|_{S^u}\exp\{\lambda^M t\}\end{aligned} \tag{3.11}$$

Finally, set $\varepsilon = \ln(\frac{\varepsilon_1}{\varepsilon_2})$, $\varepsilon_1 = \max\delta(T)$ and $\varepsilon_2 = \min\delta(T)$. This together with (3.10) and (3.11) completes the proof. □

Subsequently, we define $\lambda_p^m = \{-\max_k\{\lambda_k(A_p)\} \mid Re(\lambda_k(A_p)) < 0, k \in \mathbb{I}_n, p \in \mathscr{I}\}$, and $\lambda_p^M = \{\max_k\{\lambda_k(A_p)\} \mid Re(\lambda_k(A_p)) \geq 0, k \in \mathbb{I}_n, p \in \mathscr{I}\}$ for switched system (1). Then, Lemma 3.4 can be trivially extended to the following result for switched system (3.1).

Lemma 3.5 *Consider the switched linear system (3.1). There exist some constants $\varepsilon_p > 0$, $p \in \mathscr{I}$, such that*

$$\left\|\exp\{A_p t\}\right\|_{S_p^s} \leq \exp\{\varepsilon_p - \lambda_p^m t\} \tag{3.12}$$

$$\left\|\exp\{A_p t\}\right\|_{S_p^u} \leq \exp\{\varepsilon_p + \lambda_p^M t\} \tag{3.13}$$

Theorem 3.1 *Consider the switched linear system (3.1). For given constants $\alpha_p > \lambda_p^M > 0$, $\lambda_p^m > \beta_p > 0$, $p \in \mathscr{I}$, and η_p, if there exist two sets $\mathscr{I}_1, \mathscr{I}_2 \subset \mathscr{I}$ ($\mathscr{I}_1 \cup \mathscr{I}_2 = \mathscr{I}$) such that $\Omega_1 = \sum_{p\in\mathscr{I}_1} S_p^u$ and $\Omega_2 = \cap_{p\in\mathscr{I}_1} S_p^s$ are A_p-invariant, $p \in \mathscr{I}$, and*

$$\Omega_1 \subseteq \cap_{p\in\mathscr{I}_2} S_p^s \tag{3.14}$$

then, the system (3.1) is GUES for any switching signal $\sigma(t) \in F_{MDADT}[N_{0p}, \tau_{ap}]$ *satisfying*

$$\tau_{ap} \geq \frac{\varepsilon_p}{\alpha_p - \lambda_p^M}, \forall p \in \mathscr{I} \tag{3.15}$$

$$\tau_{ap} \geq \frac{\varepsilon_p}{\lambda_p^m - \beta_p}, \forall p \in \mathscr{I} \tag{3.16}$$

$$\sum_{p\in\mathscr{I}_1} (\alpha_p \mathscr{T}_p(T,0) + \eta_p \mathscr{T}_p(T,0)) \leq \sum_{p\in\mathscr{I}_2} (\beta_p \mathscr{T}_p(T,0) - \eta_p \mathscr{T}_p(T,0)) \tag{3.17}$$

$$\sum_{p\in\mathscr{I}_2} (\alpha_p \mathscr{T}_p(T,0) + \eta_p \mathscr{T}_p(T,0)) \leq \sum_{p\in\mathscr{I}_1} (\beta_p \mathscr{T}_p(T,0) - \eta_p \mathscr{T}_p(T,0)) \tag{3.18}$$

Proof By Lemmas 3.1 and 3.2, it is obvious that Ω_1 and Ω_2 are two subspaces in $\mathbb{R}^n$, and it is also clear from the definitions of Ω_1 and Ω_2 that,

$$\Omega_1 \cap \Omega_2 = \varnothing \tag{3.19}$$

It is also true that,

$$\mathscr{C}(\Omega_2) = \mathscr{C}(\cap_{p\in\mathscr{I}_1} S_p^s) = \sum_{p\in\mathscr{I}_1} \mathscr{C}(S_p^s) = \sum_{p\in\mathscr{I}_1} S_p^u = \Omega_1 \tag{3.20}$$

which implies

$$\Omega_1 \oplus \Omega_2 = \mathbb{R}^n \tag{3.21}$$

Next, for any sufficiently large $T > 0$, let $t_0 = 0$ and $t_1, t_2 \ldots t_i, t_{i+1}, \ldots t_{N_\sigma(T,0)}$ denote the switching times on the interval $[0, T]$, where $N_\sigma(T, 0) = \sum_{p=1}^{\Bbbk} N_{\sigma p}(T, 0)$. Then, when the initial condition $\mathbf{x}(0) \in \Omega_1$, it yields from Lemma 3.3 that, $\forall T > 0$,

$$\begin{aligned}\mathbf{x}(T) &= \exp\{A_{\sigma(t_{N_\sigma(T,0)})}(T - t_{N_\sigma(T,\,0)})\}\cdots\exp\{A_{\sigma(t_i)}(t_{i+1} - t_i)\}\cdots \\ &\quad \exp\{A_{\sigma(t_0)}(t_1 - t_0)\}\mathbf{x}(0) \\ &\in \Omega_1\end{aligned} \tag{3.22}$$

Therefore, by (3.14), (3.22), Definition 3.3 and Lemma 3.5, it arrives at, $\forall T > 0$,

$$\begin{aligned}
\|\mathbf{x}(T)\| &\le \prod_{s\in\Phi_1} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_1} \prod_{s\in\Phi_2} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_1} \|\mathbf{x}(0)\| \\
&\le \prod_{s\in\Phi_1} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_1} \prod_{s\in\Phi_2} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{S_p^s} \|x(0)\| \\
&\le \prod_{p\in\mathscr{I}_1} \exp\{N_{\sigma p}(T,0)\varepsilon_p\}\exp\{\lambda_p^M \mathscr{T}_p(T,0)\} \\
&\quad \prod_{p\in\mathscr{I}_2} \exp\{N_{\sigma p}(T,0)\varepsilon_p\}\exp\{-\lambda_p^m \mathscr{T}_p(T,0)\}\,\|\mathbf{x}(0)\| \\
&= \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{\sigma p}(T,0)\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}_1} \lambda_p^M \mathscr{T}_p(T,0)\right. \\
&\quad \left. -\sum\nolimits_{p\in\mathscr{I}_2} \lambda_p^m \mathscr{T}_p(T,0)\right\} \|x(0)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}_1} \lambda_p^M \mathscr{T}_p(T,0) - \sum\nolimits_{p\in\mathscr{I}_2} \lambda_p^m \mathscr{T}_p(T,0)\right. \\
&\quad \left. + \sum\nolimits_{p\in\mathscr{I}} \frac{\varepsilon_p \mathscr{T}_p(T,0)}{\tau_{ap}}\right\} \|\mathbf{x}(0)\| \qquad (3.23) \\
&= \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}_1} (\lambda_p^M + \frac{\varepsilon_p}{\tau_{ap}})\mathscr{T}_p(T,0)\right. \\
&\quad \left. -\sum\nolimits_{p\in\mathscr{I}_2} (\lambda_p^m - \frac{\varepsilon_p}{\tau_{ap}})\mathscr{T}_p(T,0)\right\} \|\mathbf{x}(0)\| \qquad (3.24)
\end{aligned}$$

where Φ_1 and Φ_2 denote the sets of s satisfying $\sigma(t_s) \in \mathscr{I}_1$ and $\mathscr{I}_2$, respectively. Therefore, if we specify

$$\tau_{ap} \ge \frac{\varepsilon_p}{\alpha_p - \lambda_p^M},\ p \in \mathscr{I}_1 \qquad (3.25)$$

$$\tau_{ap} \ge \frac{\varepsilon_p}{\lambda_p^m - \beta_p},\ p \in \mathscr{I}_2 \qquad (3.26)$$

then, it is clear from (3.17) and (3.23) that

$$\begin{aligned}
&\|\mathbf{x}(T)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}_1} \alpha_p \mathscr{T}_p(T,0) - \sum\nolimits_{p\in\mathscr{I}_2} \beta_p \mathscr{T}_p(T,0)\right\} \|\mathbf{x}(0)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}} -\eta_p \mathscr{T}_p(T,0)\right\} \|\mathbf{x}(0)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{-\min_{p\in\mathscr{I}}\{\eta_p\}T\right\} \|\mathbf{x}(0)\| \qquad (3.27)
\end{aligned}$$

which means that the system is GUES under MDADT satisfying (3.18), (3.24) and (3.25).

On the other hand, when the initial condition $\mathbf{x}(0) \in \Omega_2$, it is true that $\mathbf{x}(T) \in \Omega_2, \forall T > 0$, and

$$
\begin{aligned}
\|\mathbf{x}(T)\| &\le \prod_{s\in\Phi_1} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_2} \prod_{s\in\Phi_2} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_2} \|\mathbf{x}(0)\| \\
&\le \prod_{s\in\Phi_1} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{S_p^s} \prod_{s\in\Phi_2} \left\|\exp\{A_{\sigma(t_s)}(t_{s+1}-t_s)\}\right\|_{\Omega_2} \|\mathbf{x}(0)\| \\
&\le \prod_{p\in\mathscr{I}_1} \exp\{N_{\sigma p}(T,0)\varepsilon_p\}\exp\{-\lambda_p^m \mathscr{T}_p(T,0)\} \\
&\quad \prod_{p\in\mathscr{I}_2} \exp\{N_{\sigma p}(T,0)\varepsilon_p\}\exp\{\lambda_p^M \mathscr{T}_p(T,0)\}\|\mathbf{x}(0)\| \\
&= \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{\sigma p}(T,0)\varepsilon_p\right\}\exp\left\{-\sum\nolimits_{p\in\mathscr{I}_1} \lambda_p^m \mathscr{T}_p(T,0)\right. \\
&\quad \left. +\sum\nolimits_{p\in\mathscr{I}_2} \lambda_p^M \mathscr{T}_p(T,0)\right\}\|\mathbf{x}(0)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{-\sum\nolimits_{p\in\mathscr{I}_1} \lambda_p^m \mathscr{T}_p(T,0)\right. \\
&\quad \left. +\sum\nolimits_{p\in\mathscr{I}_2} \lambda_p^M \mathscr{T}_p(T,0) + \sum\nolimits_{p\in\mathscr{I}} \frac{\varepsilon_p \mathscr{T}_p(T,0)}{\tau_{ap}}\right\}\|\mathbf{x}(0)\| \\
&= \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{-\sum\nolimits_{p\in\mathscr{I}_1} \left(\lambda_p^m - \frac{\varepsilon_p}{\tau_{ap}}\right)\mathscr{T}_p(T,0)\right. \\
&\quad \left. +\sum\nolimits_{p\in\mathscr{I}_2} \left(\lambda_p^M + \frac{\varepsilon_p}{\tau_{ap}}\right)\mathscr{T}_p(T,0)\right\}\|\mathbf{x}(0)\|
\end{aligned}
\tag{3.28}
$$

Similarly, if we choose

$$
\tau_{ap} \ge \frac{\varepsilon_p}{\alpha_p - \lambda_p^M},\ p \in \mathscr{I}_2 \tag{3.29}
$$

$$
\tau_{ap} \ge \frac{\varepsilon_p}{\lambda_p^m - \beta_p},\ p \in \mathscr{I}_1 \tag{3.30}
$$

then, it is immediate from (3.18) and (3.27) that

$$
\begin{aligned}
&\|\mathbf{x}(T)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{-\sum\nolimits_{p\in\mathscr{I}_1} \beta_p \mathscr{T}_p(T,0) + \sum\nolimits_{p\in\mathscr{I}_2} \alpha_p \mathscr{T}_p(T,0)\right\}\|\mathbf{x}(0)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{\sum\nolimits_{p\in\mathscr{I}} -\eta_p \mathscr{T}_p(T,0)\right\}\|\mathbf{x}(0)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\}\exp\left\{-\min_{p\in\mathscr{I}}\{\eta_p\}T\right\}\|\mathbf{x}(0)\|
\end{aligned}
\tag{3.31}
$$

Thus, the system is GUES with MDADT satisfying (3.18), (3.28) and (3.29).

Now, we consider the case that the initial condition $\mathbf{x}(0) \in \Omega_3 = \overline{\Omega_1 \cup \Omega_2}$. By (3.21), for any $\mathbf{x}(0) \in \Omega_3$, one can always find

$$
\bar{\mathbf{x}}(0) \in \Omega_1 \tag{3.32}
$$

and

$$\tilde{\mathbf{x}}(0) \in \Omega_2 \tag{3.33}$$

such that

$$\mathbf{x}(0) = \bar{\mathbf{x}}(0) + \tilde{\mathbf{x}}(0) \tag{3.34}$$

It yields from (3.33) that

$$\begin{aligned}
\mathbf{x}(T) &= \exp\{A_{\sigma(t_{N_\sigma(T,0)})}(T - t_{N_\sigma(T,\,0)})\} \cdots \exp\{A_{\sigma(t_i)}(t_{i+1} - t_i)\} \cdots \\
&\quad \exp\{A_{\sigma(t_0)}(t_1 - t_0)\}\mathbf{x}(0) \\
&= \exp\{A_{\sigma(t_{N_\sigma(T,0)})}(T - t_{N_\sigma(T,\,0)})\} \cdots \exp\{A_{\sigma(t_0)}(t_1 - t_0)\}\bar{\mathbf{x}}(0) \\
&\quad + \exp\{A_{\sigma(t_{N_\sigma(T,0)})}(T - t_{N_\sigma(T,\,0)})\} \cdots \exp\{A_{\sigma(t_0)}(t_1 - t_0)\}\tilde{\mathbf{x}}(0) \\
&= \bar{\mathbf{x}}(T) + \tilde{\mathbf{x}}(T)
\end{aligned} \tag{3.35}$$

where $\bar{\mathbf{x}}(T) \in \Omega_1$ and $\tilde{\mathbf{x}}(T) \in \Omega_2$ are the state responses of initial conditions $\bar{\mathbf{x}}(0)$ and $\tilde{\mathbf{x}}(0)$, respectively. It then follows from (3.26) and (3.30) that the underlying system is stabilized by MDADT satisfying (3.17)–(3.18), (3.24)–(3.25) and (3.28)–(3.29).

Finally, we can conclude from (3.24)–(3.26), (3.27)–(3.29) and (3.34) that if (3.14) holds, the switched system (3.1) is GUES under MDADT meeting (3.15)–(3.18), which completes the proof. □

Remark 3.3 It is noted from the proof of Theorem 3.1 that switched system (3.1) is stabilized via the designed MDADT switching, and the decay rate of the state can be set in advance via a scalar $\eta = \min_{p \in \mathscr{I}}\{\eta_p\}$.

As a special case, if all the subsystems of switched system (3.1) are Hurwitz stable, then the sufficient condition for stabilization via MDADT switching is addressed in the following corollary.

Corollary 3.1 *Consider the switched linear system (3.1) composed of all Hurwitz stable subsystems. The system is GUES for any switching signal $\sigma(t) \in \digamma_{MDADT}[N_{0p}, \tau_{ap}]$ satisfying*

$$\tau_{ap} \geq \frac{\varepsilon_p}{\lambda_p^m},\ p \in \mathscr{I} \tag{3.36}$$

Proof Note the fact that all the subsystems are stable. Therefore, in Theorem 3.1, $\mathscr{I}_1 = \varnothing$, $\mathscr{I}_2 = \mathscr{I}$, $\Omega_1 = \varnothing$, and $\Omega_2 = \mathbb{R}^n$. Then, $\forall T > 0$,

$$
\begin{aligned}
\|\mathbf{x}(T)\| &= \left\| \exp\left\{ \sum_{s\in\Phi_1\cup\Phi_2} A_{\sigma(t_s)}(t_{s+1}-t_s) \right\} \right\| \|\mathbf{x}(0)\| \\
&\le \prod_{p\in\mathscr{I}} \exp\{N_{\sigma p}(T,0)\varepsilon_p - \lambda_p^m \mathscr{T}_p(T,0)\} \|\mathbf{x}(0)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\sum\nolimits_{p\in\mathscr{I}} \left(\frac{\varepsilon_p}{\tau_{ap}} - \lambda_p^m\right) \mathscr{T}_p(T,0)\right\} \|\mathbf{x}(0)\| \\
&\le \exp\left\{\sum\nolimits_{p\in\mathscr{I}} N_{0p}\varepsilon_p\right\} \exp\left\{\max_{p\in\mathscr{I}} \left(\frac{\varepsilon_p}{\tau_{ap}} - \lambda_p^m\right) T\right\} \|\mathbf{x}(0)\|
\end{aligned} \tag{3.37}
$$

Thus, we can see from Definition 3.2 and (3.36) that the underlying system is exponentially stabilized via MDADT satisfying (3.35). □

Remark 3.4 The above theorem and corollary provide sufficient conditions of switching stabilization for switched system (3.1) comprising all unstable subsystems and all stable subsystems, respectively. An example in the next section will show the validity of the obtained criteria.

3.2.3 Simulation Results

In this section, a numerical example of switched linear systems with all unstable subsystems is presented to show the effectiveness of the developed approaches.

Example 3.1 Consider the switched linear systems consisting of three subsystems described by:

$$
A_1 = \begin{bmatrix} -20 & -12.5 & -12.5 \\ 0 & -7.5 & 12.5 \\ 0 & 12.5 & -7.5 \end{bmatrix}, A_2 = \begin{bmatrix} -7.5 & 15 & -2.5 \\ 17.5 & -5 & 2.5 \\ -17.5 & -15 & -22.5 \end{bmatrix}, A_3 = \begin{bmatrix} -7.5 & -12.5 & 0 \\ -12.5 & -7.5 & 0 \\ 12.5 & 12.5 & 5 \end{bmatrix}.
$$

First, the state responses of each subsystem with the same initial condition $\mathbf{x}(0) = [\,5\ -5\ 10\,]^T$ are depicted in Fig. 3.1 from which it is seen that all the three subsystems are unstable. Furthermore, the simulation results with four random switching signals are given in Fig. 3.2 which shows that the above switched system is unstable under these switching signals.

Then, our purpose here is to design a set of mode-dependent average dwell time switching to exponentially stabilize the above switched systems. It is clear that $\lambda(A_1) = \{5, -20, -20\}$, $\lambda(A_2) = \{-20, -25, 10\}$, $\lambda(A_3) = \{5, 5, -20\}$, $\lambda_1^M = 5$, $\lambda_2^M = 10$, $\lambda_3^M = 5$, $\lambda_1^m = 20$, $\lambda_2^m = 20$, $\lambda_3^m = 20$. We choose $\mathscr{I}_1 = \{1, 3\}$, $\mathscr{I}_2 = \{2\}$. Therefore,

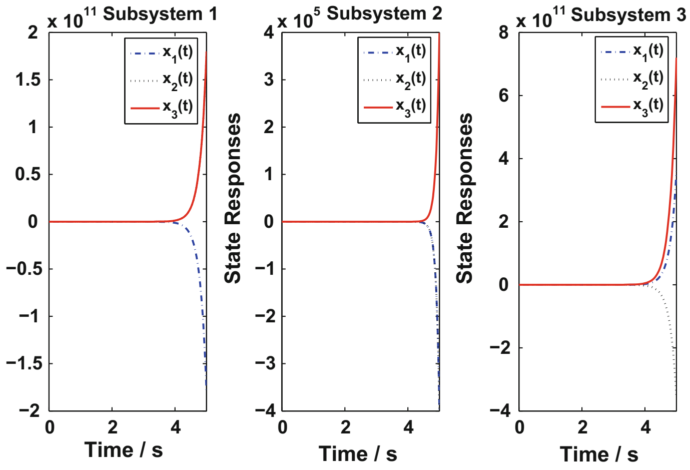

Fig. 3.1 The state responses of each subsystem

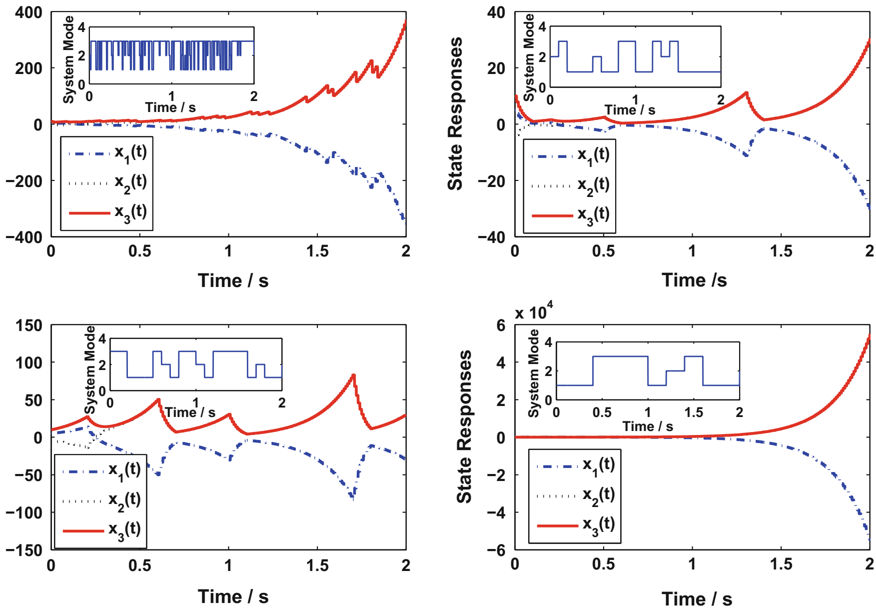

Fig. 3.2 The state responses of the system with different random switching signals

$$\Omega_1 = span\left\{\begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \end{bmatrix}\right\}, \Omega_2 = span\left\{\begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}\right\},$$

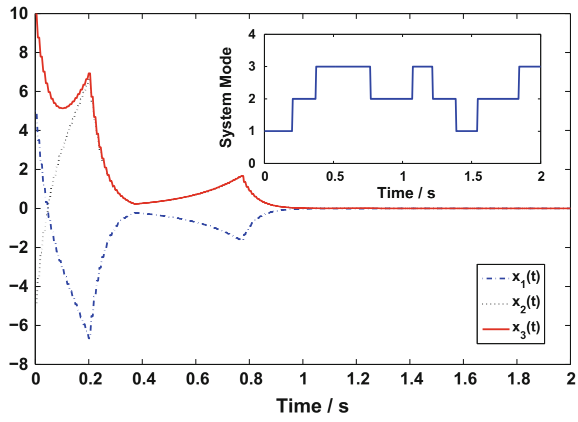

Fig. 3.3 The state responses of the system under the designed MDADT switching signals

On the other hand, it is not hard to get that Ω_1 and Ω_2 are A_p-invariant, $p \in \{1, 2, 3\}$, and satisfy the condition (3.14).

Set $\eta_p = 0.1$, $\varepsilon_p = 0.69$, $p = \{1, 2, 3\}$, $\alpha_1 = \alpha_3 = 10$, $\alpha_2 = 14$, $\beta_1 = \beta_3 = 15$, $\beta_2 = 16$. Based on Theorem 3.1, one can get a MDADT switching signal satisfying (3.17), (3.18) and

$$\tau_{a1} \geq 0.14, \tau_{a2} \geq 0.17, \tau_{a3} \geq 0.14 \tag{3.38}$$

To illustrate the correctness of the theoretical results, we now generate one possible switching sequences with the MDADT property (3.37). Then, one can obtain the corresponding state responses of the system as shown in Fig. 3.3, for the same initial state condition. It can be concluded from the curves that the underlying system is stabilized by the designed MDADT switching signal.

Finally, from the above demonstrations, we obtain that Theorem 3.1 provides an effective stabilization approach via MDADT switching for switched linear systems composed of unstable subsystems.

3.2.4 Conclusions

This section is concerned with switching stabilization for switched linear systems consisting of unstable modes. Based on the invariant subspace theory, the advanced mode-dependent average dwell time (MDADT) switching, is introduced to stabilize the systems under consideration. Then, the corresponding result is extended to switched systems composed of all Hurwitz stable subsystems. Finally, a numerical example is provided to demonstrate the correctness and effectiveness of the obtained results.

3.3 Switching Stabilization of Switched Nonlinear Systems

3.3.1 Problem Formulation and Preliminaries

This section presents some definitions and preliminary results that will be used throughout the remainder of this chapter. Consider the following switched nonlinear systems,

$$\dot{\mathbf{x}}(t) = \sum_{p=1}^{m} \delta_p(\sigma(t)) f_p(\mathbf{x}(t), t), \mathbf{x}(t_0) = \mathbf{x}_0, t \geq t_0, \tag{3.39}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, and $\mathbf{x}_0$ and $t_0 \geq 0$ denote the initial state and initial time, respectively; $\sigma(t)$ is a switching signal which is a piecewise constant function from the right of time and takes its values in the finite set $S = \{1, \ldots, m\}$, where $m > 1$ is the number of subsystems. $f_p : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ are smooth functions for any $\sigma(t) = p \in S$. Moreover, all the subsystems in system (3.39) may be unstable.

For a switching sequence, $0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $\sigma(t)$ may be either autonomous or controlled. When $t \in [t_k, t_{k+1})$, we say $\sigma(t_k)^{th}$ mode is active; i.e., the indication functions $\delta_p(\sigma(t))$ satisfy:

$$\delta_p(\sigma(t)) = \begin{cases} 1, & if\ \sigma(t) = p, \\ 0, & otherwise. \end{cases} \tag{3.40}$$

The switched nonlinear system (3.39) can be described by fuzzy systems, and the p^{th} fuzzy subsystem is represented as follows.

Model rule R_p^i : IF $\theta_1(t)$ is M_{p1}^i and $\cdots$ and $\theta_l(t)$ is M_{pl}^i, THEN

$$\dot{\mathbf{x}}(t) = A_{pi}\mathbf{x}(t),\ t \geq t_0,\ i \in R = \{1, 2, \cdot \cdot \cdot, r\}, p \in S, \tag{3.41}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector; M_{pj}^i $(j = 1, 2, \ldots, l)$ is the fuzzy set, and r is the number of IF-THEN rules; $\theta_1(t), \theta_2(t) \cdots \theta_p(t)$ are the premise variables; Furthermore, $A_{pi}, i \in R,\ p \in S$ is a real matrix with appropriate dimensions. Thus, through fuzzy blending, the global model of the p^{th} subsystem can be given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(h(t))\mathbf{x}(t) \\ &= \sum_{i=1}^{r} h_{pi}(\theta(t)) A_{pi}\mathbf{x}(t),\ p \in S. \end{aligned} \tag{3.42}$$

$h_{pi}(\theta(t))$ are the normalized membership functions satisfying:

$$h_{pi}(\theta(t)) = \frac{\prod_{j=1}^{l} M_{pj}^i(\theta_j(t))}{\sum_{i=1}^{r}\prod_{j=1}^{l} M_{pj}^i(\theta_j(t))} \geq 0,\ \sum_{i=1}^{r} h_{pi}(\theta(t)) = 1, \tag{3.43}$$

where $M^i_{pj}(\theta_j(t))$ represent the grade of the membership function of premise variable $\theta_j(t)$ in M^i_{pj}. Finally, we can describe switched nonlinear system (3.39) in the following form,

$$\dot{\mathbf{x}}(t) = \sum_{p=1}^{m}\sum_{i=1}^{r}\delta_p(\sigma(t))h_{pi}(\theta(t))A_{pi}\mathbf{x}(t). \tag{3.44}$$

Next, we introduce the following definition for later use.

Definition 3.6 [15] The equilibrium $x = 0$ of switched system (3.39) is globally asymptotically stable (GAS) under a certain switching signal $\sigma(t)$ if there exists a $\mathscr{K}\mathscr{L}$ function β such that the solution of the system satisfies the inequality $\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(t_0)\|, t),\ \forall t \geq t_0$, with any initial conditions $x(t_0)$.

In the following, our goal is to find a set of switching signals with the ADT property, such that the switched system (3.39) is GAS. For this purpose, we first define a new class of ADT switching signals.

Definition 3.7 For a switching signal $\sigma(t)$ and each $T \geq t \geq 0$, let $N_\sigma(T, t)$ denote the number of discontinuities of $\sigma(t)$ in the interval (t, T). We say that $\sigma(t)$ has an average dwell time τ_a if there exist two positive numbers N_0 (we call N_0 the chatter bound here) and τ_a such that

$$N_\sigma(T, t) \geq N_0 + \frac{T - t}{\tau_a}, \quad \forall T \geq t \geq 0. \tag{3.45}$$

3.3.2 Main Results

In this section, we consider the switching stabilization for switched nonlinear systems described in the previous section. Next, we are in a position to provide the first switching stabilization condition for switched nonlinear systems (3.39) in the following theorem by designing ADT switching signals defined in Definition 3.7.

Theorem 3.2 *Consider switched nonlinear system (3.39). Suppose that there exist a switching sequence $\xi = \{t_0, t_1, \ldots t_k, \ldots t_{N_{\sigma(t)}}\}$ satisfying (3.45), a set of C^1 non-negative functions $V_p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $p \in S$, two class K_∞ functions α_1 and α_2, and two positive numbers $\lambda > 0$ and $0 < \mu < 1$ such that*

$$\alpha_1(\|\mathbf{x}(t)\|) \leq V_p(\mathbf{x}(t), t) \leq \alpha_2(\|\mathbf{x}(t)\|), \forall p \in S, \tag{3.46}$$

$$\dot{V}_p(\mathbf{x}(t), t)) \leq \lambda V_p(\mathbf{x}(t), t), \forall p \in S, \tag{3.47}$$

$$V_q(\mathbf{x}(t_k^+), t_k^+) \leq \mu V_p(\mathbf{x}(t_k^-), t_k^-), \quad \forall p, q \in S \tag{3.48}$$

$$\tau_a \leq \frac{-ln\mu}{\lambda}. \tag{3.49}$$

Then switched system (3.39) is globally asymptotically stable under the switching sequence ξ generated by $\sigma(t)$.

Proof Without loss of generality, we denote $\xi = \{t_0, t_1, \dots t_k, \dots t_{N_{\sigma(t)}}\}$ as the switching sequence on time interval $[0, T]$ for any $T > 0,\ t_0 = 0$.

Next, we establish a multiple Lyapunov function (MLF) for switched nonlinear system (3.39) as follows,

$$V(\mathbf{x}(t), t) = \sum_{p=1}^{m} \delta_p(\sigma(t)) V_p(\mathbf{x}(t), t). \tag{3.50}$$

Then we consider the function

$$W(t) = e^{-\lambda t} \sum_{p=1}^{m} \delta_p(\sigma(t)) V_p(\mathbf{x}(t), t). \tag{3.51}$$

It is clear that it is piecewise differentiable along solutions of (3.39). When $t \in [t_k, t_{k+1})$, we get from (3.47) that

$$\begin{aligned} \dot{W}(t) &= -\lambda e^{-\lambda t} V_p(\mathbf{x}(t), t) + e^{-\lambda t} \dot{V}_p(\mathbf{x}(t), t) \\ &\le -\lambda e^{-\lambda t} V_p(\mathbf{x}(t), t) + e^{-\lambda t} \lambda V_p(\mathbf{x}(t), t) \\ &= 0. \end{aligned} \tag{3.52}$$

Thus $W(t)$ is nonincreasing when $t \in [t_k, t_{k+1})$. This together with (3.48) gives that

$$\begin{aligned} W(t_{k+1}^{+}) &= e^{-\lambda t_{k+1}^{+}} V_p(\mathbf{x}(t_{k+1}^{+}), t_{k+1}^{+}) \\ &\le \mu e^{-\lambda t_{k+1}^{-}} V_p(x(t_{k+1}^{-}), t_{k+1}^{-}) \\ &= \mu W(t_{k+1}^{-}) \\ &\le \mu W(t_k). \end{aligned} \tag{3.53}$$

By integrating this for $t \in [t_k, t_{k+1})$, it yields that

$$\begin{aligned} W(T^{-}) &\le W(t_{N_\delta}) \\ &\le \mu W(t_{N_\delta}^{-}) \\ &\le \mu W(t_{N_\delta - 1}) \\ &\quad \dots \\ &\le \mu^{N_\delta} W(t_0). \end{aligned} \tag{3.54}$$

One can easily obtain from the definition of $W(t)$ that

$$e^{-\lambda T} V_{\delta(T^-)}(\mathbf{x}(T), T) \le \mu^{N_\delta} V_{\delta(t_0)}(\mathbf{x}(t_0), t_0). \tag{3.55}$$

Moreover, it can be derived from (3.45) and (3.55) that

$$\begin{aligned} V_{\delta(T^-)}(\mathbf{x}(T), T) &\le e^{\lambda T} e^{N_\delta \ln \mu} V_{\delta(t_0)}(\mathbf{x}(t_0), t_0) \\ &\le e^{\lambda T} e^{(N_0 + \frac{T}{\tau_a}) \ln \mu} V_{\delta(t_0)}(\mathbf{x}(t_0), t_0) \\ &= e^{N_0 \ln \mu} e^{(\lambda + \frac{\ln \mu}{\tau_a})T} V_{\delta(t_0)}(\mathbf{x}(t_0), t_0). \end{aligned} \tag{3.56}$$

Finally, we can conclude from (3.56) that, if τ_a satisfies the condition in (3.49), then $V_{\delta(T^-)}(\mathbf{x}(T), T)$ exponentially converges to zero as $T \to \infty$,

By (3.46), we can get that

$$\|\mathbf{x}(T)\| = \alpha_1^{-1}(\mu^{N_0} e^{\lambda T} \alpha_2(\|\mathbf{x}_0\|),$$

which verifies the global asymptotic stability by Definition 3.6. Therefore, switched nonlinear system (3.39) is asymptotically stabilized by our proposed ADT switching signals (3.45) with (3.49) if the conditions (3.46)–(3.48) hold. This completes the proof.

In the following, we utilize the T-S fuzzy modeling approach to represent nonlinear system (3.39), to develop more applicable results.

Note that the traditional linear multiple quadratic Lyapunov function $V_p(\mathbf{x}(t)) = \mathbf{x}^T(t) P_p \mathbf{x}(t)$, where $P_p > 0$, $\forall p \in S$, will not satisfy the condition $P_q \le \mu P_p \; \forall p, q \in S$ because $0 < \mu < 1$. Hence, we choose a time-variant (TV) positive definite matrix $P_p(t)$ to construct a TV-MQLF for switched T-S fuzzy system (3.44) as follows,

$$V_p(\mathbf{x}(t), t) = \mathbf{x}^T(t) P_p(t) \mathbf{x}(t), \; \forall p \in S. \tag{3.57}$$

Then it is immediately clear that $V_q(\mathbf{x}(t_k^+), t_k^+) \le \mu V_p(\mathbf{x}(t_k^-), t_k^-), \; \forall p, q \in S$ can be expressed by $P_q(t_k^+) \le \mu P_p(t_k^-), \; p \ne q, \; \forall p, q \in S$. Next, we resort to the discretized Lyapunov function technique to numerically check the existence of such a matrix function $P_p(t)$ which is, however, difficult to be checked in the continuous case.

First of all, giving τ_a a sufficient small lower bound $\tau^* > 0$, we divide the interval $[t_k, t_k + \tau^*)$ into K segments. The length of each section is equal to $l = \frac{\tau^*}{K}$, and then the interval $[t_k, t_k + \tau^*)$ can be described as $G_{p,n} = [t_k + H_n, t_k + H_{n+1})$, $H_n = nl, \; n = 1, 2, \ldots, K-1$, Next, we use a linear interpolation formula to describe the continuous-time matrix function $P_p(t)$ which is chosen to be linear within each segment $G_{p,n} = [t_k + H_n, t_k + H_{n+1}), \; n = 1, 2, \ldots, K-1$. When $t \in G_{p,n}, \; n = 1, 2, \ldots, K-1$

$$\begin{aligned}P_p(t) &= \frac{t-t_k-H_{n+1}}{t_k+H_n-t_k-H_{n+1}}P_{p,n}+\frac{t-t_k-H_n}{t_k+H_{n+1}-t_k-H_n}P_{p,n+1}\\ &= \frac{t-t_k-H_{n+1}}{-l}P_{p,n}+\frac{t-t_k-H_n}{l}P_{p,n+1}\\ &= (1-\gamma)P_{p,n}+\gamma P_{p,n+1}\\ &= P_p^{(n)}(\gamma),\end{aligned} \tag{3.58}$$

where $P_{p,n} = P_p(t_k + H_n)$, $P_{p,n+1} = P_p(t_k + H_{n+1})$, $0 < \gamma = \frac{t-t_k-H_n}{l} < 1$. In the interval $[t_k, t_k + \tau^*)$, the continuous-time matrix function $P_p(t),\ p \in S$, is determined by $P_{p,n}\ n = 1, 2, \ldots, K,\ p \in S$. On the other hand, in the interval $[t_k + \tau^*, t_{k+1})$, the matrix function $P_p(t),\ p \in S$ is fixed by a constant matrix $P_p(t) = P_{p,K},\ p \in S$. Thus, the TV-MQLF for switched T-S fuzzy system (3.44) for mode $p \in S$ can be described as

$$V_p(\mathbf{x}(t),t) = \begin{cases}\mathbf{x}^T(t)P_p^{(n)}\mathbf{x}(t), & t \in G_{p,n},\ n = 1,2,\ldots,K-1\\ \mathbf{x}^T(t)P_{p,K}\mathbf{x}(t), & t \in [t_k+\tau^*, t_{k+1}).\end{cases} \tag{3.59}$$

Moreover, it can be derived from (3.59) that for any $t \in G_{p,n},\ n = 1, 2, \ldots, K-1$

$$\begin{aligned}\dot V_p(\mathbf{x}(t),t) &= \dot{\mathbf{x}}^T(t)P_p(t)\mathbf{x}(t)+\mathbf{x}^T(t)\dot P_p(t)\mathbf{x}(t)+\mathbf{x}^T(t)P_{pi}(t)\dot{\mathbf{x}}(t)\\ &= \sum_{i=1}^r h_{pi}(\theta(t))[(A_{pi}\mathbf{x}(t))^T P_p(t)\mathbf{x}(t)+\mathbf{x}^T(t)\dot P_p(t)\mathbf{x}(t)+\mathbf{x}^T(t)P_p(t)A_{pi}\mathbf{x}(t)]\\ &= \sum_{i=1}^r h_{pi}(\theta(t))\mathbf{x}^T(t)[A_{pi}{}^T P_p(t)+P_p(t)A_{pi}+\dot P_p(t)]\mathbf{x}(t).\end{aligned} \tag{3.60}$$

When $t \in G_{p,n},\ n = 1, 2, \ldots, K-1$, one can immediately get from (3.58) that

$$\begin{aligned}\dot P_p(t) &= -\dot\gamma P_{p,n}+\dot\gamma P_{p,n+1}\\ &= (P_{p,n+1}-P_{p,n})\frac{K}{\tau^*}\\ &= \Pi_p^n.\end{aligned} \tag{3.61}$$

In the sequel, we can obtain from (3.58), (3.60) and (3.61) that for any $t \in G_{p,n},\ n = 1, 2, \ldots, K-1$,

$$\begin{aligned}\dot V_p(\mathbf{x}(t),t) &= \sum_{i=1}^r h_{pi}(\theta(t))\mathbf{x}^T(t)[A_{pi}^T P_p^{(n)}+P_p^{(n)}A_{pi}+\Pi_{pi}^n]\mathbf{x}(t)\\ &= \sum_{i=1}^r h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)(A_{pi}^T P_{p,n}+P_{p,n}A_{pi}+\Pi_{pi}^n)\\ &\quad +\gamma(A_{pi}^T P_{p,n+1}+P_{p,n+1}A_{pi}+\Pi_{pi}^n)]\mathbf{x}(t)\end{aligned}$$

$$= \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)\Phi_{pi,1}^{(n)} + \gamma\Phi_{pi,2}^{(n)}]\mathbf{x}(t), \tag{3.62}$$

where $\Phi_{pi,1}^{(n)} = A_{pi}^T P_{p,n} + P_{p,n}A_{pi} + \Pi_p^n$ and $\Phi_{pi,2}^{(n)} = A_{pi}^T P_{p,n+1} + P_{p,n+1}A_{pi} + \Pi_p^n$.

Thus, a switching stabilization condition for switched T-S fuzzy system (3.44) can be obtained on the basis of the above developments.

Theorem 3.3 *Consider switched T-S fuzzy system (3.44), and let* $\lambda > 0$, $0 < \mu < 1$, *and* $\tau^* > 0$ *be given constants. If there exists a set of matrices* $P_{p,n} > 0$, $n = 0, 1, 2, \ldots, K$, $p \in S$, *such that* $\forall n = 0, 1, 2, \ldots, K, \forall i \in R$, $p \neq q$, $\forall (p \times q) \in S \times S$,

$$\Phi_{pi,1}^{(n)} - \lambda P_{p,n} < 0, \tag{3.63}$$

$$\Phi_{pi,2}^{(n)} - \lambda P_{p,n+1} < 0, \tag{3.64}$$

$$A_{pi}^T P_{p,K} + P_{p,K}A_{pi} - \lambda P_{p,K} < 0, \tag{3.65}$$

$$P_{q,0} - \mu P_{p,K} \leq 0, \tag{3.66}$$

where $\Phi_{pi,1}^{(n)}$ *and* $\Phi_{pi,2}^{(n)}$ *are defined in (3.62), then, the system is GAS for any switching signal with ADT satisfying*

$$\tau^* \leq \tau_a \leq \frac{-ln\mu}{\lambda}. \tag{3.67}$$

Proof. When $t \in G_{p,n}$, $n = 1, 2, \ldots, K-1$, by the discussions in (3.62), it can be seen that if (3.63) and (3.64) hold, then,

$$\begin{aligned}
&\dot{V}_p(\mathbf{x}(t), t) - \lambda V_p(\mathbf{x}(t), t) \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)\Phi_{pi,1}^{(n)} + \gamma\Phi_{pi,2}^{(n)} - \lambda P_p^{(n)}(\gamma)]\mathbf{x}(t) \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)(A_{pi}^T P_{pi,n} + P_{pi,n}A_{pi} + \Pi_p^n - \lambda P_{p,n}) \\
&\quad + \gamma(A_{pi}^T P_{p,n+1} + P_{p,n+1}A_{pi} + \Pi_p^n - \lambda P_{p,n+1})]\mathbf{x}(t) \\
&= \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)[(1-\gamma)(\Phi_{p,1}^{(n)} - \lambda P_{p,n}) + \gamma(\Phi_{pi,2}^{(n)} - \lambda P_{p,n+1})]\mathbf{x}(t) \\
&< 0.
\end{aligned} \tag{3.68}$$

Moreover, when $t \in [t_k + \tau^*, t_{k+1})$, we have from (3.59), (3.65) and (3.68) that

$$\dot{V}_p(\mathbf{x}(t), t) - \lambda V_p(\mathbf{x}(t), t) = \sum_{i=1}^{r} h_{pi}(\theta(t))\mathbf{x}^T(t)(A_{pi}^T P_{p,K} + P_{p,K}A_{pi} - \lambda P_{pi,K})\mathbf{x}(t) < 0. \tag{3.69}$$

Thus, we can get that (3.68) and (3.69) hold, which means that

$$\dot{V}_p(\mathbf{x}(t), t) \leq \lambda V_p(\mathbf{x}(t), t).$$

Then, according to (3.59) and (3.65), it can be obtained that

$$V_q(t_k^+, t^+) \leq \mu V_p(t_k^-, t^-).$$

Finally, one can readily conclude from Theorem 3.2 that switched T-S fuzzy system (3.44) is GAS for any switching signal with our proposed ADT (3.45).

Remark 3.5 Compared with Theorem 3.2, the advantage of Theorem 3.3 lies in that the obtained stability condition is formulated in terms of linear matrix inequalities that can be efficiently solved by the LMI toolbox.

3.3.3 Simulation Results

We provide the following example to verify the main results developed in this Sect. 3.2. By using a T-S fuzzy model to represent a given switched nonlinear system composed of all unstable subsystems, a switching signal with our proposed ADT property is designed to asymptotically stabilize the system.

Example 3.2 Consider the switched nonlinear system composed of the following two subsystems,

$$\Sigma_1 = \begin{cases} \dot{x}_1(t) = -7.64x_1(t) + 5.03\sin^2(x_1(t))x_2(t) + 5.84x_2(t) - 6.66\sin^2(x_1(t))x_1(t) \\ \dot{x}_2(t) = -6.44x_1(t) + 4.94x_2(t) - 5.58\sin^2(x_1(t))x_1(t) + 4.21\sin^2(x_1(t))x_2(t), \end{cases}$$

$$\Sigma_2 = \begin{cases} \dot{x}_1(t) = 7.23x_1(t) + 5.031.9\sin^2(x_1(t))x_2(t) - 8.58x_2(t) + 2.96\sin^2(x_1(t))x_1(t) \\ \dot{x}_2(t) = 9.48x_1(t) - 11.28x_2(t) + 3.82\sin^2(x_1(t))x_1(t) - 4.52\sin^2(x_1(t))x_2(t). \end{cases}$$

The state trajectories shown in Figs. 3.4 and 3.5 demonstrate that both subsystems $\sum_1$ and $\sum_2$ are unstable.

Next, we are interested in designing a class of switching signal $\sigma(t)$ with property (3.45) to asymptotically stabilize the above switched system. First, we formulate the T-S fuzzy model of the switched nonlinear system in the following.

When $p = 1$, the Σ_1 can be written as

$$\dot{x}(t) = \begin{bmatrix} -7.64 - 6.66\sin^2(x_1(t)) & 5.84 + 5.03\sin^2(x_1(t)) \\ -6.44 - 5.58\sin^2(x_1(t)) & 4.94 + 4.21\sin^2(x_1(t)) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

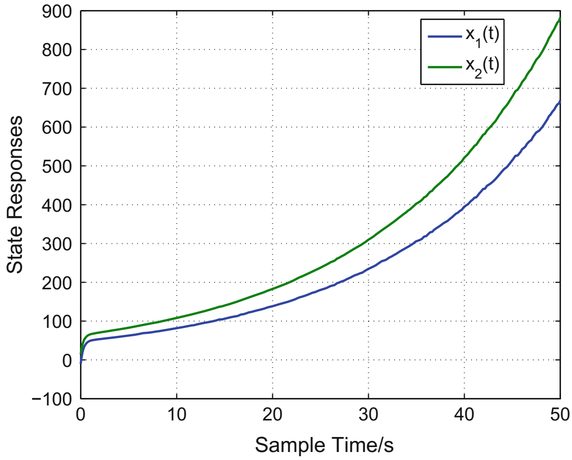

Fig. 3.4 State response of the subsystem $\sum_1$

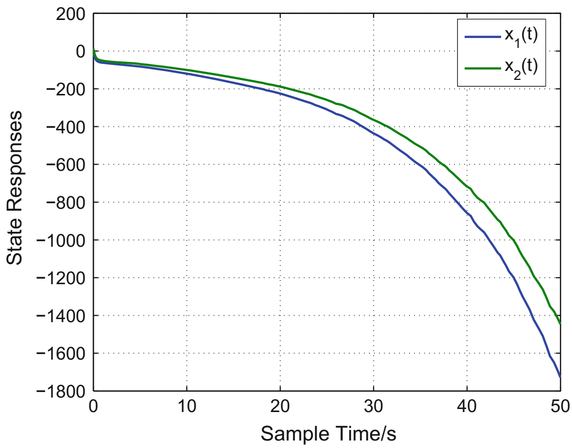

Fig. 3.5 State response of the subsystem $\sum_2$

For the nonlinear term $\sin^2(x_1(t))$, define $\theta(t) = \sin^2(x_1(t))$. Then we have

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -7.64 - 6.66\theta(t) & 5.84 + 5.03\theta(t) \\ 0.6 + 0.4\theta(t) & -0.1 + 3.1\theta(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Next, calculate the minimum and maximum values of $\theta(t)$. The minimum and maximum values of $\theta(t)$ are 0 and 1, respectively. From the minimum and maximum values, $\theta(t)$ can be represented by

$$\theta(t) = \sin^2(x_1(t)) = M_{11}(\theta(t)) \times 0 + M_{12}(\theta(t)) \times 1,$$

where

$$M_{11}(\theta(t)) + M_{12}(\theta(t)) = 1.$$

Therefore the membership functions can be secleted as

$$M_{11}(\theta(t)) = 1 - \sin^2(x_1(t)),\, M_{12}(\theta(t)) = \sin^2(x_1(t)).$$

Then, the first nonlinear subsystem Σ_1 is represented by the following fuzzy model.

Model rule R_1^1 : If $\theta(t)$ is 0, THEN

$$\dot{\mathbf{x}}(t) = A_{11}\mathbf{x}(t),$$

Model rule R_1^2 : If $\theta(t)$ is 1, THEN

$$\dot{\mathbf{x}}(t) = A_{12}\mathbf{x}(t).$$

Its normalized membership functions are $h_1(\theta(t)) = 1 - \sin^2(x_1(t))$, $h_2(\theta(t)) = \sin^2(x_1(t))$, and here,

$$A_{11} = \begin{pmatrix} -7.64 & 5.84 \\ -6.44 & 4.94 \end{pmatrix},\, A_{12} = \begin{pmatrix} -14.3 & 10.87 \\ -12.02 & 9.15 \end{pmatrix}.$$

Thus, through the use of fuzzy blending, the global mode of the 1^{st} fuzzy subsystem can be given by

$$\begin{aligned} \dot{x}(t) &= A(h(t))\mathbf{x}(t) \\ &= \sum_{i=1}^{2} h_{1i}(\theta(t))A_{1i}\mathbf{x}(t), \end{aligned}$$

where

$$\begin{aligned} h_{11}(\theta(t)) &= \frac{M_{11}(\theta(t))}{M_{11}(\theta(t)) + M_{12}(\theta(t))} = 1 - \sin^2(x_1(t)), \\ h_{12}(\theta(t)) &= \frac{M_{12}(\theta(t))}{M_{11}(\theta(t)) + M_{12}(\theta(t))} = \sin^2(x_1(t)). \end{aligned}$$

Similarly, the second nonlinear subsystem Σ_2 can be represented by the following fuzzy model.

Model rule R_2^1 : If $\theta(t)$ is 0, THEN

$$\dot{\mathbf{x}}(t) = A_{21}\mathbf{x}(t),$$

Model rule R_2^2 : If $\theta(t)$ is 1, THEN

$$\dot{\mathbf{x}}(t) = A_{22}\mathbf{x}(t),$$

where

$$A_{21} = \begin{pmatrix} 7.23 & -8.58 \\ 9.48 & -11.28 \end{pmatrix}, A_{22} = \begin{pmatrix} 10.18 & -12.05 \\ 13.30 & -15.80 \end{pmatrix}.$$

Therefore, we can describe switched nonlinear system (3.70) in the following form

$$\dot{x}(t) = \sum_{p=1}^{2}\sum_{i=1}^{2} \delta_p(\sigma(t))h_{pi}(\theta(t))A_{pi}\mathbf{x}(t)\ , i \in R = \{1, 2\},\ \ p = \{1, 2\},$$

where

$$\delta_p(\sigma(t)) = \begin{cases} 1, & if\ \sigma(t) = p, \\ 0, & otherwise. \end{cases}$$

Next, by using Theorem 3.3 and choosing $K = 1$, $\mu = 0.6$, $\eta = 0.7$, $\tau^* = 0.3$, the feasible solutions are obtained as below:

$$P_{1,0} = \begin{pmatrix} 0.5355 & -0.5210 \\ -0.5210 & 0.5436 \end{pmatrix}, P_{1,1} = \begin{pmatrix} 1.0411 & -0.8787 \\ -0.8787 & 0.7933 \end{pmatrix},$$

$$P_{2,0} = \begin{pmatrix} 0.6034 & -0.5065 \\ -0.5065 & 0.4539 \end{pmatrix}, P_{2,1} = \begin{pmatrix} 0.9275 & -0.9049 \\ -0.9049 & 0.9477 \end{pmatrix}.$$

Finally, generating one possible switching sequence by our proposed ADT switching ($\tau_a = 0.5 < -\frac{ln\lambda}{\lambda} = 0.59$), the corresponding state responses of the system under initial state condition $\mathbf{x}(0) = [-10\ 15]^T$, are shown in Fig. 3.6, from which one can see that the switched nonlinear system is stabilized by the designed ADT switching.

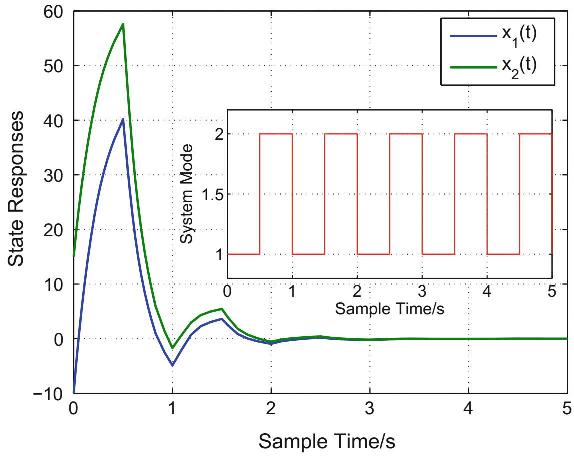

Fig. 3.6 State responses of switched nonlinear system (32) under switching signal $\sigma(t)$ with $\tau_a = 0.5$

3.3.4 Conclusions

The problem of stabilization for switched nonlinear systems composed of unstable subsystems is investigated in the above section by using ADT switching with new property. The stabilization result for the system under consideration is first derived on the basis of our proposed switching signals. After that, the T-S fuzzy modeling method together with a new type of Lyapunov function approach is also used to establish an improved stabilization condition. Finally, a numerical example is provided to verify the correctness and effectiveness of the proposed approach.

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Chapter 4
Adaptive Control of Switched Nonlinear Systems

4.1 Background and Motivation

It has been shown in [1–5] that the adaptive backstepping technique is a powerful tool which has been widely used to solve some complex optimization problems and applied in the fields of industry and engineering. Recently, many adaptive backstepping-based control methods have been used in switched nonlinear systems; see, for example, [6–10] and the references therein. The authors in [11] solved the problem of adaptive stabilization for a class of uncertain switched nonlinear systems whose non-switching part consists of feedback linearizable dynamics. In [12], the authors investigated the problem of adaptive stabilization for a class of switched nonlinearly parameterized systems where the solvability of the adaptive stabilization problem for subsystems is unnecessary.

It is well known that the stability of a switched system under arbitrary switching can be guaranteed if a CLF exists for all subsystems [13]. Therefore, CLF has been extensively used for control synthesis of switched linear systems [14–17]. Recently, there have been some results on the global stabilization problem for switched nonlinear systems in strict-feedback form under arbitrary switchings by using the backstepping technique [9, 18]. Meanwhile, [19] investigated the global stabilization problem for a class of switched nonlinear systems in p-normal form by the so-called power integrator backstepping design method.

In practice, uncertainties inevitably exist in many practical systems. In recent years, some attentions has been paid to both general nonlinear systems and switched nonlinear systems with uncertainties, but most of the obtained results require that the uncertainties should satisfy some additional conditions. However, in many cases, we cannot get the knowledge of system uncertainty a priori, which can only be described by completely unknown functions. In this case, the excellent approximation capability of neural networks (or fuzzy logic systems) has been explored in the literature to tackle the corresponding control problems for either switched systems or non-switched systems. Thus, many significant results have been proposed. To list a few, the authors in [20] investigated the control problem of nonlinear pure-

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feedback systems with unknown nonlinear functions by using the implicit function theorem and NN approximation. The adaptive tracking control problem for a class of uncertain nonlinear strict-feedback systems is solved by [21] using fuzzy logic system approximation. A practical design method is developed by [22] for cooperative tracking control of higher-order nonlinear systems with a dynamic leader. For a class of switched uncertain nonlinear systems without the measurements of the system states, the problem of adaptive neural tracking control via output-feedback was solved in [23] by using a novel switched filter.

However, few results on adaptive tracking control have been developed for lower triangular switched nonlinear systems with completely unknown uncertainties. On the other hand, most system models of the above-mentioned results about adaptive control for switched nonlinear systems are in the strict-feedback form that limits applications of the results to more general switched nonlinear systems. Therefore, considering the adaptive tracking control for switched nonstrict-feedback nonlinear systems with completely unknown uncertainties is more reasonable. In this chapter, the adaptive tracking control problem is investigated for both strict-feedback and nonstrict-feedback switched nonlinear systems with completely unknown uncertainties.

Notations: In this chapter, the notations are standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, the notation $\|\cdot\|$ refers to the Euclidean vector norm. $\mathbb{R}^+$ is the set of all nonnegative real numbers. For positive integers $1 \le i \le n,\ 1 \le j \le m,$ we also denote $\Xi_{i,\max} = \max\{\Xi_{i,j} : 1 \le j \le m\},\ \Xi_{i,\min} = \min\{\Xi_{i,j} : 1 \le j \le m\}$. $\mathscr{C}^i$ stands for a set of functions with continuous i^{th} partial derivatives. For a given matrix A (or vector v), A^T (or v^T) denotes its transpose, and $Tr\{A\}$ denotes its trace when A is a square. $\mathscr{K}$ represents the set of functions: $\mathbb{R}^+ \to \mathbb{R}^+$, which are continuous, strictly increasing and vanishing at zero; $\mathscr{K}_\infty$ denotes a set of functions which is of class $\mathscr{K}$ and unbounded.

4.2 Adaptive Control of Switched Strict-Feedback Nonlinear Systems

4.2.1 Problem Formulation and Preliminaries

Consider a class of switched nonlinear systems in the following form,

$$\begin{aligned}\dot{x}_i &= g_{i,\sigma(t)}x_{i+1} + f_{i,\sigma(t)}(\bar{x}_i),\ i = 1, 2, \ldots, n-1,\\ \dot{x}_n &= g_{n,\sigma(t)}u_{\sigma(t)} + f_{n,\sigma(t)}(\bar{x}_n),\\ y &= x_1,\end{aligned} \tag{4.1}$$

where $\bar{x}_i := (x_1, x_2, \ldots, x_i)^T \in \mathbb{R}^i,\ i = 1, 2, \ldots, n$ is the system state, y is the system output; $\sigma(t) : [0, +\infty) \to M = \{1, 2, \ldots, m\}$ is the switching signal;

$u_k \in R$ is the control input of the k^{th} subsystem, For any $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$, $f_{i,k}(\bar{x}_i)$ is an unknown smooth nonlinear function representing the system uncertainty, and $g_{i,k}$ is a positive constant.

Our control objective is to design state-feedback controllers such that the output of system (4.1) tracks a given time-varying signal $y_d(t)$ within a bounded error and all the signals of the closed-loop systems remain bounded under arbitrary switchings.

Assumption 4.1 The tracking target $y_d(t)$ and its time derivatives up to the n^{th} order are continuous and bounded.

In the controller design and stability analysis procedure, fuzzy logic systems will be used to approximate the unknown functions. Therefore, the following useful concept and lemma are first recalled.

Fuzzy logic systems include some IF-THEN rules, and the i^{th} IF-THEN rule is written as

$$\mathbb{R}_i : \text{ If } x_1 \text{ is } F_1^i \text{ and ... and } x_n \text{ is } F_n^i \text{ then } y \text{ is } B^i,$$

where $\mathbf{x} = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$, and $y \in \mathbb{R}$ are the input and output of the fuzzy logic systems, respectively. $F_1^i, F_2^i, ..., F_n^i$ and B^i are fuzzy sets in $\mathbb{R}$. By using the strategy of singleton fuzzification, the product inference and the center-average defuzzification, the fuzzy logic system can be formulated as

$$y(\mathbf{x}) = \frac{\sum_{i=1}^{N} w_i \prod_{j=1}^{n} \mu_{F_j^l}(x_j)}{\sum_{i=1}^{N} \left[\prod_{j=1}^{n} \mu_{F_j^l}(x_j) \right]},$$

where N is the number of IF-THEN rules; w_i is the point at which fuzzy membership function $\mu_{B^i}(w_i) = 1$. Let

$$s_i(\mathbf{x}) = \prod_{j=1}^{n} \mu_{F_j^l}(x_j) / \sum_{i=1}^{N} \left[\prod_{j=1}^{n} \mu_{F_j^l}(x_j) \right], \ S(\mathbf{x}) = [s_1(\mathbf{x}), \ldots, s_N(\mathbf{x})]^T$$

and $W = [w_1, w_2, \ldots, w_N]^T$. Then the fuzzy logic system can be rewritten as

$$y = W^T S(\mathbf{x}), \tag{4.2}$$

If all memberships are chosen as Gaussian functions, the following lemma holds.

Lemma 4.1 [24] *Let $f(\mathbf{x})$ be a continuous function defined on a compact set Ω. Then, for a given desired level of accuracy $\varepsilon > 0$, there exists a fuzzy logic system (4.2) such that*

$$\sup_{x \in \Omega} \left| f(\mathbf{x}) - W^T S(\mathbf{x}) \right| \le \varepsilon.$$

Remark 4.1 Lemma 4.1 plays a key role in the following design procedure and it indicates that any given real continuous function $f(x)$ can be represented by the linear combination of the basis function vector $S(\mathbf{x})$ within a bounded error ε. That is, $f(\mathbf{x}) = W^T S(\mathbf{x}) + \delta(\varepsilon)$, $|\delta(\varepsilon)| \leq \varepsilon$. It is noted that $0 < S^T S \leq 1$.

4.2.2 Main Results

In this section, we present an adaptive fuzzy control scheme for system (4.1) via the backstepping technique. In Sect. 3.1, a detailed design procedure was given. In each step, a common virtual control function α_i should be designed by using an appropriate common Lyapunov function V_i, and the control law u_k is finally designed.

4.2.2.1 Adaptive Control Design Under Multiple Adaptive Laws

In this subsection, a systemic control design procedure under multiple adaptive laws is presented. Design the control laws as

$$u_k = -\frac{1}{g_{n,k}}\left(\frac{\hat{\theta}_n}{2\zeta_{n,\min}^2} z_n + \lambda_n z_n + \frac{z_n}{2}\right), \tag{4.3}$$

where $\zeta_{n,k}$ and λ_n are positive design parameters, $\zeta_{n,\min} = \min\{\zeta_{n,k} : k \in M\}$, $\hat{\theta}_n$ is the estimation of $\theta_n = \left\| W_{n,\max} \right\|^2$, $W_{n,\max} = \max\{W_{n,k} : k \in M\}$ and $W_{n,k}$ is used in fuzzy logic system $W_{n,k}^T S_{n,k}(\mathbf{x})$ to approximate the unknown function $\hat{f}_{n,k}(\mathbf{x})$. $\hat{f}_{n,k}(\mathbf{x})$ is specified in the proof of Theorem 4.1.

The adaptive laws are defined as the solution to the following differential equations,

$$\dot{\hat{\theta}}_i = \frac{r_i}{2\zeta_{i,\min}^2} z_i^2 - \beta_i \hat{\theta}_i, \tag{4.4}$$

where r_i, $\zeta_{n,k}$ and β_i are positive design parameters, $\zeta_{n,\min} = \min\{\zeta_{n,k} : k \in M\}$, and the choice of $\hat{\theta}_j(0)$, $j = 1, 2, \ldots, n$ are required to satisfy $\hat{\theta}_j(0) \geq 0$ such that $\hat{\theta}_j \geq 0$. Now, we state one of our main results as follows.

Theorem 4.1 *Consider the closed-loop system (4.1) with the controllers (4.3) and the adaptive laws (4.4). For* $1 \leq i \leq n$, $k \in M$, *there exists* $W_{i,k}^T S_{i,k}(\mathbf{x})$ *such that* $\sup_{\mathbf{x} \in \Omega} \left| \hat{f}_{i,k}(\mathbf{x}) - W_{i,k}^T S_{i,k}(\mathbf{x}) \right| \leq \varepsilon_{i,k}$ *in the sense that the approximation error* $\varepsilon_{i,k}$ *is bounded, and all the initial values of* $\hat{\theta}_i$ *satisfy* $\hat{\theta}_i(0) \geq 0$. *Then, the tracking error and closed-loop signals are bounded.*

Proof For $1 \leq i \leq n-1$, we define the common virtual control functions as α_i which are required to be in the form:

$$\alpha_i(X_i) = -\frac{1}{g_{i,\min}}\left(\frac{\hat{\theta}_i}{2\zeta_{i,\min}^2} + \lambda_i + \frac{1}{2}\right)z_i, \tag{4.5}$$

where $\zeta_{i,k}$ is a positive design parameter, $\zeta_{i,\min} = \min\{\zeta_{i,k} : k \in M\}$, $g_{i,\min} = \min\{g_{i,k} : k \in M\}$, $\lambda_i = g_{i,\max} + c_i$, $g_{i,\max} = \max\{g_{i,k} : k \in M\}$ and c_i is a positive constant. $\hat{\theta}_i$ is the estimation of $\theta_i = \|W_{i,\max}\|^2$ where $W_{i,\max} = \max\{W_{i,k} : k \in M\}$ and $W_{i,k}$ is used in fuzzy logic system $W_{i,k}^T S_{i,k}(\mathbf{x})$ to approximate the unknown function $\hat{f}_{i,k}(\mathbf{x})$. $X_i = [\bar{x}_i^T, \bar{\hat{\theta}}_i, \bar{y}_d^{(i)}]^T$ with $\bar{x}_i^T = [x_1, x_2, \ldots, x_i]^T$, $\bar{\hat{\theta}}_i = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_i]^T$, $\bar{y}_d^{(i)} = [y_d, \dot{y}_d, \ldots, y_d^{(i)}]^T$ and $\bar{y}_d^{(i)}$ being the i^{th} derivative of y_d.

Step 1. Denote $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$. Consider a Lyapunov function candidate as

$$V_1 = \frac{1}{2}z_1^2. \tag{4.6}$$

For any $k \in M$, the derivative of V_1 is given by

$$\begin{aligned}\dot{V}_1 &= z_1(g_{1,k}\alpha_1 + g_{1,k}z_2 + f_{1,k} - \dot{y}_d)\\ &= z_1(g_{1,k}\alpha_1 + g_{1,k}z_2 + \hat{f}_{1,k}),\end{aligned} \tag{4.7}$$

where $\hat{f}_{1,k} = f_{1,k} - \dot{y}_d$. By Lemma 4.1, the following equation can be obtained,

$$\hat{f}_{1,k} = W_{1,k}^T S_{1,k}(X_1) + \delta_{1,k}(X_1),\ \left|\delta_{1,k}(X_1)\right| \le \varepsilon_{1,k}. \tag{4.8}$$

Remark 4.2 It should be pointed out that the fuzzy logic system is used to approximate the redefined unknown nonlinear function $\hat{f}_{1,k}$ that includes the unknown function $f_{1,k}$ and the derivative of the desired output rather than the unknown function $f_{1,k}$ only.

Substituting (4.8) into (4.7), one gets that

$$\begin{aligned}\dot{V}_1 &= g_{1,k}z_1\alpha_1 + g_{1,k}z_1z_2 + z_1W_{1,k}^T S_{1,k}(z_1) + z_1\delta(z_1)\\ &\le g_{1,k}z_1\alpha_1 + g_{1,k}z_1z_2 + \frac{1}{2\zeta_{1,k}^2}z_1^2\|W_{1,k}\|^2\\ &\quad + \frac{\zeta_{1,k}^2 + \varepsilon_{1,k}^2}{2} + \frac{1}{2}z_1^2,\end{aligned} \tag{4.9}$$

where $\zeta_{1,k}$ is a positive design parameter.

A feasible virtual control function can be constructed as

$$\alpha_1 = -\frac{1}{g_{1,\min}}\left(\frac{\hat{\theta}_1}{2\zeta_{1,\min}^2} + \lambda_1 + \frac{1}{2}\right)z_1, \tag{4.10}$$

where $\lambda_1 = g_{1,\max} + c_1$ with c_1 being a positive constant.

By substituting (4.1) into (4.9), one has

$$\dot{V}_1 \leq -\lambda_1 z_1^2 + \left(\frac{\|W_{1,k}\|^2}{2\zeta_{1,k}^2} - \frac{g_{1,k}\hat{\theta}_1}{2g_{1,\min}\zeta_{1,\min}^2} \right) z_1^2 + \frac{\zeta_{1,k}^2 + \varepsilon_{1,k}^2}{2} + g_{1,k} z_1 z_2. \tag{4.11}$$

Step 2. Let $z_3 = x_3 - \alpha_2$, and choose

$$V_2 = V_1 + \frac{1}{2} z_2^2. \tag{4.12}$$

For any $k \in M$, the time derivative of V_2 is given by

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + z_2(g_{2,k}\alpha_2 + g_{2,k}z_3 + f_{2,k} - \dot{\alpha}_1) \\ &= \dot{V}_1 + z_2(g_{2,k}\alpha_2 + g_{2,k}z_3 + \hat{f}_{2,k}), \end{aligned} \tag{4.13}$$

where $\hat{f}_{2,k} = f_{2,k} - \dot{\alpha}_1$, $\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 + \frac{\partial \alpha_1}{\partial \hat{\theta}_1}\dot{\hat{\theta}}_1 + \sum_{i=0}^{1} \frac{\partial \alpha_1}{\partial y_d^{(i)}} y_d^{(i+1)}$.

By Lemma 4.1, the following equation can be obtained,

$$\hat{f}_{2,k} = W_{2,k}^T S_{2,k}(X_2) + \delta_{2,k}(X_2), \ \left|\delta_{2,k}(X_2)\right| \leq \varepsilon_{2,k}. \tag{4.14}$$

Substituting (4.14) into (4.13), yields that

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + g_{2,k}z_2\alpha_2 + g_{2,k}z_2z_3 + z_2(W_{2,k}^T S(z_2) + \delta_{2,k}(z_2)) \\ &\leq \dot{V}_1 + g_{2,k}z_2\alpha_2 + g_{2,k}z_2z_3 + \frac{1}{2\zeta_{2,k}^2} z_2^2 \|W_{2,k}\|^2 + \frac{\zeta_{2,k}^2 + \varepsilon_{2,k}^2}{2} + \frac{1}{2} z_2^2, \end{aligned} \tag{4.15}$$

where $\zeta_{2,k}$ is a positive design parameter.

Design the virtual control function α_2 as

$$\alpha_2 = -\frac{1}{g_{2,\min}} \left(\frac{\hat{\theta}_2}{2\zeta_{2,\min}^2} + \lambda_2 + \frac{1}{2} \right) z_2, \tag{4.16}$$

where $\lambda_2 = g_{2,\max} + c_2$ with c_2 being a positive constant.

Then, one can get from (4.11), (4.15) and (4.16) that

$$\dot{V}_2 \leq \sum_{j=1}^{2} \left\{ -\lambda_j z_j^2 + \frac{\zeta_{j,k}^2 + \varepsilon_{j,k}^2}{2} + g_{j,k} z_j z_{j+1} + \left(\frac{\|W_{j,k}\|^2}{2\zeta_{j,k}^2} - \frac{g_{j,k}\hat{\theta}_j}{2g_{j,\min}\zeta_{j,\min}^2} \right) z_j^2 \right\}. \tag{4.17}$$

Step i. Let $z_{i+1} = x_{i+1} - \alpha_i$, and assume that we have finished the first $i-1$ ($2 \leq i \leq n$) steps. That is, for the following collection of auxiliary $(z_1, \ldots, z_{i-1})$-equations

$$\dot{z}_j = g_{j,k} x_{j+1} + \phi_{j,k}(X_j), \ \ j = 1, \ldots, i-1, \tag{4.18}$$

where

$$\phi_{j,k}(X_j) = f_{j,k}(\bar{x}_j) - \sum_{l=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_l} \dot{x}_l - \sum_{l=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}_l} \dot{\hat{\theta}}_l - \sum_{l=0}^{j-1} \frac{\partial \alpha_{j-1}}{\partial y_d^{(l)}} y_d^{(l+1)}. \tag{4.19}$$

We have a set of common virtual control functions as (4.5). A common Lyapunov function can be designed as

$$V_{i-1} = \frac{1}{2} \sum_{j=1}^{i-1} z_j^2. \tag{4.20}$$

For any $k \in M$, the time derivative of V_{i-1} satisfies

$$\begin{aligned} \dot{V}_{i-1} \leq \sum_{j=1}^{i-1} \Bigg\{ & -\lambda_j z_j^2 + \frac{\zeta_{j,k}^2 + \varepsilon_{j,k}^2}{2} + g_{j,k} z_j z_{j+1} \\ & + \left(\frac{\|W_{j,k}\|^2}{2\zeta_{j,k}^2} - \frac{g_{j,k}\hat{\theta}_j}{2 g_{j,\min} \zeta_{j,\min}^2} \right) z_j^2 \Bigg\}, \end{aligned} \tag{4.21}$$

where $\zeta_{j,k}$ is a positive design parameter.

Choose

$$V_i = V_{i-1} + \frac{1}{2} z_i^2. \tag{4.22}$$

Analogous to the procedures above, the following inequality can be obtained

$$\begin{aligned} \dot{V}_{i} \leq \sum_{j=1}^{i} \Bigg\{ & -\lambda_j z_j^2 + \frac{\zeta_{j,k}^2 + \varepsilon_{j,k}^2}{2} + g_{j,k} z_j z_{j+1} \\ & + \left(\frac{\|W_{j,k}\|^2}{2\zeta_{j,k}^2} - \frac{g_{j,k}\hat{\theta}_j}{2 g_{j,\min} \zeta_{j,\min}^2} \right) z_j^2 \Bigg\}. \end{aligned} \tag{4.23}$$

Step n. By repeatedly using the inductive argument above, a common Lyapunov function, a common virtual control function and state-feedback controllers are chosen, respectively, as

$$V_n = \sum_{j=1}^{n} \left\{ \frac{1}{2} z_l^2 + \frac{1}{2r_j} \tilde{\theta}_j^2 \right\}, \tag{4.24}$$

$$\alpha_{n-1} = -\frac{1}{g_{n-1,\min}} \left(\frac{\hat{\theta}_{n-1}}{2\zeta_{n-1,\min}^2} z_{n-1} + \lambda_{n-1} z_{n-1} + \frac{z_{n-1}}{2} \right), \tag{4.25}$$

$$u_k = -\frac{1}{g_{n,k}}\left(\frac{\hat{\theta}_n}{2\zeta_{n,\min}^2} z_n + \lambda_n z_n + \frac{z_n}{2}\right), \tag{4.26}$$

where $\theta_j = \left\| W_{j,\max} \right\|^2$, $\tilde{\theta}_j = \theta_j - \hat{\theta}_j$ $(j = 1, 2, \ldots, n)$ are the error between θ_j and its estimation $\hat{\theta}_j$.

For any $k \in M$, the time derivative of V_n satisfies

$$\begin{aligned}\dot{V}_n \le &\sum_{j=1}^{n-1}\Bigg\{-\lambda_j z_j^2 + g_{j,k} z_j z_{j+1} + \frac{\zeta_{j,k}^2 + \varepsilon_{j,k}^2}{2} - \frac{1}{r_j}\tilde{\theta}_j \dot{\hat{\theta}}_j \\ &+ \left(\frac{\left\| W_{j,k} \right\|^2}{2\zeta_{j,k}^2} - \frac{g_{j,k}\hat{\theta}_j}{2 g_{j,\min}\zeta_{j,\min}^2}\right) z_j^2\Bigg\} - \lambda_n z_n^2 \\ &+ \frac{\zeta_{n,k}^2 + \varepsilon_{n,k}^2}{2} - \frac{1}{r_n}\tilde{\theta}_n \dot{\hat{\theta}}_n + \left(\frac{\left\| W_{n,k} \right\|^2}{2\zeta_{n,k}^2} - \frac{\hat{\theta}_n}{2\zeta_{n,\min}^2}\right) z_n^2,\end{aligned} \tag{4.27}$$

where $\lambda_j = g_{j,\max} + c_j$, and c_j is a positive constant.

Substituting (4.4) into (4.27) gives that

$$\begin{aligned}\dot{V}_n \le &\sum_{j=1}^{n-1}\Bigg\{-\lambda_j z_j^2 + g_{j,k} z_j z_{j+1} + \frac{\zeta_{j,k}^2 + \varepsilon_{j,k}^2}{2} + \frac{1}{r_j}\beta_j \tilde{\theta}_j \hat{\theta}_j \\ &+ \left(\frac{\left\| W_{j,k} \right\|^2}{2\zeta_{j,k}^2} - \frac{g_{j,k}\hat{\theta}_j}{2 g_{j,\min}\zeta_{j,\min}^2} - \frac{\tilde{\theta}_j}{2\zeta_{j,\min}^2}\right) z_j^2\Bigg\} \\ &- \lambda_n z_n^2 + \frac{\zeta_{n,k}^2 + \varepsilon_{n,k}^2}{2} + \frac{1}{r_n}\beta_n \tilde{\theta}_n \hat{\theta}_n + \left(\frac{\left\| W_{n,k} \right\|^2}{2\zeta_{n,k}^2} - \frac{\theta_n}{2\zeta_{n,\min}^2}\right) z_n^2 \\ \le &\sum_{j=1}^{n}\left\{-\lambda_j z_j^2 + \frac{\zeta_{j,k}^2 + \varepsilon_{j,k}^2}{2} + \frac{1}{r_j}\beta_j \tilde{\theta}_j \hat{\theta}_j\right\} + \sum_{j=1}^{n-1} g_{j,k} z_j z_{j+1}.\end{aligned} \tag{4.28}$$

It is not difficult to see that

$$\sum_{j=1}^{n-1} g_{j,k} z_j z_{j+1} \le g_{j,\max}\sum_{j=1}^{n} z_j^2, \tag{4.29}$$

and

$$\tilde{\theta}_j \hat{\theta}_j = \tilde{\theta}_j(\theta_j - \tilde{\theta}_j) \le -\frac{1}{2}\tilde{\theta}_j^2 + \frac{1}{2}\theta_j^2. \tag{4.30}$$

One can get from (4.28), (4.29) and (4.30) that

$$\dot{V}_n \leq \sum_{j=1}^{n} \left\{ -c_j z_j^2 - \frac{1}{2r_j}\beta_j \tilde{\theta}_j^2 \right\} + \sum_{j=1}^{n} \left\{ \frac{\zeta_{j,\max}^2 + \varepsilon_{j,\max}^2}{2} + \frac{1}{2r_j}\beta_j \theta_j^2 \right\}. \tag{4.31}$$

Let $a_0 = \min\{2c_j, \beta_j : 1 \leq j \leq n\}$, $b_0 = \sum_{j=1}^{n}\{\frac{1}{2r_j}\beta_j\theta_j^2 + \frac{\zeta_{j,\max}^2+\varepsilon_{j,\max}^2}{2}\}$. One has

$$\dot{V}_n \leq -a_0 V_n + b_0. \tag{4.32}$$

According to the comparison principle, one gets

$$V_n(t) \leq \left(V_n(0) - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0},\ t \geq 0. \tag{4.33}$$

Inequality (4.33) indicates that all the signals in the closed-loop system are bounded. In particular, we have

$$\lim_{t\to\infty} |z_1| \leq \sqrt{\frac{2b_0}{a_0}}. \tag{4.34}$$

The proof is completed here. □

4.2.2.2 Adaptive Control Design Under One Adaptive Law

In this subsection, a controller design approach with one adaptive law is presented.

The control laws are chosen as

$$u_k = -\frac{1}{g_{n,k}} \left(\frac{\hat{\theta}}{2\zeta_{n,\min}^2} z_n + \lambda_n z_n + \frac{z_n}{2} \right), \tag{4.35}$$

where $\zeta_{n,k}$ and λ_n are positive design parameters, $\zeta_{n,\min} = \min\{\zeta_{n,k} : k \in M\}$, $\hat{\theta}$ is the estimation of $\theta = \sum_{i=1}^{n} \left\| W_{i,\max} \right\|^2$, $W_{i,\max} = \max\{W_{i,k} : k \in M\}$ and $W_{i,k}$ is used in fuzzy logic system $W_{i,k}^T S_{i,k}(x)$ to approximate the unknown function $\hat{f}_{i,k}(x)$.

The adaptive law is defined as the solution to the following differential equation:

$$\dot{\hat{\theta}} = \sum_{j=1}^{n} \frac{r}{2\zeta_{j,\min}^2} z_j^2 - \beta\hat{\theta}, \tag{4.36}$$

where r, $\zeta_{j,k}$ and β are positive design parameters, $\zeta_{j,\min} = \min\{\zeta_{j,k} : k \in M\}$ and the choice of $\hat{\theta}(0)$ is required to satisfy $\hat{\theta}(0) \geq 0$ such that $\hat{\theta} \geq 0$.

Next, we give another main result of the chapter.

Theorem 4.2 *Consider the closed-loop system (4.1) with the controllers (4.35) and the adaptive laws (4.36). For $1 \le i \le n$, $k \in M$, there exists $W_{i,k}^T S_{i,k}(\mathbf{x})$ such that $\sup_{x \in \Omega} \left| \hat{f}_{i,k}(\mathbf{x}) - W_{i,k}^T S_{i,k}(\mathbf{x}) \right| \le \varepsilon_{i,k}$ in the sense that the approximation error $\varepsilon_{i,k}$ is bounded, and the initial value of $\hat{\theta}$ satisfies $\hat{\theta}(0) \ge 0$. Then, the tracking error and closed-loop signals are bounded.*

Proof For $1 \le i \le n-1$, define the common virtual control functions α_i as:

$$\alpha_i(X_i) = -\frac{1}{g_{i,\min}} \left(\frac{\hat{\theta}}{2\zeta_{i,\min}^2} + \lambda_i + \frac{1}{2} \right) z_i, \tag{4.37}$$

where $\zeta_{i,k}$ is a positive design parameter, $\zeta_{i,\min} = \min\{\zeta_{i,k} : k \in M\}$, $g_{i,\min} = \min\{g_{i,k} : k \in M\}$, $\lambda_i = g_{i,\max} + c_i$, $g_{i,\max} = \max\{g_{i,k} : k \in M\}$ and c_i is a positive constant. $\hat{\theta}$ is the estimation of $\theta = \sum_{i=1}^n \left\| W_{i,\max} \right\|^2$, $X_i = [\bar{x}_i^T, \hat{\theta}, \bar{y}_d^{(i)}]^T$ where $\bar{x}_i^T = [x_1, x_2, \ldots, x_i]^T$, $\bar{y}_d^{(i)} = [y_d, \dot{y}_d, \ldots, y_d^{(i)}]^T$ and $\bar{y}_d^{(i)}$ being the i^{th} derivative of y_d.

Consider a common Lyapunov function

$$V = \sum_{j=1}^{n} \frac{1}{2} z_j^2 + \frac{1}{2r} \tilde{\theta}^2, \tag{4.38}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ is the error between θ and its estimation $\hat{\theta}$.

For any $k \in M$, the time derivative of V satisfies

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{n-1} z_i (g_{i,k}\alpha_i + g_{i,k} z_{i+1} + f_{i,k} - \dot{\alpha}_{i-1}) \\ &\quad + z_n (g_{n,k} u_k + f_{n,k} - \dot{\alpha}_{n-1}) - \frac{1}{r} \tilde{\theta} \dot{\hat{\theta}} \\ &= \sum_{i=1}^{n-1} z_i (g_{i,k}\alpha_i + g_{i,k} z_{i+1} + \hat{f}_{i,k}) \\ &\quad + z_n (g_{n,k} u_k + \hat{f}_{n,k}) - \frac{1}{r} \tilde{\theta} \dot{\hat{\theta}} \end{aligned} \tag{4.39}$$

where $\hat{f}_{i,k} = f_{i,k} - \dot{\alpha}_{i-1}$, $\dot{\alpha}_{i-1} = \sum_{l=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_l} \dot{x}_l + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \sum_{l=0}^{j-1} \frac{\partial \alpha_{j-1}}{\partial y_d^{(l)}} y_d^{(l+1)}$.

For $1 \le i \le n$, the following equation can be obtained by using Lemma 4.1.

$$\hat{f}_{i,k} = W_{i,k}^T S_{i,k}(X_i) + \delta_{i,k}(X_i), \ \left| \delta_{2,k}(X_i) \right| \le \varepsilon_{i,k}. \tag{4.40}$$

Substituting (4.36) and (4.37) into (4.39), one has

$$\begin{aligned}\dot{V} \le & \frac{\beta}{r}\tilde{\theta}\hat{\theta} + \sum_{j=1}^{n-1}\Bigg\{-\lambda_j z_j^2 + g_{j,k} z_j z_{j+1} + \frac{\zeta_{j,k}^2 + \varepsilon_{j,k}^2}{2} \\ & + \left(\frac{\|W_{j,k}\|^2}{2\zeta_{j,k}^2} - \frac{g_{j,k}\hat{\theta}}{2g_{j,\min}\zeta_{j,\min}^2} - \frac{\tilde{\theta}}{2\zeta_{j,\min}^2}\right) z_j^2\Bigg\} \\ & - \lambda_n z_n^2 + \frac{\zeta_{n,k}^2 + \varepsilon_{n,k}^2}{2} + \left(\frac{\|W_{n,k}\|^2}{2\zeta_{n,k}^2} - \frac{\theta}{2\zeta_{n,\min}^2}\right) z_n^2 \\ \le & \frac{\beta}{r}\tilde{\theta}\hat{\theta} + \sum_{j=1}^{n}\left\{-\lambda_j z_j^2 + \frac{\zeta_{j,k}^2 + \varepsilon_{j,k}^2}{2}\right\} + \sum_{j=1}^{n-1} g_{j,k} z_j z_{j+1}. \qquad (4.41)\end{aligned}$$

The rest of proof is omitted here as it is similar to (4.29)–(4.34). □

4.2.3 Simulation Results

In this section, an example is provided to demonstrate the effectiveness of our main results.

Consider the following switched nonlinear system

$$\begin{aligned}\dot{x}_1 &= g_{1,\sigma(t)} x_2 + f_{1,\sigma(t)}, \\ \dot{x}_2 &= g_{2,\sigma(t)} u_{\sigma(t)} + f_{2,\sigma(t)}, \\ y &= x_1, \\ y_d &= \sin t, \qquad (4.42)\end{aligned}$$

where $g_{1,1} = 2,\ g_{1,2} = 1,\ f_{1,1} = x_1,\ f_{1,2} = \sin x_1,\ g_{2,1} = 2,\ g_{2,2} = 1,\ f_{2,1} = x_1 x_2,\ f_{2,2} = x_1 x_2^2$. First, the controllers under multiple adaptive laws are designed by Theorem 4.1. The initial conditions are $x_1(0) = 0.05,\ x_2(0) = 0.05$, and $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$. We choose $c_1 = 2,\ c_2 = 1,\ r_1 = 10,\ r_2 = 3,\ \beta_1 = \beta_2 = 0.02,\ \varsigma_{1,1} = 0.25,\ \varsigma_{1,2} = 3,\ \varsigma_{2,1} = 0.5,\ \varsigma_{2,2} = 1.8$. Second, the controllers under one adaptive law is designed by Theorem 2, and the initial conditions are $x_1(0) = 0.05,\ x_2(0) = 0.05,\ \hat{\theta}(0) = 0$. We choose $c_1 = 2,\ c_2 = 1,\ r = 12,\ \beta = 0.025,\ \varsigma_{1,1} = 0.25,\ \varsigma_{1,2} = 3,\ \varsigma_{2,1} = 1.5,\ \varsigma_{2,2} = 1.8$. The objective is to design the controllers u_k such that y can track a desired trajectory y_d under arbitrary switchings.

According to Theorem 4.1, the adaptive laws $\hat{\theta}_1$, $\hat{\theta}_2$ and the control law u_k are chosen, respectively, as

$$\dot{\hat{\theta}}_1 = \frac{r_1}{2\zeta_{1,1}^2} z_1^2 - \beta_1 \hat{\theta}_1,\ \dot{\hat{\theta}}_2 = \frac{r_2}{2\zeta_{2,1}^2} z_2^2 - \beta_2 \hat{\theta}_2,$$

$$u_1 = -\frac{1}{g_{2,1}} \left(\frac{\hat{\theta}_2}{2\zeta_{2,1}^2} z_2 + \lambda_2 z_2 + \frac{z_2}{2} \right),$$

$$u_2 = -\frac{1}{g_{2,2}} \left(\frac{\hat{\theta}_2}{2\zeta_{2,1}^2} z_2 + \lambda_2 z_2 + \frac{z_2}{2} \right),$$

where $z_1 = x_1 - y_d,\ z_2 = x_2 - \alpha_1,\ \lambda_2 = c_2 + g_{2,1}$. The virtual control function α_1 is given by

$$\alpha_1 = -\frac{1}{g_{1,2}} \left(\frac{\hat{\theta}_1}{2\zeta_{1,1}^2} z_1 + \lambda_1 z_1 + \frac{z_1}{2} \right),$$

where $\lambda_1 = c_1 + g_{1,1}$. The controller design based on Theorem 4.1 is completed here. In the next, another design according to Theorem 2 is presented.

According to Theorem 4.2, an adaptive law $\hat{\theta}$ and the control law $u_1,\ u_2$ are chosen, respectively, as

$$\dot{\hat{\theta}} = \frac{r}{2\zeta_{1,1}^2} z_1^2 + \frac{r}{2\zeta_{2,1}^2} z_2^2 - \beta \hat{\theta},$$

$$u_1 = -\frac{1}{g_{2,1}} \left(\frac{\hat{\theta}}{2\zeta_{2,1}^2} z_2 + \lambda_2 z_2 + \frac{z_2}{2} \right),$$

$$u_2 = -\frac{1}{g_{2,2}} \left(\frac{\hat{\theta}}{2\zeta_{2,1}^2} z_2 + \lambda_2 z_2 + \frac{z_2}{2} \right),$$

where $z_1 = x_1 - y_d,\ z_2 = x_2 - \alpha_1,\ \lambda_2 = c_1 + g_{2,1}$.

The virtual control function α_1 is given as

$$\alpha_1 = -\frac{1}{g_{1,2}} \left(\frac{\hat{\theta}}{2\zeta_{1,1}^2} z_1 + \lambda_1 z_1 + \frac{z_1}{2} \right),$$

where $\lambda_1 = c_1 + g_{1,1}$.

The simulation results are shown in Figs. 4.1, 4.2, 4.3 and 4.4, respectively. Figure 4.1 shows the system output y and reference signal y_d. Figure 4.2 depicts the response of the tracking error $y - y_d$. Figure 4.3 illustrates the trajectory of the adaptive law. Figure 4.4 demonstrates the evolution of the switching signal. From Figs. 4.1, 4.2 and 4.3, it can be seen that the output y of both controllers can track the target signal y_d well, and all the closed-loop signals remain bounded.

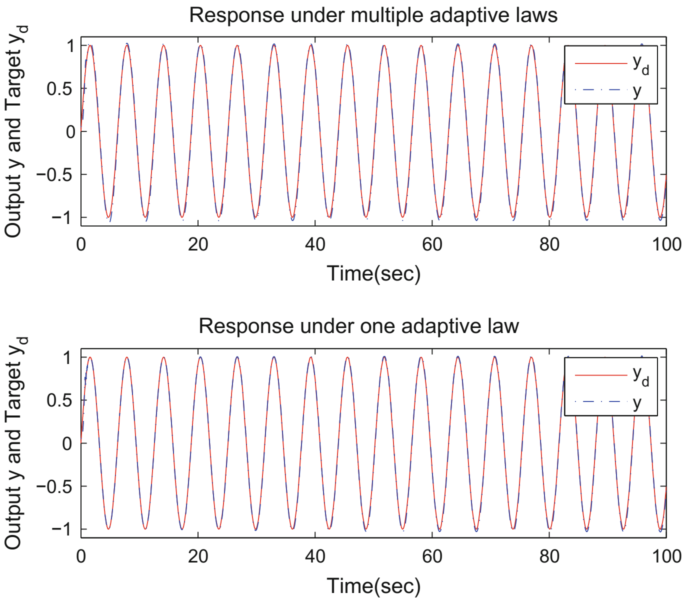
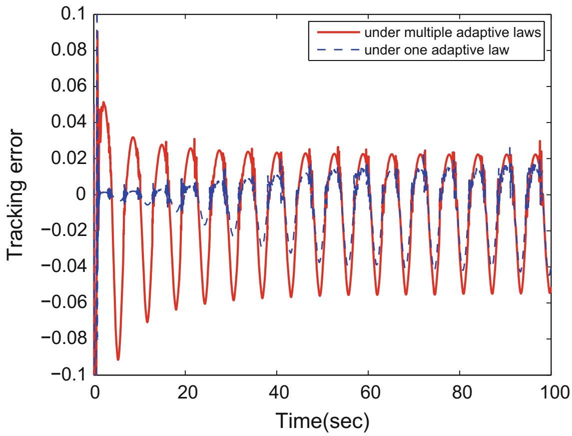


Fig. 4.1 Tracking performances

Fig. 4.2 Responses of the tracking error $y - y_d$



4.2.4 Conclusions

The tracking control problem for switched strict-feedback nonlinear systems with completely unknown nonlinear functions is given. The application of the adaptive backstepping technique is extended to a class of switched nonlinear systems with unknown uncertainties. The stability analysis shows that the designed controllers can

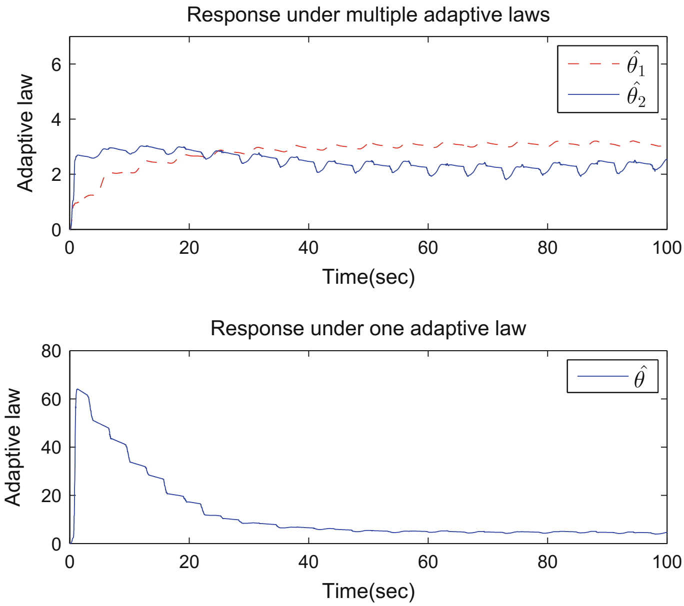
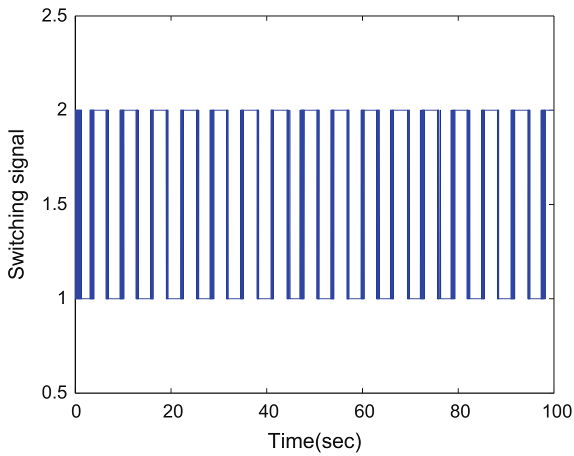


Fig. 4.3 Responses of the adaptive laws

Fig. 4.4 Switching signal



ensure all the closed-loop signals remain bounded, and the system output converges to a small neighborhood of the reference signal.

4.3 Adaptive Control of Switched Nonstrict-Feedback Nonlinear Systems

4.3.1 Problem Formulation and Preliminaries

In this section, the following nonlinear switched system in nonstrict-feedback form is considered:

$$\begin{aligned} \dot{x}_i &= g_{i,\sigma(t)}x_{i+1} + f_{i,\sigma(t)}(x) + w_{i,\sigma(t)},\ 1 \le i \le n-1 \\ \dot{x}_n &= g_{n,\sigma(t)}u_{\sigma(t)} + f_{n,\sigma(t)}(x) + w_{n,\sigma(t)} \\ y &= x_1 \end{aligned} \tag{4.43}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is the system state, y is the system output; $\sigma(t) : [0, \infty) \to M = \{1, 2, \ldots, m\}$ is the switching signal; $u_k \in \mathbb{R}$ is the control input of the k-th subsystem. For any $i = 1, 2, \ldots, n$ and $k \in M$, $f_{i,k}(x)$ are unknown smooth nonlinear functions satisfying locally Lipschitz conditions, $g_{i,k}$ are positive constants, and $w_{i,k}$ is the bounded external disturbance of the system.

Our control objective is to design state-feedback controllers such that the output of system (4.43) tracks a given time-varying signal $y_d(t)$ and all the signals of the closed-loop systems remain bounded under arbitrary switchings.

Assumption 4.2 The tracking target $y_d(t)$ and its time derivatives up to the n^{th} order are continuous and bounded. It is further assumed that there exists a positive constant d such that $|y_d| \le d$.

Assumption 4.3 There exist strictly increasing smooth functions $\phi_{i,k}(.)s : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_{i,k}(0) = 0$ such that for $i = 1, 2, \ldots, n-1,\ k \in M$,

$$|f_{i,k}(x)| \le \phi_{i,k}(\|x\|).$$

Remark 4.3 The increasing property of $\phi_{i,k}(.)$ means that if $a_i \ge 0$, $i = 1, 2, \ldots, n$, then $\phi_{i,k}(\sum_{i=1}^{n} a_i) \le \sum_{i=1}^{n} \phi_{i,k}(na_i)$. Note that $\phi_{i,k}(s)$ is a smooth function, and $\phi_{i,k}(0) = 0$. Therefore, there exists a smooth function $p_{i,k}(s)$ such that $\phi_{i,k}(s) = sp_{i,k}(s)$, which gives that

$$\phi_{i,k}\left(\sum_{i=1}^{n} a_i\right) \le \sum_{i=1}^{n} na_i p_{i,k}(na_i). \tag{4.44}$$

In the control design procedure, radial basis function (RBF) neural networks are used to approximate a continuous function $f(X)$ on a compact set $\Omega \in R^q$. For any $\varepsilon > 0$, there exists a neural network $\Phi^T P(X)$ such that

$$\sup_{x \in \Omega} \left| f(X) - \Phi^T P(X) \right| \le \varepsilon, \tag{4.45}$$

where $P(X) = [p_1(X), p_2(X), \ldots, p_l(X)]^T$ is the basis function vector, $\Phi = [\phi_1, \phi_2, \ldots, \phi_l]^T$ is the ideal constant weight vector with $l > 1$ being the number of the neural network nodes and $p_i(X)$ are chosen as the form:

$$p_i(X) = \exp\left(\frac{-(X-\mu_i)^T(X-\mu_i)}{\zeta_i^2}\right), \tag{4.46}$$

where ζ_i is the width of the Gaussian function, and $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^T$ is the center vector.

Remark 4.4 The readers may refer to [25] for more details about neural networks. Inequality (4.45) indicates that any given real continuous function $f(X)$ can be represented by the linear combination of the basis function vector $P(X)$ within a bounded error ε.

Lemma 4.2 *For any $\xi \in R$ and $\varpi > 0$, the following inequality holds,*

$$0 \le |\xi| - \xi \tanh\left(\frac{\xi}{\varpi}\right) \le \delta\varpi \tag{4.47}$$

where $\delta = 0.2785$.

4.3.2 Adaptive Control Design Based on Neural Networks

In this section, a backstepping-based adaptive control design procedure is presented. For the i^{th} subsystem, define a common virtual control function α_i as

$$\alpha_i(X_i) = -\frac{z_i}{\underline{g}_i}\left(\lambda_i + l_i^2 + \eta_i^2\hat{\theta}P_i^T(X_i)P_i(X_i)\right). \tag{4.48}$$

where λ_i, l_i and η_i are positive design parameters; $\underline{g}_i = \min\{g_{i,k},\ k \in M\}$; $\hat{\theta}$ is the estimation of θ which is an unknown constant and is specified later; $X_i = (\bar{x}_i^T, y_d, \dot{y}_d, \ldots, y_d^{(i)}, \hat{\theta})^T$, $\bar{x}_i = (x_1, x_2, \ldots, x_i)^T$; $P_i(X_i)$ represents the basis function of the ith neural network system. Subsequently, a set of the variable change of coordinates is defined as $z_i = x_i - \alpha_{i-1}$. Then, the z-system after coordinate transform is that

$$\begin{aligned} \dot{z}_i &= g_{i,k}x_{i+1} + f_{i,k}(x) + w_{i,k} - \dot{\alpha}_{i-1},\ 1 \le i \le n-1 \\ \dot{z}_n &= g_{n,k}u_k + f_{n,k}(x) + w_{n,k} - \dot{\alpha}_{n-1} \end{aligned} \tag{4.49}$$

where $\alpha_0 = y_d$.

For $i = 1, 2, \ldots, n-1$, the time derivative of α_{i-1} is given by

$$\dot{\alpha}_{i-1} = \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s}(f_{s,k} + g_{s,k}x_{s+1} + w_{s,k}) + \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} \tag{4.50}$$

where $\sum_{s=1}^{0} \frac{\partial \alpha_{i-1}}{\partial x_s}(f_{s,k} + g_{s,k}x_{s+1} + w_{s,k}) = 0$, and $\sum_{s=1}^{0} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} = 0$.

The controller can be chosen as

$$u = -\frac{z_n}{\underline{g}_n}\left(\lambda_n + l_n^2 + \eta_n^2 \hat{\theta} P_n^T(X_n) P_n(X_n)\right), \tag{4.51}$$

where λ_n, l_n and η_n are positive design parameters; $\underline{g}_n = \min\{g_{n,k},\ k \in M\}$; $X_n = (\bar{x}_i^T, y_d, \dot{y}_d, \ldots, y_d^{(n)}, \hat{\theta})^T$; $P_n(X_n)$ represents the basis function vector of the n^{th} neural network system.

The adaptive law is designed as

$$\dot{\hat{\theta}} = \sum_{i=1}^{n} r\eta_i^2 z_i^2 P_i^T P_i - \beta\hat{\theta} \tag{4.52}$$

where r and β are positive design parameters.

Lemma 4.3 *For the variable transformations* $z_i = x_i - \alpha_{i-1}$, $i = 1, 2, \ldots, n$, *the following inequality holds,*

$$\|x\| \le \sum_{i=1}^{n} |z_i|\, \varphi_i(\hat{\theta}) + d \tag{4.53}$$

where $\alpha_0 = y_d$, $\varphi_i(\hat{\theta}) = \frac{1}{\underline{g}_i}\left(-(\lambda_i + l_i^2) - \eta_i^2 \hat{\theta} P_i^T(X_i) P_i(X_i)\right) + 1$, $i = 1, 2, \ldots, n-1$, *and* $\varphi_n = 1$.

The main result is given in the following theorem.

Theorem 4.3 *Consider the closed-loop system (4.43) with the controller (4.51) and the adaptive law (4.52). For* $1 \le i \le n, k \in M$, *assume that all the unknown nonlinear functions* $\bar{f}_{i,k}(x)$ *are approximated by neural networks in the sense that the approximation error* $\varepsilon_{i,k}$ *is bounded. Then, for bounded initial conditions, the target signal can be tracked within a small bounded error and other closed-loop signals remain bounded.*

Proof Consider the common Lyapunov function candidate as

$$V = \frac{1}{2}\sum_{i=1}^{n} z_i^2 + \frac{1}{2r}\tilde{\theta}^2 \tag{4.54}$$

where $r > 0$ is a design parameter.

The time derivative of V is given by

$$\dot{V} = \sum_{i=1}^{n-1} z_i (f_{i,k} + g_{i,k} x_{i+1} + w_{1,k} - \dot{\alpha}_{i-1}) + z_n (f_{n,k} + g_{n,k} u + w_{n,k} - \dot{\alpha}_{n-1}) - \frac{1}{r}\tilde{\theta}\dot{\hat{\theta}} \tag{4.55}$$

where $\alpha_0 = y_d$.

By using (4.50), the following inequality can be obtained,

$$\begin{aligned}
\dot{V} =& \sum_{i=1}^{n-1} z_i \left\{ f_{i,k} + g_{i,k} x_{i+1} + w_{i,k} - \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} \right. \\
& \left. - \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s} (f_{s,k} + g_{s,k} x_{s+1} + w_{s,k}) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right\} \\
& + z_n \left\{ f_{n,k} + g_{n,k} u + w_{n,k} - \sum_{s=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(s)}} y_d^{(s+1)} \right. \\
& \left. - \sum_{s=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_s} (f_{s,k} + g_{s,k} x_{s+1} + w_{s,k}) - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right\} - \frac{1}{r}\tilde{\theta}\dot{\hat{\theta}} \\
=& \sum_{i=1}^{n} z_i \left\{ f_{i,k} - \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s} f_{s,k} \right\} + \sum_{i=1}^{n-1} z_i \left\{ g_{i,k} x_{i+1} + w_{i,k} \right. \\
& \left. - \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} - \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s} (g_{s,k} x_{s+1} + w_{s,k}) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right\} \\
& + z_n \left\{ g_{n,k} u + w_{n,k} - \sum_{s=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(s)}} y_d^{(s+1)} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right. \\
& \left. - \sum_{s=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_s} (g_{s,k} x_{s+1} + w_{s,k}) \right\} - \frac{1}{r}\tilde{\theta}\dot{\hat{\theta}}
\end{aligned} \tag{4.56}$$

By using Assumption 4.3, Lemma 4.3 and Remark 4.3, one has

$$\begin{aligned}
& z_i \left(f_{i,k} - \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s} f_{s,k}(x) \right) \\
& = -z_i \sum_{s=1}^{i} \frac{\partial \alpha_{i-1}}{\partial x_s} f_{s,k}(x) \le \sum_{s=1}^{i} |z_i \frac{\partial \alpha_{i-1}}{\partial x_s}| \left| f_{s,k}(x) \right|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{s=1}^{i} |z_i \frac{\partial \alpha_{i-1}}{\partial x_s}|\phi_{s,k}(\|x\|) \\
&\le \sum_{s=1}^{i}\sum_{j=1}^{n} |z_i \frac{\partial \alpha_{i-1}}{\partial x_s}||z_j|\bar{\phi}_{s,k}(|z_j|\varphi_j(\theta)) + \sum_{s=1}^{i} |z_i \frac{\partial \alpha_{i-1}}{\partial x_s}|\phi_{s,k}(d) \\
&\le \sum_{s=1}^{i}\sum_{j=1}^{n} \frac{1}{2} z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial x_s}\right)^2 + \sum_{s=1}^{i}\sum_{j=1}^{n} \frac{1}{2} z_j^2 \bar{\phi}_{s,k}^2(|z_j|\varphi_j(\theta)) \\
&\quad + \sum_{s=1}^{i} |z_i \frac{\partial \alpha_{i-1}}{\partial x_s}|\phi_{s,k}(d)
\end{aligned}
\tag{4.57}
$$

where $\bar{\phi}_{s,k}(|z_j|\varphi_j(\theta)) = \varphi_j(\theta)h_{s,k}(|z_j|\varphi_j(\theta))$.

Substituting (4.57) into (4.56) yields that

$$
\begin{aligned}
\dot{V} \le& \sum_{i=1}^{n}\sum_{s=1}^{i}\sum_{j=1}^{n} \frac{1}{2} z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial x_s}\right)^2 \\
&+ \sum_{i=1}^{n}\sum_{s=1}^{i}\sum_{j=1}^{n} \frac{1}{2} z_j^2 \bar{\phi}_{s,k}^2(|z_j|\varphi_j(\theta)) + \sum_{i=1}^{n}\sum_{s=1}^{i} |z_i \frac{\partial \alpha_{i-1}}{\partial x_s}|\phi_{s,k}(d) \\
&+ \sum_{i=1}^{n-1} z_i \left\{ g_{i,k}x_{i+1} + w_{i,k} - \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} \right. \\
&\left. - \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s}(g_{s,k}x_{s+1} + w_{s,k}) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \right\} \\
&+ z_n \left\{ g_{n,k}u + w_{n,k} - \sum_{s=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(s)}} y_d^{(s+1)} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \right. \\
&\left. - \sum_{s=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_s}(g_{s,k}x_{s+1} + w_{s,k}) \right\} - \frac{1}{r}\tilde{\theta}\dot{\hat{\theta}}
\end{aligned}
\tag{4.58}
$$

One can obtain that

$$
\sum_{i=1}^{n}\sum_{s=1}^{i}\sum_{j=1}^{n} \frac{1}{2} z_j^2 \bar{\phi}_{s,k}^2(|z_j|\varphi_j(\theta)) = \sum_{i=1}^{n} z_i^2 \sum_{s=1}^{n} q(n,s)\bar{\phi}_{s,k}^2(|z_j|\varphi_j(\theta)) \tag{4.59}
$$

where $q(n, s) = \frac{(n-(s-1))}{2}$.

By using Lemma 4.2, the following inequality holds for $\varpi_{i,k} > 0$,

$$\sum_{s=1}^{i} |z_i \frac{\partial \alpha_{i-1}}{\partial x_s}| \phi_{s,k}(d) \le z_i Z_i \tanh\left(\frac{z_i Z_i}{\varpi_{i,k}}\right) + \delta \varpi_{i,k} \tag{4.60}$$

where $Z_i = \sum_{s=1}^{i} |z_i \frac{\partial \alpha_{i-1}}{\partial x_s}| \phi_{s,k}(d)$.

It follows from (4.58)–(4.60) that

$$\begin{aligned}
\dot{V} \le & \sum_{i=1}^{n} z_i^2 \sum_{s=1}^{i} \frac{n}{2} \left(\frac{\partial \alpha_{i-1}}{\partial x_s}\right)^2 + \sum_{i=1}^{n} z_i^2 \sum_{s=1}^{n} q(n,s) \bar{\phi}_{s,k}^2(|z_j| \varphi_j(\theta)) \\
& + \sum_{i=1}^{n} z_i Z_i \tanh\left(\frac{z_i Z_i}{\varpi_{i,k}}\right) + \sum_{i=1}^{n} \delta \varpi_{i,k} + \sum_{i=1}^{n-1} z_i \Bigg\{ g_{i,k} x_{i+1} + w_{i,k} \\
& - \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} - \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s} (g_{s,k} x_{s+1} + w_{s,k}) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \Bigg\} \\
& + z_n \Bigg\{ g_{n,k} u + w_{n,k} - \sum_{s=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(s)}} y_d^{(s+1)} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \\
& - \sum_{s=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_s} (g_{s,k} x_{s+1} + w_{s,k}) \Bigg\} - \frac{1}{r} \tilde{\theta} \dot{\hat{\theta}} \\
= & \sum_{i=1}^{n} z_i \Bigg\{ z_i \sum_{s=1}^{i} \frac{n}{2} \left(\frac{\partial \alpha_{i-1}}{\partial x_s}\right)^2 + z_i \sum_{s=1}^{n} q(n,s) \bar{\phi}_{s,k}^2(|z_j| \varphi_j(\theta)) \\
& + Z_i \tanh\left(\frac{z_i Z_i}{\varpi_{i,k}}\right) - \sum_{s=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_s} (g_{s,k} x_{s+1} + w_{s,k}) \\
& - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} + w_{i,k} \Bigg\} \\
& + \sum_{i=1}^{n-1} z_i g_{i,k} x_{i+1} + z_n g_{n,k} u - \frac{1}{r} \tilde{\theta} \dot{\hat{\theta}} + \sum_{i=1}^{n} \delta \varpi_{i,k}
\end{aligned} \tag{4.61}$$

Note that

$$\sum_{i=1}^{n-1} z_i g_{i,k} x_{i+1} = \sum_{i=1}^{n-1} z_i g_{i,k} z_{i+1} + \sum_{i=1}^{n-1} z_i g_{i,k} \alpha_i \tag{4.62}$$

and define

$$\bar{f}_{i,k} = z_i \sum_{s=1}^{i} \frac{n}{2} \left(\frac{\partial \alpha_{i-1}}{\partial x_s}\right)^2 + z_i \sum_{s=1}^{n} q(n,s) \bar{\phi}_{s,k}^2(|z_j| \varphi_j(\theta))$$

$$+ Z_i \tanh\left(\frac{z_i Z_i}{\varpi_{i,k}}\right) - \sum_{s=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_s}(g_{s,k}x_{s+1} + w_{s,k})$$

$$- \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} + w_{i,k} + g_{i-1,k}z_{i-1} \quad (4.63)$$

where $g_0 = 0$ and $z_0 = 0$.

Substituting (4.62) and (4.63) into (4.61) gives that

$$\dot{V} \leq \sum_{i=1}^{n-1} z_i(\bar{f}_{i,k} + g_{i,k}\alpha_i) + z_n(\bar{f}_{n,k} + g_{n,k}u_k) + \sum_{i=1}^{n} \delta\varpi_{i,k} - \frac{1}{r}\tilde{\theta}\dot{\hat{\theta}} \quad (4.64)$$

The neural network $\Phi_{i,k}^T P_{i,k}$ is utilized to approximate the unknown function $\bar{f}_{i,k}$ such that for any given $\bar{\varepsilon}_{i,k} > 0$,

$$\bar{f}_{i,k} = \Phi_{i,k}^T P_{i,k}(X_i) + \varepsilon_{i,k}(X_i) \quad (4.65)$$

where $X_i = (\bar{x}_i^T, y_d, \dot{y}_d, \ldots, y_d^{(i)}, \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_i)^T$, $\left|\varepsilon_{i,k}\right| \leq \bar{\varepsilon}_{i,k}$, $\varepsilon_{i,k}$ denotes the approximation error. Thus, for $i = 1, 2, \ldots, n$,

$$\begin{aligned} z_i \bar{f}_{i,k} &= z_i \Phi_{i,k}^T P_{i,k}(X_i) + z_i \varepsilon_{i,k}(X_i) \\ &\leq \frac{\eta_i^2}{2} z_i^2 \left\|\Phi_{i,k}\right\|^2 P_{i,k}^T P_{i,k} + \frac{1}{2\eta_i^2} + \frac{l_{i,k}^2}{2} z_i^2 + \frac{\varepsilon_{i,k}^2}{2l_{i,k}^2} \\ &\leq \eta_i^2 z_i^2 \theta_i P_i^T P_i + l_i^2 z_i^2 + \frac{\bar{\varepsilon}_i^2}{l_i^2} + \frac{1}{\eta_i^2} \end{aligned} \quad (4.66)$$

where $\eta_i, l_i > 0$, $\theta_{i,k} = \left\|\Phi_{i,k}\right\|^2$, $\theta_i = \max\{\theta_{i,k} : k \in M\}$, $P_i(X_i)$ and $\bar{\varepsilon}_i(X_i)$ represent the basis function vector and the estimation error belongs to θ_i.

The feasible virtual control functions, adaptive laws and controllers are designed, respectively, as

$$\alpha_i = -\frac{z_i}{\underline{g}_i}\left(\lambda_i + l_i^2 + \eta_i^2 \hat{\theta}_i P_i^T P_i\right) \quad (4.67)$$

$$\dot{\hat{\theta}}_i = r_i \eta_i^2 z_i^2 P_i^T P_i - \beta_i \hat{\theta}_i \quad (4.68)$$

$$u_k = -\frac{z_n}{\underline{g}_n}\left(\lambda_n + l_n^2 + \eta_n^2 \hat{\theta}_n P_n^T P_n\right) \quad (4.69)$$

where for $i = 1, 2, \ldots, n$, λ_i, r_i, β_i are positive design parameters, and $\hat{\theta}_i$ is the estimation of θ_i.

Consider that too many adaptive parameters $(\hat{\theta}_1, \ldots, \hat{\theta}_n)$ can cause the problem of over-parameterization. Set $r_1 = r_2 = \cdots = r_n = r$, $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$, and define $\theta = \sum_{i=1}^{n} \theta_i,\ \hat{\theta} = \sum_{i=1}^{n} \hat{\theta}_i$. The adaptive laws (4.68) can be changed as follows

$$\dot{\hat{\theta}} = \sum_{i=1}^{n} \dot{\hat{\theta}}_i = \sum_{i=1}^{n} \left(r\eta_i^2 z_i^2 P_i^T P_i - \beta\hat{\theta}_i \right)$$
$$= \sum_{i=1}^{n} r\eta_i^2 z_i^2 P_i^T P_i - \beta\hat{\theta}. \tag{4.70}$$

Then, the stabilizing functions, the adaptive law and controllers can be designed as (4.48), (4.52) and (4.51) respectively.

Substituting (4.48), (4.51) and (4.52) into (4.62) one has

$$\dot{V} \leq -\sum_{i=1}^{n} \lambda_i z_i^2 + \sum_{i=1}^{n} \frac{\beta}{r}\tilde{\theta}\hat{\theta} + \sum_{i=1}^{n} \left(\frac{\bar{\varepsilon}_i^2}{l_i^2} + \frac{1}{\eta_i^2} + \delta\varpi_i \right) \tag{4.71}$$

where $\varpi_i = \max\{\varpi_{i,k}, k \in M\}$.

It is true that

$$\tilde{\theta}\hat{\theta} = \tilde{\theta}(\theta - \tilde{\theta}) \leq -\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\theta^2 \tag{4.72}$$

Then, (4.71) can be rewritten as

$$\dot{V} \leq -\frac{1}{2}\sum_{i=1}^{n} \left(2\lambda_i z_i^2 + \frac{\beta}{r}\tilde{\theta}^2 \right) + \sum_{i=1}^{n} \left(\frac{\bar{\varepsilon}_i^2}{l_i^2} + \frac{1}{\eta_i^2} + \delta\varpi_i + \frac{\beta}{2r}\theta^2 \right)$$
$$\leq -a_0 V + b_0 \tag{4.73}$$

where $a_0 = \min\{2\lambda_i,\ \beta : 1 \leq i \leq n\}$ and $b_0 = \sum_{i=1}^{n} (\bar{\varepsilon}_i^2/l_i^2 + 1/\eta_i^2 + \delta\varpi_i + \frac{\beta}{2r}\theta^2)$.

Furthermore

$$V(t) \leq \left(V(0) - \frac{b_0}{a_0} \right) e^{-a_0 t} + \frac{b_0}{a_0} \tag{4.74}$$

which means that all the signals in the closed-loop system are bounded. In particular, we have

$$\lim_{t\to\infty} |z_1| \leq \sqrt{\frac{2b_0}{a_0}}$$

The proof is completed here. □

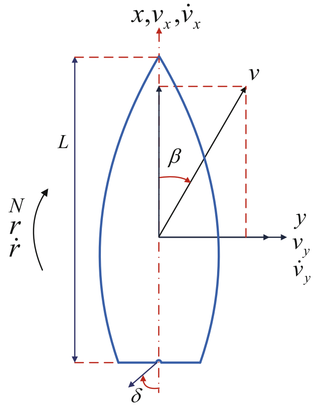


Fig. 4.5 Ship manoeuvring system

4.3.3 Simulation Results

In this section, the simulation studies for the ship manoeuvring systems shown in Fig. 4.5 are carried out to illustrates the effectiveness of our results.

L : length of ship

N : moment component on body relative to z-axis

r : yaw rate

v : speed of ship

v_x : forward velocity in x-axis

v_y : drift velocity along y-axis

β : drift angle

x, y : force components on body

ψ : yaw angle

δ : rudder angle

The ship maneuvering system can be described by the following Norrbin nonlinear model,

$$T\dot{h} + h + \tau h^3 = K\delta + \omega, \tag{4.75}$$

where T is the time constant, $h = \dot{\psi}$ denotes the yaw rate, ψ stands for the heading angle, τ is the Norrbin coefficient, K represents the rudder gain, δ is the rudder angle and ω stands for the outside disturbances. The value of τ can be determined via a spiral test. The ship's dynamic parameters are basically determined by its size and shape, and may vary with operational conditions such as ship speed, draft, trim, and water depth.

A simplified mathematical model of the rudder system can be described as:

$$T_E\dot{\delta} + \delta = K_E\delta_E, \tag{4.76}$$

where T_E represents the rudder time constant, δ stands for the actual rudder angle, K_E denotes the rudder control gain and δ_E is the rudder order.

Let $x_1 = \psi$, $x_2 = h$, $x_3 = \delta$; one has

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f + bx_3 + \omega, \\ \dot{x}_3 &= -\frac{1}{T_E}x_3 + \frac{K_E}{T_E}\delta_E, \end{aligned} \tag{4.77}$$

where $f = -\frac{1}{T}x_2 - \frac{\tau}{T}x_2^3$ is an unknown nonlinear function, $b = \frac{K}{T}$.

Note that some parameters of the aforementioned system will change when the speed of the ship changes. We adopt the following switched model to depict the dynamic behavior when the ship is at low speed, medium speed and high speed, respectively.

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f_{\sigma(v)}(x_2) + b_{\sigma(v)}x_3 + \omega_{\sigma(v)}, \\ \dot{x}_3 &= -\frac{1}{T_{E,\sigma(v)}}x_3 + \frac{K_{E,\sigma(v)}}{T_{E,\sigma(v)}}\delta_{E,\sigma(v)}, \end{aligned} \tag{4.78}$$

where $f_{\sigma(v)}(x_2) = -\frac{1}{T_{\sigma(v)}}x_2 - \frac{\tau_{\sigma(v)}}{T_{\sigma(v)}}x_2^3$, $b_{\sigma(v)} = \frac{K_{\sigma(v)}}{T_{\sigma(v)}}$ and $\sigma(v)$ is the switching signal that satisfies:

$$\sigma(v) = \begin{cases} 1, \ 0 < v \le v_L \\ 2, \ v_L < v \le v_M \\ 3, \ v_M < v \le v_T \end{cases}$$

where v_L, v_M, v_T represent the value of low speed, medium speed and top speed, respectively.

The vessel data comes from a ship that is listed in Table 4.1. The controller parameters are chosen as those in Table 4.2. Furthermore, the outside disturbances are: $w_1 = 0.01\sin t$; $w_2 = 0.015\cos t$; $w_3 = 0.013\sin t$. We construct the basis function vectors P_1, P_2 and P_3 using 7, 15 and 27 nodes, the centers μ_1, μ_2 and μ_3 evenly spaced on $[-3, 3] \times [-4, 1] \times [-2, 2]$, $[-0.5, 3.5] \times [-4, 4] \times [-8, 8]$ and $[-4, 4] \times [-30, 10] \times [-0.5, 3.5]$, and the widths $\zeta_1 = 2$, $\zeta_2 = 2.5$, and $\zeta_3 = 2$,

Table 4.1 Model parameters of ship maneuvering system

$v = 3.72$ m/s (low speed)		$v = 7.5$ m/s (medium speed)		$v = 15.3$ m/s (high speed)	
Parameter	Value	Parameter	Value	Parameter	Value
L (m)	160.9	L	160.9	L	160.9
K_1 (s^{-1})	0.32	K_2	0.114	K_3	0.051
T_1 (s)	30	T_2	63.69	T_3	80.47
τ_1 (s^2)	40	τ_2	30	τ_3	25
$T_{E,1}$ (s)	4	$T_{E,2}$	2.5	$T_{E,3}$	1
$K_{E,1}$	2	$K_{E,2}$	1	$K_{E,3}$	0.72

Table 4.2 Controller parameters

Parameter	λ_1	λ_2	λ_3	l_1	λ_2	λ_3	r
Value	2	3	5	12	14	10	0.01
Parameter	η_1	η_2	η_3	$\underline{g}_1$	$\underline{g}_2$	$\underline{g}_3$	β
Value	8	10	12	1	6.3×10^{-4}	0.4	0.1

respectively. The initial conditions are $x_1(0) = x_2(0) = x_3(0) = 0.02$, $\hat{\theta}(0) = 1$ and the target signal is $y_d = 10 \sin 0.05t$.

To illustrate the effectiveness of the proposed controller, comparison results are presented. The first one uses existing results in [26] and our results, respectively, to control the system when the ship is at a constant speed: low speed. The other one uses existing results [26] and our results respectively to control the system when the ship switches among different speeds.

According to (4.51) and (4.52), the adaptive law $\hat{\theta}$ and the control law u_k are chosen, respectively, as

$$\begin{aligned}
\dot{\hat{\theta}} &= \sum_{i=1}^{3} r\eta_i^2 z_i^2 P_i^T P_i - \beta\hat{\theta} \\
&= 0.64 z_1^2 P_1^T P_1 + z_2^2 P_2^T P_2 + 1.44 z_3^2 P_3^T P_3 - 0.1\hat{\theta} \\
u_k &= -\frac{z_3}{\underline{g}_3}\left(\lambda_3 + l_3^2 + \eta_3^2 \hat{\theta} P_3^T P_3\right) \\
&= -2.5 z_3 \left(105 + 144 P_3^T P_3\right)
\end{aligned}$$

where $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$, $z_3 = x_3 - \alpha_2$.

The virtual control functions α_1 and α_2 are given by

$$\begin{aligned}
\alpha_1 &= -\frac{z_1}{\underline{g}_1}\left(\lambda_1 + l_1^2 + \eta_1^2 \hat{\theta} P_1^T P_1\right) \\
&= -z_1(146 + 64\hat{\theta} P_1^T P_1)
\end{aligned}$$

$$\alpha_2 = -\frac{z_2}{\underline{g}_2}\left(\lambda_2 + l_2^2 + \eta_2^2\hat{\theta}P_1^T P_1\right)$$
$$= -1.6 \times 10^3 \times z_2\left(199 + 100\hat{\theta}P_2^T P_2\right)$$

Figures 4.6, 4.7 and 4.8 show the comparison results by using the existing method in [26] and our approach, respectively. It can be seen that both methods can ensure the target signal is tracked within a small bounded error.

Figures 4.6, 4.7, 4.8, 4.9, 4.10 and 4.11 depict the comparison results by using the existing method in [26] and our method under different speeds, and Fig. 4.12 gives the switching evolution among different speeds. From Figs. 4.9 and 4.10, it can be seen that the existing results in [26] cannot guarantee a good tracking performance

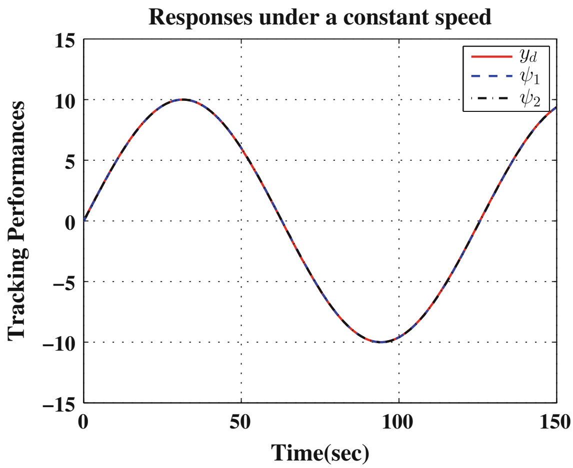


Fig. 4.6 Tracking performances under a constant speed. y_d is the target signal; ψ_1 and ψ_2 represent the outputs by using existing results in [26] and our results respectively

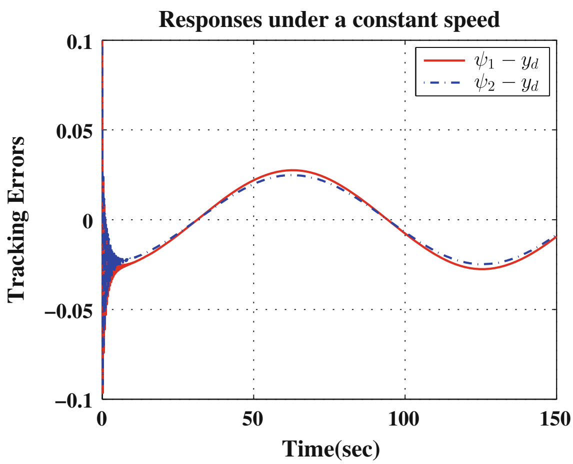


Fig. 4.7 Responses of tracking errors under a constant speed. $\psi_1 - y_d$ and $\psi_2 - y_d$ stand for the tracking error by using existing results in [26] and our results respectively

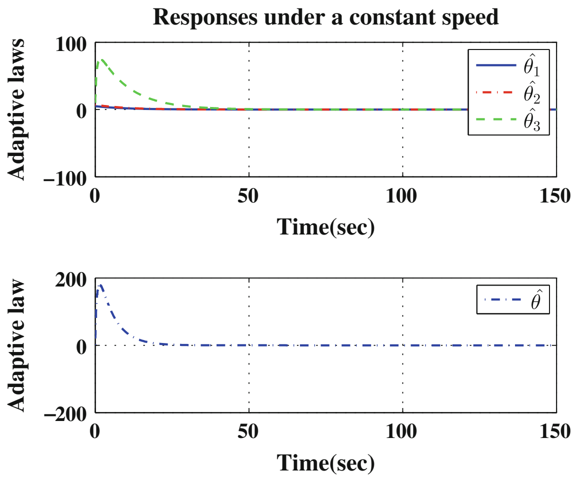


Fig. 4.8 Responses of adaptive laws under a constant speed. $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ denote the adaptive laws by existing results in [26]; $\hat{\theta}$ represents the adaptive law by our results

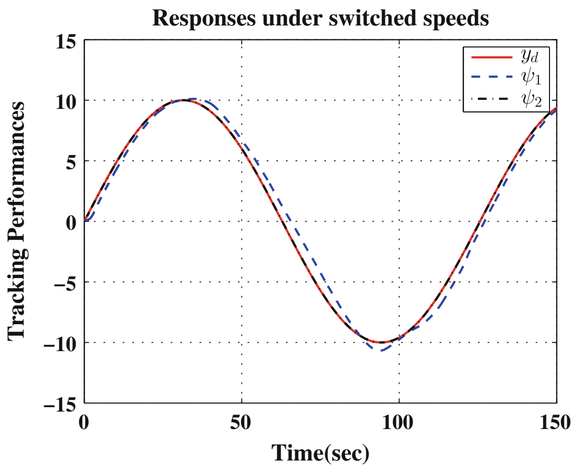


Fig. 4.9 Tracking performances under switched speeds. y_d is the target signal; ψ_1 and ψ_2 represent the outputs by using existing results in [26] and our results respectively

under switched speeds. However, our method can still ensure the target signal is tracked within a small bounded error. Figure 4.11 indicates that the adaptive law's number in our results is less than the one in [26].

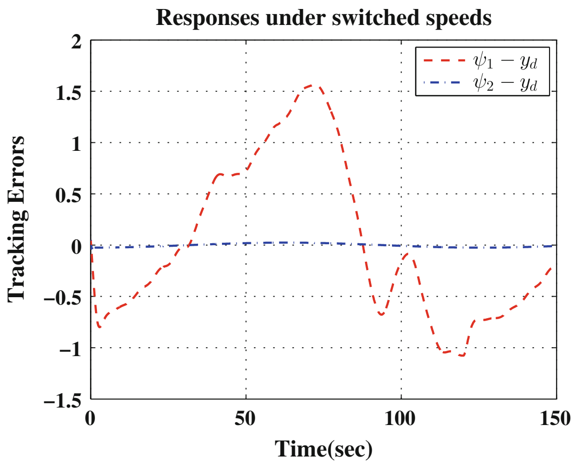


Fig. 4.10 Responses of tracking errors under switched speeds. $\psi_1 - y_d$ and $\psi_2 - y_d$ stand for the tracking error by using existing results in [26] and our results respectively

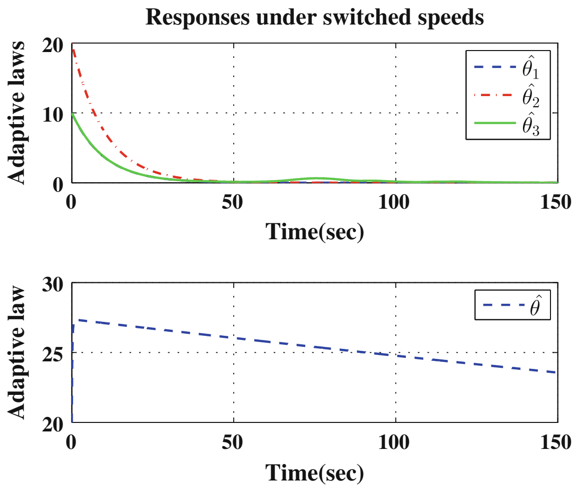


Fig. 4.11 Responses of adaptive laws under switched speeds. $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ denote the adaptive laws by existing results in [26]; $\hat{\theta}$ represents the adaptive law by our results

4.3.4 Conclusions

The problem of adaptive neural tracking control for a class of switched uncertain nonlinear systems in nonstrict-feedback form is investigated. The stability analysis in indicates that the designed controllers can ensure that the target signal can be tracked with a small bounded error and the stability of the system can be kept under arbitrary switchings.

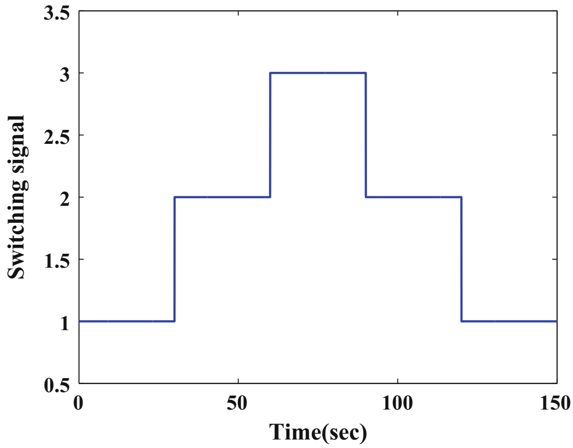


Fig. 4.12 Responses of switching signal

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Chapter 5
Adaptive Control of Switched Stochastic Nonlinear Systems

5.1 Background and Motivation

The last chapter discussed adaptive control design methods for switched nonlinear systems with uncertainties. However, the system structures considered in the last chapter are somewhat simple, which greatly limits the applications of the results in practice.

It is well known that stochastic disturbance is inevitably encountered in practical systems. Therefore, control of stochastic systems with or without switching has become an active research field and received much attention recently, see, e.g., [1–5] and the references therein. The authors in [6] considered global stabilization for high-order stochastic nonlinear systems with stochastic integral input-to-state stability inverse dynamics. The moment stability and sample path stability of switched stochastic linear systems were investigated in [7]. In [8] dissipativity-based sliding mode control for switched stochastic linear systems was adopted. Stabilization problems for stochastic nonlinear systems with Markovian switching were studied in [9]. The p^{th} moment exponential stability and global asymptotic stability in probability for a class of switched stochastic nonlinear retarded systems with asynchronous switching were solved in [10].

Moreover, dead-zone characteristics are encountered in many physical components of control systems. They are particularly common in actuators, such as hydraulic servovalves and electric servomotors. They also appear in biomedical systems. The system model is more realistic and reliable when the dead-zone nonlinearities are taken into consideration.

On the other hand, since the input-to-state stability (ISS) property was proposed in [11], it has rapidly become an important tool to investigate the stability problem of nonlinear systems. In view of the crucial importance of ISS, it is natural to introduce this concept to switched nonlinear systems. In this chapter, we consider some control problems of switched high-order nonlinear systems. Some complex dynamics such as stochastic disturbances, uncertainties, dead-zone nonlinearities and input-to-state stability inverse dynamics are considered in the systems under investigations. The

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considered mathematical models can provide a good description of a large number of practical switched nonlinear systems.

Notation $\mathbb{R}$ denotes the n-dimensional space, $\mathbb{R}^n$ is the set of all nonnegative real numbers. $\mathscr{C}^i$ stands for a set of functions with continuous i^{th} partial derivatives. For a given matrix A (or vector v), A^T (or v^T) denotes its transpose, and $Tr\{A\}$ denotes its trace when A is a square. $\mathscr{K}$ represents the set of functions: $\mathbb{R}^+ \to \mathbb{R}^+$, which are continuous, strictly increasing and vanishing at zero; $\mathscr{K}_\infty$ denotes a set of functions that is of class $\mathscr{K}$ and unbounded. In addition, $\|\cdot\|$ refers to the Euclidean vector norm. $\mathbb{R}$ denotes the n-dimensional space, $\mathbb{R}^+$ denotes the set of all nonnegative real numbers, and $\mathbb{R}^* = \{q \in \mathbb{R}^+ : q \geq 1 \text{ is an odd integer}\}$. $\mathscr{C}^i$ denotes a set of all functions with continuous i^{th} partial derivatives. For a given matrix A (or vector v), A^T (or v^T) denotes its transpose, and $Tr\{A\}$ denotes its trace when A is a square. $\mathscr{K}$ denotes the set of all functions: $\mathbb{R}^+ \to \mathbb{R}^+$, which are continuous, strictly increasing and vanishing at zero; $\mathscr{K}_\infty$ denotes a set of functions that are of class $\mathscr{K}$ and unbounded. In addition, $\|\cdot\|$ refers to the Euclidean vector norm.

5.2 Adaptive Tracking Control for Switched Stochastic Nonlinear Systems with Unknown Actuator Dead-Zone

5.2.1 Problem Formulation and Preliminaries

Consider the following switched stochastic nonlinear system in nonstrict-feedback form.

$$\begin{aligned} dx_i &= (g_{i,\sigma(t)}x_{i+1} + f_{i,\sigma(t)}(x))dt + \psi^T_{i,\sigma(t)}(x)dw, \\ &\quad 1 \leq i \leq n-1, \\ dx_n &= (g_{n,\sigma(t)}v_{\sigma(t)} + f_{n,\sigma(t)}(x))dt + \psi^T_{n,\sigma(t)}(x)dw, \\ v_{\sigma(t)} &= D_{\sigma(t)}(u_{\sigma(t)}), \\ y &= x_1, \end{aligned} \tag{5.1}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is the system state, w is an r-dimensional independent standard Brownian motion defined on the complete probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P\right)$ with Ω being a sample space, $\mathscr{F}$ being a σ-field, $\{\mathscr{F}_t\}_{t\geq 0}$ being a filtration, and P being a probability measure, and y is the system output; $\sigma(t) : [0, \infty) \to M = \{1, 2, \ldots, m\}$ represents the switching signal; $v_{\sigma(t)}, u_{\sigma(t)} \in \mathbb{R}$ are the actuator output and input. For any $i = 1, 2, \ldots, n$ and $k \in M$, $f_{i,k}(x) : \mathbb{R}^n \to \mathbb{R}$, $\psi_{i,k} : \mathbb{R}^n \to \mathbb{R}^r$ are locally Lipschitz unknown nonlinear functions and $g_{i,k}$ are positive known constants.

The nonsymmetric dead-zone nonlinearity is considered in the chapter, which is defined as the form in [12]:

$$v_k = D_k(u_k) = \begin{cases} m_{r_k}(u_k - b_{r_k}), & u_k \geq b_{r_k} \\ 0, & -b_{l_k} < u_k < b_{r_k} \\ m_{l_k}(u_k + b_{l_k}), & u_k \leq -b_{l_k} \end{cases} \tag{5.2}$$

Here, $m_{r_k} > 0$ and $m_{l_k} > 0$ represent the right and the left slopes of the dead-zone characteristic. $b_{r_k} > 0$ and $b_{l_k} > 0$ stand for the breakpoints of the input nonlinearity.

It is assumed that the nonsymmetric dead-zone nonlinearity can be reformulated as:

$$v_k = D_k^{'}(u_k) + \iota_k, \tag{5.3}$$

where $D_k^{'}(u_k)$ is a smooth function, ι_k is the error between $D_k(u_k)$ and $D_k^{'}(u_k)$ with $|\iota_k| \leq \bar{\iota}_k$.

Moreover, we have

$$\begin{aligned} v_k &= u_k + (D_k^{'}(u_k) - u_k + \iota_k) \\ &= u_k + \eta_k^{'}(u_k) + \iota_k, \end{aligned} \tag{5.4}$$

where $\eta_k^{'}(u_k) = D_k^{'}(u_k) - u_k$ is an unknown function.

The controller can be designed as

$$u_k = u_{c_k} - u_{\phi_k}. \tag{5.5}$$

Then (5.4) can be rewritten as

$$v_k = u_{c_k} + \eta_k^{'}(u_k) - u_{\phi_k} + \iota_k. \tag{5.6}$$

where u_{ϕ_k} is the compensator of dead-zone nonlinearity and u_{c_k} is a main controller of system (5.1).

Our control objective is to design a state-feedback controller such that the output of system (5.1) can track a given time-varying signal $y_d(t)$, and the problem of the actuator dead-zone can be solved. The following assumptions are supposed to be true.

Assumption 5.1 The tracking target $y_d(t)$ and its time derivatives up to n^{th} order $y_d^{(n)}(t)$ are continuous and bounded; it is further assumed that $|y_d(t)| \leq d$.

Assumption 5.2 There exist strictly increasing smooth functions $\phi_{i,k}(\cdot)$, $\rho_{i,k}(\cdot)$: $R^+ \to R^+$ with $\phi_{i,k}(0) = \rho_{i,k}(0) = 0$ such that for $i = 1, 2, \ldots, n$ and $k \in M$,

$$|f_{i,k}(x)| \leq \phi_{i,k}(\|x\|). \tag{5.7}$$

$$\left|\psi_{i,k}(x)\right| \leq \rho_{i,k}(\|x\|). \tag{5.8}$$

Remark 5.1 The increasing properties of $\phi_{i,k}(\cdot)$, $\rho_{i,k}(\cdot)$ imply that if $a_i, b_i \geq 0$, for $i = 1, 2, \ldots, n$, then $\phi_{i,k}(\sum_{i=1}^n a_i) \leq \sum_{i=1}^n \phi_{i,k}(na_i)$, $\rho_{i,k}(\sum_{i=1}^n b_i) \leq \sum_{i=1}^n \rho_{i,k}(nb_i)$. Notice that $\phi_{i,k}(s)$, $\rho_{i,k}(s)$ are smooth functions, and $\phi_{i,k}(0) =$

$\rho_{i,k}(0) = 0$. Therefore, there exist smooth functions $h_{i,k}(s)$, $\eta_{i,k}(s)$ such that $\phi_{i,k}(s) = sh_{i,k}(s)$, $\rho_{i,k}(s) = s\eta_{i,k}(s)$ which results in

$$\phi_{i,k}\left(\sum_{j=1}^{n} a_j\right) \leq \sum_{j=1}^{n} na_j h_{i,k}(na_j). \tag{5.9}$$

$$\rho_{i,k}\left(\sum_{j=1}^{n} b_j\right) \leq \sum_{j=1}^{n} nb_j \eta_{i,k}(nb_j). \tag{5.10}$$

We use the radial basis function (RBF) neural networks to approximate any a real function $f(Z)$ over a compact set $\Omega_Z \subset \mathbb{R}^q$. For arbitrary $\bar{\varepsilon} > 0$, there exists a neural network $W^T S(Z)$ such that

$$f(Z) = W^T S(Z) + \varepsilon(Z),\ \varepsilon(Z) \leq \bar{\varepsilon}, \tag{5.11}$$

where $Z \in \Omega_Z \subset R^q$, $W = [w_1, w_2, \ldots, w_l]^T$ is the ideal constant weight vector, and $S(Z) = [s_1(Z), s_2(Z), \ldots, s_l(Z)]^T$ is the basis function vector, with $l > 1$ being the number of the neural network nodes and $s_i(Z)$ being chosen as Gaussian functions; i.e., for $i = 1, 2, \ldots, l$,

$$s_i(Z) = \exp\left[\frac{-(Z-\mu_i)^T(Z-\mu_i)}{\zeta_i^2}\right], \tag{5.12}$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^T$ is the center vector, and ζ_i is the width of the Gaussian function.

Definition 5.1 For any given $V(x_i, t) \in \mathscr{C}^{2,1}$ associated with system (5.1), define the differential operator $\mathscr{L}$ as follows;

$$\mathscr{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i} F_{i,k} + \frac{1}{2} Tr\left\{\psi_{i,k}^T \frac{\partial^2 V}{\partial x_i^2} \psi_{i,k}\right\}, \tag{5.13}$$

where $F_{i,k} = g_{i,k}x_{i+1} + f_{i,k}(x)$.

Definition 5.2 The trajectory $\{x(t), t \geq 0\}$ of switched stochastic system (5.1) is said to be semi-globally uniformly ultimately bounded (SGUUB) in the p^{th} moment, if for some compact set $\Omega \in \mathbb{R}^n$ and any initial state $x_0 = x(t_0)$, there exist a constant $\varepsilon > 0$, and a time constant $T = T(\varepsilon, x_0)$ such that $E(|x(t)|^p) < \varepsilon$, for all $t > t_0 + T$. Especially, when $p = 2$, it is usually called SGUUB in mean square.

Lemma 5.1 ([13]) *Suppose that there exist a $C^{2,1}$ function $V(x,t): \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$, two constants $c_1 > 0$ and $c_2 > 0$, class $\mathscr{K}_\infty$ functions $\bar{\alpha}_1$ and $\bar{\alpha}_2$ such that*

$$\begin{cases} \bar{\alpha}_1(|x|) \leq V(x,t) \leq \bar{\alpha}_2(|x|) \\ \mathscr{L}V \leq -c_1 V(x,t) + c_2 \end{cases}$$

for all $x \in \mathbb{R}^n$ and $t > t_0$. Then, there is an unique strong solution of system (5.1) for each $x_0 \in \mathbb{R}^n$, that satisfies

$$E[V(x,t)] \leq V(x_0)e^{-c_1 t} + \frac{c_2}{c_1}, \ \forall t > t_0$$

Lemma 5.2 ([14]) *For any $\xi \in \mathbb{R}$ and $\varpi > 0$, the following inequality holds:*

$$0 \leq |\xi| - \xi \tanh\left(\frac{\xi}{\varpi}\right) \leq \delta\varpi, \tag{5.14}$$

with $\delta = 0.2785$.

5.2.2 Main Results

Based on the backstepping technique, a control design and stability analysis procedure is presented in this section. For $i = 1, 2, \ldots, n-1$, define a common virtual control function α_i as

$$\alpha_i = \frac{1}{g_{i,\min}}\left[-\left(\lambda_i + \frac{3}{4}\right)z_i - \frac{1}{2a_i^2} z_i^3 \hat{\theta} S_i^T S_i\right], \tag{5.15}$$

where λ_i, $a_i > 0$ are design parameters, $g_{i,\min} = \min\{g_{i,k} : k \in M\}$, z_i is the new state variable after the coordinate transformation: $z_i = x_i - \alpha_{i-1}$, $\alpha_0 = y_d$. $\hat{\theta}$ is an unknown constant that is specified later. $S_i = S_i(X_i)$ is the basis function vector. $X_i = [\bar{x}_i^T, \bar{\hat{\theta}}_i, \bar{y}_d^{(i)}]^T$ with $\bar{x}_i = [x_1, x_2, \ldots, x_i]^T$, $\bar{\hat{\theta}}_i = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_i]^T$, $\bar{y}_d^{(i)} = [y_d, \dot{y}_d, \ldots, y_d^{(i)}]^T$. The z-system is obtained as

$$\begin{aligned} dz_i &= (g_{i,k}x_{i+1} + f_{i,k} - \mathscr{L}\alpha_{i-1})dt + \left(\psi_{i,k} - \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_{j,k}\right)^T dw, \ 1 \leq i \leq n-1 \\ dz_n &= (g_{n,k}v_k + f_{n,k} - \mathscr{L}\alpha_{n-1})dt + \left(\psi_{n,k} - \sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\psi_{j,k}\right)^T dw, \end{aligned} \tag{5.16}$$

where the differential operator $\mathscr{L}$ is defined in Definition 5.1; $\mathscr{L}\alpha_{i-1}$ is given by:

$$\mathscr{L}\alpha_{i-1} = \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s}(f_{s,k} + g_{s,k}x_{s+1})$$
$$+ \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} + \frac{1}{2}\sum_{p,q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q}\psi_{p,k}^T \psi_{q,k}. \tag{5.17}$$

Consider the following common stochastic Lyapunov function candidate

$$V = \sum_{i=1}^{n} \frac{1}{4} z_i^4 + \frac{1}{2r_1}\tilde{\theta}^2 + \frac{1}{2r_2}\tilde{\vartheta}^2, \tag{5.18}$$

where $r_1,\ r_2 > 0$ are design parameters; θ and ϑ are specified later. $\hat{\theta}$ and $\hat{\vartheta}$ stand for the estimations of θ and ϑ, respectively; $\tilde{\theta} = \theta - \hat{\theta}$, $\tilde{\vartheta} = \vartheta - \hat{\vartheta}$.

Lemma 5.3 *From the coordinate transformations $z_i = x_i - \alpha_{i-1}$, $i = 1, 2, \ldots, n$, $\alpha_0 = y_d$, the following results hold,*

$$\|x\| \le \sum_{i=1}^{n} |z_i| \varphi_i(z_i, \hat{\theta}) + d, \tag{5.19}$$

where $\varphi_i(z_i, \hat{\theta}) = \frac{1}{g_{i,\min}}[(\lambda_i + \frac{3}{4}) + \frac{1}{2a_i^2} z_i^2 \hat{\theta} S_i^T S_i] + 1$, *for* $i = 1, 2, \ldots, n-1$, *and* $\varphi_n = 1$.

Proof From Assumption 5.1 and (5.15), one can get that

$$\begin{aligned}
\|x\| &\le \sum_{i=1}^{n} |x_i| \\
&\le \sum_{i=1}^{n} (|z_i| + |\alpha_{i-1}|) \\
&\le \sum_{i=1}^{n} |z_i| + y_d + \sum_{i=1}^{n-1} \left(\frac{1}{g_{i,\min}}[(\lambda_i + \frac{3}{4}) + \frac{1}{2a_i^2} z_i^2 \hat{\theta} S_i^T S_i] \right) |z_i| \\
&\le \sum_{i=1}^{n} |z_i| \varphi_i(z_i, \hat{\theta}) + d.
\end{aligned}$$

The proof of Lemma 5.3 is completed here. □

The $\mathscr{L}V$ can be given by

$$\begin{aligned}\mathscr{L}V =&\sum_{i=1}^{n-1}\Bigg\{z_i^3\Bigg(f_{i,k}+g_{i,k}x_{i+1}-\sum_{s=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(s)}}y_d^{(s+1)}-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\\&-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}(f_{s,k}+g_{s,k}x_{s+1})-\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\psi_{p,k}^T\psi_{q,k}\Bigg)\\&+\frac{3}{2}z_i^2\left\|\psi_{i,k}-\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_{j,k}\right\|^2\Bigg\}+z_n^3\Bigg(f_{n,k}+g_{n,k}v_k-\sum_{s=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_d^{(s)}}y_d^{(s+1)}\\&-\sum_{s=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_s}(f_{s,k}+g_{s,k}x_{s+1})-\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}-\frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\psi_{p,k}^T\psi_{q,k}\Bigg)\\&-\frac{1}{r_1}\tilde{\theta}\dot{\hat{\theta}}+\frac{3}{2}z_n^2\left\|\psi_{n,k}-\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\psi_{n,k}\right\|^2-\frac{1}{r_2}\tilde{\vartheta}\dot{\hat{\vartheta}}\end{aligned}$$

$$\begin{aligned}=&\sum_{i=1}^{n}\Bigg\{z_i^3\Bigg(f_{i,k}-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}f_{s,k}-\sum_{s=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(s)}}y_d^{(s+1)}-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}g_{s,k}x_{s+1}\\&-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}-\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\psi_{p,k}^T\psi_{q,k}\Bigg)+\frac{3}{2}z_i^2\left\|\psi_{i,k}-\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_{j,k}\right\|^2\Bigg\}\\&-\frac{1}{r_1}\tilde{\theta}\dot{\hat{\theta}}-\frac{1}{r_2}\tilde{\vartheta}\dot{\hat{\vartheta}}+\sum_{i=1}^{n}z_i^3g_{i,k}x_{i+1}+z_n^3g_{n,k}v_k.\end{aligned}\tag{5.20}$$

By resorting to Assumption 5.2 and Lemma 5.3, one has that

$$\begin{aligned}&z_i^3(f_{i,k}-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}f_{s,k}(x))\\=&-z_i^3\sum_{s=1}^{i}\frac{\partial\alpha_{i-1}}{\partial x_s}f_{s,k}(x)\\\le&\frac{3}{4}nz_i^4\sum_{s=1}^{i}\left(\frac{\partial\alpha_{i-1}}{\partial x_s}\right)^{\frac{4}{3}}+\sum_{s=1}^{i}\sum_{l=1}^{n}z_l^4\bar{\phi}_{s,k}^4(z_l,\hat{\theta})+\left|z_i^3\right|\sum_{s=1}^{i}\left|\frac{\partial\alpha_{i-1}}{\partial x_s}\right|\phi_{s,k}((n+1)d),\end{aligned}\tag{5.21}$$

where $\bar{\phi}_{s,k}^4(z_l,\hat{\theta})=\frac{1}{4}(n+1)^4\varphi_l^4(z_l,\hat{\theta})h_{s,k}^4((n+1)|z_l|\varphi_l(z_l,\hat{\theta}))$, $\frac{\partial\alpha_0}{\partial x_i}=0$ and $\frac{\partial\alpha_{i-1}}{\partial x_i}=-1$.

Then, the following inequality can be obtained,

$$
\begin{aligned}
&\frac{3}{2}z_i^2\left\|\psi_{i,k}-\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_{j,k}\right\|^2\\
&\quad\le\frac{9}{8}i^2(n+1)^2nz_i^4+\sum_{l=1}^{n}z_l^4\bar{\rho}_{i,k}^4(z_l,\hat{\theta})+\sum_{j=1}^{i-1}\sum_{l=1}^{n}z_l^4\bar{\rho}_{j,k}^4(z_l,\hat{\theta})\\
&\quad\quad+\frac{9}{8}i^2(n+1)^2nz_i^4\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j})^4+\frac{9}{8}i^2(n+1)^2z_i^4l_{ii}^{-2}\rho_{i,k}^4((n+1)d)\\
&\quad\quad+\sum_{j=1}^{i}l_{ij}^2+\frac{9}{8}i^2(n+1)^2z_i^4\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j})^4l_{ij}^{-2}\rho_{j,k}^4((n+1)d),
\end{aligned}
\tag{5.22}
$$

where l_{ij} is a positive constant, and $\frac{\partial\alpha_0}{\partial x_j}=0$ because $\alpha_0=y_d$, and

$$
\begin{aligned}
&-\frac{1}{2}z_i^3\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\psi_{p,k}^T\psi_{q,k}\\
&\le(i-1)\sum_{s=1}^{i-1}\sum_{l=1}^{n}z_l^4\bar{\rho}_{s,k}^4(z_l,\hat{\theta})\\
&\quad+\frac{1}{8}(n+1)^2nz_i^6\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\left(\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right)^2\\
&\quad+\frac{1}{2}(n+1)\left|z_i^3\right|\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\left|\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right|\rho_{s,k}^2((n+1)d),
\end{aligned}
\tag{5.23}
$$

where $\bar{\rho}_{s,k}^4(z_l,\hat{\theta})=\frac{1}{2}(n+1)^4\varphi_l^4(z_l,\hat{\theta})\eta_{s,k}^4((n+1)|z_l|\varphi_l(z_l,\hat{\theta})),s=1,2,\ldots,i-1.$

Substituting (5.21), (5.22) and (5.23) into (5.20) gives that

$$
\begin{aligned}
\mathscr{L}V&\le\sum_{i=1}^{n}\frac{3}{4}nz_i^4\sum_{s=1}^{i}\left(\frac{\partial\alpha_{i-1}}{\partial x_s}\right)^{\frac{4}{3}}+\sum_{i=1}^{n}\sum_{s=1}^{i}\sum_{l=1}^{n}z_l^4\bar{\phi}_{s,k}^4(z_l,\hat{\theta})\\
&+\sum_{i=1}^{n}\left|z_i^3\right|\sum_{s=1}^{i}\left|\frac{\partial\alpha_{i-1}}{\partial x_s}\right|\phi_{s,k}((n+1)d)+\sum_{i=1}^{n}\sum_{s=1}^{i-1}\sum_{l=1}^{n}(i-1)z_l^4\bar{\rho}_{s,k}^4(z_l,\hat{\theta})\\
&+\sum_{i=1}^{n}\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\frac{1}{8}(n+1)^2nz_i^6\left(\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right)^2\\
&+\sum_{i=1}^{n}\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\frac{1}{2}(n+1)\left|z_i^3\right|\rho_{s,k}^2((n+1)d)\left|\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right|
\end{aligned}
$$

$$+\sum_{i=1}^{n}\Bigg\{\frac{9}{8}i^2(n+1)^2nz_i^4+\sum_{l=1}^{n}z_l^4\bar{\rho}_{i,k}^4(z_l,\hat{\theta})+\sum_{j=1}^{i-1}\sum_{l=1}^{n}z_l^4\bar{\rho}_{j,k}^4(z_l,\hat{\theta})$$
$$+\frac{9}{8}i^2(n+1)^2nz_i^4\sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\right)^4+\frac{9}{8}i^2(n+1)^2z_i^4l_{ii}^{-2}\rho_{i,k}^4((n+1)d)+\sum_{j=1}^{i}l_{ij}^2$$
$$+\frac{9}{8}i^2(n+1)^2z_i^4\sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\right)^4 l_{ij}^{-2}\rho_{j,k}^4((n+1)d)\Bigg\}$$
$$+\sum_{i=1}^{n}z_i^3\left(-\sum_{s=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(s)}}y_d^{(s+1)}-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}-\sum_{s=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_s}g_{s,k}x_{s+1}\right)$$
$$+\sum_{i=1}^{n-1}z_i^3g_{i,k}x_{i+1}+z_n^3g_{n,k}v_k-\frac{1}{r_1}\tilde{\theta}\dot{\hat{\theta}}-\frac{1}{r_2}\tilde{\vartheta}\dot{\hat{\vartheta}}. \tag{5.24}$$

Define $U_{i,k}$ as

$$U_{i,k}=\sum_{s=1}^{i}|\frac{\partial\alpha_{i-1}}{\partial x_s}|\phi_{s,k}((n+1)d)$$
$$+\frac{1}{2}(n+1)\sum_{s=1}^{i-1}\sum_{j=1}^{i-1}\left|\frac{\partial^2\alpha_{i-1}}{\partial x_s\partial x_j}\right|\rho_{s,k}^2((n+1)d). \tag{5.25}$$

By using Lemma 5.2 one has

$$\left|z_i^3\right|U_{i,k}\le z_i^3U_{i,k}\tanh(\frac{z_i^3U_{i,k}}{\varpi_{i,k}})+\delta\varpi_{i,k}. \tag{5.26}$$

Note that

$$\sum_{i=1}^{n-1}z_i^3g_{i,k}x_{i+1}=\sum_{i=1}^{n-1}z_i^3g_{i,k}z_{i+1}+\sum_{i=1}^{n-1}g_{i,k}z_i^3\alpha_i, \tag{5.27}$$

Therefore, one has

$$\sum_{i=1}^{n}\sum_{s=1}^{i}\sum_{l=1}^{n}z_l^4\bar{\phi}_{s,k}^4(z_l,\hat{\theta})=\sum_{i=1}^{n}z_i^4\sum_{s=1}^{n}(n-s+1)\bar{\phi}_{s,k}^4(z_i,\hat{\theta}),$$
$$\sum_{i=1}^{n}(i-1)\sum_{s=1}^{i-1}\sum_{l=1}^{n}z_l^4\bar{\rho}_{s,k}^4(z_l,\hat{\theta})=\sum_{i=1}^{n}z_i^4\sum_{s=1}^{n-1}(n-s)(i-1)\bar{\rho}_{s,k}^4(z_i,\hat{\theta}),$$
$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{l=1}^{n}z_l^4\bar{\rho}_{j,k}^4(z_l,\hat{\theta})=\sum_{i=1}^{n}z_i^4\sum_{j=1}^{n}(n-j+1)\bar{\rho}_{j,k}^4(z_i,\hat{\theta}).$$

For any $i = 1, 2, \ldots, n$ and $k \in M$, define $\bar{f}_{i,k}$ as

$$\begin{aligned}
\bar{f}_{i,k} = {} & \frac{3}{4} n z_i \sum_{s=1}^{i} \left(\frac{\partial \alpha_{i-1}}{\partial x_s} \right)^{\frac{4}{3}} + z_i \sum_{s=1}^{n} (n - s + 1) \bar{\phi}_{s,k}^4 (z_i, \hat{\theta}) \\
& + z_i \sum_{s=1}^{n-1} (n - s)(i - 1) \bar{\rho}_{s,k}^4 (z_i, \hat{\theta}) + \sum_{s=1}^{i-1} \sum_{j=1}^{i-1} \frac{1}{8} (n+1)^2 n z_i^3 \left(\frac{\partial^2 \alpha_{i-1}}{\partial x_s \partial x_j} \right)^2 \\
& + \frac{9}{8} i^2 (n+1)^2 z_i \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_j} \right)^4 l_{ij}^{-2} \rho_{j,k}^4 ((n+1)d) \\
& + z_i \sum_{j=1}^{n} (n - j + 1) \bar{\rho}_{j,k}^4 (z_i, \hat{\theta}) + \frac{9}{8} i^2 (n+1)^2 n z_i \\
& + \frac{9}{8} i^2 (n+1)^2 z_i l_{ii}^{-2} \rho_{i,k}^4 ((n+1)d) + \frac{9}{8} i^2 (n+1)^2 n z_i \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_j} \right)^4 \\
& - \sum_{s=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(s)}} y_d^{(s+1)} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \sum_{s=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_s} g_{s,k} x_{s+1} \\
& + U_{i,k} \tanh\left(\frac{z_i^3 U_{i,k}}{\varpi_{i,k}}\right) + g_{i,k} z_{i+1},
\end{aligned} \tag{5.28}$$

with $z_{n+1} = 0$.
Substituting (5.6) and (5.26)–(5.28) into (5.24) yields that

$$\begin{aligned}
\mathscr{L} V \le {} & \sum_{i=1}^{n-1} z_i^3 (\bar{f}_{i,k} + g_{i,k} \alpha_i) + z_n^3 \bar{f}_{n,k} + z_n^3 g_{n,k} (u_{c_k} + \eta_k^{'} - u_{\phi_k} + \iota_k) \\
& - \frac{1}{r_1} \tilde{\theta} \dot{\hat{\theta}} - \frac{1}{r_2} \tilde{\vartheta} \dot{\hat{\vartheta}} + \sum_{i=1}^{n} \left(\delta \varpi_{i,k} + \sum_{j=1}^{i} l_{ij}^2 \right).
\end{aligned} \tag{5.29}$$

By exploring the neural networks' approximation capability and Young's inequality, one can get the following inequalities.

$$\begin{aligned} z_i^3 \bar{f}_{i,k} &= z_i^3 W_{i,k}^T S_{i,k} + z_i^3 \varepsilon_{i,k} \\ &\le \frac{1}{2a_i^2} z_i^6 \left\| W_{i,k} \right\|^2 S_{i,k}^T S_{i,k} + \frac{a_i^2}{2} + \frac{3}{4} z_n^4 + \frac{\bar{\varepsilon}_{i,k}^4}{4}, \\ &\le \frac{1}{2a_i^2} z_i^6 \theta_i S_i^T S_i + \frac{a_i^2}{2} + \frac{3}{4} z_i^4 + \frac{\bar{\varepsilon}_i^4}{4}, \end{aligned} \tag{5.30}$$

$$\begin{aligned} z_n^3(\eta_k^{'} + \iota_k) &= z_n^3 W_{\eta,k}^T S_{\eta,k} + z_n^3 (\varepsilon_{\eta,k} + \iota_k) \\ &\le \frac{1}{2a_\eta^2} z_n^6 \vartheta_\eta S_\eta^T S_\eta + \frac{a_\eta^2}{2} + \frac{3z_n^4 + \bar{\varepsilon}_\eta^4}{4}, \end{aligned} \tag{5.31}$$

where $\theta_{i,k} = \left\| W_{i,k} \right\|^2$, $\vartheta_{\eta,k} = \left\| W_{\eta,k} \right\|^2$, $\theta_i = \max\{\theta_{i,k} : k \in M\}$, $\vartheta_\eta = \max\{\vartheta_{\eta,k} : k \in M\}$, $\left|\varepsilon_{i,k}\right| \le \bar{\varepsilon}_i$, $\left|\varepsilon_{\eta,k} + \iota_k\right| \le \bar{\varepsilon}_\eta$.

Substituting (5.30) and (5.31) into (5.29) gives

$$\begin{aligned} \mathscr{L}V \le& \sum_{i=1}^{n-1} z_i^3 \left(\frac{z_i^3 \theta_i}{2a_i^2} S_i^T S_i + g_{i,k} \alpha_i \right) + z_n^3 \left(\frac{z_n^3 \theta_n}{2a_n^2} S_n^T S_n + g_{n,k} u_{c_k} \right) \\ &+ z_n^3 g_{n,k} \left(\frac{1}{2a_\eta^2} z_n^3 \theta_\eta S_\eta^T S_\eta - u_{\phi_k} \right) + g_{n,k} \left(\frac{a_\eta^2}{2} + \frac{3}{4} z_n^4 + \frac{\bar{\varepsilon}_\eta^4}{4} \right) \\ &+ \sum_{i=1}^{n} \left(\frac{2a_i^2 + 3z_i^4 + \bar{\varepsilon}_i^4}{4} \right) - \frac{1}{r_1} \tilde{\theta} \dot{\hat{\theta}} - \frac{1}{r_2} \tilde{\vartheta} \dot{\hat{\vartheta}} + \sum_{i=1}^{n} \left(\delta \varpi_i + \sum_{j=1}^{i} l_{ij}^2 \right), \end{aligned} \tag{5.32}$$

where $\varpi_i := \max\{\varpi_{i,k},\ k \in M\}$.

Design the virtual control function as

$$\alpha_i = \frac{1}{g_{i,\min}} \left[-\left(\lambda_i + \frac{3}{4} \right) z_i - \frac{1}{2a_i^2} z_i^3 \hat{\theta} S_i^T S_i \right], \tag{5.33}$$

where $\hat{\theta} = \sum_{i=1}^{n} \hat{\theta}_i$ is the estimation of θ; $\lambda_i > 0$ is a design parameter.

The actual actuator input is given as

$$u_k = u_{c_k} - u_{\phi_k}, \tag{5.34}$$

where

$$u_{c_k} = \frac{1}{g_{n,k}} \left[-\left(\lambda_n + \frac{3}{4} \right) z_n - \frac{1}{2a_n^2} z_n^3 \hat{\theta} S_n^T S_n \right], \tag{5.35}$$

$$u_{\phi_k} = \left(\lambda_\eta + \frac{3}{4} \right) z_n + \frac{g_{n,\max}}{2a_\eta^2 g_{n,k}} z_n^3 \hat{\vartheta} S_\eta^T S_\eta, \tag{5.36}$$

λ_n, λ_η, a_n, $a_\eta > 0$ are design parameters, $g_{n,\max} = \max\{g_{n,k}, k \in M\}$, $g_{n,\min} = \min\{g_{n,k}, k \in M\}$, $\hat{\vartheta}$ is the estimation of ϑ.

The adaptive laws can be designed as

$$\dot{\hat{\theta}} = \sum_{i=1}^{n} \frac{r_1}{2a_{i,\min}^2} z_i^6 S_i^T S_i - \beta_1 \hat{\theta}, \tag{5.37}$$

$$\dot{\hat{\vartheta}} = \frac{g_{n,\max} r_2}{2a_{\eta,\min}^2} z_n^6 S_\eta^T S_\eta - \beta_2 \hat{\vartheta}. \tag{5.38}$$

Then, one can get from (5.32)–(5.38) that

$$\begin{aligned} \mathscr{L}V \leq & -\sum_{i=1}^{n} \lambda_i z_i^4 - \lambda_\eta z_n^4 + g_{n,k}\left(\frac{a_\eta^2}{2} + \frac{\bar{\varepsilon}_\eta^4}{4}\right) + \sum_{i=1}^{n}\left(\frac{a_i^2}{2} + \frac{\bar{\varepsilon}_i^4}{4}\right) + \frac{\beta_1}{r_1}\tilde{\theta}\hat{\theta} \\ & + \frac{\beta_2}{r_2}\tilde{\vartheta}\hat{\vartheta} + \sum_{i=1}^{n}\left(\delta\varpi_i + \sum_{j=1}^{i} l_{ij}^2\right). \end{aligned} \tag{5.39}$$

It is clear that

$$\tilde{\theta}\hat{\theta} = \tilde{\theta}(\theta - \tilde{\theta}) \leq -\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\theta^2, \tag{5.40}$$

$$\tilde{\vartheta}\hat{\vartheta} = \tilde{\vartheta}(\vartheta - \tilde{\vartheta}) \leq -\frac{1}{2}\tilde{\vartheta}^2 + \frac{1}{2}\vartheta^2. \tag{5.41}$$

Combining (5.39) with (5.40) and (5.41) yields that

$$\begin{aligned} \mathscr{L}V \leq & -\sum_{i=1}^{n} \lambda_i z_i^4 - \frac{\beta_1}{2r_1}\tilde{\theta}^2 - \frac{\beta_2}{2r_2}\tilde{\vartheta}^2 + g_{n,k}\left(\frac{a_\eta^2}{2} + \frac{\bar{\varepsilon}_\eta^4}{4}\right) + \sum_{i=1}^{n}\left(\frac{a_i^2}{2} + \frac{\bar{\varepsilon}_i^4}{4}\right) \\ & + \sum_{i=1}^{n}\left(\delta\varpi_i + \sum_{j=1}^{i} l_{ij}^2\right) + \frac{\beta_1\theta^2}{2r_1} + \frac{\beta_2\vartheta^2}{2r_2} \\ \leq & - p_0 V + q_0, \end{aligned} \tag{5.42}$$

where $\lambda_n := \lambda_n + \lambda_\eta$, $p_0 = \min\{4\lambda_i, \beta_1, \beta_2 : 1 \leq i \leq n\}$, $q_0 = \sum_{i=1}^{n}(\frac{a_i^2}{2} + \frac{\bar{\varepsilon}_i^4}{4}) + \sum_{i=1}^{n}\left(\delta\varpi_i + \sum_{j=1}^{i} l_{ij}^2\right) + \frac{\beta_1\theta^2}{2r_1} + \frac{\beta_2\vartheta^2}{2r_2} + g_{n,k}(\frac{a_\eta^2}{2} + \frac{\bar{\varepsilon}_\eta^4}{4})$.

By using Lemma 5.1, we have

$$\frac{dE[V(t)]}{dt} \leq -p_0 E[V(t)] + q_0; \tag{5.43}$$

Then, the following inequality holds

$$0 \le E[V(t)] \le V(0)e^{-p_0 t} + \frac{q_0}{p_0}, \tag{5.44}$$

where $V(0) = \sum_{j=1}^{n} \frac{z_j^2(0)}{4} + \frac{1}{2r_1}\tilde{\theta}(0)^2 + \frac{1}{2r_2}\tilde{\vartheta}(0)^2$. Equation (5.44) implies that all the signals in the closed-loop system are bounded in probability. In particular, we have

$$E[|z_i|^4] \le \frac{4q_0}{p_0}, \quad t \to \infty. \tag{5.45}$$

Now, we are ready to provide our main result in the following theorem.

Theorem 5.1 *Consider the closed-loop system (5.1) with unknown nonsymmetric actuator dead-zone (5.2). Suppose that for* $1 \le i \le n$, $k \in M$, *the packaged unknown functions* $\bar{f}_{i,k}$ *can be approximated by neural networks in the sense that the approximation error* $\varepsilon_{i,k}$ *are bounded. Under the state feedback controller (5.34) and the adaptive laws (5.37), (5.38), the following statements hold,*

(i) All the signals of the closed-loop z-system (5.17) are SGUUB in the fourth moment and

$$P\left\{\lim_{t\to\infty} \sum_{i=1}^{n} E[|z_i|^4] \le \frac{4q_0}{p_0}\right\} = 1.$$

(ii) The output y of the closed-loop system (5.1) can be almost surely regulated to a small neighborhood of the target signal.

Proof It is not difficult to complete the proof by using the above developments. □

5.2.3 Simulation Results

In this section, an example about the control of a ship manoeuvring system are used to illustrate the effectiveness of the obtained results.

The ship maneuvering system can be described by the following Norrbin nonlinear model [15].

$$T_{\sigma(v_s)}\dot{h} + h + \alpha_{\sigma(v_s)}h^3 = K_{\sigma(v_s)}\delta + \phi^T_{\sigma(v_s)}(\psi, h, \delta)w,$$

where $T_{\sigma(v_s)}$ is the time constant, $h = \dot{\psi}$ denotes the yaw rate, ψ stands for the heading angle, $\alpha_{\sigma(v_s)}$ is the Norrbin coefficient, $K_{\sigma(v_s)}$ represents the rudder gain, δ is the rudder angle and w stands for an r-dimensional independent standard Brownian motion, $\phi_{\sigma(v_s)}(\psi, h, \delta) : \mathbb{R}^3 \to \mathbb{R}^{3\times r}$ is an unknown function, and $\sigma(v_s)$ is the switching signal that satisfies:

$$\sigma(v_s) = \begin{cases} 1, & 0 < v_s \le v_L \\ 2, & v_L < v_s \le v_M \\ 3, & v_M < v_s \le v_T \end{cases}$$

v_L, v_M, v_T represent the value of low speed, middle speed and top speed, respectively.

A simplified mathematical model of the rudder system can be described as follows,

$$T_{E,\sigma(v_s)}\dot{\delta} + \delta = K_{E,\sigma(v_s)}\delta_{E,\sigma(v_s)},$$

where $T_{E,\sigma(v_s)}$ represents the rudder time constant, δ stands for the actual rudder angle, $K_{E,\sigma(v_s)}$ denotes the rudder control gain and $\delta_{E,\sigma(v_s)}$ is the rudder order.

Let $x_1 = \psi$, $x_2 = h$, $x_3 = \delta$, $v_{\sigma(v_s)} = \delta_{E,\sigma(v_s)}$; we can get the following switched nonlinear system model with actuator dead-zone to describe the dynamic behavior of the ship with low speed, middle speed and high speed, respectively.

$$\begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= (f_{\sigma(v_s)} + b_{\sigma(v_s)}x_3)dt + \phi^T_{\sigma(v_s)}d\omega, \\ dx_3 &= \left(-\frac{1}{T_{E,\sigma(v_s)}}x_3 + \frac{K_{E,\sigma(t)}}{T_{E,\sigma(v_s)}}v_{\sigma(v_s)}\right)dt, \\ v_{\sigma(v_s)} &= D(u_{\sigma(v_s)}) \end{aligned}$$

where $f_{\sigma(v_s)} = -\frac{1}{T_{\sigma(v_s)}}x_2 - \frac{\tau_{\sigma(v_s)}}{T_{\sigma(v_s)}}x_2^3$, $b_{\sigma(v_s)} = \frac{K_{\sigma(v_s)}}{T_{\sigma(v_s)}}$.

The vessel data comes from a ship that has a length overall of 160.9 m. $v_L = 3.7$ m/s, $K_1 = 32$ s^{-1}, $T_1 = 30$ s, $\tau_1 = 40$ s^2, $T_{E,1} = 4$ s, $K_{E,1} = 2$; $v_M = 7.5$ m/s, $K_2 = 11.4$ s^{-1}, $T_2 = 63.69$ s, $\tau_2 = 30$ s^2, $T_{E,2} = 2.5$ s, $K_{E,2} = 1$; $v_T = 15.3$ m/s, $K_3 = 5.1$ s^{-1}, $T_3 = 80.47$ s, $\tau_3 = 25$ s^2, $T_{E,3} = 1$ s, $K_{E,3} = 0.72$. The initial conditions are $x_1(0) = 2, x_2(0) = -0.05, x_3(0) = 0.03, \hat{\theta}(0) = 10, \hat{\vartheta}(0) = 1$. We construct the basis function vectors S_1, S_2, S_3 and S_η using 11, 15, 21 and 48 nodes, the centers μ_1, μ_2, μ_3, μ_η evenly spaced on $[-1.5, 4.5] \times [-3, 4] \times [-10, 8]$, $[-5, 4] \times [-30, 20] \times [-0.5, 5.5]$, $[-5.5, 8] \times [-12, 25] \times [-0.1, 2]$ and $[-10, 2] \times [-60, 2] \times [-0.2, 10.5]$, and the widths $\zeta_1 = 1.2$, $\zeta_2 = 2.2$, $\zeta_3 = 2$, $\zeta_\eta = 1.8$. The design parameters are $a_1 = a_2 = a_3 = a_\eta = 10$, $r_1 = 2$, $r_2 = 10$, $\beta_1 = 0.5$, $\beta_2 = 0.1$, $\lambda_1 = \lambda_2 = \lambda_3 = 5$, and $\lambda_\eta = 3$. The desired trajectory is $y_d = 10\sin 0.08t$.

According to Theorem 5.1, the adaptive laws $\dot{\hat{\theta}}$, $\dot{\hat{\vartheta}}$ and the control laws u_{c_k}, u_{ϕ_k} are chosen, respectively, as

$$\begin{aligned} \dot{\hat{\theta}} &= \sum_{i=1}^{3} 0.01 z_i^6 S_i^T S_i - 0.5\hat{\theta}, \\ \dot{\hat{\vartheta}} &= 0.036 z_3^6 S_\eta^T S_\eta - 0.1\hat{\vartheta}, \\ u_{c_k} &= \frac{1}{g_{3,k}}[-5.75z_3 - 0.005z_3^3\hat{\theta}S_3^T S_3], \\ u_{\phi_k} &= 3.75z_3 + \frac{0.00057}{g_{3,k}}z_3^3\hat{\vartheta}S_\eta^T S_\eta, \end{aligned}$$

where $u_k = u_{c_k} - u_{\phi_k}$, $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$, $z_3 = x_3 - \alpha_2$ and α_1, α_2 are given by

$$\alpha_1 = -5.75z_1 - 0.005z_1^3\hat{\theta}S_1^T S_1,$$
$$\alpha_2 = -92z_2 - 0.08z_2^3\hat{\theta}S_2^T S_2.$$

In order to give the simulation results, we assume that

$$v_k = D(u_k) = \begin{cases} 10(u_k - 50), & u_k \geq 50 \\ 0, & -60 < u_k < 50 \\ 20(u_k + 60), & u_k \leq -60 \end{cases}$$

and $\phi_1 = 0.5x_1 \sin x_2 x_3$, $\phi_2 = 0.25x_1^2 x_2 \cos x_2$, $\phi_3 = 0.1x_1x_3$. The simulation results are shown in Figs. 5.1–5.4. Figure 5.1 depicts the responses of system output ψ and target signal y_d. Figure 5.2 shows the trajectories of adaptive laws. Figure 5.3 demonstrates the responses of $D(u_{c_k})$ (without dead-zone compensation controller) and $D(u_{c_k} - u_{\phi_k})$ (with dead-zone compensation controller) and Fig. 5.4 illustrates the evolution of the switching signal. From Fig. 5.1, it can be seen that the output ψ can track the target signal y_d within a small bounded error. On the other hand, Fig. 5.3 proves that the dead-zone nonlinearity can be compensated by u_{ϕ_k}.

5.2.4 Conclusions

The tracking control problem for a class of stochastic switched nonlinear systems under arbitrary switchings has been investigated, where the unknown nonsymmetric actuator dead-zone is taken into account. A state feedback controller is designed for the systems under consideration. It is shown that the target signal can be almost

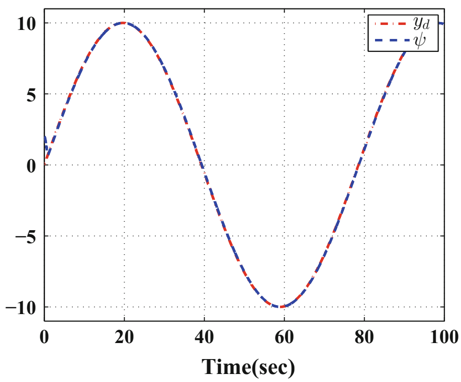


Fig. 5.1 Tracking performance

Fig. 5.2 The responses of adaptive laws

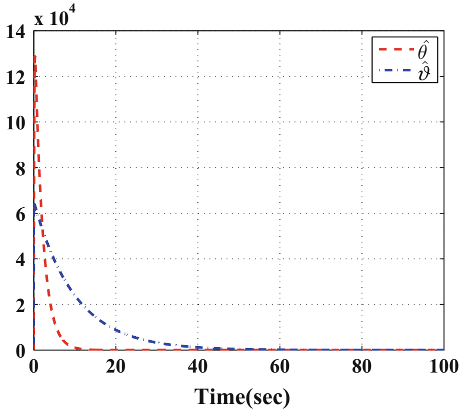


Fig. 5.3 The responses of $D(u_{c_k} - u_{\phi_k})$ and $D(u_{c_k})$

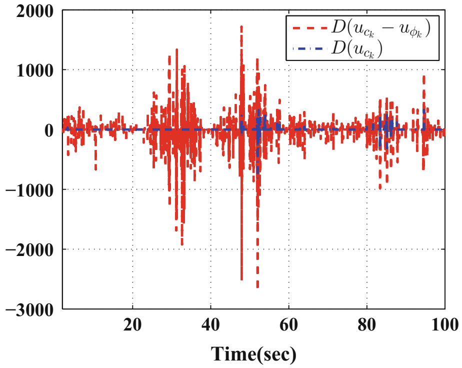
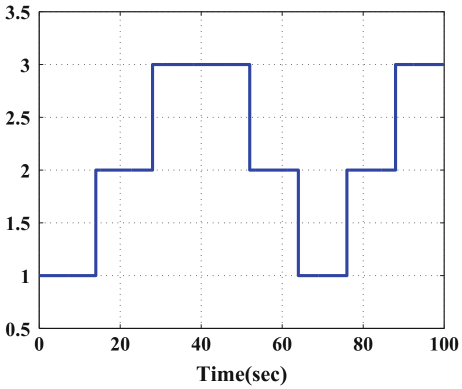


Fig. 5.4 The response of switching signal



surely tracked by the system output within a small bounded error, and the tracking error is SGUUB in 4^{th} moment.

5.3 Adaptive Neural Control for Switched Stochastic High-Order Uncertain Nonlinear Systems with SISS Inverse Dynamic

5.3.1 Problem Formulation and Preliminaries

Here, we consider the following stochastic switched high-order nonlinear systems with SISS inverse dynamic,

$$\begin{aligned} d\zeta &= f_{0,\sigma(t)}(\zeta, x_1)\,dt + \psi^T_{0,\sigma(t)}(\zeta, x_1)\,d\omega, \\ dx_i &= \left(g_{i,\sigma(t)}(\zeta, x)x_{i+1}^{p_i} + f_{i,\sigma(t)}(\zeta, x)\right)dt + \psi^T_{i,\sigma(t)}(\zeta, x)\,d\omega,\ i = 1, 2, \ldots, n-1, \\ dx_n &= \left(g_{n,\sigma(t)}(\zeta, x)u_{\sigma(t)}^{p_n} + f_{n,\sigma(t)}(\zeta, x)\right)dt + \psi^T_{n,\sigma(t)}(\zeta, x)\,d\omega, \\ y &= x_1, \end{aligned} \tag{5.46}$$

where $\zeta \in \mathbb{R}^r$ are immeasurable stochastic inverse dynamics; $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and $y \in R$ are the system state and output, respectively; p_i is a positive odd integer and ω is an m-dimensional standard Wiener process defined on the complete probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P\right)$ with Ω being a sample space, $\mathscr{F}$ being a σ-field, $\{\mathscr{F}_t\}_{t\geq 0}$ being a filtration, and P being a probability measure; $\sigma(t)$: $[0, +\infty) \to M = \{1, 2, \ldots, m\}$ is the switching signal; $u_k \in R$ is the control input of the k-th subsystem; $f_{0,k} : \mathbb{R}^r \times R \to \mathbb{R}^r$, $\psi_{0,k} : \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^{m\times r}$; For any $i = 1, 2 \ldots, n$ and $k = 1, 2, \ldots, m$, $f_{i,k} : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$, $\psi_{i,k} : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^m$ are unknown nonlinear functions assumed to be locally Lipschitz with $f_{i,k}(0) = 0$, $\psi_{i,k}(0) = 0$, and $g_{i,k} : R^r \times \mathbb{R}^n \to R$ is a strictly either positive or negative known function.

Remark 5.2 System (5.46) reduces to the well-known normal form when $p_i = 1$, $\zeta = 0$ and $m = 1$. In the case that $p_i > 1$, $\zeta = 0$ and $m = 1$, the Jacobian linearization of the system is neither controllable nor feedback linearizable. This makes the control design very challenging. To solve this problem, Lin and Qian [16] proposed a fruitful deterministic technique: adding a power integrator. Subsequently, many excellent results are proposed based on the adding a power integrator technique, see, e.g., [17–19] and the references therein.

Definition 5.3 For any given $V(x_i, t) \in \mathscr{C}^{2,1}$ associated with system (5.46), define the differential operator $\mathscr{L}$ as follows,

$$\mathscr{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i} F_{i,k} + \frac{1}{2} Tr \left\{ \psi_{i,k}^T \frac{\partial^2 V}{\partial x_i^2} \psi_{i,k} \right\}, \tag{5.47}$$

where $F_{i,k} = g_{i,\sigma(t)}(\zeta, x) x_{i+1}^{p_i} + f_{i,\sigma(t)}(\zeta, x)$.

Assumption 5.3 The sign and the upper bound of function $g_{i,k}$ for $1 \le i \le n$ and $k \in M$, are known, and without loss of generality, it is assumed that

$$0 < \underline{d}_i \le g_{i,k}(\zeta, x) \le \bar{d}_i,$$

where $\underline{d}_i$ and $\bar{d}_i$ stand for the lower and upper bound values of $g_{i,k}(\zeta, x)$, respectively.

Assumption 5.4 For $1 \le i \le n$ and $k \in M$, there exists a $\mathscr{C}^2$ function $V_0(\zeta)$, which is positive definite and proper, such that $\mathscr{L}V_0 \le -\lambda_0 \zeta^4 + \bar{\lambda}_0 x_1^{p+3}$, where λ_0 and $\bar{\lambda}_0$ are positive constants.

Lemma 5.4 *Let $p \in \mathbb{R}^*$ and x, y be real-valued functions. There exists a constant $c > 0$ such that*

$$\left| x^p - y^p \right| \le c \left| x - y \right| \left| (x-y)^{p-1} + y^{p-1} \right|.$$

Lemma 5.5 *Suppose that there exists a $\mathscr{C}^{2,1}$ function $V(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$, two constants $c_1 > 0$, $c_2 > 0$, and $\mathscr{K}_\infty$ functions $\bar{c}_1$, $\bar{c}_2$ such that*

$$\begin{cases} \bar{c}_1(|x|) \le V(x,t) \le \bar{c}_2(|x|) \\ \mathscr{L}V(x,t) \le -c_1 V(x,t) + c_2 \end{cases}$$

for all $x \in R^n$ and $t > t_0$. Then, there is an unique strong solution for each $x_0 \in \mathbb{R}^n$ and it satisfies:

$$E[V(x,t)] \le V(x_0) e^{-c_1 t} + \frac{c_2}{c_1}, \ \forall t > t_0.$$

In the following control design procedure, radial basis function (RBF) neural networks are used to approximate a continuous real function $f(X)$. For arbitrary $\varepsilon > 0$, there exists a neural network $W^T S(X)$ such that

$$f(X) = W^T S(X) + \delta(X), \ \delta(X) \le \varepsilon,$$

where $X \in \Omega_X \subset \mathbb{R}^q$ is the input vector with q dimension, $S(X) = [s_1(X), s_2(X), \ldots, s_l(X)]^T$ is the basis function vector, and $W = [w_1, w_2, \ldots, w_l]^T$ is the ideal constant weight vector with $l > 1$ being the number of the neural network nodes, and $s_i(X)$ are chosen as Gaussian functions; i.e., for $i = 1, 2, \ldots, l$,

$$s_i(X) = \exp\left(-\frac{(X - \mu_i)^T (X - \mu_i)}{\zeta_i^2} \right),$$

where ζ_i is the width of the Gaussian function, and $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^T$ is the center vector.

Lemma 5.6 *Consider the Gaussian RBF networks. Let* $\rho := \frac{1}{2}\min_{i\neq j} ||\mu_i - \mu_j||$*; then an upper bound of* $||S(X)||$ *is taken as*

$$||S(X)|| \leq \sum_{k=0}^{\infty} 3q(k+2)^{q-1} e^{-2\rho^2 k^2/\zeta^2} := D.$$

It has been proven in [20] that the constant D in Lemma 5.6 is a limited value and is independent of the variable X.

5.3.2 Main Results

In the following, the adaptive tracking control design is carried out by using a standard backstepping procedure. Firstly, define $p = \max_{i=1,\ldots,n}\{p_i\}$. The following lemma is also given.

Lemma 5.7 *Suppose that the Lyapunov function*

$$V(\xi_1,\ldots,\xi_n) = \sum_{i=1}^{n} \frac{\xi_i^{p-p_i+4}}{p-p_i+4}$$

is positive-definite and proper, satisfying

$$\mathscr{L}V \leq -\sum_{i=1}^{n} \xi_i^{p+3} + \phi. \tag{5.48}$$

Then, the following inequality holds

$$\mathscr{L}V \leq -a_0 V + b_0,$$

where

$$a_0 = \min\left(\phi^{(p_i-1)/(p+3)}\right),\ b_0 = (n+1)\phi.$$

Proof Let $a = \phi^{1/(p+3)}$ and $b = \xi_i$. Then, by using Young's inequality

$$\begin{aligned} a^{p_i-1} b^{p-p_i+4} &\leq \frac{p_i-1}{p+3} a^{p+3} + \frac{p-p_i+4}{p+3} b^{p+3} \\ &\leq a^{p+3} + b^{p+3}, \end{aligned}$$

which implies that

$$-\xi_i^{p+3} \leq -\phi^{(p_i-1)/(p+3)} \xi_i^{p-p_i+4} + \phi. \tag{5.49}$$

Substituting (5.49) into (5.48) yields that

$$\mathscr{L}V \leq -\sum_{i=1}^{n} \phi^{(p_i-1)/(p+1)} \xi_i^{p-p_i+4} + (n+1)\phi.$$

The proof of Lemma 5.7 is completed here. □

Step 1: Define the variable $z_1 = x_1$. Then, consider the following Lyapunov function candidate

$$V_1 = \frac{\zeta^4}{4} + \frac{z_1^{p-p_1+4}}{p-p_1+4}.$$

It follows from (5.47) and Assumption 5.4 that

$$\mathscr{L}V_1 = -\lambda_0\zeta^4 + \bar{\lambda}_0 z_1^{p+3} + z_1^{p-p_1+3}\left(g_{1,k}x_2^{p_1} + f_{1,k}\right) + \frac{p-p_1+3}{2}\left\|\psi_{1,k}\right\|^2 z_1^{p-p_1+2}, \tag{5.50}$$

where $f_{1,k}$ and $\left\|\psi_{1,k}\right\|^2$ are unknown. Then, two neural networks $W_{1,k}S_1$ and $\Phi_{1,k}P_{1,k}$ are used to approximate the unknown function $f_{1,k}$ and the norm $\left\|\psi_{1,k}\right\|$ such that for any given $\varepsilon_{1,k} > 0$ and $\tau_{1,k} > 0$,

$$\begin{aligned} f_{1,k} &= W_{1,k}^T S_{1,k}(X_1) + \delta_{1,k}(X_1), \\ \left\|\psi_{1,k}\right\|^2 &= \Phi_{1,k}^T P_{1,k}(X_1) + \bar{\delta}_{1,k}(X_1), \end{aligned}$$

where $X_1 := [\zeta^T, x^T]^T \in R^{r+n}$, $\left|\delta_{1,k}(X)\right| \leq \varepsilon_{1,k}$, $\bar{\delta}_{1,k}(X_1) \leq \tau_{1,k}$.

One can get from the Young's inequality and Lemma 5.6 that

$$\begin{aligned} & z_1^{p-p_1+3} f_{1,k} \\ &= z_1^{p-p_1+3}(W_{1,k}^T S_{1,k}(X_1) + \delta_{1,k}(X_1)) \\ &\leq \frac{p-p_1+3}{p+3} l_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} \left\|W_{1,k}\right\|^{\frac{p+3}{p-p_1+3}} \left\|S_{1,k}\right\|^{\frac{p+3}{p-p_1+3}} + \frac{p_1}{p+3} l_1^{-\frac{p+3}{p_1}} \\ &+ \frac{p-p_1+3}{p+3} \eta_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} + \frac{p_1}{p+3} \eta_1^{-\frac{p+3}{p_1}} \varepsilon_{1,k}^{\frac{p+3}{p_1}} \\ &\leq l_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} \left\|W_{1,k}\right\|^{\frac{p+3}{p-p_1+3}} \left\|S_{1,k}\right\|^{\frac{p+3}{p-p_1+3}} + \eta_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} + l_1^{-\frac{p+3}{p_1}} + \eta_1^{-\frac{p+3}{p_1}} \varepsilon_{1,k}^{\frac{p+3}{p_1}} \\ &\leq z_1^{p+3}\left(l_1^{\frac{p+3}{p-p_1+3}} \theta_1 D_1^{\frac{p+3}{p-p_1+3}} + \eta_1^{\frac{p+3}{p-p_1+3}}\right) + b_1, \end{aligned} \tag{5.51}$$

where $l_1, \eta_1 > 0$ are design parameters; $\|S_1\| \leq D_1$; $\theta_1 := \max\{\left\|W_{1,k}\right\|^{(p+3)/(p-p_1+3)} : k \in M\}$; $b_1 = l_1^{-(p+3)/p_1} + \eta_1^{-(p+3)/p_1}\varepsilon_1^{(p+3)/p_1}$.

Moreover, one has that

$$
\begin{aligned}
&\left\|\psi_{1,k}\right\|^2 z_1^{p-p_1+2} \\
&= z_1^{p-p_1+2}\left(\Phi_{1,k}^T P_{1,k}(X_1)+\bar{\delta}_{1,k}(X_1)\right) \\
&\le z_1^{p-p_1+2}\Phi_{1,k}^T P_{1,k} + z_1^{p-p_1+2}\bar{\delta}_{1,k} \\
&\le \frac{p-p_1+2}{p+3}\xi_1^{\frac{p+3}{p-p_1+2}} z_1^{p+3}\left\|\Phi_{1,k}\right\|^{\frac{p+3}{p-p_1+2}}\left\|P_{1,k}\right\|^{\frac{p+3}{p-p_1+2}} + \frac{p_1+1}{p+3}\xi_1^{-\frac{p+3}{p_1+1}} \\
&+ \frac{p-p_1+2}{p+3} m_1^{\frac{p+3}{p-p_1+2}} z_1^{p+3} + \frac{p_1+1}{p+3} m_1^{-\frac{p+3}{p_1+1}} \tau_{1,k}^{\frac{p+3}{p_1+1}} \\
&\le z_1^{p+3}\left(\xi_1^{\frac{p+3}{p-p_1+2}}\varphi_1 Q_1^{\frac{p+3}{p-p_1+2}} + m_1^{\frac{p+3}{p-p_1+2}}\right) + \bar{b}_1,
\end{aligned}
\tag{5.52}
$$

where $\xi_1,\ m_1 > 0$ are design parameters; $\|P_1\| \le Q_1$; $\bar{b}_1 = \xi_1^{-(p+3)/(p_1+1)} + m_1^{-(p+3)/(p_1+1)}\tau_1^{(p+3)/(p_1+1)}$; $\varphi_1 = \max\{\left\|\Phi_{1,k}\right\|^{(p+3)/(p-p_1+2)} : k \in M\}$.

Substituting (5.51) and (5.52) into (5.50), yields that

$$
\begin{aligned}
\mathscr{L}V_1 \le &-\lambda_0\zeta^4 + \bar{\lambda}_0 z_1^{p+3} + g_{1,k} z_1^{p-p_1+3} x_2^{p_1} + z_1^{p+3}\left(l_1^{\frac{p+3}{p-p_1+3}}\theta_1 D_1^{\frac{p+3}{p-p_1+3}} + \eta_1^{\frac{p+3}{p-p_1+3}}\right) \\
&+ z_1^{p+3}\left(\frac{1}{2}(p-p_1+3)\xi_1^{\frac{p+3}{p-p_1+2}}\varphi_1 Q_1^{\frac{p+3}{p-p_1+2}} + \frac{1}{2}(p-p_1+3) m_1^{\frac{p+3}{p-p_1+2}}\right) + \tilde{b}_1,
\end{aligned}
\tag{5.53}
$$

where $\tilde{b}_1 := b_1 + 0.5(p-p_1+3)\bar{b}_1$.

Then, the common virtual control function can be designed as

$$
\begin{aligned}
\alpha_1 = &-z_1\left\{\frac{1}{\underline{d}_1}\left(\bar{\lambda}_1 + l_1^{\frac{p+3}{p-p_1+3}}\hat{\theta}_1 D_1^{\frac{p+3}{p-p_1+3}} + \frac{1}{2}(p-p_1+3)\xi_1^{\frac{p+3}{p-p_1+2}}\hat{\varphi}_1 Q_1^{\frac{p+3}{p-p_1+2}}\right.\right. \\
&\left.\left.+\ \eta_1^{\frac{p+3}{p-p_1+3}} + \frac{1}{2}(p-p_1+3) m_1^{\frac{p+3}{p-p_1+2}}\right)\right\}^{\frac{1}{p_1}} \\
= &-z_1\beta_1,
\end{aligned}
\tag{5.54}
$$

where $\hat{\theta}_1$, $\hat{\varphi}_1$ are the estimations of θ_1, φ_1 respectively; $\bar{\lambda}_1 > 1 + \bar{\lambda}_0$ is a positive design parameter; $\underline{d}_1$ is defined in Assumption 5.3.

It follows from (5.53) and (5.54) that

$$
\begin{aligned}
\mathscr{L}V_1 \le &-\lambda_0\zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0) z_1^{p+3} + g_{1,k} z_1^{p-p_1+3}\left(x_2^{p_1} - \alpha_1^{p_1}\right) \\
&+ l_1^{\frac{p+3}{p-p_1+3}}\tilde{\theta}_1 D_1^{\frac{p+3}{p-p_1+3}} z_1^{p+3} + 0.5(p-p_1+3)\xi_1^{\frac{p+3}{p-p_1+2}}\tilde{\varphi}_1 Q_1^{\frac{p+3}{p-p_1+2}} z_1^{p+3} + \tilde{b}_1,
\end{aligned}
\tag{5.55}
$$

where $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$, $\tilde{\varphi}_1 = \varphi_1 - \hat{\varphi}_1$.

Step 2: Denote $z_2 = x_2 - \alpha_1$, and define $d\alpha_1$ as

$$
\begin{aligned}
d\alpha_1 &= \left(\sum_{j=1}^{n}\frac{\partial \alpha_1}{\partial x_j}\left(g_{j,k}x_{j+1}^{p_2}+f_{1,k}\right)+\frac{\partial \alpha_1}{\partial \hat{\theta}_1}\dot{\hat{\theta}}_1+\frac{\partial \alpha_1}{\partial \hat{\varphi}_1}\dot{\hat{\varphi}}_1\right)dt+\frac{\partial \alpha_1}{\partial x_1}\psi_{1,k}^T d\omega \\
&= \bar{a}_1 dt+\frac{\partial \alpha_1}{\partial x_1}\psi_1^T d\omega, \qquad (5.56)
\end{aligned}
$$

where $\dot{\hat{\theta}}_1$ and $\dot{\hat{\varphi}}_1$ will be specified later, $x_{n+1} := u$ will given at final step.
Choose the Lyapunov function as

$$
V_2 = V_1 + \frac{z_2^{p-p_2+4}}{p-p_2+4}.
$$

Then, $\mathscr{L}V_2$ is given by

$$
\begin{aligned}
\mathscr{L}V_2 \le &-\lambda_0\zeta^4-(\bar{\lambda}_1-\bar{\lambda}_0)z_1^{p+3}+g_{1,k}z_1^{p-p_1+3}\left(x_2^{p_1}-\alpha_1^{p_1}\right)\\
&+l_1^{\frac{p+3}{p-p_1+3}}\tilde{\theta}_1 D_1^{\frac{p+3}{p-p_1+3}}z_1^{p+3}+0.5(p-p_1+3)\xi_1^{\frac{p+3}{p-p_1+2}}\tilde{\varphi}_1 Q_1^{\frac{p+3}{p-p_1+2}}z_1^{p+3}+\tilde{b}_1\\
&+z_2^{p-p_2+3}\left(g_{2,k}x_3^{p_2}+f_{2,k}-\bar{a}_1\right)+\frac{p-p_2+3}{2}\left\|\psi_{2,k}-\frac{\partial \alpha_1}{\partial x_1}\psi_{1,k}\right\|^2 z_2^{p-p_2+2}. \qquad (5.57)
\end{aligned}
$$

By using Lemma 5.4 and Young's inequality, one can obtain that

$$
\begin{aligned}
&\left|g_{1,k}z_1^{p-p_1+3}(x_2^{p_1}-\alpha_1^{p_1})\right|\\
&\le c_1\bar{d}_1\left|z_1^{p-p_1+3}\right||z_2|\left|z_2^{p_1-1}+(z_1\beta_1)^{p_1-1}\right|\\
&\le c_1\bar{d}_1\frac{p-p_1+3}{p+3}z_1^{p+3}+c_1\bar{d}_1\frac{p_1}{p+3}z_2^{p+3}+c_1\bar{d}_1\frac{p+2}{p+3}z_1^{p+3}\\
&\quad +c_1\bar{d}_1\frac{1}{p+3}z_2^{p+3}\beta_1^{(p_1-1)(p+3)}\\
&\le z_1^{p+3}+z_2^{p+3}\left(1+\beta_1^{(p_1-1)(p+3)}\right)\\
&= z_1^{p+3}+z_2^{p+3}\bar{\beta}_1, \qquad (5.58)
\end{aligned}
$$

where $\bar{\beta}_1 = 1+\beta_1^{(p_1-1)(p+1)}$; c_1 is chosen as $1/2\bar{d}_1$; $\bar{d}_1$ is defined in Assumption 5.3.
Substituting (5.58) into (5.57), gives that

$$
\begin{aligned}
\mathscr{L}V_2 \le &-\lambda_0\zeta^4-(\bar{\lambda}_1-\bar{\lambda}_0-1)z_1^{p+3}+z_2^{p+3}\bar{\beta}_1+l_1^{\frac{p+3}{p-p_1+3}}\tilde{\theta}_1 D_1^{\frac{p+3}{p-p_1+3}}z_1^{p+3}\\
&+\frac{1}{2}(p-p_1+3)\xi_1^{\frac{p+3}{p-p_1+2}}\tilde{\varphi}_1 Q_1^{\frac{2p+6}{p-p_1+2}}z_1^{p+3}+\tilde{b}_1+z_2^{p-p_2+3}\left(g_{2,k}x_3^{p_2}+\bar{f}_{2,k}\right)\\
&+\frac{1}{2}(p-p_2+3)\,\bar{\psi}_{2,k}z_2^{p-p_2+2}, \qquad (5.59)
\end{aligned}
$$

where $\bar{f}_{2,k} = f_{2,k} - \bar{\alpha}_1$, $\bar{\psi}_{2,k} := \left\| \psi_{2,k} - \frac{\partial \alpha_1}{\partial x_1}\psi_{1,k} \right\|^2$. Then, neural networks $W_{2,k}^T S_{2,k}(X_2)$ and $\Phi_{2,k}^T P_{2,k}(X_2)$ are used to approximate the unknown functions $\bar{f}_{2,k}$ and $\bar{\psi}_{2,k}$ such that for any given $\varepsilon_{2,k} > 0$ and $\tau_{2,k} > 0$,

$$\bar{f}_{2,k} = W_{2,k}^T S_{2,k}(X_2) + \delta_{2,k}(X_1),$$
$$\bar{\psi}_{2,k} = \Phi_{2,k}^T P_{2,k}(X_2) + \bar{\delta}_{2,k}(X_1),$$

where $X_2 := [\zeta^T, x^T, \hat{\theta}_1, \hat{\varphi}_1]^T \in R^{r+n+2}$, $\left|\delta_{2,k}(X_2)\right| \le \varepsilon_{2,k}, \bar{\delta}_{2,k}(X_2) \le \tau_{2,k}$.

Similar to the procedure in (5.51), one can obtain that

$$\begin{aligned}
&z_2^{p-p_2+3} f_{2,k} \\
&= z_2^{p-p_2+3}(W_{2,k}^T S_{2,k}(X_2) + \delta_{2,k}(X_2)) \\
&\le \frac{p-p_2+3}{p+3} l_2^{\frac{p+3}{p-p_2+3}} z_2^{p+3} \left\| W_{2,k} \right\|^{\frac{p+3}{p-p_2+3}} \left\| S_{2,k} \right\|^{\frac{p+3}{p-p_2+3}} + \frac{p_2}{p+3} l_2^{-\frac{p+3}{p_2}} \\
&\quad + \frac{p-p_2+3}{p+3} \eta_2^{\frac{p+3}{p-p_2+3}} z_2^{p+3} + \frac{p_2}{p+3} \eta_2^{-\frac{p+3}{p_2}} \varepsilon_{2,k}^{\frac{p+3}{p_2}} \\
&\le l_2^{\frac{p+3}{p-p_2+3}} z_2^{p+3} \left\| W_{2,k} \right\|^{\frac{p+3}{p-p_2+3}} \left\| S_{2,k} \right\|^{\frac{p+3}{p-p_2+3}} + \eta_2^{\frac{p+3}{p-p_2+3}} z_2^{p+3} + l_2^{-\frac{p+3}{p_2}} + \eta_2^{-\frac{p+3}{p_2}} \varepsilon_{2,k}^{\frac{p+3}{p_2}} \\
&\le z_2^{p+3} \left(l_2^{\frac{p+3}{p-p_2+3}} \theta_2 D_2^{\frac{p+3}{p-p_2+3}} + \eta_2^{\frac{p+3}{p-p_2+3}} \right) + b_2,
\end{aligned} \tag{5.60}$$

where l_2, $\eta_2 > 0$ are design parameters, $\theta_2 := \max\{\left\| W_{2,k} \right\|^{(p+3)/(p-p_2+3)} : k \in M\}$, $b_2 = l_2^{-(p+3)/p_2} + \eta_2^{-(p+3)/p_2} \varepsilon_2^{(p+3)/p_2}$.

Using a similar way to (5.52), one gets that

$$\begin{aligned}
&\bar{\psi}_{2,k} z_2^{p-p_2+2} \\
&= z_2^{p-p_2+2} \left(\Phi_{2,k}^T P_{2,k}(X_2) + \bar{\delta}_{2,k}(X_2) \right) \\
&\le z_2^{p-p_2+2} \Phi_{2,k}^T P_{2,k} + z_2^{p-p_2+2} \bar{\delta}_{2,k} \\
&\le \frac{p-p_2+2}{p+3} \xi_2^{\frac{p+3}{p-p_2+2}} z_2^{p+3} \left\| \Phi_{2,k} \right\|^{\frac{p+3}{p-p_2+2}} \left\| P_{2,k} \right\|^{\frac{p+3}{p-p_2+2}} + \frac{p_2+1}{p+3} \xi_2^{-\frac{p+3}{p_2+1}} \\
&\quad + \frac{p-p_2+2}{p+3} m_2^{\frac{p+3}{p-p_2+2}} z_2^{p+3} + \frac{p_2+1}{p+3} m_2^{-\frac{p+3}{p_2+1}} \tau_{2,k}^{\frac{p+3}{p_2+1}} \\
&\le z_2^{p+3} \left(\xi_2^{\frac{p+3}{p-p_2+2}} \varphi_2 Q_2^{\frac{p+3}{p-p_2+2}} + m_2^{\frac{p+3}{p-p_2+2}} \right) + \bar{b}_2,
\end{aligned} \tag{5.61}$$

where ξ_2, $m_2 > 0$ are design parameters; $\bar{b}_2 = \xi_2^{-\frac{p+3}{p_2+1}} + m_2^{-\frac{p+3}{p_2+1}} \tau_2^{\frac{p+3}{p_2+1}}$; $\varphi_2 = \max\{\left\| \Phi_{2,k} \right\|^{\frac{p+3}{p-p_2+2}} : k \in M\}$; $P_2(X_2)$ and $\tau_2(X_2)$ represent the basis function vector and the estimation error of φ_2.

Design the common virtual control function as

$$\alpha_2 = -z_2\left\{\frac{1}{\underline{d}_2}\left(\bar{\beta}_1+\lambda_2+l_2^{\frac{p+3}{p-p_2+3}}\hat{\theta}_2 D_2^{\frac{p+3}{p-p_2+3}}+\frac{1}{2}(p-p_2+3)\xi_2^{\frac{p+3}{p-p_2+2}}\hat{\varphi}_2 Q_2^{\frac{p+3}{p-p_2+2}}\right.\right.$$
$$\left.\left.+\eta_2^{\frac{p+3}{p-p_2+3}}+\frac{1}{2}(p-p_2+3)m_2^{\frac{p+3}{p-p_2+2}}\right)\right\}^{\frac{1}{p_2}}$$
$$= -z_2\beta_2, \tag{5.62}$$

where $\hat{\theta}_2, \hat{\varphi}_2$ are the estimation of θ_2 and φ_2 respectively; $\lambda_2 > 1$ is a design parameter; $\underline{d}_2$ is defined in Assumption 5.3.

By substituting (5.60)–(5.62) into (5.59), one has

$$\begin{aligned}
&\mathscr{L}V_2\\
&\le -\lambda_0\zeta^4-(\bar{\lambda}_1-\bar{\lambda}_0-1)z_1^{p+3}-\lambda_2 z_2^{p+3}+g_{2,k}z_2^{p-p_2+3}(x_3^{p_2}-\alpha_2^{p_2})\\
&\quad+\sum_{j=1}^{2}\left(l_j^{\frac{p+3}{p-p_j+3}}\tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}}z_j^{p+3}+0.5(p-p_j+3)\xi_j^{\frac{p+3}{p-p_j+2}}\tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}}z_j^{p+3}+\tilde{b}_j\right),
\end{aligned}$$

where $\tilde{b}_j := b_j + 0.5(p-p_j+3)\bar{b}_j, \tilde{\theta}_j = \theta_j - \hat{\theta}_j, \tilde{\varphi}_j = \varphi_j - \hat{\varphi}_j, j = 1, 2.$

Step i: Suppose at step i $(3 \le i \le n-1)$ that, there is a set of virtual control functions $\alpha_3, \ldots, \alpha_{n-1}$, defined by

$$\alpha_i = z_i\beta_i,\ z_i = x_{i+1}-\alpha_i \tag{5.63}$$

and assume that a set of unknown nonlinear functions $\bar{f}_{i,k}$ and $\bar{\psi}_{i,k}$ can be approximated by neural networks $W_{i,k}^T S_{i,k}(X_i)$ and $\Phi_{i,k}^T P_{i,k}(X_i)$ for any given $\varepsilon_{i,k} > 0$, $\tau_{i,k} > 0$.

$$\begin{aligned}
\bar{f}_{i,k} &= W_{i,k}^T S_{i,k}(X_i)+\delta_{i,k},\ \left|\delta_{i,k}(X_i)\right| \le \varepsilon_{i,k},\\
\bar{\psi}_{i,k} &= \Phi_{i,k}^T P_{i,k}(X_i)+\bar{\delta}_{i,k},\ \left|\bar{\delta}_{i,k}(X_i)\right| \le \tau_{i,k},
\end{aligned}$$

where $X_i := [\zeta^T, x^T, \hat{\theta}_1, \ldots, \hat{\theta}_i, \hat{\varphi}_1, \ldots, \hat{\varphi}_i]^T \in R^{r+n+2i}$.

A straightforward calculation gives that

$$\begin{aligned}
&\mathscr{L}V_i\\
&\le -\lambda_0\zeta^4-(\bar{\lambda}_1-\bar{\lambda}_0-1)z_1^{p+3}-\sum_{j=2}^{i-1}(\lambda_j-1)z_j^{p+3}\\
&-\lambda_i z_i^{p+3}+g_{i,k}z_i^{p-p_i+3}(x_{i+1}^{p_i}-\alpha_i^{p_i})\\
&+\sum_{j=1}^{i}\left(l_j^{\frac{p+3}{p-p_j+3}}\tilde{\theta}_j\left\|S_j\right\|^{\frac{p+3}{p-p_j+3}}z_j^{p+3}+0.5(p-p_j+3)\xi_j^{\frac{p+3}{p-p_j+2}}\tilde{\varphi}_j\left\|P_j\right\|^{\frac{p+3}{p-p_j+2}}z_j^{p+3}+\tilde{b}_j\right),
\end{aligned} \tag{5.64}$$

where $\tilde{\theta}_j = \theta_j - \hat{\theta}_j$, $\tilde{\varphi}_j = \varphi_j - \hat{\varphi}_j$; $[\hat{\theta}_j, \hat{\varphi}_j]$ is the estimation of

$$[\theta_j,\ \varphi_j] := \max\{\left\| W_{j,k} \right\|^{\frac{p+3}{p-p_j+3}},\ \left\| \Phi_{j,k} \right\|^{\frac{p+3}{p-p_j+2}} : k \in M\};$$
$$\tilde{b}_j = b_j + 0.5(p - p_j + 3)\bar{b}_j,$$
$$b_j = l_j^{-(p+3)/p_j} + \eta_j^{-(p+3)/p_j} \varepsilon_j^{(p+3)/p_j}$$

and $\bar{b}_j = \xi_j^{-(p+3)/(p_j+1)} + m_j^{-(p+3)/(p_j+1)} \tau_j^{(p+3)/(p_j+1)}$.

Step n: Let $z_n = x_n - \alpha_{n-1}$, define $d\alpha_{n-1}$ as

$$\begin{aligned} d\alpha_{n-1} = & \left(\sum_{j=1}^{n} \frac{\partial \alpha_{n-1}}{\partial x_j} \left(g_{j,k} x_{j+1}^{p_2} + f_{j,k} \right) + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_j} \dot{\hat{\theta}}_j + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\varphi}_j} \dot{\hat{\varphi}}_j \right) dt \\ & + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_{j,k}^T d\omega \\ & = \bar{a}_{n-1} dt + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_{j,k}^T d\omega. \end{aligned}$$

where $x_{n+1} := u$ is provided later.

We construct the Lyapunov function as

$$V_n = V_{n-1} + \frac{z_n^{p-p_n+4}}{p - p_n + 4}.$$

By using (5.64), $\mathscr{L} V_n$ is given by

$$\mathscr{L} V_n \tag{5.65}$$
$$\leq -\lambda_0 \zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0 - 1) z_1^{p+3} - \sum_{j=2}^{n-2} (\lambda_j - 1) z_j^{p+3} \tag{5.66}$$
$$\begin{aligned} & - \lambda_{n-1} z_{n-1}^{p+3} + g_{n-1,k} z_{n-1}^{p-p_n+3} (x_n^{p_{n-1}} - \alpha_{n-1}^{p_{n-1}}) \\ & + \sum_{j=1}^{n-1} \left(l_j^{\frac{p+3}{p-p_j+3}} \tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p - p_j + 3) \xi_j^{\frac{p+3}{p-p_j+2}} \tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} + \tilde{b}_j \right) \\ & + z_n^{p-p_n+3} \left(g_{n,k} u^{p_n} + \bar{f}_{n,k} \right) + \frac{p - p_n + 3}{2} \bar{\psi}_{n,k} z_n^{p-p_n+2}, \end{aligned} \tag{5.67}$$

where $\bar{f}_{n,k} = f_{n,k} - \bar{\alpha}_{n-1}$, $\bar{\psi}_{n,k} = \left\| \psi_{n,k} - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} \psi_{i,k} \right\|^2$. Then, neural networks $W_{n,k}^T S_{n,k}(X_n)$ and $\Phi_{n,k}^T P_{n,k}(X_n)$ are used to approximate unknown functions $\bar{f}_{n,k}$ and $\bar{\psi}_{n,k}$ such that for any given $\varepsilon_{n,k} > 0$ and $\tau_{n,k} > 0$,

$$\bar{f}_{n,k} = W_{n,k}^T S_{n,k}(X_n) + \delta_{n,k}(X_n),$$
$$\bar{\psi}_{n,k} = \Phi_{n,k}^T P_{n,k}(X_n) + \bar{\delta}_{n,k}(X_n),$$

where $X_n := [\zeta^T, x^T, \hat{\theta}_1, \ldots, \hat{\theta}_n, \hat{\varphi}_1, \ldots, \hat{\varphi}_n]^T \in R^{r+3n}$, $\left|\delta_{n,k}(X_n)\right| \le \varepsilon_{n,k}$, $\bar{\delta}_{n,k}(X_n) \le \tau_{n,k}$.

Similar to (5.60) and (5.61), one has

$$z_n^{p-p_n+3}\bar{f}_{n,k} \le z_n^{p+3}\left(l_n^{\frac{p+3}{p-p_n+3}}\theta_n D_n^{\frac{p+3}{p-p_n+3}} + \eta_n^{\frac{p+3}{p-p_n+3}}\right) + b_n,$$
$$z_n^{p-p_n+3}\bar{\psi}_{n,k} \le z_n^{p+3}\left(\xi_n^{\frac{p+3}{p-p_n+2}}\varphi_n Q_n^{\frac{p+3}{p-p_n+2}} + m_n^{\frac{p+3}{p-p_n+2}}\right) + \bar{b}_n, \tag{5.68}$$

where $l_n,\ \eta_n,\ \xi_n,\ m_n > 0$ are design parameters; $\theta_n := \max\{\left\|W_{n,k}\right\|^{(p+3)/(p-p_n+3)} : k \in M\}$; $b_n = l_n^{-\frac{p+3}{p_n}} + \eta_n^{-\frac{p+3}{p_n}}\varepsilon_n^{\frac{p+3}{p_n}}$, $\varphi_n := \max\{\left\|\Phi_{n,k}\right\|^{(p+3)/(p-p_n+2)} : k \in M\}$; $\bar{b}_n = \xi_n^{-(p+3)/(p_n+1)} + m_n^{-(p+3)/(p_n+1)}\tau_n^{(p+3)/(p_n+1)}$.

Furthermore, it is not hard to get that

$$\begin{aligned}
&\left|g_{n-1,k}z_{n-1}^{p-p_n+3}(x_n^{p_{n-1}} - \alpha_{n-1}^{p_{n-1}})\right| \\
&\le c_{n-1}\bar{d}_{n-1}\left|z_{n-1}^{p-p_{n-1}+3}\right||z_n|\left|z_n^{p_{n-1}-1} + (z_{n-1}\beta_{n-1})^{p_{n-1}-1}\right| \\
&\le c_{n-1}\bar{d}_{n-1}\frac{p-p_{n-1}+3}{p+3}z_{n-1}^{p+3} + c_{n-1}\bar{d}_{n-1}\frac{p_{n-1}}{p+3}z_n^{p+3} + c_{n-1}\bar{d}_{n-1}\frac{p+2}{p+3}z_{n-1}^{p+3} \\
&\quad + c_{n-1}\bar{d}_{n-1}\frac{1}{p+3}z_n^{p+3}\beta_{n-1}^{(p_{n-1}-1)(p+3)} \\
&\le z_{n-1}^{p+3} + z_n^{p+3}\left(1 + \beta_{n-1}^{(p_{n-1}-1)(p+3)}\right) \\
&= z_{n-1}^{p+3} + z_n^{p+3}\bar{\beta}_{n-1},
\end{aligned} \tag{5.69}$$

where $\bar{\beta}_{n-1} = 1 + \beta_{n-1}^{(p_{n-1}-1)(p+1)}$, c_{n-1} is chosen as $1/2\bar{d}_{n-1}$, and $\bar{d}_{n-1}$ is defined in Assumption 5.3.

Substituting (5.68) and (5.69) into (5.67), the following inequality can be obtained.

$$\begin{aligned}
&\mathscr{L}V_n \\
&\le -\lambda_0\zeta^4 - (\bar{\lambda}_1 - \bar{\lambda}_0 - 1)z_1^{p+3} - \sum_{j=2}^{n-1}(\lambda_j - 1)z_j^{p+3} + z_n^{p+3}\bar{\beta}_{n-1} \\
&+ \sum_{j=1}^{n-1}\left(l_j^{\frac{p+3}{p-p_j+3}}\tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p-p_j+3)\xi_j^{\frac{p+3}{p-p_j+2}}\tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3}\right) + \sum_{j=1}^{n}\tilde{b}_j \\
&+ z_n^{p+3}\left(l_n^{\frac{p+3}{p-p_n+3}}\theta_n D_n^{\frac{p+3}{p-p_n+3}} + \eta_n^{\frac{p+3}{p-p_n+3}} + 0.5(p-p_n+3)\xi_n^{\frac{p+3}{p-p_n+2}}\varphi_n Q_n^{\frac{p+3}{p-p_n+2}}\right.
\end{aligned} \tag{5.70}$$

$$+ 0.5(p - p_n + 3)m_n^{\frac{p+3}{p-p_n+2}}\Bigg) + z_n^{p-p_n+3} g_{n,k} u^{p_n}. \quad (5.71)$$

Design the controller u as

$$u = - z_n \left\{ \frac{1}{\underline{d}_n} \left(\lambda_n + \bar{\beta}_{n-1} + l_n^{\frac{p+3}{p-p_n+3}} \hat{\theta}_n D_n^{\frac{p+3}{p-p_n+3}} + \eta_n^{\frac{p+3}{p-p_n+3}} \right.\right.$$
$$\left.\left. + \ 0.5(p - p_n + 3)\xi_n^{\frac{p+3}{p-p_n+2}} \hat{\varphi}_n Q_n^{\frac{p+3}{p-p_n+2}} + 0.5(p - p_n + 3)m_n^{\frac{p+3}{p-p_n+2}} \right) \right\}^{\frac{1}{p_n}}$$
$$= - z_n \beta_n, \quad (5.72)$$

where $\hat{\theta}_n$ is the estimation of θ_n; $\lambda_n > 1$ is a positive design parameter; d_n is defined in Assumption 5.3.

It follows from (5.71) and (5.72) that

$$\mathscr{L}V_n \leq \quad (5.73)$$
$$- \lambda_0 \zeta^4 - \sum_{j=1}^{n} (\lambda_j - 1) z_j^{p+3}$$
$$+ \sum_{j=1}^{n} \left(l_j^{\frac{p+3}{p-p_j+3}} \tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p - p_j + 3)\xi_j^{\frac{p+3}{p-p_j+2}} \tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} + \tilde{b}_j \right) \quad (5.74)$$

where $\lambda_1 := \bar{\lambda}_1 - \bar{\lambda}_0$.

Last Step: Choose the final Lyapunov function as

$$V = V_n + \sum_{j=1}^{n} \left(\frac{1}{2r_j} \tilde{\theta}_j^2 + \frac{1}{2\bar{r}_j} \tilde{\varphi}_j^2 \right) \quad (5.75)$$

where r_j is a positive design parameter.

$\mathscr{L}V$ is given by

$$\mathscr{L}V \leq \quad (5.76)$$
$$- \lambda_0 \zeta^4 - \sum_{j=1}^{n} (\lambda_j - 1) z_j^{p+3} - \sum_{j=1}^{n} \left(\frac{1}{r_j} \tilde{\theta}_j \dot{\hat{\theta}}_j + \frac{1}{\bar{r}_j} \tilde{\varphi}_j \dot{\hat{\varphi}}_j \right)$$
$$+ \sum_{j=1}^{n} \left(l_j^{\frac{p+3}{p-p_j+3}} \tilde{\theta}_j D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} + 0.5(p - p_j + 3)\xi_j^{\frac{p+3}{p-p_j+2}} \tilde{\varphi}_j Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} + \tilde{b}_j \right). \quad (5.77)$$

The adaptive laws are defined as the solutions to the following differential equations

$$\dot{\hat{\theta}}_j = r_j l_j^{\frac{p+3}{p-p_j+3}} D_j^{\frac{p+3}{p-p_j+3}} z_j^{p+3} - B_j\hat{\theta}_j,$$
$$\dot{\hat{\varphi}}_j = \frac{1}{2}(p-p_j+3)\bar{r}_j \xi_j^{\frac{p+3}{p-p_j+2}} Q_j^{\frac{p+3}{p-p_j+2}} z_j^{p+3} - \bar{B}_j\hat{\varphi}_j, \tag{5.78}$$

where $j = 1, 2, \ldots, n,\ B_j,\ \bar{B}_j > 0$ are design parameters.

This, together with (5.77), means that

$$\mathscr{L}V \le -\lambda_0\zeta^4 - \sum_{j=1}^{n}(\lambda_j - 1)z_j^{p+3} + \sum_{j=1}^{n}\frac{B_j\hat{\theta}_j\tilde{\theta}_j}{r_j} + \sum_{j=1}^{n}\frac{\bar{B}_j\hat{\varphi}_j\tilde{\varphi}_j}{\bar{r}_j} + \sum_{j=1}^{n}\tilde{b}_j. \tag{5.79}$$

Notice that

$$\tilde{\theta}_j\hat{\theta}_j = \tilde{\theta}_j(\theta_j - \tilde{\theta}_j) \le -\frac{1}{2}\tilde{\theta}_j^2 + \frac{1}{2}\theta_j^2,$$
$$\tilde{\varphi}_j\hat{\varphi}_j = \tilde{\varphi}_j(\varphi_j - \tilde{\varphi}_j) \le -\frac{1}{2}\tilde{\varphi}_j^2 + \frac{1}{2}\varphi_j^2, \tag{5.80}$$

By using (5.79), (5.80) and Lemma 5.7, one has

$$\begin{aligned}\mathscr{L}V \le& -\lambda_0\zeta^4 - \sum_{j=1}^{n}\left((\lambda_j - 1)z_j^{p+3} + \frac{B_j}{2r_j}\tilde{\theta}_j^2 + \frac{\bar{B}_j}{2\bar{r}_j}\tilde{\varphi}_j^2\right)\\ &+ \sum_{j=1}^{n}\left(\frac{B_j}{2r_j}\theta_j^2 + \frac{\bar{B}_j}{2\bar{r}_j}\varphi_j^2 + \tilde{b}_j\right)\\ \le& -q_0V + q_1,\end{aligned}$$

where $q_0 = \min\{(p - p_i + 4)(\lambda_j - 1)\phi^{(p_i-1)/(p+3)},\ B_j,\ \bar{B}_j,\ 2\lambda_0 : 1 \le j \le n\}$, $\phi = \sum_{j=1}^{n}(\frac{B_j}{2r_j}\theta_j^2 + \frac{\bar{B}_j}{2\bar{r}_j}\varphi_j^2 + \tilde{b}_j)$, $q_1 = (\lambda_j - 1)(n + 1)\phi$.

According to Lemma 5.5, we have that

$$E[V(x,t)] \le V(x_0)e^{-q_0t} + \frac{q_1}{q_0},\ \forall t \ge 0, \tag{5.81}$$

which indicates that all the signals in the closed-loop system are bounded. The design is completed here. Next, we address our main result.

Theorem 5.2 *For* $1 \le i \le n, k \in M$ *assume that all the unknown nonlinear functions* $\bar{f}_{i,k}$ *and* $\bar{\psi}_{i,k}$ *can be approximated by neural networks in the sense that the approximation errors are bounded, and all the initial values of* $\hat{\theta}_i$ *and* $\hat{\varphi}_i$ *satisfy* $\hat{\theta}_i(0) \ge 0$ *and* $\hat{\varphi}_i(0) \ge 0$*, respectively. Then, under the state feedback controller*

(5.72) and the adaptive laws (5.78), the equilibrium at the origin of the closed-loop system is boundedly stable in probability and

$$P\left\{\lim_{t\to\infty}\left(\frac{|\zeta|^4}{4}+\sum_{i=1}^{n}\frac{|z_i|^{p-p_i+4}}{p-p_i+4}\right)\le\frac{q_1}{q_0}\right\}=1.$$

Proof It is not difficult to complete the proof by the above discussions, and thus we omit the proof here.

In the following, a corollary is given by using only two adaptive laws.

Corollary 5.1 *For $1 \le i \le n, k \in M$ assume that all the unknown nonlinear functions $\bar{f}_{i,k}$ and $\bar{\psi}_{i,k}$ can be approximated by neural networks in the sense that the approximation errors are bounded, and all the initial values of $\hat{\theta}_i$ and $\hat{\varphi}_i$ satisfy $\hat{\theta}_i(0) \ge 0$ and $\hat{\varphi}_i(0) \ge 0$, respectively. Consider the following controller and adaptive laws:*

$$\begin{aligned}
u =& -z_n\Bigg\{\frac{1}{\underline{d}_n}\Bigg(\lambda_n+\bar{\beta}_{n-1}+l_n^{\frac{p+3}{p-p_n+3}}\hat{\theta}D_n^{\frac{p+3}{p-p_n+3}}+\eta_n^{\frac{p+3}{p-p_n+3}}\\
&+\frac{1}{2}(p-p_n+3)\xi_n^{\frac{p+3}{p-p_n+2}}\hat{\varphi}Q_n^{\frac{p+3}{p-p_n+2}}+0.5(p-p_n+3)m_n^{\frac{p+3}{p-p_n+2}}\Bigg)\Bigg\}^{\frac{1}{p_n}},\\
\dot{\hat{\theta}} =& \sum_{j=1}^{n} r l_j^{\frac{p+3}{p-p_j+3}}D_j^{\frac{p+3}{p-p_j+3}}z_j^{p+3}-B\hat{\theta},\\
\dot{\hat{\varphi}} =& \frac{1}{2}\sum_{j=1}^{n}(p-p_j+3)\bar{r}\xi_j^{\frac{p+3}{p-p_j+2}}Q_j^{\frac{p+3}{p-p_j+2}}z_j^{p+3}-\bar{B}\hat{\varphi},
\end{aligned}$$

where $\lambda_n > 1$, l_j, ξ_j, m_n, η_n, r, B, $\bar{r}$, $\bar{B} > 0$ are positive design parameters, $\hat{\theta}=\sum_{j=1}^{n}\hat{\theta}_j$, $\hat{\varphi}=\sum_{j=1}^{n}\hat{\varphi}_j$. Then, the equilibrium at the origin of the closed-loop system is boundedly stable in probability and

$$P\left\{\lim_{t\to\infty}\left(\frac{|\zeta|^4}{4}+\sum_{i=1}^{n}\frac{|z_i|^{p-p_i+4}}{p-p_i+4}\right)\le\frac{q_1}{q_0}\right\}=1.$$

Proof It should be pointed out that $\hat{\theta} \ge \hat{\theta}_j \ge 0$, $\hat{\varphi} \ge \hat{\varphi}_j \ge 0$, $j = 1, \ldots, n$. Therefore, we can use $\hat{\theta}$ and $\hat{\varphi}$ instead of $\hat{\theta}_j$ and $\hat{\varphi}_j$ in (5.54), (5.62), (5.63) and (5.72). In (5.75), the parameters $\tilde{\theta}_j$ and $\tilde{\varphi}_j$ in Lyapunov function V should be rewritten as $\tilde{\theta}$ and $\tilde{\varphi}$. The detailed proof is omitted here because it is similar to the one of Theorem 5.2. □

5.3.3 Simulation Results

An example with two controllers (multiple adaptive laws and two adaptive laws respectively) is presented in the following to demonstrate the effectiveness of our main results.

Consider the following switched stochastic high-order nonlinear systems with SISS inverse dynamics:

$$\sum_1 = \begin{cases} d\zeta = f_{0,1}(\zeta, x_1)\,dt + \psi_{0,1}^T(\zeta, x_1)\,d\omega, \\ dx_1 = \left[g_{1,1}(\zeta, x_1, x_2)x_2^{p_1} + f_{1,1}(\zeta, x_1, x_2)\right]dt + \psi_{1,1}^T(\zeta, x_1, x_2)\,d\omega, \\ dx_2 = \left[g_{2,1}(\zeta, x_1, x_2)u^{p_2} + f_{2,1}(\zeta, x_1, x_2)\right]dt + \psi_{2,1}^T(\zeta, x_1, x_2)\,d\omega, \end{cases}$$

$$\sum_2 = \begin{cases} d\zeta = f_{0,2}(\zeta, x_1)\,dt + \psi_{0,2}^T(\zeta, x_1)\,d\omega, \\ dx_1 = \left[g_{1,2}(\zeta, x_1, x_2)x_2^{p_1} + f_{1,2}(\zeta, x_1, x_2)\right]dt + \psi_{1,2}^T(\zeta, x_1, x_2)\,d\omega, \\ dx_2 = \left[g_{2,2}(\zeta, x_1, x_2)u^{p_2} + f_{2,2}(\zeta, x_1, x_2)\right]dt + \psi_{2,2}^T(\zeta, x_1, x_2)\,d\omega, \end{cases}$$

where $g_{1,1}$, $f_{1,1}$, $\psi_{1,1}$, $g_{2,1}$, $f_{2,1}$, $\psi_{2,1}$, $g_{1,2}$, $f_{1,2}$, $\psi_{1,2}$, $g_{2,2}$, $f_{2,2}$, and $\psi_{2,2}$ are all completely unknown functions; $p_1 = 3$, $p_2 = 5$. First, a controller under multiple adaptive laws is designed by Theorem 5.2. The initial conditions are $\zeta(0) = 1$ $x_1(0) = 0.5$, $x_2(0) = -0.5$ and $\hat{\theta}_1(0) = 2$, $\hat{\theta}_2(0) = 3.5$, $\hat{\varphi}_1(0) = 3$, $\hat{\varphi}_2(0) = 4$. The controller parameters are chosen as: $\lambda_1 = \lambda_2 = 5$, $l_2 = l_2 = \eta_1 = \eta_2 = \xi_1 = \xi_2 = m_1 = m_2 = 4$, $r_1 = r_2 = \bar{r}_1 = \bar{r}_2 = 1$, $B_1 = B_2 = \bar{B}_1 = \bar{B}_2 = 0.1$. We apply three nodes for each input dimension of $W_1^T S_1$, $W_2^T S_2$, $\Phi_1^T P_1$ and $\Phi_2^T P_2$. Therefore, each of them contains 81 nodes with centers spaced evenly in the interval $[-0.5, 0.5] \times [-0.5, 0.5] \times [-0.5, 0.5] \times [-0.5, 0.5]$, and the widths still being equal to 2.5. Second, a controller under two adaptive laws is designed by Corollary 5.1 with same conditions except $\hat{\theta}(0) = 3$, $\hat{\varphi}(0) = 4$, $r = 1$, $B = 0.1$.

In order to give the simulation results, it is assumed that $f_{0,1} = -15\zeta + 0.1x_1^2$, $\psi_{0,1} = \left(\zeta^2 + 0.3x_1^4\right)^{1/2}$, $g_{1,1} = \sin(x_1x_2+\zeta)+2$, $f_{1,1} = x_1x_2+\zeta$, $\psi_{1,1} = \sin(x_1x_2+\zeta)$, $g_{2,1} = \cos(x_1 + x_2^2 + \zeta) + 2$, $f_{2,1} = x_1x_2^2 + \zeta\sin\zeta$, $\psi_{2,1} = x_1\cos x_2 + \zeta^2$;

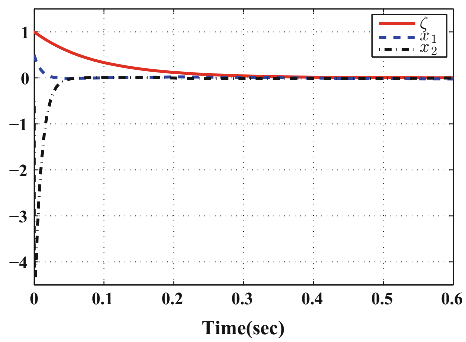


Fig. 5.5 Responses of system states by using multiple adaptive laws

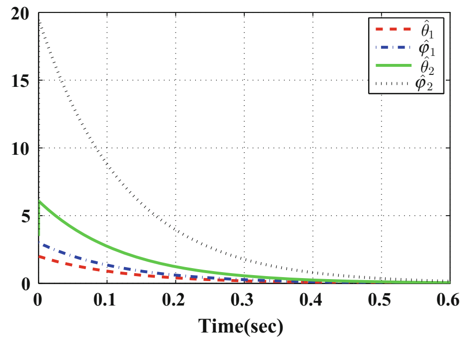


Fig. 5.6 Responses of the multiple adaptive laws

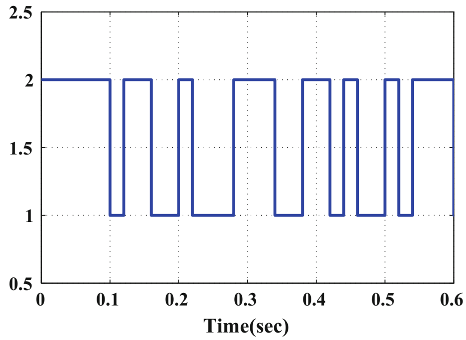


Fig. 5.7 Response of switching signal

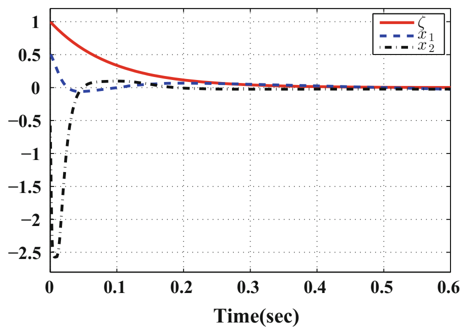


Fig. 5.8 Responses of system states by using two adaptive laws

$f_{0,2} = -13\zeta + 0.3x_1^2$, $\psi_{0,2} = \left(0.17\zeta^2 + 0.13x_1^4\right)^{1/2}$, $g_{1,2} = \sin(x_1^2 + x_2 + \zeta) + 2$, $f_{1,2} = x_1^2 x_2 + \zeta^2$, $\psi_{1,2} = \sin(x_1 + x_2) + \zeta^3$, $g_{2,2} = \cos(x_1 x_2^2 + \zeta) + 2$, $f_{2,2} = x_1 x_2 + \zeta \cos\zeta$, $\psi_{2,2} = x_1 \sin(x_1 x_2) + \zeta^2$.

The simulation results based on Theorem 5.2 are shown in Figs. 5.5, 5.6 and 5.7, respectively. Figure 5.5 depicts the responses of system states. The trajectories of adaptive laws are shown in Figs. 5.6, and 5.7 describes the switching signal. From Fig. 5.5, it can be seen that all the system states eventually converge to a small neighborhood of the origin. The simulation results based on Corollary 5.1 are shown in

Fig. 5.9 Responses of two adaptive laws

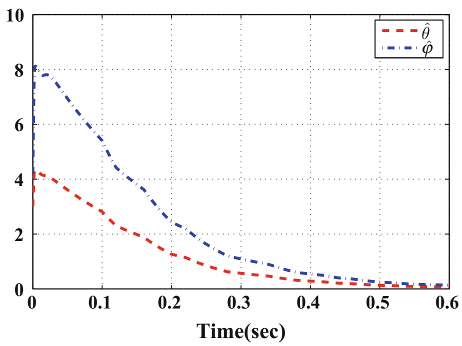
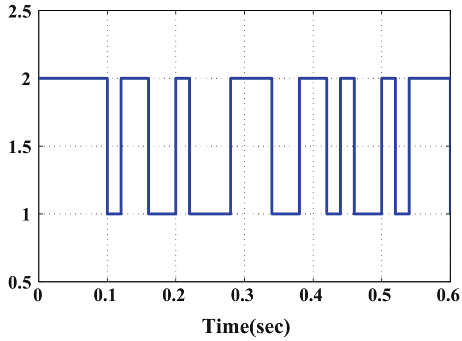


Fig. 5.10 Response of switching signal



Figs. 5.8, 5.9 and 5.10, respectively. It can be seen that all the system states eventually converge to a small neighborhood of the origin by using only two adaptive laws.

5.3.4 Conclusions

The adaptive neural control for a class of stochastic high-order switched nonlinear systems with SISS inverse dynamic is studied. An adaptive neural control algorithm is proposed. It can be shown that the equilibrium at the origin of the closed-loop system is BIBO stable in probability.

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Chapter 6
Output Tracking Control of Constrained Switched Nonlinear Systems

6.1 Background and Motivation

Control systems often suffer from various limits or constraints in the operation space [1, 2], that may arise out of performance requirements or physical constraints imposed on the system by its environmental regulations. For instance, the restoring torque of an aircraft certainly has a maximum value, as has the armature of a DC motor [3]. If the constraints are destroyed during operation, then serious consequences causing performance degradation, hazards or system damage will happen. Therefore, tackling constraints in control design has attracted much attention from various fields in science and engineering.

In the study of constrained linear or nonlinear systems, different approaches have been presented over the last a few years. To handle both state and input constraints in linear systems, many techniques have been developed (see, e.g., [4–6]), most of which are based on the notions of set invariance and admissible set control [7, 8]. Model Predictive Control that represents an effective control design methodology for handling both constraints and performance issues has been investigated in [9, 10]. In addition, reference governors have also been proposed to tackle the problem of constraints for nonlinear systems in [11]. The approaches mentioned above are numerical in nature or depend heavily on computationally intensive algorithms to solve the control problems.

It is worth pointing out that Barrier Lyapunov Functions (BLFs), which have been proposed in [12, 13], can be used to handle constraints. In the method, output constraints are handled directly during the controller design procedure. The proposed design procedure is flexible and can handle bounded uncertainties in the system. However, a resulting problem is that the constructed asymmetric BLF is of a switching type, a C^1 function. Consequently, the subsequent stabilizing functions must be of a high power. Furthermore, p-times differentiable unbounded functions are first introduced in [14] to handle the output tracking error constraints for a class of nonlinear systems in a lower triangular form. The advantage of the p-times differentiable

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unbounded function method is that in the controller design procedure, switching is not needed despite the asymmetrical limit range.

Note that control problems for switched systems with constraints have been investigated recently. Time optimal control for a class of integrator switched systems with state constraints was considered in [15]. A predictive control framework for a class of nonlinear switched systems subject to state and control constraints was presented in [16].

In this chapter, we aim at the problem of output tracking control for a class of constrained nonlinear switched systems in lower triangular form. By ensuring boundedness of the employed BLFs in the closed-loop, we assure that the constraints are not exceeded. Under the simultaneous domination assumption, we construct continuous feedback controllers for the switched system, which render that asymptotic output tracking is achieved, the limits are not transgressed and all closed-loop signals keep bounded. Moreover, we also explore the use of p-times differentiable unbounded functions to deal with asymmetric output constraints.

Notations: We use the following notations throughout this chapter. $\mathbb{R}_+$ denotes the set of nonnegative real numbers, $\mathbb{R}^n$ represents the n-dimensional real Euclidean vector space and $\| \bullet \|$ stands for the Euclidean vector norm. For positive integers i, j, we also denote $\bar{x}_i = [x_1, x_2, \ldots, x_i]^T$,$\bar{z}_i = [z_1, z_2, \ldots, z_i]^T$,$z_{i:j} = [z_i, z_{i+1}, \ldots, z_j]^T$, $\tilde{y}_{d_i} = [y_d, y_d^{(1)}, y_d^{(2)}, \ldots, y_d^{(i)}]^T$, $\tilde{b}_1^{(i)} = [b_1, b_1^{(1)}, b_1^{(2)}, \ldots, b_1^{(i)}]^T$ and $\tilde{b}_2^{(i)} = [b_2, b_2^{(1)}, b_2^{(2)}, \ldots, b_2^{(i)}]^T$, respectively.

6.2 Barrier Lyapunov Functions-Based Control Design

6.2.1 Problem Formulation and Preliminaries

Consider a class of switched nonlinear systems described by:

$$
\begin{aligned}
\dot{x}_1 &= f_1^{\sigma(t)}(x_1) + x_2, \\
&\cdots \\
\dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i) + x_{i+1}, \\
&\cdots \\
\dot{x}_{n-1} &= f_{n-1}^{\sigma(t)}(\bar{x}_{n-1}) + x_n, \\
\dot{x}_n &= f_n^{\sigma(t)}(\bar{x}_n) + g^{\sigma(t)}(\bar{x}_n)u, \\
y &= x_1,
\end{aligned}
\tag{6.1}
$$

where $x_1, x_2, \ldots, x_n$ are the states, $u = [u_1, u_2, \ldots, u_q]^T \in \mathbb{R}^q$ and $y \in \mathbb{R}$ are the input and output, respectively. $\sigma(t)$ is the switching signal, which takes its values in a finite set $I_m = \{1, 2, \ldots, m\}$ where $m > 1$ is the number of subsystems. $\forall i =$

$1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$; functions f_i^k, g^k are smooth with $g^k(\bar{x}_n) \neq 0, \forall \bar{x}_n \in R^n$. The output is required to satisfy certain constraints that are specified later.

For system (6.1), we design a feedback controller by using the information of all the states and a desired trajectory $y_d(t)$ such that $\lim_{t\to\infty}(y(t) - y_d(t)) = 0$ under arbitrary switchings.

The control objective is to solve the output tracking control problem guaranteeing all closed-loop signals to be bounded without exceeding the constraints.

To avoid the violation of the constraints, we employ a BLF with the following definition.

Definition 6.1 ([13]) A BLF is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region D containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of D, has the property $V(x) \to \infty$ as x approaches the boundary of D, and satisfies $V(x) \leq b, \forall t \geq 0$ along the solution $\dot{x} = f(x)$ for $x(0) \in D$ and some positive constant b.

It is worth pointing out that the Lyapunov function $V(x)$ in Definition 6.1 can be extended to be time-varying when the constraints are time-varying.

The following lemma that establishes a result of barrier function is first proposed for the subsequent developments.

Lemma 6.1 *For any positive constants $b_i, i = 1, 2, \ldots, n$, let $Z = \{\bar{z}_n \in \mathbb{R}^n : |z_i| < b_i, i = 1, \ldots, n\} \subset \mathbb{R}^n$ be an open set. Consider the switched system:*

$$\dot{\bar{z}}_n = h_{\sigma(t)}(t, \bar{z}_n), \tag{6.2}$$

where $\sigma(t)$ is the same as in (6.1); $h_i : \mathbb{R}_+ \times Z \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in η, uniformly in t, on $\mathbb{R}_+ \times Z$. We assume that the state of the system (6.2) does not jump at the switching instants. Let $Z_i = \{z_i \in \mathbb{R} : |z_i| < b_i\} \subset \mathbb{R}$. Suppose that there exist functions $V_i : z_i \to \mathbb{R}_+, i = 1, 2, \ldots, n$ continuously differentiable and positive definite in their respective domains, such that

$$V_i(z_i) \to \infty, \quad \text{as } z_i \to -b_i \text{ or } z_i \to b_i. \tag{6.3}$$

Let $\overline{V}(\bar{z}_n) = \sum_{i=1}^{n} V_i(z_i)$ and $z_i(0) \in Z_i$. If the inequality

$$\dot{\overline{V}}(\bar{z}_n) = \frac{\partial \overline{V}(\bar{z}_n)}{\partial \bar{z}_n} h_i(t, \bar{z}_n) < 0, \quad \forall \bar{z}_n \neq 0, i \in I_m \tag{6.4}$$

holds, then under arbitrary switchings, $z_i(t) \in Z_i, \forall t \in [0, \infty)$.

Proof The conditions on h_i and the trajectory of the system (6.2) is continuous at the switching instants ensuring the existence and uniqueness of a maximal solution $\bar{z}_n(t)$ on the time interval $[0, \tau_{\max})$. This implies that $\overline{V}(\bar{z}_n(t))$ exists for $\forall t \in [0, \tau_{\max})$.

From the fact that $z_i(0) \in Z_i$ and $V_i(z_i(0))$, $i = 1, 2, \ldots, n$ are known, we have that $\overline{V}(z_n(0))$ exists. Since $\overline{V}(\bar{z}_n)$ is positive definite and $\dot{\overline{V}}(\bar{z}_n) < 0$, therefore we obtain that $\overline{V}(\bar{z}_n(t)) < \overline{V}(\bar{z}_n(0))$ for $\forall t \in [0, \tau_{\max})$. Because $\overline{V}(\bar{z}_n) = \sum_{i=1}^{n} V_i(z_i)$ and the fact that $V_i(z_i)$ are positive functions, it is clear that each $V_i(z_i)$ is also bounded for $\forall t \in [0, \tau_{\max})$. Thus, we conclude from (6.3) that $z_i \neq -b_i$ and $z_i \neq b_i$. Given $-b_i < z_i(0) < b_i$, we know that $z_i(t)$ remains in the set Z_i for $\forall t \in [0, \tau_{\max})$.

Therefore, there is a compact subset $K \subseteq Z$ such that the maximal solution of (6.2) satisfies $\bar{z}_n(t) \in K$ for $\forall t \in [0, \tau_{\max})$. As a direct consequence of [38, p.481 Proposition C.3.6], we can infer that $\bar{z}_n(t) \in K$ is established for $\forall t \in [0, \infty)$. It follows that $|z_i(t)| \in Z_i, \forall t \in [0, \infty)$. In addition, it is clear that $V(\bar{z}_n)$ is a common Lyapunov function for the system (6.2), then the result holds under arbitrary switchings. □

Lemma 6.2 (Barbalat's Lemma) *Consider a differentiable function* $h(t)$. *If* $\lim_{t\to\infty} h(t)$ *is finite and* $\dot{h}(t)$ *is uniformly continuous, then* $\lim_{t\to\infty} \dot{h}(t) = 0$.

6.2.2 Control Design for Full State Constraints

We consider the full state constraints in the following; that is, for system (1), $x_i(t)$ is required to remain in the set $|x_i| \le c_i, \forall t \ge 0$, where c_i are positive constants, for all $i = 1, 2, \ldots, n$. The controller is designed to achieve asymptotic output tracking while ensuring that the full state constraints are not violated.

First, the following assumptions are used in the backstepping design procedures.

Assumption 6.1 For any $c_1 > 0$, there exist positive constants $\underline{B}_0$, $\overline{B}_0$, A_0, B_1, B_2, $\ldots$, B_n satisfying $\max\{\underline{B}_0, \overline{B}_0\} \le A_0 < c_1$ such that the desired trajectory $y_d(t)$ and its time derivatives satisfy $-\underline{B}_0 \le y_d(t) \le \overline{B}_0$, $|\dot{y}_d(t)| < B_1$, $|\ddot{y}_d(t)| < B_2, \ldots,$ $|y_d^{(n)}(t)| < B_n, \forall t \ge 0$.

Assumption 6.2 The functions $g^k(\bar{x}_n) = [g^{k,1}(\bar{x}_n), g^{k,2}(\bar{x}_n), \ldots, g^{k,q}(\bar{x}_n)]$, $k = 1, 2, \ldots, m$ are known. Furthermore, for $\forall j \in \{1, 2, \ldots, q\}$, assume that $\min_{k\in\{1,2,\ldots,m\}} g^{k,j}(\bar{x}_n) \ge 0, \forall \overline{x}_n \in \mathbb{R}^n$ or $\max_{k\in\{1,2,\ldots,m\}} g^{k,j}(\bar{x}_n) \le 0, \forall \overline{x}_n \in \mathbb{R}^n$. For ease of analysis, denote

$$
\begin{aligned}
M &= \{j \in \{1, 2, \ldots, q\} | min_{k\in\{1,2,\ldots,m\}} g^{k,j}(\bar{x}_n) \ge 0\}, \\
F &= \{j \in \{1, 2, \ldots, q\} | j \notin M\}.
\end{aligned} \tag{6.5}
$$

In what follows, the control design is proposed based on the simultaneous domination assumption with a barrier function in each step of the backstepping procedure.

Denote $z_1 = x_1 - y_d$ and $z_i = x_i - \phi_{i-1}$, $i = 2, \ldots, n$. Consider the Lyapunov function candidate:

$$\overline{V}_i(\bar{z}_i) = \sum_{l=1}^{i} V_l(z_l), \quad V_i(z_i) = \frac{1}{2}\log\frac{b_i^2}{b_i^2 - z_i^2}, \quad i = 1, 2, \ldots, n, \tag{6.6}$$

where $\phi_{i-1}, i = 2, \ldots, n$ stand for virtual controls, $\log(\bullet)$ denotes the natural logarithm of $\bullet$, $b_1 = c_1 - A_0$ and $b_i, i = 2, \ldots, n$ are positive constants. It is easy to know that $\overline{V}_n(\bar{z}_n) = \sum_{i=1}^{n} V_i(z_i)$ is positive definite and continuously differentiable in the set $|z_i| < b_i$ for all $i = 1, 2, \ldots, n$.

Step 1. Consider the following collection of auxiliary first-order subsystems.

$$\dot{z}_1 = f_1^k(x_1) + x_2 - \dot{y}_d, \qquad k = 1, 2, \ldots, m. \tag{6.7}$$

With the candidate Lyapunov function $V_1(z_1)$ and taking x_2 as the virtual control, we say that these first-order subsystems are simultaneously dominant if there exists a differentiable feedback law $\phi_1(x_1, z_1, \tilde{y}_{d_1}) = \phi_1^*(x_1, y_d) + \dot{y}_d$ such that, along the solutions of the subsystems in (6.7),

$$\dot{V}(z_1) = \frac{z_1\dot{z}_1}{b_1^2 - z_1^2} = \frac{z_1(\phi_1^*(x_1, y_d) + f_1^k(x_1))}{b_1^2 - z_1^2} < 0, \quad \forall z_1 \neq 0, k = 1, 2, \ldots, m. \tag{6.8}$$

Define

$$d_1^k(x_1, z_1, \tilde{y}_{d_1}) = \frac{z_1(\phi_1^*(x_1, y_d) + f_1^k(x_1))}{b_1^2 - z_1^2}, \qquad k = 1, 2, \ldots, m. \tag{6.9}$$

With $V_1(z_1)$, the control design for the first step is completed if a simultaneously dominating feedback law $x_2 = \phi_1(x_1, z_1, \tilde{y}_{d_1})$ is found.

Step i (for $i = 2, \ldots, n-1$). Consider the collection of auxiliary ith-order subsystems:

$$\begin{aligned}
\dot{z}_1 &= f_1^k(x_1) + z_2 + \phi_1^*(x_1, y_d),\\
&\cdots\\
\dot{z}_i &= f_i^k(\bar{x}_i) + x_{i+1} - \sum_{j=1}^{i-1}\tfrac{\partial\phi_{i-1}}{\partial x_j}(x_{j+1} + f_j^k(\bar{x}_j)) - \sum_{j=0}^{i-1}\tfrac{\partial\phi_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)},\\
k &= 1, 2, \ldots, m.
\end{aligned} \tag{6.10}$$

With the candidate Lyapunov function $\overline{V}_i(\bar{z}_i)$ and taking x_{i+1} as the virtual control, we say that the ith-order subsystems are simultaneously dominatable if there exists a continuously differentiable feedback law $x_{i+1} = \phi_i(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i})$ such that, along the solutions of the subsystems in (6.10),

$$\dot{\overline{V}}_i(\bar{z}_i) = \frac{z_1\dot{z}_1}{b_1^2 - z_1^2} + \sum_{j=2}^{i} \frac{z_j\dot{z}_j}{b_j^2 - z_j^2} = \sum_{j=1}^{i} d_j^k(\bar{x}_j, \bar{z}_j, \tilde{y}_{d_j}) < 0, \quad \forall \bar{z}_i \neq 0, k = 1, 2, \ldots, m, \tag{6.11}$$

where, for $j = 2, \ldots, i$,

$$d_j^k(\bar{x}_j, \bar{z}_j, \tilde{y}_{d_j}) = z_j \left[\frac{z_{j-1}}{b_{j-1}^2 - z_{j-1}^2} + \frac{1}{b_j^2 - z_j^2} \left(\phi_j + f_j^k(\bar{x}_j) \right. \right. \tag{6.12}$$

$$\left. \left. - \sum_{l=1}^{j-1} \frac{\partial \phi_{j-1}}{\partial x_l}(x_{l+1} + f_l^k(\bar{x}_l)) - \sum_{l=0}^{j-1} \frac{\partial \phi_{j-1}}{\partial y_d^{(l)}} y_d^{(l+1)} \right) \right]. \tag{6.13}$$

With the constructed $\overline{V}_i(\bar{z}_i)$, the control design for the ith step is completed if a simultaneously dominating feedback law $x_{i+1} = \phi_i(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i})$ is found.

By using repeatedly the inductive argument above, we say that the subsystems of (6.1) are simultaneously dominant if the control design for the $(n-1)$th step can be completed. Then, we construct a controller for the final step.

Step n. The derivative of $\overline{V}_n(\bar{z}_n)$ in (6.6) along the trajectory of the kth subsystem is

$$\begin{aligned}
\dot{\overline{V}}_n =& \frac{z_1\dot{z}_1}{b_1^2 - z_1^2} + \sum_{i=2}^{n} \frac{z_i\dot{z}_i}{b_i^2 - z_i^2} \\
=& \sum_{i=1}^{n-1} d_i^k(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i}) + z_n \left[\frac{z_{n-1}}{b_{n-1}^2 - z_{n-1}^2} + \frac{1}{b_n^2 - z_n^2} \left(f_n^k(\bar{x}_n) \right. \right. \\
& \left. \left. + g^k(\bar{x}_n)u - \sum_{j=1}^{n-1} \frac{\partial \phi_{n-1}}{\partial x_j}(x_{j+1} + f_j^k(\bar{x}_j)) - \sum_{j=0}^{n-1} \frac{\partial \phi_{n-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \right) \right] \\
=& a_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) + b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})u,
\end{aligned} \tag{6.14}$$

where

$$\begin{aligned}
a_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) =& \sum_{i=1}^{n-1} d_{i,k}(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i}) + z_n \left[\frac{z_{n-1}}{b_{n-1}^2 - z_{n-1}^2} + \frac{1}{b_n^2 - z_n^2} \left(f_n^k(\bar{x}_n) \right. \right. \\
& \left. \left. - \sum_{j=1}^{n-1} \frac{\partial \phi_{n-1}}{\partial x_j}(x_{j+1} + f_j^k(\bar{x}_j)) - \sum_{j=0}^{n-1} \frac{\partial \phi_{n-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \right) \right],
\end{aligned} \tag{6.15}$$

$$b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) = \frac{z_n}{b_n^2 - z_n^2} g^k(\bar{x}_n). \tag{6.16}$$

In view of the above discussions and the simultaneous domination condition, a controller for systems (6.1) can be established:

$$u(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) = [u_1(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), u_2(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), \ldots, u_q(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})]^T, \tag{6.17}$$

where

$$u_j = \begin{cases} \min_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right\}, & if\ z_n > 0, \\ \max_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right\}, & if\ z_n < 0, \quad for\ j \in M, \\ 0, & if\ z_n = 0, \end{cases} \tag{6.18}$$

and

$$u_j = \begin{cases} \max_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right\}, & if\ z_n > 0, \\ \min_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right\}, & if\ z_n < 0, \quad for\ j \in F, \\ 0, & if\ z_n = 0, \end{cases} \tag{6.19}$$

with

$$\begin{aligned} p_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) &= \left[p_{k,1}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), p_{k,2}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), \ldots, p_{k,q}(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})\right]^T \\ &= \begin{cases} -b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) \dfrac{\max\left\{a_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) + b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) b_k^T(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}), 0\right\}}{b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) b_k^T(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})}, if\ z_n \neq 0, \\ 0, if\ z_n = 0. \end{cases} \end{aligned} \tag{6.20}$$

Lemma 6.3 *Consider switched system (6.1). Suppose that the subsystems of (6.1) are simultaneously dominatable. Then, the continuous controller (6.17) can be constructed such that, along the solutions of all the closed-loop subsystems,*

$$\dot{\overline{V}}_n(\bar{z}_n) < 0, \quad \forall \bar{z}_n \neq 0, \tag{6.21}$$

where $\overline{V}_n(\bar{z}_n)$ is the Lyapunov function obtained in (6.6).

Proof For the sake of simplicity, we rewrite the system (6.1) as

$$\dot{\bar{x}}_n = \hat{f}_k(\bar{x}_n) + \hat{g}_k(\bar{x}_n) u, \quad k \in I_m. \tag{6.22}$$

In what follows, we will show that, $\forall k = 1, 2, \ldots, m$,

$$\begin{aligned} &\frac{\partial \overline{V}_n(\bar{z}_n)}{\partial \bar{z}_n}(\hat{f}_k(\bar{x}_n) + \hat{g}_k(\bar{x}_n) u(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})) \\ =& a_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) + b_k(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) u(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}) < 0, \quad \forall \bar{z}_n \neq 0, \end{aligned} \tag{6.23}$$

where $u(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n})$ is the controller presented in (6.17).

For simplicity, we shall omit the dependence on $\bar{x}_n, \bar{z}_n$ and $\tilde{y}_{d_n}$ for functions wherever no confusion will be caused. 1. Consider $z_n = 0$. In this case, $b_k = 0$, $u = 0$, and

$$a_k + b_k u = a_k < 0, \quad k = 1, 2, \ldots, m. \tag{6.24}$$

2. Consider $z_n > 0$. In this case, by the definitions of (6.18) and (6.19), we have

$$u_j = \begin{cases} \min_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}\right\}, \, j \in M, \\ \max_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}\right\}, \, j \in F. \end{cases} \tag{6.25}$$

If $j \in M$, then $b_{k,j} \geq 0$. Therefore, we have $b_{k,j}u_j = b_{k,j} \min_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}\right\} \leq b_{k,j}p_{k,j}$. Similarly, if $j \in F$, we have $b_{k,j}u_j \leq b_{k,j}p_{k,j}$. Therefore,

$$\begin{aligned} a_k + b_k u &= a_k + \sum_{i\in M} b_{k,i}u_i + \sum_{j\in F} b_{k,j}u_j \leq a_k + \sum_{j=1}^{q} b_{k,j}p_{k,j} \\ &= a_k + b_k p_k = \begin{cases} -b_k b_k^T, & if \quad a_k + b_k b_k^T \geq 0 \\ a_k, \; if \quad a_k + b_k b_k^T < 0 \end{cases} \\ &< 0, \quad k = 1, 2, \ldots, m. \end{aligned} \tag{6.26}$$

3. Consider $z_n < 0$. Similarly, in this case we can show that

$$u_j = \begin{cases} \max_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}\right\}, \, j \in M, \\ \min_{i\in\{1,2,\ldots,m\}} \left\{p_{i,j}\right\}, \, j \in F. \end{cases} \tag{6.27}$$

and

$$a_k + b_k u_k < 0, \quad k = 1, 2, \ldots, m. \tag{6.28}$$

Therefore, we conclude that, $\forall k = 1, 2, \ldots, m$, (6.23) is true. Thus, $\overline{V}_n(\bar{z}_n)$ is a common Lyapunov function for all subsystems of (1). □

Based on the above discussions, we are now in a position to give the following result.

Theorem 6.1 *Consider the closed-loop system (6.1), (6.17) under Assumptions 6.1–6.2. Let A_i be an upper bound for ϕ_i in compact set Ω_i:*

$$A_i \geq \sup_{\left(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i}\right)\in\Omega_i} \left|\phi_i\left(\bar{x}_i, \bar{z}_i, \tilde{y}_{d_i}\right)\right|, \quad i = 1, \ldots, n-1, \tag{6.29}$$

where $\Omega_i = \{\bar{x}_i \in R^i, \bar{z}_i \in R^i, \tilde{y}_{d_i} \in R^{i+1} : \left|x_j\right| \le D_{z_j} + A_{j-1}, \left|z_j\right| \le D_{z_j}, |y_d| < A_0, |y_d^{(j)}| \le B_j, j = 1, \ldots, i\}$, $D_{z_j} = b_j\sqrt{1 - \frac{\Pi_{k=1}^n \left(b_k^2 - z_k^2(0)\right)}{\Pi_{k=1}^n b_k^2}}, i = 1, \ldots, n-1.$
Given that the following conditions are satisfied,

(1) $c_{i+1} > A_i + b_{i+1}$ *holds for* $\forall i = 1, 2, \ldots, n-1$.

(2) The initial conditions $\bar{z}_n(0)$ *belong to the set* $\Omega_{z_0} = \{\ \bar{z}_n \in R^n : |z_i| < b_i, i = 1, \ldots, n\ \}$.

Under arbitrary switching signals, closed-loop system (1) has the following properties:

(i) The signals $z_i(t), i = 1, 2, \ldots, n$, *remain in the compact set* $\Omega_z = \{\bar{z}_n \in R^n : |z_i| < D_{z_i}, i = 1, 2, \ldots, n\}$.

(ii) $x_i(t)$ *remains in the set* $\Omega_x = \{\bar{x}_n \in R^n : |x_i| < D_{z_i} + A_{i-1} < c_i, i = 1, \ldots, n\}, \forall t \ge 0$; *i.e., the full state constraints are never violated.*

(iii) All closed-loop signals are bounded.

(iv) The output tracking error $z_1(t)$ *asymptotically converges to zero, i.e.,* $y(t) \to y_d(t)$ *as* $t \to \infty$.

Proof (i) By $\dot{\overline{V}}_n < 0$, it is clear that $\overline{V}_n(t) < \overline{V}_n(0), \forall t \ge 0$. Because $z_i^2(0) < b_i^2$ from condition (2), we have that $\overline{V}_n(0) < \sum_{i=1}^n \frac{1}{2}\log\frac{b_i^2}{b_i^2 - z_i^2(0)}$, which means

$$\frac{1}{2}\log\frac{b_i^2}{b_i^2 - z_i^2} < \sum_{i=1}^n \frac{1}{2}\log\frac{b_i^2}{b_i^2 - z_i^2(0)} \tag{6.30}$$

for $i = 1, \ldots, n$. Because $\log a + \log b = \log ab$, we rewrite (6.30) as

$$\log\frac{b_i^2}{b_i^2 - z_i^2} < \log\frac{\Pi_{i=1}^n b_i^2}{\Pi_{i=1}^n \left(b_i^2 - z_i^2(0)\right)} \tag{6.31}$$

for $i = 1, \ldots, n$. Furthermore, we obtain from Lemma 1 that $b_i^2 - z_i^2(t) > 0, \forall t \ge 0$. Then, (6.31) is equivalent to $|z_i(t)| < D_{z_i}, \forall t \ge 0$.

(ii) Because $|z_1(t)| < D_{z_1} < c_1 - A_0$, we obtain

$$|x_1(t)| < D_{z_1} + |y_d(t)| < c_1 - A_0 + |y_d(t)|. \tag{6.32}$$

Noting that $|y_d(t)| < A_0$, we thus conclude from Assumption 6.1 that $|x_1(t)| < D_{z_1} + A_0 < c_1, \forall t \ge 0$.

To show that $|x_2(t)| < c_2$, we first verify that there exists a positive constant A_1 such that $|\phi_1(t)| \le A_1, \forall t \ge 0$. Because $|x_1(t)| < D_{z_1} + A_0$, $|z_1(t)| \le D_{z_1}$ and $|\dot{y}_d(t)| \le B_1$, it is clear that $\left(x_1(t), z_1(t), \tilde{y}_{d_1}(t)\right) \in \Omega_1$, and thus, the stabilizing function ϕ_1 is bounded because it is a continuous function. As a result, $\sup_{(x_1, z_1, \tilde{y}_{d_1}) \in \Omega_i} \left|\phi_1\left(x_1, z_1, \tilde{y}_{d_1}\right)\right|$ exists, and an upper bound A_1 can be found. Then, we can see from $|z_2(t)| \le D_{z_2} < b_2$ that

$$|x_2(t)| \le D_{z_2} + |\phi_1(t)| < b_2 + |\phi_1(t)|. \tag{6.33}$$

Since $|\phi_1(t)| < A_1$, therefore we deduce that $|x_2(t)| \le D_{z_2} + A_1 < b_2 + A_1 < c_2, \forall t \ge 0$.

We can get that $|x_{i+1}(t)| \le c_{i+1}, i = 2, \ldots, n-1$, after verifying that there exist positive constants A_i such that $|\phi_i(t)| \le A_i, \forall t \ge 0$. Because $|x_i(t)| \le D_{z_i} + A_{i-1}, |z_i(t)| \le D_{z_i}$ and $\left|y_d^{(i)}(t)\right| \le Y_i$, it is clear that $\left(\bar{x}_i(t), \bar{z}_i(t), \tilde{y}_{d_i}(t)\right) \in \Omega_i$, and thus, the stabilizing function ϕ_i is bounded because it is a continuously differentiable function. As a result, we have that $\sup_{(\bar{x}_i, \bar{z}_i, \tilde{y}_i) \in \Omega_i} |\phi_i(\bar{x}_i, \bar{z}_i, \tilde{y}_i)|$ exists, and an upper bound A_i can be found. Then, from $|z_{i+1}(t)| \le D_{z_{i+1}} < b_{i+1}$, we can show that $|x_{i+1}(t)| < D_{z_{i+1}} + |\phi_i(t)| < b_{i+1} + |\phi_i(t)|$. Because $|\phi_i(t)| \le A_i$, therefore we have that $|x_{i+1}(t)| < D_{z_{i+1}} + A_i < b_{i+1} + A_i < c_{i+1}, \forall t \ge 0$.

(iii) By virtue of the boundedness of $\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}$, it is clear that stabilizing functions $\phi_i(\bar{x}_i, \bar{z}_i, \tilde{y}_i)$ and control $u_n\left(\bar{x}_n, \bar{z}_n, \tilde{y}_{d_n}\right)$ are bounded. Therefore, all closed-loop signals are bounded.

(iv) Based on the fact that $\bar{x}_i(t), \bar{z}_i(t), i = 1, 2, \ldots, n$ are bounded, it can be obtained that $\ddot{V}$ is bounded, which means that $\dot{V}$ is uniformly continuous. Then, by Lemma 6.2, we obtain that $z_i(t) \to 0$ as $t \to 0$. Because $z_1(t) = x_1(t) - y_d(t)$ and $y(t) = x_1(t)$, we finally have $y(t) \to y_d(t)$ as $t \to \infty$. □

6.2.3 *Control Design for Time-Varying Output Constraints*

In this section, we consider the case that the output is required to satisfy $-\bar{c}_1(t) < y(t) < \bar{c}_2(t), \forall t \ge 0$, where $\bar{c}_1(t), \bar{c}_2(t)$ are positive-valued time-varying functions. By incorporating an appropriate barrier function in the backstepping design, we show that the output constraints are not violated at any time and asymptotic output tracking is realized while ensuring boundedness of all closed-loop signals.

Assumption 6.3 There exist positive constants $K_l^i, i = 0, 1, \ldots, n, l = 1, 2$ such that the time-varying functions $\bar{c}_l(t)$ and their time derivatives satisfy $\bar{c}_l(t) \le K_l^0, \bar{c}_l^{(i)}(t) \le K_l^i, i = 1, 2, \ldots, n, l = 1, 2, \forall t \ge 0$.

Assumption 6.4 There exist functions $B_l : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $B_l(t) < \bar{c}_l(t), l = 1, 2, \forall t \ge 0$ and positive constants $B_l^i, i = 1, 2, \ldots, n$ such that the desired trajectory $y_d(t)$ and its time derivatives satisfy $-B_1(t) \le y_d(t) \le B_2(t), -B_1^i < y_d^{(i)}(t) < B_2^i, i = 1, 2, \ldots, n, \forall t \ge 0$.

Lemma 6.4 *For any positive constants a_0, b_0, let $\Pi = \{\xi \in \mathbb{R} : -a_0 < \xi < b_0\} \subset \mathbb{R}$ and $X = \mathbb{R}^v \times \Pi \subset \mathbb{R}^{v+1}$ be open sets. Consider the switched system:*

$$\dot{\eta} = h_{\sigma(t)}(t, \eta), \tag{6.34}$$

where $\eta := [\xi, z] \in X, z \in \mathbb{R}^v$, $\sigma(t)$ is the same as in (1), and $h_i : \mathbb{R}_+ \times X \to \mathbb{R}^{v+1}$ is piecewise continuous in t and locally Lipschitz in η, uniformly in t, on $\mathbb{R}_+ \times X$.

We also assume that the state of the switched system (6.34) does not jump at switching instants. Suppose that there exist functions $V_1 : \Pi \to \mathbb{R}_+$ *and* $V_2 : \mathbb{R}^v \to \mathbb{R}_+$ *continuously differentiable and positive definite in their individual domains, such that*

$$V_1(\xi) \to \infty, \quad as\, \xi \to -a_0\, or\ \ \xi \to b_0, \tag{6.35}$$

$$\gamma_1(\|z\|) \le V_2(z) \le \gamma_2(\|z\|), \tag{6.36}$$

where γ_1 *and* γ_2 *are class* K_∞ *functions. Let* $V(\eta) = V_1(\xi) + V_2(z)$*, and* $\xi(0)$ *belong to the set* $(-a_0, b_0)$*. If the inequality*

$$\dot{V}(\eta) = \frac{\partial V(\eta)}{\partial \eta} h_i(t, \eta) < 0, \quad \forall \eta \ne 0, i \in I_m \tag{6.37}$$

holds, then under arbitrary switchings, $\xi(t)$ *remains in the open set* $(-a_0, b_0)$, $\forall t \in [0, \infty)$.

Proof The proof is similar to Lemma 6.1. □

Noting that the output constraints are asymmetric and time-varying, we construct the following asymmetric barrier function, which explicitly depends on time.

$$V_1(z_1(t), b_1(t)) = \frac{1}{2}(1 - q(z_1(t))) \log \frac{b_1^2(t)}{b_1^2(t) - z_1^2(t)} + \frac{1}{2} q(z_1(t)) \log \frac{b_2^2(t)}{b_2^2(t) - z_1^2(t)}, \tag{6.38}$$

where $z_1 = x_1 - y_d$, $b_1(t) = \bar{c}_1(t) - B_1(t)$ and $b_2(t) = \bar{c}_2(t) - B_2(t)$ are the constraints on z_1; that is, we require $-b_1(t) < z_1(t) < b_2(t)$, and

$$q(\bullet) = \begin{cases} 0, & if \quad \bullet \le 0, \\ 1, & if \quad \bullet > 0. \end{cases} \tag{6.39}$$

Lemma 6.5 *The Lyapunov function candidate* V_1 *in (6.38) is positive definite and* C^1 *in the set* $(-b_1(t), b_2(t))$.

Proof For $-b_1(t) < z_1(t) < b_2(t)$, we have that $V_1 \ge 0$ and $V_1 = 0$ if and only if $z_1(t) = 0$. This means that V_1 is positive definite. Furthermore, V_1 is piecewise smooth among intervals $z_1(t) \in (-b_1(t), 0]$ and $z_1(t) \in (0, b_2(t))$. Noting that $\lim_{z_1 \to 0^-} \frac{dV_1}{dz_1} = \lim_{z_1 \to 0^+} \frac{dV_1}{dz_1} = 0$, we conclude that V_1 is C^1. This completes the proof. □

Then, to remove the explicit dependence on time in (6.38), we use a coordinate transformation:

$$\xi_1 = \frac{z_1(t)}{b_1(t)}, \quad \xi_2 = \frac{z_1(t)}{b_2(t)}, \quad \xi = (1 - q(z_1))\xi_1 + q(z_1)\xi_2. \tag{6.40}$$

Therefore, we can rewrite V_1 in (6.38) as

$$V_1(\xi) = \frac{1}{2}\log\frac{1}{1-\xi^2}. \tag{6.41}$$

It is clear that $V_1(\xi)$ is positive definite and continuously differentiable in the set $|\xi| < 1$.

Now, consider the Lyapunov function candidate:

$$\overline{V}_i(\xi, \bar{z}_{2:i}) = V_1(\xi) + \sum_{l=2}^{i} V_l(z_l),\ V_i(z_i) = \frac{1}{2}z_i^2, \quad i = 2, 3, \ldots, n, \tag{6.42}$$

where $z_i = x_i - \phi_{i-1}$, $i = 2, \ldots, n$, and $\phi_1 = (1 - q(z_1))\phi_1^1(x_1, \xi_1, z_1, \tilde{b}_1^{(1)}, \tilde{y}_{d_1}) + q(z_1)\phi_1^2(x_1, \xi_2, z_1, \quad \tilde{b}_2^{(1)}, \tilde{y}_{d_1})$, $\phi_j = \phi_j(\bar{x}_j, \xi_1, \xi_2, \bar{z}_j, \tilde{b}_1^{(j)}, \tilde{b}_2^{(j)}, \tilde{y}_{d_j})$, $j = 2, \ldots, n-1$ are the virtual controls.

Using the backstepping design technique in Sect. 6.3, we can then get

$$d_1^k \tag{6.43}$$
$$= (1 - q(z_1))\frac{\xi_1(\phi_1^1 + f_1^k(x_1) - \dot{y}_d - \xi_1\dot{b}_1)}{b_1(1-\xi_1^2)} + q(z_1)\frac{\xi_2(\phi_1^2 + f_1^k(x_1) - \dot{y}_d - \xi_2\dot{b}_2)}{b_2(1-\xi_2^2)}.$$

$$d_2^k \tag{6.44}$$
$$= (1 - q(z_1))z_2\left(\frac{\xi_1}{b_1(1-\xi_1^2)} + \phi_2 + f_2^k(\bar{x}_2) - \frac{\partial\phi_1}{\partial t} - \frac{\partial\phi_1}{\partial x_1}(x_2 + f_2^k(\bar{x}_2)) - \sum_{l=0}^{1}\frac{\partial\phi_1}{\partial y_d^l}y_d^{l+1}\right)$$
$$+ q(z_1)z_2\left(\frac{\xi_2}{b_2(1-\xi_2^2)} + \phi_2 + f_2^k(\bar{x}_2) - \frac{\partial\phi_1}{\partial t} - \frac{\partial\phi_1}{\partial x_1}(x_2 + f_2^k(\bar{x}_2)) - \sum_{l=0}^{1}\frac{\partial\phi_1}{\partial y_d^l}y_d^{l+1}\right),$$

$$d_j^k \tag{6.45}$$
$$= z_j\left(z_{j-1} + \phi_j + f_j^k(\bar{x}_j) - \frac{\partial\phi_{j-1}}{\partial t} - \sum_{l=1}^{j-1}\frac{\partial\phi_{j-1}}{\partial x_l}(x_{l+1} + f_l^k(\bar{x}_l)) - \sum_{l=0}^{j-1}\frac{\partial\phi_{j-1}}{\partial y_d^l}y_d^{l+1}\right).$$

$$k = 1, 2, \ldots, m,\ j = 3, 4, \ldots, n-1. \tag{6.46}$$

and

$$a_k = \sum_{i=1}^{n-1} d_{i,k} + z_n\left(z_{n-1} + f_n^k(\bar{x}_n) - \frac{\partial\phi_{n-1}}{\partial t}\right. \tag{6.47}$$
$$\left. - \sum_{j=1}^{n-1}\frac{\partial\phi_{n-1}}{\partial x_j}(x_{j+1} + f_j^k(\bar{x}_j)) - \sum_{j=0}^{n-1}\frac{\partial\phi_{n-1}}{\partial y_d^{(j)}}y_d^{(j+1)}\right),$$
$$b_k = z_n g^k(\bar{x}_n). \tag{6.48}$$

We design the following controller for the system (6.1).

$$\begin{aligned} & u(\bar{x}_n, \xi_1, \xi_2, \bar{z}_n, \tilde{b}_1^{(n)}, \tilde{b}_2^{(n)}, \tilde{y}_{d_n}) \\ =& [u_1(\bar{x}_n, \xi_1, \xi_2, \bar{z}_n, \tilde{b}_1^{(n)}, \tilde{b}_2^{(n)}, \tilde{y}_{d_n}), u_2(\bar{x}_n, \xi_1, \xi_2, \bar{z}_n, \tilde{b}_1^{(n)}, \tilde{b}_2^{(n)}, \tilde{y}_{d_n}), \ldots, \\ & u_q(\bar{x}_n, \xi_1, \xi_2, \bar{z}_n, \tilde{b}_1^{(n)}, \tilde{b}_2^{(n)}, \tilde{y}_{d_n})]. \end{aligned} \tag{6.49}$$

Applying Lemma 6.3, we can conclude that $\dot{\overline{V}}_n(\xi, \bar{z}_{2:n}) < 0, \forall (\xi, z_{2:n})^T \neq 0$ along the solutions of closed-loop system (6.1).

Based on the above discussions, we can obtain the following theorem.

Theorem 6.2 *Consider the switched system (6.1) under Assumptions 6.2–6.4. If the subsystems are simultaneously dominatable with the controller (6.49), then the closed-loop system (6.1) has the following properties under arbitrary switching, where the initial conditions are* $\bar{z}_n(0) \in \Omega_{z_0} = \{\bar{z}_n \in R^n : -b_1(0) < z_1(0) < b_2(0)\}$.

(i) The signals $\xi_1(t), \xi_2(t)$ *and* $z_i(t), i = 1, 2, \ldots, n$ *are bounded, for* $\forall t \geq 0$, *as follows.*

$$\begin{aligned} & -\sqrt{1 - e^{-2V_n(0)}} < \xi_1(t) < 0, \\ & 0 \leq \xi_2(t) < \sqrt{1 - e^{-2V_n(0)}}, \\ & -b_1(t) < -\underline{D}_{z_1}(t) < z_1(t) < \overline{D}_{z_1}(t) < b_2(t), \\ & \|\bar{z}_{2:n}(t)\| < \sqrt{2V_n(0)}, \end{aligned} \tag{6.50}$$

where $\underline{D}_{z_1}(t) = b_1(t)\left(1 - e^{-2V_n(0)}\right)^{\frac{1}{2}}, \overline{D}_{z_1}(t) = b_2(t)\left(1 - e^{-2V_n(0)}\right)^{\frac{1}{2}}$.

(ii) The output $y(t)$ *remains in the set* $\Omega_y = \{y \in R : -\bar{c}_1(t) < -b_2(t) - B_2(t) < y(t) < b_1(t) + B_1(t) < \bar{c}_2(t)\}$; *i.e., the output constraints are never violated.*

(iii) All closed-loop signals are bounded.

(iv) The output tracking error asymptotically converges to zero; i.e., $y(t) \to y_d(t)$ *as* $t \to \infty$.

Proof (i) Applying Lemma 6.4 yields that $|\xi_i(t)| < 1, i = 1, 2$, from which we have that $-b_1(t) < z_1(t) < b_2(t), \forall t \geq 0$. Furthermore, it follows from $\overline{V}_n(t) < \overline{V}_n(0), \forall t \geq 0$, that

$$\overline{V}_n(0) > \begin{cases} \log \frac{b_1^2(t)}{b_1^2(t) - z_1^2(t)}, & -b_1(t) < z_1(t) < 0, \\ \log \frac{b_2^2(t)}{b_2^2(t) - z_1^2(t)}, & 0 \leq z_1(t) < b_2(t). \end{cases} \tag{6.51}$$

Then, we get that

$$z_1^2(t) < \begin{cases} b_1^2(t)\left(1 - e^{-2\overline{V}_n(0)}\right), -b_1(t) < z_1(t) < 0, \\ b_2^2(t)\left(1 - e^{-2\overline{V}_n(0)}\right), \quad 0 \leq z_1(t) < b_2(t). \end{cases} \tag{6.52}$$

This implies that $z_1(t) > -b_1(t)\left(1-e^{-2\overline{V}_n(0)}\right)^{\frac{1}{2}}$ for negative $z_1(t)$, and $z_1(t) < b_2(t)\left(1-e^{-2\overline{V}_n(0)}\right)^{\frac{1}{2}}$ for nonnegative $z_1(t)$. Therefore, it is obvious that $-\underline{D}_{z_1}(t) < z_1(t) < \overline{D}_{z_1}(t), \forall t \geq 0$.

Similarly, from the fact that $\frac{1}{2}\sum_{j=2}^{n} z_j^2(t) \leq \overline{V}_n(0)$, we can obtain that $|z_{2:n}(t)| \leq \sqrt{2\overline{V}_n(0)}, \forall t \geq 0$.

(ii) Because $y(t) = z_1(t) + y_d, -\underline{D}_{z_1}(t) < z_1(t) < \overline{D}_{z_1}(t)$, and $|y_d(t)| \leq B_l(t), l = 1, 2, \forall t \geq 0$. Then, we can conclude that

$$-\underline{D}_{z_1}(t) - B_1(t) < z_1(t) + y_d(t) < \overline{D}_{z_1}(t) + B_2(t). \tag{6.53}$$

$\underline{D}_{z_1}(t) < b_1(t)$ and $\overline{D}_{z_1}(t) < b_2(t)$, therefore we know that

$$\begin{aligned}\underline{D}_{z_1}(t) + y_d(t) &< b_1(t) + B_1(t) = \bar{c}_1(t),\\ \overline{D}_{z_1}(t) + y_d(t) &< b_2(t) + B_2(t) = \bar{c}_2(t).\end{aligned} \tag{6.54}$$

Hence, we can deduce that $y(t) \in \Omega_y, \forall t \geq 0$.

(iii) From (i), we know that the error signals $z_1(t), z_2(t), \ldots, z_n(t)$ are bounded. The boundedness of $z_1(t)$ and $y_d(t)$ implies that the state $x_1(t)$ is bounded. From (6.38), we see that $\dot{b}_i(t)$ are bounded, because $\dot{\bar{c}}_i(t) \leq K_i^1$ and $|\dot{y}_d(t)| \leq B_i^1, i = 1, 2$, where K_i^1 and B_i^1 are some positive constants. Therefore, the virtual control ϕ_1 is also bounded. This leads to the boundedness of $x_2(t)$, because $x_2 = z_2 + \phi_1$. Furthermore, it is not hard to check that all variables of continuous function ϕ_2 are bounded, and thus we get that ϕ_2 is bounded. This leads to the boundedness of state $x_3(t)$, because $x_3 = z_3 + \phi_2$. Following the same procedures, one can know that each ϕ_i, for $i = 3, \ldots, n-1$, is bounded. Hence, the boundedness of state $x_{i+1}(t)$ can be ensured. With $\bar{x}_n(t)$ and $\bar{z}_n(t)$ being bounded, and $-b_1(t) < z_1(t) < b_2(t), \forall t \geq 0$, we deduce that the control $u(t)$ is bounded. Thus, all closed-loop signals are bounded.

(iv) Let $d_1 = d_1^k, k = 1, 2, \ldots, m$, which is differentiable in the set $|\xi| < 1$. Because $|\xi(t)| < 1, \forall t \geq 0$ from Lemma 6.1, we can integrate both sides of $\dot{\overline{V}}_n = a_k + b_k u$ with the controller (6.49) to obtain

$$\lim_{t\to\infty}\int_0^t d_1(\tau)\,d\tau < V(0) < \infty, \forall k = 1, 2, \ldots, m. \tag{6.55}$$

Meanwhile, one can also derive from d_1^k that $\dot{d}_1(t)$ is bounded, which implies that $d_1(t)$ is uniformly continuous. By Lemma 6.2, one can get that $d_1(t) \to 0,\ as\ t \to \infty$, which means $\xi_i(t) \to 0,\ as\ t \to \infty$, because $\xi_i(t) = z_1(t)/b_i(t)$ and $b_i(t) > 0, i = 1, 2, \forall t \geq 0$. Subsequently, one can obtain $z_1(t) \to 0,\ as\ t \to \infty$. Therefore, we finally have $y(t) \to y_d(t),\ as\ t \to \infty$. □

6.2.4 Simulation Results

In this section, two examples are presented to demonstrate the effectiveness of the obtained results.

Consider the following switched nonlinear system,

$$\begin{aligned} \dot{x}_1 &= g_1^{\sigma(t)}(x_1)\,x_2, \\ \dot{x}_2 &= f_2^{\sigma(t)}(\bar{x}_2, d(t)) + g_2^{\sigma(t)}(\bar{x}_2)\,u_{\sigma(t)}, \\ y &= x_1, \end{aligned} \tag{6.56}$$

where $\sigma : [0, +\infty) \to \{1, 2\}$, $g_1^1(x_1) = g_1^2(x_1) = 1$, $f_2^1(\bar{x}_2, d(t)) = \theta x_2^2, \theta \in [0.4, 0.8]$, $f_2^2(\bar{x}_2, d(t)) = x_2 \cos\left(2x_1x_2^2\right)$. The control objective is to design a state feedback controller such that the output x_1 of the system can track the given signal $y_c = 0.2$, and does not destroy a symmetric constraint $\underline{L} = \overline{L} = 0.25$.

Due to the symmetric constraint $\underline{L} = \overline{L} = 0.25$, one can set $z_1 = \Psi(x_{1d}, -0.25, 0.25) = tan[2\pi(x_1 - 0.2)]$ and $V_1(z_1) = \frac{1}{2}z_1^2$. By using the proposed method, the common stabilizing function $\phi_1(z_1)$ can be obtained for each subsystem at the initial step:

$$\phi_1(z_1) = -z_1\left[1 + \frac{2}{2\pi \sec^2[2\pi(x_1 - 0.2)]}\right]. \tag{6.57}$$

Next, set $z_2 = x_2 - \phi_1(z_1)$, and $\overline{V}_2(\bar{z}_2) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ is the CLF for system (6.56). We can give the following state feedback controller.

$$\begin{aligned} u(\bar{z}_2) = -z_2\Bigg[&1.6\pi \sec^2[2\pi(x_1 - 0.2)] + \left(1 + \frac{2}{2\pi \sec^2[2\pi(x_1 - 0.2)]}\right)^4 \\ &+\left(1 + \frac{2}{2\pi \sec^2[2\pi(x_1 - 0.2)]}\right)^2 (1 + x_2^2) + (1 + x_2^2)^{\frac{1}{2}} \\ &+\left(1 + \frac{2}{2\pi \sec^2[2\pi(x_1 - 0.2)]}\right) + 1\Bigg]. \end{aligned} \tag{6.58}$$

Choose the initial values as $x_1(0) = 0.449$, $x_2(0) = -2.2$. Figure 6.1 demonstrates that asymptotic tracking performance can be achieved under a randomly generated switching signal in Fig. 6.2. From Fig. 6.3, it can be seen that the output tracking error x_{1d} converges to zero while remaining in the set $(-0.25, 0.25)$. Finally, the state response of the p-times differentiable unbounded function $z_1 = tan[2\pi(x_1 - 0.2)]$ is shown in Fig. 6.4 demonstrating the validity of the designed state feedback controller (6.58).

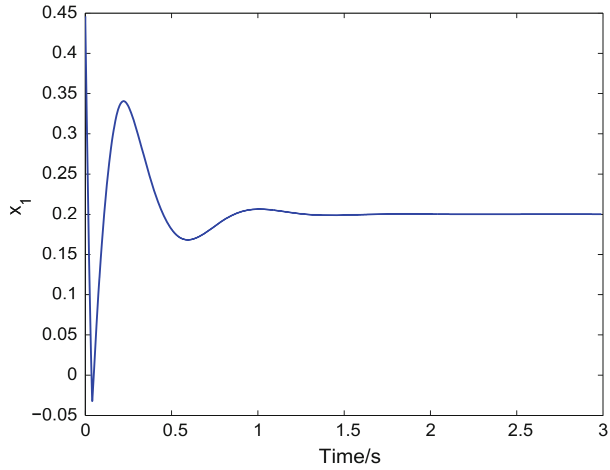

Fig. 6.1 Output tracking for the desired signal $y_d = 0.2$

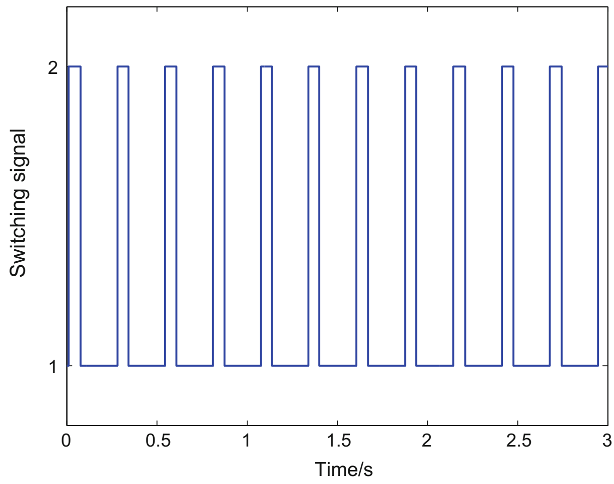

Fig. 6.2 The given switching signal for the system (6.56)

6.2.5 Conclusions

Based on the BLF approach, a control design method for constrained nonlinear switched systems in lower triangular form has been developed to achieve the output tracking objective. By guaranteeing the boundedness of the BLF in the closed-loop, the restrictions are not transgressed. Furthermore, asymptotic output tracking is achieved without violating the constraints, and all closed-loop signals are bounded.

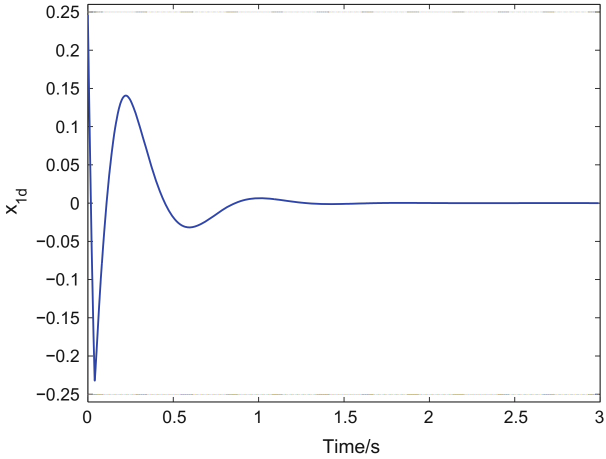

Fig. 6.3 The state response of the output tracking error x_{1d}

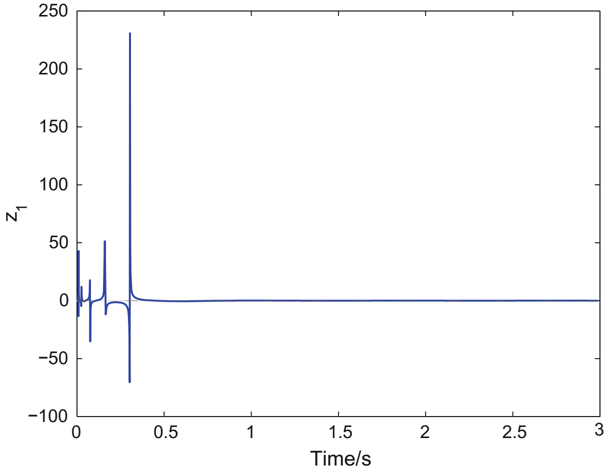

Fig. 6.4 The state response of the p-times differentiable unbounded function z_1

In particular, the issue of output tracking control with full state constraints and asymmetric time-varying output constraints are considered for switched nonlinear systems.

6.3 *p*-Times Differentiable Unbounded Functions-Based Control Design

6.3.1 Problem Formulation and Preliminaries

Consider uncertain switched nonlinear systems with the following lower triangular form,

$$\begin{aligned}\dot{x}_1 &= g_1^{\sigma(t)}(x_1)\,x_2,\\ \dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i, d(t)) + g_i^{\sigma(t)}(\bar{x}_i)\,x_{i+1},\ i = 2, 3, \ldots, n-1,\\ \dot{x}_n &= f_n^{\sigma(t)}(\bar{x}_n, d(t)) + g_n^{\sigma(t)}(\bar{x}_n)\,u_{\sigma(t)},\\ y &= x_1,\end{aligned} \tag{6.59}$$

where $x_1, x_2, \ldots, x_n$ are system states, y is the output; $\sigma(t)$ is the switching signal, which takes its values in a finite set $I_m = \{1, 2, \ldots, m\}$ and $m > 1$ is the number of subsystems; $d(t)$ is an unknown piecewise continuous disturbance belonging to a known compact set $\Omega \in R^s$; $u_k \in R$ is the control input of the k-th subsystem. For $\forall i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$, functions $f_i^k(\bar{x}_i, d(t))$ and $g_i^k(\bar{x}_i)$ are known and smooth with $0 < \underline{g} \le g_i^k(\bar{x}_i) \le \bar{g}$, where $\underline{g}$ and $\bar{g}$ are positive constants. As commonly assumed in the literature, the Zeno behavior for all types of switching signals is not considered. In addition, we assume that the state of system (6.59) is continuous at switching instants.

Remark 6.1 For non-switched nonlinear systems, the structure of (6.59) has been widely investigated (see, e.g., [12, 14, 17–19]). For switched nonlinear systems, the considered system structure of (6.59) was restricted to the design of stabilizing controllers [20–23].

Here, we consider the following output-constrained tracking control problem.

The output-constrained tracking control problem: For the system (6.59) under arbitrary switchings, design state feedback controllers to ensure the output of system (6.59) to track a given constant reference signal y_c such that:

(1) Asymptotic tracking is achieved; i.e.,

$$\lim_{t\to\infty}(y(t) - y_c) = 0. \tag{6.60}$$

(2) The output tracking error is confined to be a pre-specified limit range; i.e.,

$$-\underline{L} \le y(t) - y_c \le \overline{L} \tag{6.61}$$

for all $t \geq t_0 \geq 0$, where $\underline{L}$ and $\overline{L}$ are strictly positive constants. If $\underline{L} = \overline{L}$, the constraint (6.61) is referred to as a symmetric constraint. If $\underline{L} \neq \overline{L}$, the constraint (6.61) is referred to as an asymmetric constraint.

(3) All signals of the closed-loop system (6.59) are bounded.

The following assumptions are needed to develop the main results.

Assumption 6.5 For $i = 2, 3, \ldots, n$,

$$\left| f_i^k(\bar{x}_i, d(t)) \right| \leq (|x_2| + \cdots + |x_i|)\mu_i^k(\bar{x}_i), \forall k \in I_m, \tag{6.62}$$

where $\mu_{i,k}(\bar{x}_i)$ is a set of known non-negative smooth functions.

Assumption 6.6 At t_0, there exist strictly positive constants $\underline{L}_1 < \underline{L}$ and $\overline{L}_1 < \overline{L}$ such that

$$-\underline{L}_1 \leq x_{1d}(t_0) \leq \overline{L}_1, \tag{6.63}$$

where $x_{1d}(t_0) = x_1(t_0) - y_c$ is the initial output tracking error.

Two definitions and two relevant lemmas are addressed in the following for later use.

Definition 6.2 ([2]) A scalar function $h(x, a, b)$ is said to be a p-times differentiable step function if it satisfies the following properties.

(1) $h(x, a, b) = 0, \quad \forall -\infty < x \leq a$,

(2) $h(x, a, b) = 1, \quad \forall b \leq x < +\infty$,

(3) $0 < h(x, a, b) < 1, \quad \forall x \in (a, b)$,

(4) $h(x, a, b)$ is p times differentiable with respect to x,

(5) $h'(x, a, b) > 0, \quad \forall x \in (a, b)$,

(6) $h'(x, a, b) \geq \delta_1(\rho_1) > 0, \quad \forall x \in (a + \rho_1, b - \rho_1)$ with $0 < \rho_1 < \frac{b-a}{2}$,

where p is a positive integer, $x \in R$, a and b are constants such that $a < 0 < b$, $h'(x, a, b) = \frac{\partial h'(x,a,b)}{\partial x}$, and $\delta_1(\rho_1)$ is a positive constant depending on the positive constant ρ_1. Moreover, if the function $h(x, a, b)$ is infinite times differentiable with respect to x, then it is said to be a smooth step function.

Lemma 6.6 ([14]) *Let the scalar function $h(x, a, b)$ be defined as*

$$h(x, a, b) = \frac{\int_{\mathrm{a}}^{x} f(\tau - a) f(b - \tau) d\tau}{\int_{\mathrm{a}}^{b} f(\tau - a) f(b - \tau) d\tau} \tag{6.64}$$

where a and b are constants such that $a < 0 < b$, and the function $f(y)$ is defined below:

$$\begin{aligned} f(y) &= 0, \quad && if \ \ y \leq 0, \\ f(y) &= g(y), && if \ \ y > 0, \end{aligned} \tag{6.65}$$

where $g(y)$ is a single-valued function satisfying the following properties,

(a) $g(\tau - a) f(b-\tau) > 0, \quad \forall \tau \in (a, b)$,

(b) $g(\tau - a) f(b-\tau) \geq \delta_2(\rho_2) > 0, \quad \forall \tau \in (a+\rho_2, b-\rho_2)$, *with* $0 < \rho_2 < \frac{b-a}{2}$,

(c) $g(y)$ is p times differentiable with respect to y, and $\lim_{y \to 0^+} \frac{\partial^k g(y)}{\partial y^k} = 0, \quad k = 1, 2, \ldots, p-1$, *with p being a positive integer, and $\delta_2(\rho_2)$ is a positive constant depending on the positive constant ρ_2. Then, the function $h(x, a, b)$ is a p-times differentiable step function. Furthermore, if $g(y)$ in (6.65) is replaced by $g(y)$=$\mathrm{e}^{-1/y}$, then property (4) in Definition 6.2 is replaced by* $(4)'$*; i.e., $h(x, a, b)$ is a smooth step function.*

Definition 6.3 ([2]) A function $\Psi(x, a, b)$ is said to be a p-times differentiable unbounded function if it holds the following properties.

(1) $x = 0 \Leftrightarrow \Psi(x, a, b) = 0$,

(2) $\lim_{x \to a^-} \Psi(x, a, b) = -\infty$, $\lim_{x \to b^+} \Psi(x, a, b) = \infty$,

(3) $\Psi(x, a, b)$ is p times differentiable with respect to x, for all $x \in (a, b)$,

(4) $\Psi'(x, a, b) > 0, \quad \forall x \in (a, b)$,

(5) $\Psi'(x, a, b) \geq \delta_3(\rho_3) > 0, \quad \forall x \in (a+\rho_3, b-\rho_3)$, with $0 < \rho_3 < \frac{b-a}{2}$,

where p is a positive integer, a and b are constants such that $a < 0 < b$, $\Psi'(x, a, b) = \frac{\partial \Psi(x,a,b)}{\partial x}$, and $\delta_3(\rho_3)$ is a positive constant depending on the positive constant ρ_3. Furthermore, if $p = \infty$, then the function $\Psi(x, a, b)$ is said to be a smooth unbounded function.

Lemma 6.7 ([2]) *Let the scalar function $\Psi(x, a, b)$ be defined as*

$$\Psi(x, a, b) = \bar{\Psi}(\varphi(x, a, b)) - \bar{\Psi}(\varphi(0, a, b)), \tag{6.66}$$

where the function $\varphi(x, a, b)$ is defined as follows.

$$\varphi(x, a, b) = \varepsilon(2h(x, a, b) - 1) \tag{6.67}$$

with ε being a positive constant, and $h(x, a, b)$ being the p-times differentiable step functions in Definition 6.2. The function $\bar{\Psi}(\xi)$ is such that

(1) $\xi = 0 \Leftrightarrow \bar{\Psi}(\xi) = 0$,

(2) $\lim_{\xi \to -\varepsilon^-} \bar{\Psi}(\xi) = -\infty$, $\lim_{\xi \to \varepsilon^+} \bar{\Psi}(\xi) = \infty$,

(3) $\bar{\Psi}(\xi)$ is p times differentiable with respect to ξ, for all $\xi \in (-\varepsilon, \varepsilon)$,

(4) $\bar{\Psi}'(\xi) > 0, \quad \forall \xi \in (-\varepsilon, \varepsilon)$,

(5) $\bar{\Psi}'(\xi) > \delta_4(\rho_4) > 0, \quad \forall \xi \in (a+\rho_4, b-\rho_4)$, *with* $0 < \rho_4 < \frac{b-a}{2}$,

where $\bar{\Psi}'(\xi) = \frac{\partial \bar{\Psi}(\xi)}{\partial \xi} > 0, \forall \xi \in (-\varepsilon, \varepsilon)$, *and $\delta_4(\rho_4)$ is a positive constant depending on the positive constant ρ_4. Then the function $\Psi(x, a, b)$ is a p-times differentiable unbounded function. Moreover, if $h(x, a, b)$ is a smooth step function, then the function $\Psi(x, a, b)$ is a smooth unbounded function.*

Remark 6.2 For Lemma 6.6, it can be seen that several functions satisfy properties (a)–(c) of the function $g(y)$, such as $g(y)$=y^p, $g(y)$= $\tanh(\mathrm{y})^p$, $g(y)$= $\arctan(\mathrm{y}^p)$, etc.

Remark 6.3 In Definition 6.3, if $a = -b$, then many p-times differentiable unbounded functions can be obtained. An example is the function $\tan(-\frac{\pi}{2a}x)$. If $a \neq -b$, it is difficult to give a p-times differentiable unbounded function. However, we can construct a p-times differentiable unbounded function by using the p-times differentiable step function in Definition 6.2 with Lemma 6.7. For example, $\Psi(x, a, b) = \tan[\frac{\pi}{2}(2h(x, a, b) - 1)] - \tan[\frac{\pi}{2}(2h(0, a, b) - 1)]$.

Lemma 6.8 ([18]) *For any positive real numbers c, d and any real-valued function $\rho(x, y) > 0$,*

$$|x|^a |y|^d \leq \frac{a}{a+d}\rho(x, y)|x|^{a+d} + \frac{d}{a+d}\rho^{-a/d}(x, y)|y|^{a+d}. \tag{6.68}$$

Lemma 6.9 ([24]) (Barbalat's Lemma) *Consider a differentiable function $h(t)$. If $\lim_{t\to\infty} h(t)$ is finite and $\dot{h}(t)$ is uniformly continuous, then $\lim_{t\to\infty} \dot{h}(t) = 0$.*

6.3.2 Main Results

In what follows, a systematic design procedure for output-constrained tracking control of the system (6.59) is presented by using the CLF approach and the p-times differentiable unbounded functions in Definition 6.3.

First, give a coordinate transformation:

$$z_1 = \Psi(x_{1d}, a, b), \tag{6.69}$$

where $x_{1d} = x_1 - y_c = y - y_c$ is the output tracking error, $\Psi(x_{1d}, a, b)$ is a p-times differentiable unbounded function with $p \geq n - 1$, and the constants a and b are chosen such that

$$-\underline{L} \leq a < -\underline{L}_1, \overline{L}_1 < b \leq \bar{L}. \tag{6.70}$$

On the basis of the properties of $\Psi(x_{1d}, a, b)$ presented in Definition 6.3, it is clear that if we design a control input u ensuring $\lim_{t\to\infty} z_1(t) = 0$ and keeping all signals of the corresponding closed-loop system bounded for a bounded $z_1(t_0)$, then the output-constrained tracking control problem of system (6.59) is solved. Note that $z_1(t_0)$ is bounded under the constants a and b in (6.70), the assumption on the initial output tracking error in (6.63), and the properties of the function $\Psi(x_{1d}, a, b)$ listed in Definition 6.3.

Differentiating both sides of (6.69) in conjunction with system (6.59), we can rewrite them as

$$\begin{aligned}
\dot{z}_1 &= \Psi'(x_{1d}, a, b)g_1^{\sigma(t)}(x_1)x_2, \\
\dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i, d(t)) + g_i^{\sigma(t)}(\bar{x}_i)x_{i+1}, i = 2, 3, \ldots, n-1,
\end{aligned}$$

$$\dot{x}_n = f_n^{\sigma(t)}(\bar{x}_n, d(t)) + g_n^{\sigma(t)}(\bar{x}_n)u,$$
$$y = x_1, \tag{6.71}$$

Next, the steps of designing controllers are given below.

Step 1. Choose $V_1(z_1) = \frac{1}{2}z_1^2$ and let $z_2 = x_2 - \phi_1(z_1)$, where $\phi_1(z_1)$ is the common stabilizing function to be designed.

The derivative of $V_1(z_1)$ is

$$\dot{V}_1(z_1) = z_1\Psi'(x_{1d}, a, b)g_1^k(x_1)(z_2 + \phi_1(z_1)). \tag{6.72}$$

Choose the common stabilizing function as

$$\phi_1(z_1) = z_1\left[-\frac{1}{\underline{g}}(((n-2)/\Psi'(x_{1d}, a, b) + 1)\overline{g} + n/\Psi'(x_{1d}, a, b))\right]. \tag{6.73}$$

Substituting (6.73) into (6.72) yields that

$$\begin{aligned}\dot{V}_1(z_1) =& -z_1^2\Psi'(x_{1d}, a, b)\frac{g_1^k(x_1)}{\underline{g}}\frac{n}{\Psi'(x_{1d}, a, b)} \\ & -z_1^2\Psi'(x_{1d}, a, b)\frac{g_1^k(x_1)}{\underline{g}}\left(\frac{(n-2)}{\Psi'(x_{1d}, a, b)} + 1\right)\overline{g} \\ & + \Psi'(x_{1d}, a, b)g_1^k(x_1)z_1z_2 \\ \leq & -nz_1^2 - (\Psi'(x_{1d}, a, b) + n - 2)\overline{g}z_1^2 \\ & + \Psi'(x_{1d}, a, b)g_1^k(x_1)z_1z_2,\end{aligned} \tag{6.74}$$

where the coupling term $\Psi'(x_{1d}, a, b)g_1^k(x_1)z_1z_2, \forall k \in I_m$ can be canceled by following the steps below.

Step 2. Let $z_3 = x_3 - \phi_2(\bar{z}_2)$, where $\phi_2(\bar{z}_2)$ is the common stabilizing function to be designed.

Choose $\overline{V}_2(\bar{z}_2) = V_1(z_1) + \frac{1}{2}z_2^2$, and then the time derivative of $\overline{V}_2(\bar{z}_2)$ can be given by

$$\begin{aligned}\dot{\overline{V}}_2(\bar{z}_2) =& -nz_1^2 - (\Psi'(x_{1d}, a, b) + n - 2)\bar{g}z_1^2 \\ & + z_2\left(f_2^k(\bar{x}_2, d(t)) - \frac{\partial\phi_1(z_1)}{\partial z_1}\dot{z}_1 + g_2^k(\bar{x}_2)x_3\right) \\ & + \Psi'(x_{1d}, a, b)g_1^k(x_1)z_1z_2 \\ \leq & -nz_1^2 - (\Psi'(x_{1d}, a, b) + n - 2)\bar{g}z_1^2 \\ & + \bar{g}\Psi'(x_{1d}, a, b)|z_1z_2| + \left|z_2\Phi_2^k(\bar{z}_2, d(t))\right| \\ & + g_2^k(\bar{x}_2)(z_3 + \phi_2(\bar{z}_2)),\end{aligned} \tag{6.75}$$

where $\Phi_2^k(\bar{z}_2, d(t)) = f_2^k(\bar{x}_2, d(t)) - \frac{\partial \phi_1(z_1)}{\partial z_1}\Gamma_1^k(\bar{z}_2)$, $\Gamma_1^k(\bar{z}_2) = g_1^k(x_1)(z_2 + \phi_1(z_1))$, $\forall k \in I_m$.

Furthermore, one has $\left|f_2^k(\bar{x}_2, d(t))\right| \leq |x_2|\,\mu_2^k(\bar{x}_2) \leq (|z_1| + |z_2|)\hat{\mu}_2^k(\bar{z}_2)$, $\forall k \in I_m$, where $\hat{\mu}_2^k(\bar{z}_2)$ are a set of smooth non-negative functions. It means that

$$\left|\Phi_2^k(\bar{z}_2, d(t))\right| \leq (|z_1| + |z_2|)\widetilde{\mu}_2^k(\bar{z}_2), \tag{6.76}$$

where $\widetilde{\mu}_2^k(\bar{z}_2)$ are a set of smooth non-negative functions, $\forall k \in I_m$.

According to Lemma 6.8 and (6.76), it holds that $|z_1 z_2| \leq z_1^2 + z_2^2\tilde{\varphi}_2(\bar{z}_2)$, $|z_2\Phi_2^k(\bar{z}_2, d(t))| \leq z_1^2 + z_2^2\tilde{\varphi}_2^k(\bar{z}_2)$, $\forall k \in I_m$, where $\tilde{\varphi}_2(\bar{z}_2) \geq 1$, $\tilde{\varphi}_2^k(\bar{z}_2) \geq 1$ are some smooth functions. Thus, we get that

$$\begin{aligned}
\dot{V}_2(\bar{z}_2) = & -nz_1^2 - (n-2)\bar{g}z_1^2 - \bar{g}\Psi'(x_{1d}, a, b)z_1^2 + z_1^2 \\
& + \bar{g}\Psi'(x_{1d}, a, b)z_1^2 + \bar{g}\Psi'(x_{1d}, a, b)z_2^2\tilde{\varphi}_2(\bar{z}_2) \\
& + z_2^2\tilde{\varphi}_2^k(\bar{z}_2) + g_2^k(\bar{x}_2)z_2\phi_2(\bar{z}_2) + g_2^k(\bar{x}_2)z_2z_3 \\
\leq & -(n-1)z_1^2 - (n-2)\bar{g}z_1^2 + z_2^2\varphi_2^k(\bar{z}_2) \\
& + \bar{g}\Psi'(x_{1d}, a, b)z_1^2 + g_2^k(\bar{x}_2)z_2\phi_2(\bar{z}_2) \\
& + g_2^k(\bar{x}_2)z_2z_3 \\
\leq & -(n-1)z_1^2 - (n-2)\bar{g}z_1^2 + z_2^2\varphi_2^{\max}(\bar{z}_2) \\
& + g_2^k(\bar{x}_2)z_2\phi_2(\bar{z}_2) + g_2^k(\bar{x}_2)z_2z_3,
\end{aligned} \tag{6.77}$$

where $\varphi_2^{\max}(\bar{z}_2) \geq \varphi_2^k(\bar{z}_2) = \bar{g}\Psi'(x_{1d}, a, b)\tilde{\varphi}_2(\bar{z}_2) + \tilde{\varphi}_2^k(\bar{z}_2)$, $\forall k \in I_m$ is a smooth function.

Design the common stabilizing function as

$$\phi_2(\bar{z}_2) = z_2\left[-\frac{1}{\underline{g}}(\varphi_2^{\max}(\bar{z}_2) + (n-2)\bar{g} + (n-1))\right]. \tag{6.78}$$

Substituting (6.78) into (6.77) yields that

$$\begin{aligned}
\dot{V}_2(\bar{z}_2) \leq & -(n-1)z_1^2 - (n-2)\bar{g}z_1^2 + z_2^2\varphi_2^{\max}(\bar{z}_2) \\
& - \frac{g_2^k(\bar{x}_2)}{\underline{g}}z_2^2\varphi_2^{\max}(\bar{z}_2) - \frac{g_2^k(\bar{x}_2)}{\underline{g}}(n-2)\bar{g}z_2^2 \\
& - \frac{g_2^k(\bar{x}_2)}{\underline{g}}(n-1)z_2^2 + g_2^k(\bar{x}_2)z_2z_3 \\
\leq & -(n-1)(z_1^2 + z_2^2) - (n-2)\bar{g}(z_1^2 + z_2^2) \\
& + g_2^k(\bar{x}_2)z_2z_3,
\end{aligned} \tag{6.79}$$

where the coupling term $g_2^k(\bar{x}_2)z_2z_3$ can be canceled by the following steps.

Step i. Let $z_{i+1} = x_{i+1} - \phi_i(\bar{z}_i)$, where $\phi_i(\bar{z}_i)$ is a common stabilizing function to be designed.

Assume that the first $i-1$ $(2 \le i \le n)$ steps are finished, that is, for the following auxiliary $(z_1, \ldots, z_{i-1})$-equations:

$$\dot{z}_j = \Phi_j^k(\bar{z}_j, d(t)) + g_j^k\left(\bar{x}_j\right)x_{j+1},\ j = 1, \ldots, i-1, \tag{6.80}$$

where $\Phi_j^k(\bar{z}_j, d(t)) = f_j^k(\bar{z}_j, d(t)) - \sum_{l=1}^{j-1} \frac{\partial \phi_{j-1}(\bar{z}_{j-1})}{\partial z_l}\Gamma_l^k(\bar{z}_{l-1})$, we have a set of common stabilizing functions (6.73), (6.78) and

$$\phi_j(\bar{z}_j) = z_j\left[-\frac{1}{\underline{g}}(\varphi_j^{\max}\left(\bar{z}_j\right) + (n-j)\bar{g} + (n-j+1))\right], \tag{6.81}$$

where $3 \le j \le i-1$, such that there exists a CLF for system (6.80),

$$\overline{V}_{i-1}(\bar{z}_{i-1}) = V_1(z_1) + \frac{1}{2}\sum_{l=2}^{i-1} z_l^2, \tag{6.82}$$

and the time derivative of $\overline{V}_{i-1}(\bar{z}_{i-1})$ fulfills $\dot{\overline{V}}_{i-1}(\bar{z}_{i-1}) \le -(n-i+2)(z_1^2 + \cdots + z_{i-1}^2) - (n-i+1)\bar{g}(z_1^2 + \cdots + z_{i-1}^2) + g_{i-1}^k(\bar{x}_{i-1})\, z_{i-1}z_i$.

Choosing $\overline{V}_i(\bar{z}_i) = \overline{V}_{i-1}(\bar{z}_{i-1}) + \frac{1}{2}z_i^2$, then we can derive that

$$\begin{aligned}\dot{\overline{V}}_i(\bar{z}_i) \le\ & -(n-i+2)(z_1^2 + \cdots + z_{i-1}^2)\\ & -(n-i+1)\bar{g}(z_1^2 + \cdots + z_{i-1}^2)\\ & +z_i(\Phi_i^k(\bar{z}_i, d(t)) + g_i^k(\bar{x}_i)\, x_{i+1})\\ & +g_{i-1}^k(\bar{z}_{i-1})\, z_{i-1}z_i\\ \le\ & -(n-i+2)(z_1^2 + \cdots + z_{i-1}^2)\\ & -(n-i+1)\bar{g}(z_1^2 + \cdots + z_{i-1}^2)\\ & +\bar{g}\left|z_{i-1}z_i\right| + \left|z_i(\Phi_i^k(\bar{z}_i, d(t))\right|\\ & +g_i^k(\bar{z}_i)\, z_i\phi_i(\bar{z}_i) + g_i^k(\bar{z}_i)\, z_i z_{i+1},\end{aligned} \tag{6.83}$$

where $\Phi_i^k(\bar{z}_i, d(t)) = f_i^k(\bar{z}_i, d(t)) - \sum_{l=1}^{i-1} \frac{\partial \phi_{l-1}(\bar{z}_{l-1})}{\partial z_i}\Gamma_l^k(\bar{z}_{l-1}),\ \forall k \in I_m$.

Similar to Step 2, one has $\left|z_{i-1}z_i\right| \le z_1^2 + \cdots + z_{i-1}^2 + z_i^2\tilde{\varphi}_i(\bar{z}_i)$, $\left|z_i\Phi_i^k(\bar{z}_i, d(t))\right| \le z_1^2 + \cdots + z_{i-1}^2 + z_i^k\tilde{\varphi}_i^k(\bar{z}_i), \forall k \in I_m$, where $\tilde{\varphi}_i(\bar{z}_i) \ge 1, \tilde{\varphi}_i^k(\bar{z}_i) \ge 1$ are some smooth functions. Therefore, we can arrive at

$$\begin{aligned}\dot{\overline{V}}_i(\bar{z}_i) &\le -(n-i+2)(z_1^2+\cdots+z_{i-1}^2)\\ &\quad -(n-i+1)\bar{g}(z_1^2+\cdots+z_{i-1}^2)\\ &\quad +\bar{g}(z_1^2+\cdots+z_{i-1}^2)+z_1^2+\cdots+z_{i-1}^2\\ &\quad +z_i^2\tilde{\varphi}_i^k(\bar{z}_i)+\bar{g}z_i^2\tilde{\varphi}_i(\bar{z}_i)+g_i^k(\bar{x}_i)z_i\phi_i(\bar{z}_i)\\ &\quad +g_i^k(\bar{x}_i)z_iz_{i+1}\\ &\le -(n-i+1)(z_1^2+\cdots+z_{i-1}^2)\\ &\quad -(n-i)\bar{g}(z_1^2+\cdots+z_{i-1}^2)\\ &\quad +z_i^2\varphi_i^{\max}(\bar{z}_i)+g_i^k(\bar{x}_i)z_i\phi_i(\bar{z}_i)\\ &\quad +g_i^k(\bar{x}_i)z_iz_{i+1},\end{aligned} \tag{6.84}$$

where $\varphi_i^{\max}(\bar{z}_i) \ge \varphi_i^k(\bar{z}_i) = \bar{g}\tilde{\varphi}_i(\bar{z}_i) + \tilde{\varphi}_i^k(\bar{z}_i), \forall k \in I_m$ are some smooth functions.

Design the common stabilizing function as

$$\phi_i(\bar{z}_i) = z_i\left[-\frac{1}{\underline{g}}(\varphi_i^{\max}(\bar{z}_i) + (n-i)\bar{g} + (n-i+1))\right]. \tag{6.85}$$

Then, substituting (6.85) into (6.84) yields that

$$\begin{aligned}\dot{\overline{V}}_i(\bar{z}_i) &\le -(n-i+1)(z_1^2+\cdots+z_{i-1}^2)\\ &\quad -(n-i)\bar{g}(z_1^2+\cdots+z_{i-1}^2)+z_i^2\varphi_i^{\max}(\bar{z}_i)\\ &\quad -\frac{g_i^k(\bar{x}_i)}{\underline{g}}z_i^2\varphi_i^{\max}(\bar{z}_i)-\frac{g_i^k(\bar{x}_i)}{\underline{g}}(n-i)\bar{g}z_i^2\\ &\quad -\frac{g_i^k(\bar{x}_i)}{\underline{g}}(n-i+1)z_i^2+g_i^k(\bar{x}_i)z_iz_{i+1}\\ &\le -(n-i+1)(z_1^2+\cdots+z_i^2)\\ &\quad -(n-i)\bar{g}(z_1^2+\cdots+z_i^2)+g_i^k(\bar{x}_i)z_iz_{i+1},\end{aligned} \tag{6.86}$$

where the coupling term $g_i^k(\bar{x}_i)z_iz_{i+1}$ can be canceled by the following steps.

Step n. Repeating the procedures above, it is straightforward to see that there exists a CLF of system (6.59)

$$\overline{V}_n(\bar{z}_n) = V_1(z_1) + \frac{1}{2}\sum_{l=2}^{n} z_l^2. \tag{6.87}$$

Then, we can explicitly design an individual controller for each subsystem

$$u_k(\bar{z}_n) = z_n\left[-\frac{1}{g_{n,k}}(\varphi_{n,k}(\bar{z}_n)+1)\right], \forall k \in I_m \tag{6.88}$$

such that

$$\dot{\overline{V}}_n(\bar{z}_n) \leq -(z_1^2 + \cdots + z_n^2). \tag{6.89}$$

Remark 6.4 In fact, we can also design a common state feedback controller for the system (6.59) as

$$u(\bar{z}_n) = z_n \left[-\frac{1}{\underline{g}}(\varphi_n^{max}(\bar{z}_n) + 1) \right], \tag{6.90}$$

where $\varphi_n^{max}(\bar{z}_n) \geq \varphi_n^k(\bar{z}_n) = \bar{g}\tilde{\varphi}_n(\bar{z}_n) + \tilde{\varphi}_n^k(\bar{z}_n)$ is a smooth function. It can be seen that (6.90) can be extended from (6.88), which illustrates the less conservativeness of the controller to be developed.

Based on the above discussions, we now provide the main result.

Theorem 6.3 *Suppose that Assumption 6.5 holds. The output-constrained tracking controller for system (6.59) under arbitrary switching can be designed as (6.88), and the output tracking error $x_{1d}(t)$ locally exponentially converges to zero.*

Proof (i) Forward completeness. From (6.89) and $\Psi'(x_{1d}, a, b) > 0$ for all $x_{1d}(t) \in (a, b)$, noticing Property (6.58) of the function $\Psi(x_{1d}, a, b)$ in Definition 6.3, one obtains that

$$\dot{\overline{V}}_n \leq 0 \Rightarrow \overline{V}_n(t) \leq \overline{V}(t_0), \forall t \geq t_0 \geq 0. \tag{6.91}$$

This means that

$$\sum_{i=1}^{n} z_i(t) \leq \sum_{i=1}^{n} z_i(t_0) \tag{6.92}$$

for all $t \geq t_0 \geq 0$. Under the initial condition specified in (6.61), and the choice of the constants a and b in (6.70), the right-hand side of (6.92) is bounded. This means that the left-hand side of (6.92) must be bounded. Boundedness of the left-hand side of (6.92) implies that all z_i, $i = 1, 2, \ldots, n$ are bounded. Because $|z_1(t)|$ is bounded for all $t \geq t_0 \geq 0$, the output tracking error $x_{1d}(t)$ never reaches its boundary values a and b; i.e., $x_{1d}(t) \in (a, b)$ for all $t \geq t_0 \geq 0$. This together with (6.70), $\underline{L}_1 < \underline{L}$ and $\overline{L}_1 < \overline{L}$ (Assumption 6.2) implies that $x_{1d}(t)$ is always in its constraint range, i.e. $\underline{L} < x_{1d}(t) < \overline{L}$ for all $t \geq t_0 \geq 0$. Boundedness of all x_i, $i = 1, 2, \ldots, n$ follows from the boundedness of all z_i, and smooth property of all functions $f_i^k(\bar{x}_i, d(t))$, $g_i^k(\bar{x}_i)$ and $\Psi(x_{1d}, a, b)$. Boundedness of all x_i, $i = 1, 2, \ldots, n$ also denotes that the closed-loop system (6.55) is forward complete.

(ii) Asymptotic convergence. Noting that $x_i(t)$, $z_i(t)$, $i = 1, 2, \ldots, n$ are bounded, it is not hard to deduce that $\ddot{\overline{V}}_n(\bar{z}_n)$ is bounded, which gives that $\dot{\overline{V}}_n(\bar{z}_n)$ is uniformly continuous. Then, we get from Lemma 6.9 that $\lim_{t\to\infty} z_i(t) = 0$, $i =$

$1, 2, \ldots, n$. Therefore, it follows from Property (6.55) of function $\Psi(x_{1d}, a, b)$. that $\lim_{t\to\infty} x_{1d}(t) = 0$.

(iii) Local exponential convergence of the output tracking error $x_{1d}(t)$. It follows from (6.89) that

$$V_n(t) \le V_n(t_0)e^{-(t-t_0)}, \forall t \ge t_0. \tag{6.93}$$

One can get from (6.93) that

$$|z_1(t)| \le \sqrt{2V_n(t_0)}e^{-\frac{1}{2}(t-t_0)}, \forall t \ge t_0, \tag{6.94}$$

which implies that $z_1(t)$ locally exponentially converges to 0. Now, with the help of Taylor expansion of function $\Psi(x_{1d}, a, b)$ around $x_{1d} = 0$ and noticing Property (6.59) of the function $h(x_{1d}, a, b)$, Property (6.60) of the function $\Psi(x_{1d}, a, b)$, and the construction of the function $\Psi(x_{1d}, a, b)$ (see Lemma 6.7), it can be shown that there exists a strictly positive constant $\delta_5(\rho_5)$ depending on the positive constant ρ_5 with $0 < \rho_5 < \frac{b-a}{2}$ such that

$$|\Psi(x_{1d}(t), a, b)| \ge \delta_5(\rho_5)\,|x_{1d}(t)|\,, \forall t \ge t_1, \tag{6.95}$$

where the time instance $t_1 > t_0$. By definition $z_1(t) = \Psi(x_{1d}(t), a, b)$, a combination of (6.93) and (6.95) gives

$$|x_{1d}(t)| \le \frac{\sqrt{2V_n(t_0)}e^{-\frac{1}{2}(t-t_0)}}{\delta_5(\rho_5)}, \forall t \ge t_1, \tag{6.96}$$

which shows the local exponential convergence of $x_{1d}(t)$ to 0. □

Remark 6.5 When the output-constrained tracking control problem is considered, it is required that $|x_1|$ be absent in Assumption 6.5, and thus $f_1^k(x_1)$ cannot appear in x_1-equation of system (6.59), $k = 1, 2, \ldots, m$. It seems that Assumption 6.1 appears to be conservative. However, if the stabilization problem is considered, $|x_1|$ can be presented in Assumption 6.5, which leads to: $f_1^k(x_1)$ exists in the x_1-equation of (1), $k = 1, 2, \ldots, m$. We give the design procedures for the stabilization problem in the next section.

In what follows, we consider the robust state-constrained stabilization problem for the following uncertain switched nonlinear system,

$$\begin{aligned} \dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i, d(t)) + g_i^{\sigma(t)}(\bar{x}_i)\,x_{i+1}, i = 1, 2, \ldots, n-1, \\ \dot{x}_n &= f_n^{\sigma(t)}(\bar{x}_n, d(t)) + g_n^{\sigma(t)}(\bar{x}_n)\,u_{\sigma(t)}, \end{aligned} \tag{6.97}$$

where all functions are smooth with $f_{i,k}(0, d(t)) = 0$ for all $d(t) \in \Omega$ and $0 < \underline{g} < g_{i,k}(\bar{x}_i) \le \bar{g}$, $\underline{g}$, $\bar{g}$ are positive constants, respectively, $i = 1, 2, \ldots, n$, $\forall k \in I_m$.

The robust state-constrained stabilization problem: For system (6.97) under arbitrary switching, design state feedback controllers for all subsystems to ensure that:

(1) System (6.97) is asymptotically stabilizable.

(2) The state x_1 is within a pre-specified limit range; i.e.,

$$-\underline{L} \leq x_1 \leq \overline{L} \tag{6.98}$$

for all $t \geq t_0 \geq 0$, where $\underline{L}$ and $\overline{L}$ are strictly positive constants.

(3) All signals of the closed-loop system (6.97) are bounded.

In addition, it is assumed that the following conditions hold.

Assumption 6.7 For $i = 1, 2, \ldots, n$,

$$\left| f_i^k(\bar{x}_i, d(t)) \right| \leq (|x_1| + |x_2| + \cdots + |x_i|)\mu_i^k(\bar{x}_i), \forall k \in I_m, \tag{6.99}$$

where $\mu_i^k(\bar{x}_i)$ are a set of known non-negative smooth functions.

Assumption 6.8 The p-times differentiable unbounded function in Definition 6.2 satisfies

$$\Psi(x, a, b) = x[1 + \chi(x)], \tag{6.100}$$

where $\chi(x)$ is a non-negative smooth function.

Similar to Theorem 6.3, we apply a coordinate transformation:

$$z_1 = \Psi(x_1, a, b), \tag{6.101}$$

where $\Psi(x_1, a, b)$ is a p-times differentiable unbounded function with $p \geq n - 1$.

Differentiating both sides of (6.101) in conjunction with system (6.97), one can rewrite them in the form:

$$\begin{aligned}
\dot{z}_1 &= \Psi'(x_1, a, b)(f_1^{\sigma(t)}(x_1, d(t)) + g_1^{\sigma(t)}(x_1)x_2), \\
\dot{x}_i &= f_i^{\sigma(t)}(\bar{x}_i, d(t)) + g_i^{\sigma(t)}(\bar{x}_i)x_{i+1}, i = 2, 3, \ldots, n-1, \\
\dot{x}_n &= f_n^{\sigma(t)}(\bar{x}_n, d(t)) + g_n^{\sigma(t)}(\bar{x}_n)u_{\sigma(t)},
\end{aligned} \tag{6.102}$$

Step 1. Choose $V_1(z_1) = \frac{1}{2}z_1^2$ and let $z_2 = x_2 - \phi_1(z_1)$, where $\phi_1(z_1)$ is the common stabilizing function to be designed.

The derivative of $V_1(z_1)$ is given by

$$\begin{aligned}
\dot{V}_1(z_1) &= \Psi'(x_1, a, b)z_1[f_1^k(x_1, d(t)) + g_1^k(x_1)(z_2 + \phi_1(z_1))] \\
&= \Psi'(x_1, a, b)z_1 f_1^k(x_1, d(t)) \\
&\quad + \Psi'(x_{1d}, a, b)z_1 g_1^k(x_1)(z_2 + \phi_1(z_1)) \\
&\leq \Psi'(x_1, a, b)|z_1 \Phi_1^k(x_1)| \\
&\quad + \Psi'(x_{1d}, a, b)z_1 g_1^k(x_1)(z_2 + \phi_1(z_1)),
\end{aligned} \tag{6.103}$$

where $\Phi_1^k(x_1) = f_1^k(x_1, d(t)), \forall k \in I_m$.

Under Assumptions 6.5 and 6.8, one can find that

$$\left| f_1^k\left(x_1, d(t)\right)\right| \le |x_1|\mu_1^k\left(x_1\right) \le |z_1|\hat{\mu}_1^k\left(z_1\right), \forall k \in I_m, \tag{6.104}$$

where $\hat{\mu}_1^k\left(z_1\right)$ are a set of smooth non-negative functions.

Then, we can get that

$$\left| z_1 \Phi_1^k\left(z_1\right)\right| \le z_1^2 \tilde{\varphi}_1^k\left(z_1\right), \forall k \in I_m. \tag{6.105}$$

where $\tilde{\varphi}_1^k\left(z_1\right) \ge 1, \forall k \in I_m$ is a smooth function.

Then, one can see that

$$\begin{aligned}\dot{V}_1\left(z_1\right) &\le z_1^2 \Psi'(x_1, a, b)\tilde{\varphi}_1^k\left(z_1\right) \\ &\quad + z_1 \Psi'(x_1, a, b) g_1^k\left(x_1\right)\left(z_2 + \phi_1(z_1)\right) \\ &\le z_1^2 \Psi'(x_1, a, b)\tilde{\varphi}_1^{max}\left(z_1\right) \\ &\quad + z_1 \Psi'(x_1, a, b) g_1^k\left(x_1\right)\left(z_2 + \phi_1(z_1)\right),\end{aligned} \tag{6.106}$$

where $\tilde{\varphi}_1^{max}\left(z_1\right) \ge \tilde{\varphi}_1^k\left(z_1\right) \ge 1, \forall k \in I_m$ is a smooth function.

The common stabilizing function is designed as

$$\begin{aligned}\phi_1(z_1) = z_1 \bigg[&-\frac{1}{\underline{g}}\left(\tilde{\varphi}_1^{max}\left(z_1\right) + ((n-2)/\Psi'(x_1, a, b) + 1)\overline{g}\right. \\ &\left. + n/\Psi'(x_1, a, b)\right) \bigg].\end{aligned} \tag{6.107}$$

Substituting (6.107) into (6.106), one can get that

$$\begin{aligned}\dot{V}_1\left(z_1\right) = \; & z_1^2 \Psi'(x_1, a, b)\tilde{\varphi}_1^{max}\left(z_1\right) \\ & - z_1^2 \Psi'(x_1, a, b)\frac{g_1^k\left(x_1\right)}{\underline{g}}\tilde{\varphi}_1^{max}\left(z_1\right) \\ & - z_1^2 \Psi'(x_1, a, b)\frac{g_1^k\left(x_1\right)}{\underline{g}}\frac{n}{\Psi'(x_1, a, b)} \\ & - z_1^2 \Psi'(x_1, a, b)\frac{g_1^k\left(x_1\right)}{\underline{g}}\left(\frac{(n-2)}{\Psi'(x_1, a, b)} + 1\right)\overline{g} \\ & + \Psi'(x_1, a, b) g_1^k\left(x_1\right) z_1 z_2 \\ \le \; & -n z_1^2 - (\Psi'(x_1, a, b) + n - 2)\overline{g} z_1^2 \\ & + \Psi'(x_1, a, b) g_1^k\left(x_1\right) z_1 z_2,\end{aligned} \tag{6.108}$$

where the coupling term $\Psi'(x_1, a, b) g_1^k\left(x_1\right) z_1 z_2, \forall k \in I_m$ can be canceled by using the steps below.

Similar to the procedures in the above section, we design the individual controllers for the subsystems as

$$u_k(\bar{z}_n) = z_n \left[-\frac{1}{g_{n,k}} (\varphi_{n,k}(\bar{z}_n) + 1) \right], k \in I_m \tag{6.109}$$

such that

$$\dot{\bar{V}}_n(\bar{z}_n) \le -(z_1^2 + \cdots + z_n^2). \tag{6.110}$$

Now, we give the following result focusing on robust state-constrained stabilization problem of system (6.97).

Theorem 6.4 *Suppose that Assumptions 6.6–6.8 are satisfied; then the robust state-constrained stabilization problem of system (6.97) under arbitrary switchings can be solved by the controller in (6.109).*

Proof The proof is similar to the one of Theorem 6.3. □

6.3.3 Simulation Results

In this section, the following example is provided to illustrate the effectiveness of the proposed results.

Consider the switched nonlinear system:

$$\begin{cases} \dot{x}_1 = f_1^{\sigma(t)}(x_1) + x_2, \\ \dot{x}_2 = f_2^{\sigma(t)}(\bar{x}_2) + g^{\sigma(t)}(\bar{x}_2)\, u, \\ \ y \ = x_1, \quad \sigma(t) : [0, \infty) \to \{1, 2\}, \end{cases} \tag{6.111}$$

where $f_1^1(x_1) = 0$, $f_2^1(\bar{x}_2) = 3x_1^2 x_2^3$, $f_1^2(x_1) = 2x_1 - 0.4$, $f_2^2(\bar{x}_2) = x_1 x_2 \left(1 + x_1^2\right)$, $g^1(\bar{x}_2) = \left[- \sin^2\left(x_1^3 + 2x_2\right), 1.4 - \cos(x_1 x_2)\right]$, $g^2(\bar{x}_2) = \left[- 4x_1^4 x_2^2, 1.2\right]$. The objective is enable $y(t)$ to track the desired trajectory $y_d = 0.2$ subject to asymmetric time-varying output constraints $-(0.2 + 0.1\cos(t)) < y(t) < 0.7 + 0.1\cos(t)$.

According to Assumption 6.4, we choose $B_1(t) = 0.1 + 0.1\cos(t)$ and $B_2(t) = 0.3 + 0.1\cos(t)$. Based on (6.38), we can get an asymmetric barrier Lyapunov function:

$$V_1 = (1 - q(z_1)) \log \frac{0.09}{\left(0.09 - z_1^2\right)} + q(z_1) \log \frac{0.16}{\left(0.16 - z_1^2\right)}. \tag{6.112}$$

Defining $z_1 = x_1 - 0.2$, one can see that $\phi_1 = (1 - q(z_1))(-2z_1 - z_1(0.09 - z_1^2)) + q(z_1)(-2z_1 - z_1(0.16 - z_1^2))$ is a dominating feedback law for the auxiliary first-order subsystems. In that scenario

$$d_1^1 = (1 - q(z_1))(-z_1^2) + q(z_1)(-z_1^2),$$
$$d_1^2 = (1 - q(z_1))\left(-z_1^2 - \frac{2z_1^2}{\left(0.09 - z_1^2\right)}\right) + q(z_1)\left(-z_1^2 - \frac{2z_1^2}{\left(0.16 - z_1^2\right)}\right). \tag{6.113}$$

Define $z_2 = x_2 - \phi_1$. Then, $\overline{V}_2 = V_1 + \frac{1}{2}z_2^2$ is continuously differentiable and positive definite when $-0.3 < z_1(t) < 0.4$. Furthermore, $\overline{V}_2$ is a common Lyapunov function for the two subsystems in (6.111). For $k = 1, 2,$ let

$$a_k = (1 - q(z_1))(d_1^k + z_2\left(\frac{z_1}{0.09 - z_1^2} + f_2^k(\bar{x}_2) - \frac{\partial \phi_1}{\partial x_1}(x_2 + f_1^k(\bar{x}_1))\right)$$
$$+ q(z_1)(d_1^k + z_2\left(\frac{z_1}{0.16 - z_1^2} + f_2^k(\bar{x}_2) - \frac{\partial \phi_1}{\partial x_1}\left(x_2 + f_1^k(\bar{x}_1)\right)\right),$$
$$b_k = z_2 g^k(\bar{x}_2). \tag{6.114}$$

It is clear that $M = \{2\}$, $F = \{1\}$. From (6.49), we can obtain the controller:

$$u = [u_1, u_2]^T \tag{6.115}$$

with

$$u_1 = \begin{cases} \max_{i\in\{1,2\}}\left\{p_{i,1}\right\}, & if\ z_2 > 0 \\ \min_{i\in\{1,2\}}\left\{p_{i,1}\right\}, & if\ z_2 < 0 \\ z_2 = 0, \ \ if\ z_2 = 0 \end{cases} \tag{6.116}$$

and

$$u_2 = \begin{cases} \min_{i\in\{1,2\}}\left\{p_{i,2}\right\}, & if\ \ z_2 > 0 \\ \max_{i\in\{1,2\}}\left\{p_{i,2}\right\}, & if\ \ z_2 < 0 \\ z_2 = 0, \ \ \ if\ z_2 = 0 \end{cases} \tag{6.117}$$

where

$$p_k = \left[p_{k,1}, p_{k,2}\right] = \begin{cases} -b_k^T \frac{\max\left\{a_k + b_k b_k^T, 0\right\}}{b_k b_k^T}, & if\ \ z_2 \neq 0, \\ 0, \ \ if\ z_2 \neq 0. \end{cases} \tag{6.118}$$

Given $x_1(0) = -0.05$ and $x_2(0) = -2.2$, Fig. 6.5 shows that asymptotic output tracking performance is achieved and the output stays within the set $(-0.2 - 0.1cos(t), 0.7 + 0.1cos(t))$ when the Lyapunov function obtained in (6.112) is used. The switching signal for switched system (6.111) is shown in Fig. 6.6. Moreover, given different initial values of z_1, Fig. 6.7 indicates that the error z_1 converges to 0 while remaining in the set $(-0.4 - 0.1cos(t), 0.5 + 0.1cos(t)),\ \forall t \geq 0$. The phase portraits of z_1 and z_2 are depicted in Fig. 6.8, from which we can see that the error $z_1(t)$ does not transgress its barriers as long as its initial value satisfies $-0.3 < z_1(0) < 0.4$.

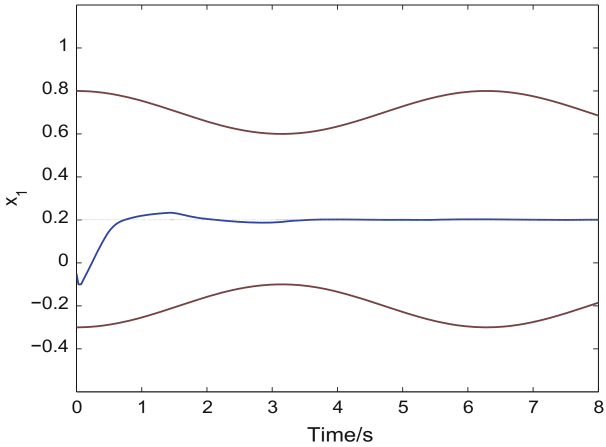

Fig. 6.5 Output tracking for the desired signal $y_d = 0.2$

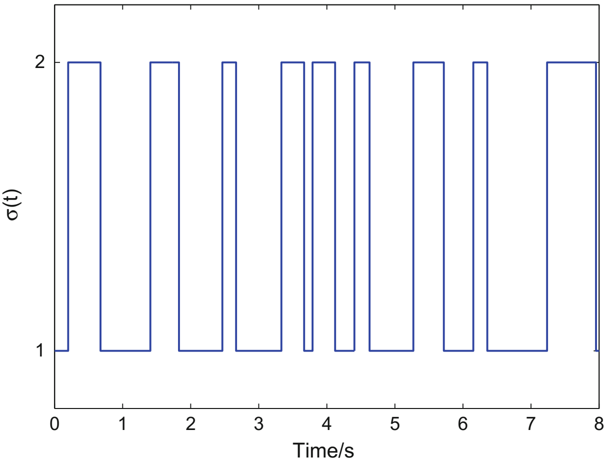

Fig. 6.6 The switching signals for the switched system (6.111)

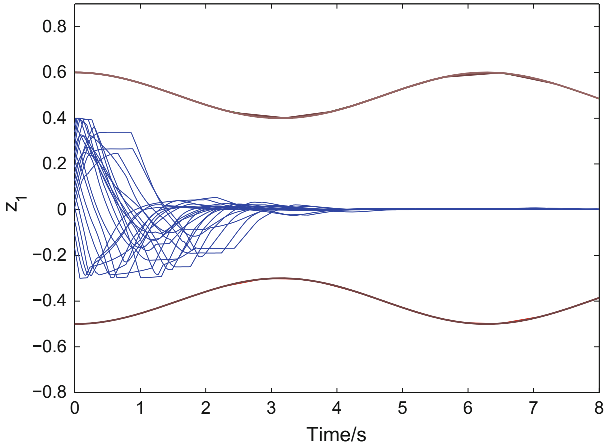

Fig. 6.7 Tracking error z_1 for various initial values satisfying $-0.3 < z_1(0) < 0.4$

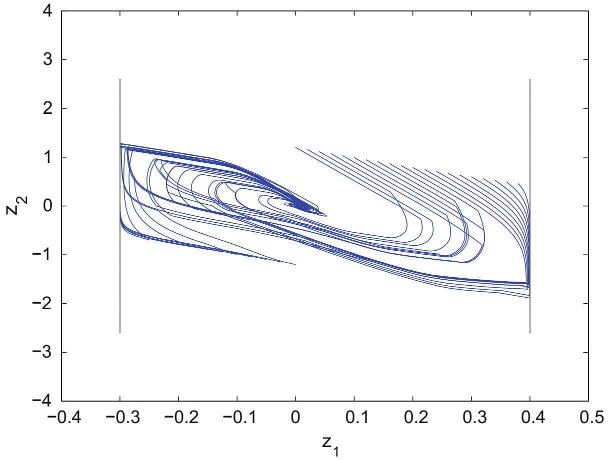

Fig. 6.8 Phase portraits of z_1, z_2 for the closed-loop system (6.111)

6.3.4 Conclusions

The problems of robust output-constrained tracking control and state-constrained stabilization for uncertain switched nonlinear systems in lower triangular form have been respectively studied. In the proposed approach, the p-times differentiable unbounded functions are introduced and incorporated in output tracking error transformations to convert the problem of controlling the switched systems with output tracking error constraints to a new problem of regulating the converted systems without a constraint. The backstepping technique is resorted to design controllers for the transformed systems.

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Chapter 7
Conclusions and Future Study Directions

Control synthesis of switched systems is always a hot study topic in the control field for its significance of both theory aspect and practical application. In the past few years, some control problems of switched systems have been successfully solved, but there still are quite many interesting topics deserving further investigation; some of them have been considered in the book. This book has presented some stabilization approaches for both switched linear systems and switched nonlinear systems, and the considered systems can be composed of unstable subsystems. The adaptive control design methods for some classes of switched nonlinear systems have also been developed. In addition, the book also probes the tracking control problem of switched constrained switched nonlinear systems. Most contents of the book are extracted from Refs. [1–9].

Finally, we conclude the paper by providing some future study directions:

(1) Investigations on obtaining tighter bounds on time-dependent switching signals for switched systems. The time-dependent switching stabilization for switched systems has been studied in the book. For time-dependent switching stabilization design of switched systems, obtaining a tighter bound on the switching signal will give the designer additional flexibility. Therefore, proposing a new switching signal design method to achieve stabilization with a tighter bound deserves further investigations which is practically important but theoretically challenging.

(2) Investigations on asymptotic tracking control of switched systems with unknown uncertainties. This book has investigated the tracking control problem for some classes of switched systems with unknown uncertainties. However, it is noted that the obtained results can only achieve bounded tracking performance. Therefore, how to further extend the results to achieve asymptotic tracking performance is not only theoretically important but of practical significance.

(3) Investigations on intelligent switching control. The switching signal adopted in this book is piecewise constant, and thus the designed controllers are suddenly switched at the switching moments. Such a hard switching mechanism may dete-

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riorate the system performance or even cause instability of the system. Therefore, proposing intelligent switching strategies has broad applications.

(4) Investigations on control of switched non-smooth systems. The dynamics of the subsystems considered in this book are assumed to be Lipschitz continuous or even smooth. However, there often exist many practical switched systems whose subsystem dynamics are not smooth. Some classical techniques developed for general switched systems will fail to be applied to switched non-smooth systems. It is reasonable to carry out studies on control synthesis of switched non-smooth systems.

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