# Geometric Scalar Theory of Gravity

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**Abstract** The present article introduces a new scalar theory of gravity based on the Einstein's assumption that gravitation is an expression of the geometrical structure of the spacetime. In the geometric scalar theory of gravity all kind of matter and energy interacts with the gravitational (scalar) field only through a metric structure that naturally arises with the non linear dynamics of the scalar field. This allows us to overcome the problems from the previous scalar theories and construct a new scalar theory for gravitation which is in accordance at least with the observational data coming from our solar system.

## 1 Introduction

Since its formulation until the present days, the Einstein's theory of general relativity (GR) remains consistent with all experimental tests performed, the so called classical tests of gravitation (Turyshev 2009). Notwithstanding, over all these years, there have always been open questions that led physicists to seek alternative paths in the description of gravitational phenomena. Alternative theories of gravitation exist in large numbers and in the most diverse formulations, whereas those following Einstein's ideas, choosing describe gravitation as a geometric phenomenon, are those that obtained greatest success. Inside this extensive group, scalar-tensor theories and $f(R)$ theories are the ones that most currently stand (Clifton 2006).

In the class of the purely scalar metric theories, i.e. where the gravitational field is represented by one or more scalar functions that generate a gravitational metric, much was done up to mid-seventies, but all formulations failed to comply with all classical tests. In 1972, Wei-Tou Ni wrote a compendium of metric theories containing a broad review and analysis of scalar theories (Ni 1972).

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**Table 1** Different proposals for scalar theories of gravitation according to Eqs. (1) and (2)

| Scalar theories of gravitation | |
|---|---|
| Author (year) | Basic functions |
| Nordström (1912) | $f = \Phi$ |
| | $k = 1$ |
| Nordström (1913–1914) | $f = -\ln \Phi$ |
| | $k = \Phi$ |
| Littlewood (1953) | $f = -2\ln(1-\Phi)$ |
| Bergmann (1956) | $k = 1$ |

Following Ni, these various proposals have the common property of being conformally flat. Its gravitational metrics have the general form,

$$g_{\mu\nu} = e^{-2f(\Phi)}\, \eta_{\mu\nu}\,, \tag{1}$$

where $\Phi$ is the gravitational potential and $\eta_{\mu\nu}$ is the Minkowski metric. The field equations of these theories can be summarized in the expression,

$$\Box\Phi \propto k(\Phi)\, T\,, \tag{2}$$

with the $\Box$ being the d'Alembertian operator constructed with the metric (22) and $T$ the trace of the energy-momentum tensor of the source of the gravitational field. The $f(\Phi)$ and $k(\Phi)$ functions have distinct forms according to the theory which one wants to describe. The table below shows the main scalar theories and its correspondent functions (Table 1).

The fact that all these theories are conformally flat is the main cause why one can not couple gravity and electromagnetism, since the Maxwell equations are conformally invariants. Also, with the source of the gravitational field being the trace of the energy-momentum tensor, which is zero for the electromagnetic field, shows that this fields can not produce gravitation. Thus none of the theories in the table above are in agreement with the measurement of the bending of light. Further, all these theories fail to provide the correct advance of the perihelion of Mercury. However, Ni's paper does not cite the theory proposed by Dowker in 1965, which although not predicting the bending of light, gives the right answer for the Mercury's perihelion precession (Dowker 1965).

Though, the role of the scalar field representing the gravitational potential is not fully determined, as I will show here. A recent study of effective metrics in non linear scalar theories shows that is possible to establish a metric structure, not conformally flat, which describe the dynamic of the field itself (Goulart et al. 2011). In the next section I show how this mathematical property emerges. The physical aspects of such property can only be determined if one introduces a way by which this metric will interact with the other fields of nature. In other words, in order to interpret the scalar field as the gravitational potential and the metric generated by it as the physical metric, one needs to say how matter/energy interacts with it. This will constitute the grounds of the geometric scalar gravity (GSG).

## 2 Geometrization of a Nonlinear Scalar Theory

Consider a relativistic scalar field $\Phi$ with a nonlinear dynamic in the Minkowski spacetime. The action describing its dynamic is given by,

$$S = \int L(\Phi, w)\sqrt{-\eta}\, d^4x\,, \tag{3}$$

where $\eta$ is the determinant of the Minkowski metric and,

$$w \equiv \eta^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi\,. \tag{4}$$

The notation $\partial_\mu$ indicates a simple derivative in relation with the coordinate $x^\mu$ . The minimal action principle returns the equation of motion of the scalar field,

$$\frac{1}{\sqrt{-\eta}}\,\partial_\mu\Big(\sqrt{-\eta}\,L_w\,\eta^{\mu\nu}\partial_\nu\Phi\Big) - \frac{1}{2}\,L_\Phi \,=\, 0\,, \tag{5}$$

where $L_X$ indicates a derivative in relation with the variable $X$ .

Introducing the metric tensor,

$$q^{\mu\nu} = \alpha\,\eta^{\mu\nu} + \frac{\beta}{w}\,\partial^\mu\Phi\,\partial^\nu\Phi\,, \tag{6}$$

with $\alpha$ and $\beta$ being functions of $\Phi$ and $w$, and the correspondent covariant expression, defined by $q^{\mu\alpha}\,q_{\alpha\nu} = \delta^\mu_\nu$ , given by

$$q_{\mu\nu} = \frac{1}{\alpha}\,\eta_{\mu\nu} - \frac{\beta}{\alpha\,(\alpha+\beta)\,w}\,\partial_\mu\Phi\,\partial_\nu\Phi\,, \tag{7}$$

Eq. (5) is rewritten as

$$\frac{L_w}{\alpha+\beta}\left[\Box_q\Phi + \frac{(\alpha+\beta)^{3/2}}{\alpha^{3/2}L_w}\,\partial_\mu\left(\frac{\alpha^{3/2}L_w}{\sqrt{\alpha+\beta}}\right)\gamma^{\mu\nu}\partial_\nu\Phi - \frac{L_\Phi}{2\,L_w}\,(\alpha+\beta)\right] = 0, \tag{8}$$

where the subscript in the d'Alembertian operator indicates that it is constructed with the metric $q_{\mu\nu}$ .

Note that by a simple choice of the coefficients $\alpha$ and $\beta$ is possible to describe the nonlinear dynamic of $\Phi$ as if it were embedded in a curved spacetime (generated by the field itself) where it interacts minimally with $q_{\mu\nu}$ . In order to do this we restrict the second order derivatives of $\Phi$ to appear only in the $\Box_q\Phi$ term of the above equation.

The simplest manner is the imposition

$$\frac{\alpha^{3/2}L_w}{\sqrt{\alpha+\beta}} = C\,, \tag{9}$$

where $C$ is a constant. The resultant equation is,

$$\Box_q\Phi = j(\Phi\,,\partial\Phi)\,, \tag{10}$$

where we have defined

$$j(\Phi\,,\partial\Phi) \equiv \frac{\alpha^3}{2C}L_\Phi\,L_w\,. \tag{11}$$

Equations (5) and (10) are equivalents, allowing us to interpret the dynamic of $\Phi$ as (1) nonlinear in the Minkowski spacetime or (2) "linear" in the metric $q_{\mu\nu}$ with a source $j(\Phi, \partial\Phi)$ . Important to emphasize that the use of the word "linear" is made here in a metaphoric sense, given that, since the metric $q_{\mu\nu}$ depends on $\Phi$ , the dynamic remains nonlinear.

A second possibility of geometrization consist in relax the condition (9) by substituting the constant $C$ by a function of $\Phi$ only,

$$\frac{\alpha^{3/2}L_w}{\sqrt{\alpha+\beta}} = F(\Phi)\,. \tag{12}$$

Using this in the Eq. (8) we get,

$$\frac{L_w}{\alpha+\beta}\left[\Box_q\Phi + (\alpha+\beta)\left(\frac{F_\Phi}{F}\,w - \frac{L_\Phi}{2L_w}\right)\right] = 0\,, \tag{13}$$

and, by appropriately choosing the function $F$ , we can write the dynamic of $\Phi$ as "free field" (again in a metaphoric way) without the source of the previous case. Thus, we have,

$$\Box_q\Phi = 0\,. \tag{14}$$

If the function $F(\Phi)$ satisfies the condition

$$\frac{F_\Phi}{F}\,w - \frac{L_\Phi}{2L_w} = 0\,. \tag{15}$$

Note that these two cases are equal when $L_\Phi = 0$ .

In GR, matter/energy curves the spacetime where it propagates and, in this sense that we understand how the metric structure $q_{\mu\nu}$ can be associated with a gravitational process. The scalar field itself curves the spacetime around it. But if we want to assign to $\Phi$ the role of a gravitational potential, with $q_{\mu\nu}$ being the gravitational metric, we need to determine how it will interact with other fields in the nature. The next section is occupied of this task.

We will use the second geometrization method present in this section to describe the dynamic of $\Phi$ in the $q$-spacetime. The hypothesis postulated and the observational data should help us to determine the Lagrangian of the scalar field and the functional dependence of the metric coefficients $\alpha$ and $\beta$ .

## 3 The Fundamentals of the GSG

In order to propose the main properties of GSG we will follow the basic ideas of Einstein's theory. Field formulation of GR describe the gravitational metric as sum of a flat metric (Minkowski) plus a perturbation $h^{\mu\nu}$ (not necessarily small),

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} . \tag{16}$$

Although the above expression be exact, its covariant version is indeed an infinity series (Feynman et al. 1995),

$$g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + h_{\mu\alpha} h^{\alpha}{}_{\nu} - \ldots \tag{17}$$

According to this formulation we can cite the basic properties of GR as follows.

- Gravitational interaction is described by a second order tensor field $h_{\mu\nu}$ that satisfies a non linear dynamic equation (Einstein's equation);
- The theory reproduces Newton's gravity in a weak field approximation;
- Any kind of matter and energy interacts with the gravitational field only through the metric $g_{\mu\nu}$ ;
- Test particles and electromagnetic waves follows geodesics in the curved spacetime described by $g_{\mu\nu}$ ;
- The $g_{\mu\nu}$ metric interacts universally with all fields in the nature following the minimum couple principle.

Now, we postulate the basic properties of the GSG.

- Gravitational interaction is described by scalar field $\Phi$ that satisfies a non linear dynamic equation;
- The theory reproduces Newton's gravity in a weak field approximation;
- Any kind of matter and energy interacts with the gravitational field only through the metric $q^{\mu\nu}$ [cf. (6)];

- Test particles and electromagnetic waves follows geodesics in the curved spacetime described by $q_{\mu\nu}$ ;
- The $q_{\mu\nu}$ metric interacts universally with all fields in the nature following the minimum couple principle.

Note that, different from GR, the covariant version of the gravitational metric in GSG is not an infinite series, as shown in Eq. (7).

Immediately, as it is in GR, the coupling between gravitation and electromagnetism in GSG is granted by this hypothesis. The Maxwell's field, under the influence of gravity, will be described by the action,

$$S_E = -\frac{1}{16\pi c}\int F\sqrt{-g}\, d^4x\,, \tag{18}$$

where $F = F_{\mu\nu}F^{\mu\nu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell tensor. When varying $S_E$ in relation with $A_\mu$ we get precisely the Maxwell's equations in a curved spacetime, $q_{\mu\nu}$ in this case.

Assuming that the test particles follow geodesics relative to the geometry $q_{\mu\nu}$, and the Newtonian limit in the static weak field approximation and low velocities, we have that

$$\frac{d^2x^i}{dt^2} = -c^2\Gamma^i_{00} = -\partial^i\,\Phi_N\,, \qquad \text{with} \quad i = 1, 2, 3. \tag{19}$$

The last equality is relating the particle acceleration with the Newtonian gravitational force, where $\Phi_N$ represents Newton's potential.

From Eq. (7), we have

$$\Gamma^i_{00} \approx -\frac{1}{2}\partial^i\,\ln\alpha. \tag{20}$$

It follows that the Newtonian potential $\Phi_N$ is approximately given by

$$\frac{\Phi_N}{c^2} \approx -\frac{1}{2}\,\ln\alpha\,, \tag{21}$$

which yields the relation between the $q_{00}$ component and the Newtonian potential,

$$q_{00} = \frac{1}{\alpha} \approx 1 + 2\,\frac{\Phi_N}{c^2} = 1 - 2\,\frac{GM}{c^2 r}\,, \tag{22}$$

where $G$ is the Newtonian constant and $M$ is the mass of the source.

However, this relation is determined up to a first order approximation in $\Phi$, and in the development of GSG we will extrapolate the above relation by considering a more general expression for the $\alpha$ coefficient,

$$\alpha = e^{-2\Phi}\,. \tag{23}$$

The theory that we are constructing here presents three functions that have to be entirely determined by the end, the Lagrangian of the scalar field and the functions $\alpha$ and $\beta$. Since the geometrization method of the previous section gives a condition between them, and with $\alpha$ now being fixed, only remains to determine the Lagrangian of $\Phi$ .

# 4 Field Equation

Let us consider the following shape for the scalar field Lagrangian,

$$L = V(\Phi)\, w\,. \tag{24}$$

Following the second geometrization method in Sect. 2 we have that, in absence of other fields, the field equation is

$$\Box\Phi = 0\,, \tag{25}$$

and conditions (12) and (15) reduce to the expression

$$\alpha + \beta = \alpha^3\, V\,. \tag{26}$$

Important to note that we are not using the subscript $q$ in the d'Alembertian operator anymore. Since in GSG Minkowski metric appears only as an auxiliary structure, we assume that all relevant quantities are constructed with the gravitational metric $q_{\mu\nu}$ .

To select among all possible Lagrangians of the above form we look for indications from the various circumstances in which reliable experiments have been performed. In this vein, we initiate the discussion by analyzing the consequences of GSG for the solar system.

## 4.1 *The Static and Spherically Symmetric Solution*

Any theory of gravity must account for planetary orbits. In general relativity this motion is described by geodesics of the Schwarzschild geometry. In the GSG particles follow geodesics in the $q_{\mu\nu}$ metric.

Let us start by rewriting the auxiliary Minkowski background metric in spherical coordinates

$$ds_M^2 = dt^2 - dR^2 - R^2\, d\Omega^2. \tag{27}$$

Changing the radial coordinate to $R = \sqrt{\alpha}\, r$, where $\alpha = \alpha(r)$ we get

$$ds_M^2 = dt^2 - \alpha \left( \frac{r}{2\alpha} \frac{d\alpha}{dr} + 1 \right)^2 dr^2 - \alpha\, r^2\, d\Omega^2. \tag{28}$$

Since we are looking for static spherically symmetric solution we assume that the field depends only on the radial variable $\Phi = \Phi(r)$. Then the gravitational metric (7) takes the form

$$ds^2 = \frac{1}{\alpha}\, dt^2 - B\, dr^2 - r^2\, d\Omega^2, \tag{29}$$

where we have defined

$$B \equiv \frac{\alpha}{\alpha^3 V} \left( \frac{r}{2\alpha} \frac{d\alpha}{dr} + 1 \right)^2 . \tag{30}$$

From the PPN analysis of the classical tests of gravitation (Will 2006) we know that the agreement with observations will be satisfied if we have

$$q_{00} \approx 1 - 2GM/c^2 r - 2(GM/c^2 r)^2 \quad \text{and} \quad q_{11} \approx 1 + 2GM/c^2 r\,. \tag{31}$$

Then, looking to Eq. (22), we can guarantee the correspondence between GSG and observations if we assume $B \approx \alpha$. However, we will again extrapolate this condition choosing a more general expression where $B = \alpha$. Using this the field equation can be easily solved, returning

$$\Phi = \frac{1}{2} \ln \left( 1 - 2 \frac{GM}{c^2 r} \right), \tag{32}$$

where we have used the asymptotic behavior to determine the integration constants and, from Eq. (30), we get

$$V(\Phi) = \frac{(\alpha - 3)^2}{4\, \alpha^3}\,. \tag{33}$$

With these results the line element of the static and spherically symmetric vacuum solution in GSG is given by

$$ds^2 = \left( 1 - \frac{r_H}{r} \right) dt^2 - \left( 1 - \frac{r_H}{r} \right)^{-1} dr^2 - r^2 d\Omega^2. \tag{34}$$

This geometry has the same form as in general relativity and yields the observed regime for solar tests. Thus, the present geometric scalar gravity is a good description of planetary orbits and also for light rays trajectories that follow geodesics (time-like and null-like, respectively) in the $q_{\mu\nu}$ geometry. If new observations would require a modification of the metric in the neighborhood of a massive body this should be made by adjusting the form of the potential $V(\Phi)$.

### 4.2 Action Principle

Now that we have defined all functions for the theory we are in position to write its dynamical equation. Let us start by the action of the scalar field written in the auxiliary Minkowski background. From variational principle

$$\delta S_\Phi = \frac{1}{\kappa c}\,\delta \int \sqrt{-\eta}\, V(\Phi) w\, d^4x\,, \tag{35}$$

we get,

$$\delta S_\Phi = -\frac{2}{\kappa c}\int \sqrt{-\eta}\left(\frac{1}{2}\,V' w + V\,\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi\right)\delta\Phi\, d^4x\,, \tag{36}$$

where $\kappa$ is a constant with dimensions of distance/energy and the prime indicates a derivative in relation to $\Phi$ . The expression in parentheses above is just the left hand side of Eq. (5) and, by comparing with (8) using (26), it returns $\Box\Phi/\alpha^3$ . Rewriting $\eta$ in terms of $q$ we finally get,

$$\delta S_\Phi = -2\int \sqrt{-q}\,\sqrt{V}\,\Box\Phi\,\delta\Phi\, d^4x\,. \tag{37}$$

In presence of matter we add a corresponding term $L_m$ to the total action,

$$S_m = \frac{1}{c}\int \sqrt{-q}\, L_m\, d^4x\,. \tag{38}$$

The first variation of this term as usual yields

$$\delta S_m = -\frac{1}{2}\int \sqrt{-q}\; T^{\mu\nu}\delta q_{\mu\nu}\, d^4x\,, \tag{39}$$

where we have defined the energy-momentum tensor in the standard way

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-q}}\,\frac{\delta(\sqrt{-q}\,L_m)}{\delta q^{\mu\nu}}.$$

General covariance leads to conservation of the energy-momentum tensor $T^{\mu\nu}{}_{;\nu} = 0$ .

The equation of motion is obtained by the action principle

$$\delta S_1 + \delta S_m = 0\,. \tag{40}$$

However, in the GSG theory, the metric $q_{\mu\nu}$ is not the fundamental quantity. We have to write the variation $\delta q_{\mu\nu}$ as function of $\delta\Phi$. After some calculation we get

$$\delta S_m = -\frac{1}{c}\int\left[T + \left(2 - \frac{V'}{2V}\right)E + C^{\lambda}{}_{;\lambda}\right]\delta\Phi\,\sqrt{-q}\,d^4x\,, \tag{41}$$

where we have defined some quantities as follows,

$$T \equiv T^{\mu\nu}\,q_{\mu\nu}, \quad E \equiv \frac{T^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi}{\Omega}\,, \tag{42}$$

$$C^{\lambda} \equiv \frac{\beta}{\alpha\,\Omega}\left(T^{\lambda\mu} - E\,q^{\lambda\mu}\right)\partial_\mu\Phi\,, \tag{43}$$

and " ; " means the covariant derivative in respect to the $q$-metric.

Finally, the equation of motion for the gravitational field $\Phi$ takes the form

$$\sqrt{V}\,\Box\Phi = \kappa\,\chi, \tag{44}$$

with the notation simplified by writing

$$\chi = -\frac{1}{2}\left[T + \left(2 - \frac{V'}{2V}\right)E + C^{\lambda}{}_{;\lambda}\right]. \tag{45}$$

Equation (44) describes the dynamics of GSG in presence of matter, under the assumptions (23) and (33). The quantity $\chi$ involves a non-trivial coupling between the gradient of the scalar field and the complete energy-momentum tensor of the matter, and not uniquely its trace. This property allows the electromagnetic field to interact with the gravitational field. The Newtonian limit gives the identification

$$\kappa \equiv \frac{8\pi G}{c^4}\,. \tag{46}$$

## 5 Final Comments

GSG is an alternative propose to describe the gravitational process using a single scalar field, but it still treats gravity as a geometrical effect and all kind of matter and energy interact with gravitational potential only through metric $q_{\mu\nu}$ in Eq. (7). With different premises from that previous scalars theories, GSG overcomes the problems surrounding the scalar gravity.

Guided by observations too, we develop GSG choosing the Lagrangian of $\Phi$ as $L = Vw$, with

$$V = \frac{(3-\alpha)^2}{4\alpha^3}, \tag{47}$$

where the Newtonian limit of the theory led us to work with

$$\alpha = e^{-2\Phi}. \tag{48}$$

The geometrization technique is what gives the relation between the $\beta$ coefficient of the metric and these two other functions, namely

$$\alpha + \beta = \alpha^3 V. \tag{49}$$

Therewith, the field equation of the theory is given by

$$\sqrt{V}\,\Box\Phi = \kappa\,\chi, \tag{50}$$

with $\chi$ defined in (45).

Even so, the GSG can be seen as a little more than an unique theory in the sense that it represents a way in which is possible to develop scalar theories of gravitation. Relaxing the expressions for $\alpha$ and $V$ can still be in agreement with observations while given a very different gravitational theory.

GSG is a result of a wonderful work with Mario Novello, Ugo Moschella, Eduardo Bittencourt and others. The ideas here can be found with more details in Novello et al. (2013). Also, in Bittencourt et al. (2014), there is the consequences of this theory for the cosmology.

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