Chapter 4
Relative Fractal Drums and Their Complex Dimensions

Only yesterday the practical things of today were decried as impractical, and the theories which will be practical tomorrow will always be branded as valueless games by the practical man of today.

William Feller (1906–1970)

Abstract In this chapter, we introduce the notion of relative fractal drums (or RFDs, in short). They represent a simple and natural extension of two fundamental objects of fractal analysis, simultaneously: that of bounded sets in $\mathbb{R}^N$ (i.e., of fractals) and that of bounded fractal strings (introduced by the first author and Carl Pomerance in the early 1990s). Furthermore, there is a natural way to define their associated Minkowski contents and relative distance as well as tube zeta functions. We stress a new phenomenon exhibited by relative fractal drums: namely, their box dimensions can be negative as well (and even equal to $-\infty$). This can be viewed as a property of their 'flatness', since it is related to the loss of the cone property. In short, a relative fractal drum (RFD) consists of an ordered pair (A,Ω), where A is an arbitrary (possibly unbounded) subset of $\mathbb{R}^N$ and Ω is an open subset of $\mathbb{R}^N$ of finite volume and such that $\Omega \subseteq A_\delta$, for some $\delta > 0$. The corresponding zeta function, either a distance or tube zeta function, is denoted by $\zeta_{A,\Omega}$ or $\tilde{\zeta}_{A,\Omega}$, respectively. We show that $\zeta_{A,\Omega}$ and $\tilde{\zeta}_{A,\Omega}$ are connected via a key functional equation, which implies that their poles (i.e., the *complex dimensions* of the RFD (A,Ω)) are the same. We also extend to this general setting the main results of Chapters 2 and 3 concerning the holomorphicity and meromorphicity of the fractal zeta functions. We introduce the notion of transcendentally quasiperiodic relative fractal drums, using their tube functions. One way of constructing such drums is based on a carefully chosen sequence of generalized Cantor sets, as well as on the use of a classic result by Alan Baker from transcendental number theory. This construction and result extend the corresponding ones obtained in Chapter 3, in which we studied transcendentally quasiperiodic fractal sets. Furthermore, some explicit constructions of RFDs lead us naturally to introduce a new class of fractals, which we call *hyperfractals*. Particulary noteworthy among them are the maximal hyperfractals, for which the critical line $\{\mathrm{Re}\, s = \overline{\dim}_B(A,\Omega)\}$, where $\overline{\dim}_B(A,\Omega)$ is the relative upper box dimension of (A,Ω) and coincides with the abscissa of convergence of $\zeta_{A,\Omega}$ or $\tilde{\zeta}_{A,\Omega}$, consists solely of nonisolated singularities of the corresponding fractal zeta function (i.e., of the relative distance or tube zeta function), $\zeta_{A,\Omega}$ or $\tilde{\zeta}_{A,\Omega}$.

M.L. Lapidus et al., *Fractal Zeta Functions and Fractal Drums*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-319-44706-3_4

Key words: relative fractal drum (RFD), relative Minkowski dimension, relative Minkowski content, relative distance and tube zeta functions, relative fractal zeta functions, scaling, relative fractal spray, flatness of RFDs, compact sets of positive reach, spectral zeta function, modified Weyl–Berry conjecture, meromorphic extension, abscissae of meromorphic and absolute convergence, residue, inhomogeneous Sierpiński N-gasket, relative Sierpiński N-carpet, spectral zeta functions of fractal drums, transcendentally ∞-quasiperiodic RFD, hyperfractals, embeddings of RFDs into higher-dimensional spaces.

In this chapter, we introduce the notion of relative fractal drums. They represent a simple and natural extension of two fundamental objects of fractal analysis, simultaneously: that of bounded sets in $\mathbb{R}^N$ (i.e., of fractals) and that of bounded fractal strings (introduced by the first author and Carl Pomerance in the early 1990s). Furthermore, there is a natural way to define their associated Minkowski contents and relative distance zeta functions. We stress a new phenomenon exhibited by relative fractal drums: namely, their box dimensions can be negative as well (and even equal to $-\infty$). This can be viewed as a property of their 'flatness', since it is related to the loss of the cone property; see Proposition 4.1.33.

In short, a relative fractal drum (RFD) consists of an ordered pair (A,Ω), where A is an arbitrary (possibly unbounded) subset of $\mathbb{R}^N$ and Ω is an open subset of $\mathbb{R}^N$ of finite volume and such that $\Omega \subseteq A_\delta$ for some $\delta > 0$; see Definition 4.1.2.

In Section 4.6, we introduce the notion of transcendentally quasiperiodic relative fractal drums, using their tube functions. One way of constructing such drums, described in Theorem 4.6.9, is based on a carefully chosen sequence of generalized Cantor sets, as well as on a classic result by Alan Baker from transcendental number theory; see Theorem 3.1.14. This construction and result extend the corresponding ones obtained in Section 3.1, in which we studied transcendentally quasiperiodic fractal sets.

Furthermore, some explicit constructions of RFDs lead us naturally to introduce a new class of fractals, which we call *hyperfractals*; see Definition 4.6.23. Particulary noteworthy among them are the maximal hyperfractals, for which the critical line $\{\mathrm{Re}\, s = \overline{\dim}_B(A,\Omega)\}$, where $\overline{\dim}_B(A,\Omega)$ is the relative upper box dimension of (A,Ω) and coincides with the abscissa of convergence of the corresponding zeta function, consists solely of nonisolated singularities; see Corollary 4.6.17. Therefore, for such a (relative) maximally hyperfractal drum, the critical line is a (meromorphic) natural boundary (in the sense of part (*ii*) of Definition 1.3.8) for each of the associated fractal zeta functions $\zeta_{A,\Omega}$ and $\tilde{\zeta}_{A,\Omega}$.

4.1 Zeta Functions of Relative Fractal Drums

We discuss here several natural generalizations of various notions which are central to this and related works, including notably relative distance zeta functions (in which the region of integration need not be bounded but is of finite volume), the associated relative box (and complex) dimensions, and RFDs. As is illustrated in a number of examples, this additional flexibility enables us to account for a broad range of situations and phenomena, including the case of fractal strings (Example 4.1.3) and of unbounded geometric chirps (Example 4.4.1). We also provide sufficient conditions ensuring the existence of a (necessarily unique) meromorphic continuation of the relative distance zeta function.

4.1.1 Relative Minkowski Content, Relative Box Dimension, and Relative Zeta Functions

In this subsection, we introduce the notion of a relative zeta function, associated to an appropriate ordered pair (A,Ω) of two suitable subsets of $\mathbb{R}^N$, which may be unbounded. The relative distance zeta function (see (4.1.1)), is a natural generalization of the standard distance zeta function defined by (2.1.1). We have already briefly encountered it in Section 2.1.5 in a less general context (see especially, Definition 2.1.75, Proposition 2.1.76 and Theorem 2.1.78, where Ω was assumed to be bounded), but we will now significantly relax our earlier assumptions.

Definition 4.1.1. Let Ω be an open subset of $\mathbb{R}^N$, not necessarily bounded, but of finite N-dimensional Lebesgue measure. Let $A \subseteq \mathbb{R}^N$, also possibly unbounded, such that Ω is contained in A_δ for some $\delta > 0$.[1] The *distance zeta function* $\zeta_{A,\Omega}$ *of* A *relative to* Ω (or the *relative distance zeta function*) is defined by

$$\zeta_{A,\Omega}(s) := \int_\Omega d(x,A)^{s-N}\mathrm{d}x, \tag{4.1.1}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.

Unlike in (2.1.102), the closure of A is allowed here to intersect the boundary of Ω. (The closures of A and Ω may even be disjoint; see Example 4.1.22.) For this reason, the abscissa of convergence of this new zeta function will depend not only on the set A, but on Ω as well; see Theorem 4.1.7 and Example 4.1.25 below.

Definition 4.1.2. We propose to call the ordered pair (A,Ω), appearing in Definition 4.1.1, a *relative fractal drum* (RFD). Therefore, we shall also use the phrase *zeta functions of relative fractal drums* instead of relative zeta functions.

Example 4.1.3. Any bounded fractal string $\mathscr{L} = (\ell_j)_{j\geq 1}$ (initially defined, as usual, as an infinite nonincreasing sequence of positive numbers $(\ell_j)_{j=1}^\infty$ such that $\sum_{j=1}^\infty \ell_j < \infty$) can also be viewed as a relative fractal drum $(A_\mathscr{L},\Omega_\mathscr{L})$. Indeed, the associated sets $A_\mathscr{L}$ and $\Omega_\mathscr{L}$ are

$$A_\mathscr{L} = \Big\{a_k = \sum_{j=k}^\infty \ell_j : k \in \mathbb{N}\Big\}, \quad \Omega_\mathscr{L} = \bigcup_{k=1}^\infty (a_{k+1},a_k); \tag{4.1.2}$$

[1] We need this technical condition on A and Ω in order to ensure that the integral defined by Equation (4.1.1) is well defined for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ large enough.

see Subsection 2.1.4. Therefore, the notion of relative fractal drum (A,Ω) is a natural extension of the notion of bounded fractal string $\mathscr{L}=(\ell_j)_{j\geq 1}$. Here, we point out that the notion of "generalized fractal string" already exists, and has a different meaning; see [Lap-vFr3, Chapter 4]. See also Remark 4.1.4 just below.

Remark 4.1.4. In short, a *generalized fractal string* (in the sense of [Lap-vFr3]) is a local positive or complex measure on $(0,+\infty)$ which does not have mass near 0. A local positive measure is simply a locally bounded positive Borel measure on $(0,+\infty)$, while a local complex measure is a locally bounded set-function on (the Borel σ-algebra of) $(0,+\infty)$ whose restriction to every compact subinterval J of $(0,+\infty)$ is a (necessarily bounded) Borel complex measure on J. (See Definition A.1.1 in Appendix A.) For example, an ordinary fractal string is represented by the positive measure $\eta_{\mathscr{L}}:=\sum_{j=1}^{\infty}\delta_{\ell_j^{-1}}$, where for $x>0$, δ_x denotes the unit Dirac mass (or measure) concentrated at x. Note that clearly, since $\ell_j\downarrow 0$ as $j\to\infty$, $\eta_{\mathscr{L}}$ is a generalized fractal string because it does not have any mass near 0. More generally, one could consider *generalized fractal strings* which are discrete but with noninteger multiplicities, say, $\eta=\eta_{\mathscr{L}}:=\sum_{l\in\mathscr{L}}b_l\delta_{l^{-1}}$, where $\mathscr{L}=\{l\}$ is an ordinary fractal string (now consisting of distinct lengths l) and 'multiplicities' or 'weights' b_l (with $b_l\geq 0$ or $b_l\in\mathbb{C}$ for each $l\in\mathscr{L}$); so that its 'geometric zeta function'

$$\zeta_\eta(s):=\int_0^{+\infty}x^{-s}\eta(\mathrm{d}x)$$

is the *Mellin transform* of the generalized Dirichlet series $\sum_{l\in\mathscr{L}}b_l l^s$. Of course, one can also consider continuous analogs, say, $\eta(\mathrm{d}x)=\varphi(x)\mathrm{d}x$, with φ a suitable real-valued function on $(0,+\infty)$. See, especially, [Lap-vFr3, Chapters 4, 5, 9, 10] and [Lap-vFr3, Section 6.3 and 11.1] for a variety of examples and applications of the theory of generalized fractal strings.

Remark 4.1.5. For results and conjectures concerning the spectra and the vibrations of (ordinary) fractal drums (or 'drums with fractal boundary'), we refer, e.g., to [Berr1, Berr2], [BroCar], [SapGoMar], [Lap1–3], [Ger], [GerSc], [FlVa], [Cae], [LapNeuReGr], [LapPa], [MolVai], [HeLap], [vBGilk], [Lap-vFr1–2], [HamLap], as well as [Lap-vFr3, Section 12.5] and the relevant references therein. We note that in the present monograph, however, we study mainly the geometry (rather than the eigenvalue spectrum) of (relative) fractal drums. A short discussion of the spectral zeta functions of a simple class of RFDs in $\mathbb{R}^N$ can be found in Section 4.3.1.

We can define the *relative complex dimensions* of A with respect to Ω (and with respect to a given window $\boldsymbol{W}$) as the set of poles (in $\boldsymbol{W}$) of the meromorphic extension of the relative distance zeta function $\zeta_{A,\Omega}$.

In particular, when $\zeta_{A,\Omega}$ has a meromorphic continuation to an open (connected) neighborhood of $\{\mathrm{Re}\,s=\overline{\dim}_B(A,\Omega)\}$, one can define (much as in Definition 2.1.67) the *set of relative principal complex dimensions* of (A,Ω), which is denoted by $\mathscr{P}(\zeta_{A,\Omega})$ or by $\dim_{PC}(A,\Omega)$, and consists of the poles of $\zeta_{A,\Omega}$ which lie on the critical line $\{\mathrm{Re}\,s=\overline{\dim}_B(A,\Omega)\}$. (We will see in Theorem 4.1.7 and

Remark 4.1.8 that the abscissa of convergence of $\zeta_{A,\Omega}$ coincides with $\overline{\dim}_B(A,\Omega)$ and hence, $\zeta_{A,\Omega}$ is holomorphic in the open half plane $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$.) See Definition 4.1.13.

In the previous paragraph and in the sequel, we are using the relative upper box dimension $\overline{\dim}_B(A,\Omega)$ (introduced in [Žu4] and generalizing that of [BroCar] and [Lap1–3], where $A := \partial\Omega$), instead of $\overline{\dim}_B(A)$. Its definition is analogous to that of the usual upper box dimension.

First, for any $r \in \mathbb{R}$, we define the *upper r-dimensional*[2] *Minkowski content of A relative to* Ω (or *the upper relative Minkowski content*, or *the upper Minkowski content of the RFD* (A,Ω)) by

$$\mathscr{M}^{*r}(A,\Omega) = \limsup_{t\to 0^+} \frac{|A_t \cap \Omega|}{t^{N-r}}, \tag{4.1.3}$$

and then proceed exactly as in (1.3.4) or in (1.3.5) in order to define $\overline{\dim}_B(A,\Omega)$:

$$\begin{aligned} \overline{\dim}_B(A,\Omega) &= \inf\{r \in \mathbb{R} : \mathscr{M}^{*r}(A,\Omega) = 0\} \\ &= \inf\{r \in \mathbb{R} : \mathscr{M}^{*r}(A,\Omega) < \infty\} \\ &= \sup\{r \in \mathbb{R} : \mathscr{M}^{*r}(A,\Omega) = +\infty\}. \end{aligned} \tag{4.1.4}$$

We call it the *relative upper box dimension* (or *relative upper Minkowski dimension*) of A with respect to Ω (or else the *relative upper box dimension of* (A,Ω)). Note that

$$\overline{\dim}_B(A,\Omega) \in [-\infty, N], \tag{4.1.5}$$

and the values can indeed be negative, and even equal to $-\infty$; see Proposition 4.1.35 and Corollary 4.1.38.

Naturally, $\mathscr{M}_*^r(A,\Omega)$, the *lower r-dimensional Minkowski content* of (A,Ω), is defined as in (4.1.3), except for a lower instead of an upper limit.

We define analogously the *relative lower box* (or *Minkowski*) *dimension* of (A,Ω):

$$\begin{aligned} \underline{\dim}_B(A,\Omega) &= \inf\{r \in \mathbb{R} : \mathscr{M}_*^r(A,\Omega) = 0\} \\ &= \inf\{r \in \mathbb{R} : \mathscr{M}_*^r(A,\Omega) < \infty\} \\ &= \sup\{r \in \mathbb{R} : \mathscr{M}_*^r(A,\Omega) = +\infty\}. \end{aligned} \tag{4.1.6}$$

Furthermore, when $\underline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega)$, we denote by $\dim_B(A,\Omega)$ this common value and then say that the *relative box* (or *Minkowski*) *dimension* $\dim_B(A,\Omega)$ *exists*. See Remark 4.1.6 below.

If $0 < \mathscr{M}_*^D(A,\Omega) \le \mathscr{M}^{*D}(A,\Omega) < \infty$, we say that the relative fractal drum (A,Ω) is *Minkowski nondegenerate*. It then follows that $\dim_B(A,\Omega)$ exists and is equal to D.

If $\mathscr{M}_*^D(A,\Omega) = \mathscr{M}^{*D}(A,\Omega)$, this common value is denoted by $\mathscr{M}^D(A,\Omega)$ and called the *relative Minkowski content* of (A,Ω). If $\mathscr{M}^D(A,\Omega)$ exists and is

[2] An important novelty here is that we allow *negative values* of r as well.

different from 0 and $+\infty$ (in which case $\dim_B(A,\Omega)$ exists and we necessarily have $D = \dim_B(A,\Omega)$), we say that the relative fractal drum (A,Ω) is *Minkowski measurable*. For relative box (or rather, Minkowski) dimensions and their properties, see [Lap1], [HeLap] and more generally, [Žu4].

For example, if we assume that $A = \partial\Omega$, then $\overline{\dim}_B(\partial\Omega,\Omega)$ is also called the *one-sided box dimension of the boundary* (i.e., with respect to Ω, see [HeLap]) or the *inner Minkowski dimension* of $\partial\Omega$ (see, e.g., [BroCar], [Lap1–3], [LapPo1–3], [LapMa1–2], [FlVa] and [Lap-vFr1–3]). It may be different from $\overline{\dim}_B\partial\Omega$.

Remark 4.1.6. Here and in the sequel, we use interchangeably the terms "relative box dimension" and "relative Minkowski dimension". However, strictly speaking, only the latter term is correct in this general context because we do not have a proper independent (and geometric) definition of relative box dimension. See Problem 6.2.6 on page 556.

If A is a bounded subset of $\mathbb{R}^N$ and Ω is an open subset of $\mathbb{R}^N$ of finite N-dimensional Lebesgue measure, it is clear that

$$\overline{\dim}_B(A,\Omega) \in [-\infty, \overline{\dim}_B A], \tag{4.1.7}$$

and similarly for the lower box dimension. The inequality $\overline{\dim}_B(A,\Omega) \le \overline{\dim}_B A$ may be strict; see Examples 4.1.23 and 4.1.25. An obvious example is when the distance between A to Ω is positive, in which case $\overline{\dim}_B(A,\Omega) = \dim_B(A,\Omega) = -\infty$, no matter what value is taken by $\overline{\dim}_B A$. Furthermore, there are simple examples of disjoint sets $\overline{A}$ and $\overline{\Omega}$ for which $\overline{\dim}_B(A,\Omega)$ is nonzero; see Example 4.1.22. It is interesting that the value of $\overline{\dim}_B(A,\Omega)$ may be negative, whereas $\overline{\dim}_B A$ (as well as $\underline{\dim}_B A$) is always nonnegative. See, especially, Proposition 4.1.35, Corollary 4.1.38 and Remark 4.1.39.

The following result extends Theorem 2.1.11 to the present, more general, setting. To see this, it suffices to take $\Omega = A_\delta$ for any fixed $\delta > 0$.

Theorem 4.1.7. *Let Ω be an open subset of $\mathbb{R}^N$ of finite N-dimensional Lebesgue measure, and let $A \subseteq \mathbb{R}^N$ be such that $\Omega \subseteq A_\delta$ for some $\delta > 0$. Then the following properties hold*:

(*a*) *The relative distance zeta function $\zeta_{A,\Omega}(s)$ is holomorphic in the half-plane $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$, and for those same values of s, we have*

$$\zeta'_{A,\Omega}(s) = \int_\Omega d(x,A)^{s-N} \log d(x,A)\, \mathrm{d}x.$$

(*b*) *The lower bound in the open right half-plane $\{\mathrm{Re}(s) > \overline{\dim}_B(A,\Omega)\}$ is optimal, from the point of view of the* (*absolute*) *convergence of the Dirichlet-type integral initially defining $\zeta_{A,\Omega}$ in* (4.1.1). *In other words,*

$$D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega), \tag{4.1.8}$$

where $D(\zeta_{A,\Omega})$ is the abscissa of convergence of $\zeta_{A,\Omega}$. (See also Remark 4.1.8 and part (i) of Corollary 4.1.10 below.)

(c) *If $D = \dim_B(A,\Omega)$ exists, $D < N$, and $\mathscr{M}_*^D(A,\Omega) > 0$, then $\zeta_{A,\Omega}(s) \to +\infty$ as $s \in \mathbb{R}$ converges to D from the right. See also part (ii) of Corollary 4.1.10 below.*

Proof. The proof is similar to that of Theorem 2.1.11. Instead of Lemma 2.1.3, we have to use a more general result (see [Žu4, Theorem 3.3]):

$$\text{If } \gamma < N - \overline{\dim}_B(A,\Omega), \text{ then } \int_{A_\delta \cap \Omega} d(x,A)^{-\gamma} \mathrm{d}x < \infty, \tag{4.1.9}$$

where δ is any fixed positive number. Lemma 2.1.6 can be easily adapted to the case of the relative box dimension; see [Žu2]. We omit the details. We simply note that in [Žu4, Theorem 3.3], the result is proven under the assumption that we deal with relative Minkowski contents for $r \geq 0$. Here, we allow $r < 0$ as well and it is easy to see that, nevertheless, this result still holds in this more general context. ☐

Remark 4.1.8. The claim in Theorem 4.1.7(b) follows easily from (a) and the fact that if $s \in \mathbb{R}$ and $s < \overline{\dim}_B(A,\Omega)$, then the defining integral in (4.1.1) is equal to infinity. Note that it follows from Theorem 4.1.7(b) that the relative upper box dimension, $\overline{\dim}_B(A,\Omega)$, coincides with the abscissa of convergence of the Dirichlet-type integral defining $\zeta_{A,\Omega}$ in (4.1.1). Equivalently, as was stated in Theorem 4.1.7(b), we have

$$\overline{\dim}_B(A,\Omega) = D(\zeta_{A,\Omega}), \tag{4.1.10}$$

where the latter notation is defined in Equation (2.1.92).

Remark 4.1.9. The continuity property stated in Theorem 2.1.78 also holds in the more general case of the relative zeta functions studied in the present subsection (that is, under the general assumptions of Definition 4.1.1). The proof of this fact is completely analogous to that of Theorem 2.1.78.

Since, as we have noted in Example 2.1.41, the relative distance zeta function $\zeta_{A,\Omega}$ is a Dirichlet-type integral satisfying condition (2.1.54) specified in Subsection 2.1.3.2 (i.e., it is a tamed DTI, in the sense of Definition A.1.3 of Appendix A), its abscissa of convergence $D(\zeta_{A,\Omega})$ is well defined. (For more details, see also the proof of part (1) of Proposition A.2.4 in Appendix A.) Exactly the same comment can be made about the relative tube zeta function $\tilde{\zeta}_{A,\Omega}$, to be introduced later in Subsection 4.5.1, Equation (4.5.1).

The following result is the exact analog for RFDs of Corollary 2.1.20 and Corollary 2.2.10 combined with Remark 4.1.11 and Remark 4.1.8. Note, however, that we no longer conclude that $D(\zeta_{A,\Omega}) \geq 0$, as will be further discussed in Subsection 4.1.2. Moreover, the analog of this result holds for the relative tube zeta function $\tilde{\zeta}_{A,\Omega}$.

Corollary 4.1.10. (i) *Let* (A,Ω) *be a relative fractal drum of* $\mathbb{R}^N$. *Then*:

$$D_{\rm mer}(\zeta_{A,\Omega}) \leq D_{\rm hol}(\zeta_{A,\Omega}) \leq D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega), \tag{4.1.11}$$

and each of these inequalities is sharp, in general.

(ii) *If, in addition, we assume that the hypotheses of part* (c) *of Theorem 4.1.7 are satisfied, we then have the following equalities*:

$$D_{\rm hol}(\zeta_{A,\Omega}) = D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega), \tag{4.1.12}$$

and hence, $\Pi(\zeta_{A,\Omega}) = \mathscr{H}(\zeta_{A,\Omega})$*, whereas under the assumptions of part* (i) *just above, we only have* $\Pi(\zeta_{A,\Omega}) \subseteq \mathscr{H}(\zeta_{A,\Omega})$.

Remark 4.1.11. Much as was noted in part (a) of Remark 2.1.21, we do not know whether there exist RFDs (A,Ω) for which the second inequality in (4.1.11) is strict; namely, $D_{\rm hol}(\zeta_{A,\Omega}) < D(\zeta_{A,\Omega})$. Such RFDs could not be ordinary fractal strings since we always have an equality in the latter case.

It is easy to find a relative fractal drum (A,Ω) for which $D_{\rm mer}(\zeta_{A,\Omega}) < D_{\rm hol}(\zeta_{A,\Omega})$. In fact, for every (nontrivial) fractal string, the equalities in Equation (4.1.12) always hold (without assuming the hypotheses of part (c) of Theorem 4.1.7), and with $\overline{\dim}_B(A,\Omega) \geq 0$, but, for example, for the Cantor string, we have $D_{\rm mer}(\zeta_{A,\Omega}) = -\infty$.

It is easy to see that, given any subset A and an open set Ω in $\mathbb{R}^N$ with finite N-dimensional Lebesgue measure, the relative zeta function of (A,Ω) can also be defined in the following way:

$$\zeta_{A,\Omega}(s;\delta) := \int_{\Omega\cap A_\delta} d(x,A)^{s-N}\mathrm{d}x, \tag{4.1.13}$$

where δ is a fixed positive number. Namely, for $\Omega' := \Omega \cap A_\delta$, the condition in Theorem 4.1.7 according to which $\Omega' \subseteq A_\delta$ is clearly satisfied.

In our definition of RFDs (A,Ω), we assume that Ω is an open subset of $\mathbb{R}^N$. Actually, Theorem 4.1.7 holds even in the case when Ω is a Borel set in $\mathbb{R}^N$. For example, Ω may have an empty interior and a positive Lebesgue measure. Therefore, it is natural to consider more general fractal drums (A,Ω), for which Ω is just an arbitrary Borel subset of $\mathbb{R}^N$. This issue is pursued in Appendix B, where the notion of 'local zeta function' is discussed.

In the following result, we obtain a simple sufficient condition for two RFDs to be equivalent. Its proof is similar to the proof of Proposition 2.1.76, and therefore we omit it.

Proposition 4.1.12. *Assume that* (A,Ω_1) *and* (A,Ω_2) *are RFDs in* $\mathbb{R}^N$ *such that* $f_j(s) := \int_{\Omega_j\setminus(\Omega_1\cap\Omega_2)} d(x,A)^{s-N}\mathrm{d}x$ *are entire functions, for* $j=1,2$. *Then the corresponding distance zeta functions are equivalent, that is,*

$$\zeta_{A,\Omega_1} \sim \zeta_{A,\Omega_2}.$$

Definition 4.1.13. Assume that (A,Ω) is a relative fractal drum in $\mathbb{R}^N$ such that its distance zeta function possesses a meromorphic extension to a domain which contains the critical line $\{\operatorname{Re} s = D(\zeta_{A,\Omega})\}$ in its interior. The set of poles of $\zeta_{A,\Omega}$ located on the critical line is called *the set of principal complex dimensions of the relative fractal drum* (A,Ω), or *the set of relative principal complex dimensions of* (A,Ω), and is denoted by $\dim_{PC}(A,\Omega)$ or equivalently, $\mathscr{P}_c(\zeta_{A,\Omega})$. (This extends the definition of $\dim_{PC} A = \mathscr{P}_c(\zeta_A)$ given in Definition 2.1.67.) We can analogously define the set $\dim_{PC}\mathscr{L}$ of *principal complex dimensions of any bounded* (*or unbounded*) *fractal string* $\mathscr{L} = (\ell_j)_{j\geq 1}$, as the set of poles of $\zeta_{\mathscr{L}}$ contained on the critical line $\{\operatorname{Re} s = D(\zeta_{\mathscr{L}})\}$.

In light of Theorem 2.2.3, we have the following result.

Theorem 4.1.14. *Assume that* (A,Ω) *is a Minkowski nondegenerate RFD in* $\mathbb{R}^N$, *that is,* $0 < \mathscr{M}_*^D(A,\Omega) \leq \mathscr{M}^{*D}(A,\Omega) < \infty$ (*in particular,* $\dim_B(A,\Omega) = D$), *and* $D < N$. *If* $\zeta_{A,\Omega}(s)$ *can be extended meromorphically to a connected open neighborhood of* $s = D$, *then* D *is necessarily a simple pole of* $\zeta_{A,\Omega}$, *the residue* $\operatorname{res}(\zeta_{A,\Omega}, D)$ *is independent of* δ *and*

$$(N-D)\mathscr{M}_*^D(A,\Omega) \leq \operatorname{res}(\zeta_{A,\Omega}, D) \leq (N-D)\mathscr{M}^{*D}(A,\Omega). \tag{4.1.14}$$

Furthermore, if (A,Ω) *is Minkowski measurable, then*

$$\operatorname{res}(\zeta_{A,\Omega}, D) = (N-D)\mathscr{M}^D(A,\Omega). \tag{4.1.15}$$

The next lemma follows immediately from the definition of the relative upper and lower box dimensions.

Lemma 4.1.15. *Assume that we have two RFDs* (A_j,Ω_j) *in* $\mathbb{R}^N$ $(j = 1,2)$, *where each* Ω_j *is of finite Lebesgue measure. If* $A_1 \subseteq A_2$ *and* $\Omega_1 \subseteq \Omega_2$, *then* $\overline{\dim}_B(A_1,\Omega_1) \leq \overline{\dim}_B(A_2,\Omega_2)$. *This is also true for the lower relative box dimensions.*

An immediate consequence is the following simple and useful result.

Lemma 4.1.16. *Assume that* $\Omega_1 \subseteq \Omega \subseteq \Omega_2$ *are open sets of finite Lebesgue measure in* $\mathbb{R}^N$. *If*

$$\overline{\dim}_B(A,\Omega_1) = \overline{\dim}_B(A,\Omega_2),$$

then this common value is equal to $\overline{\dim}_B(A,\Omega)$.

The following countable additivity property of zeta functions is a simple consequence of the σ-additivity property of the Lebesgue integral.

Proposition 4.1.17. *Assume that* $\Omega = \cup_{j=1}^{\infty} B_j$ *is an open subset of* $\mathbb{R}^N$ *of finite* N-*dimensional Lebesgue measure, where* $(B_j)_{j=1}^{\infty}$ *is a sequence of pairwise disjoint* *open subsets of* $\mathbb{R}^N$*. Furthermore, assume that* $A \subseteq \mathbb{R}^N$ *and there exists* $\delta > 0$ *such that* $\Omega \subseteq A_\delta$*. Then, for all* $s \in \mathbb{C}$ *such that* $\operatorname{Re} s > \overline{\dim}_B(A,\Omega)$*, we have*

$$\zeta_{A,\Omega}(s) = \sum_{j=1}^{\infty} \zeta_{A,B_j}(s). \tag{4.1.16}$$

Example 4.1.18. Let $\Omega \subset \mathbb{R}$ be a disjoint union of open intervals I_k in the real line, of lengths $1/k^2$ for each $k \geq 1$. Here, Ω may be unbounded. Let $A = \partial\Omega$. Then

$$\zeta_{A,\Omega}(s) = \sum_{k=1}^{\infty} \zeta_{A,I_k}(s) = \frac{2^{1-2s}}{s} \sum_{k=1}^{\infty} k^{-2s} \sim \sum_{k=1}^{\infty} k^{-2s} = \zeta(2s), \tag{4.1.17}$$

where $\zeta(s) = \sum_{j\geq 1} k^{-s}$ is the classical Riemann zeta function (or its meromorphic continuation). The abscissa of convergence of $\zeta_{A,\Omega}(s)$ is therefore equal to $s = 1/2$, and by using Theorem 4.1.7(b) we conclude that $\overline{\dim}_B(A,\Omega) = 1/2$. Note that by analytic continuation, $\zeta_{A,\Omega}$ has a meromorphic extension to all of $\mathbb{C}$, and that $\zeta_{A,\Omega}(s) = \frac{2^{1-2s}}{s}\zeta(2s)$ for all $s \in \mathbb{C}$.

The following example is a relative analog of Example 2.2.21.

Example 4.1.19. Let $\Omega = B_R(0)$ be the open ball in $\mathbb{R}^N$ of radius R and let $A = \partial\Omega$ be the boundary of Ω, i.e, the $N-1$-dimensional sphere of radius R. Then, introducing the new variable $\rho = R - r$, we have

$$\begin{aligned}
\zeta_{A,\Omega}(s) &= N\omega_N \int_0^R (R-r)^{s-N} r^{N-1} \mathrm{d}r = N\omega_N \int_0^R \rho^{s-N}(R-\rho)^{N-1} \mathrm{d}\rho \\
&= N\omega_N \int_0^R \rho^{s-N} \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} R^{N-1-k} \rho^k \mathrm{d}\rho \\
&= N\omega_N R^s \sum_{k=0}^{N-1} \binom{N-1}{k} \frac{(-1)^k}{s-(N-k-1)} \\
&= N\omega_N R^s \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{(-1)^{N-j-1}}{s-j}
\end{aligned}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s > N-1$, where ω_N is the N-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$; see Equation (1.3.22) on page 40. (Note that we have also used the well-known symmetry of the binomial coefficients, $\binom{N-1}{N-1-j} = \binom{N-1}{j}$.) In particular, $\zeta_{A,\Omega}$ can be meromorphically extended to the whole complex plane and is given by

$$\zeta_{A,\Omega}(s) = N\omega_N R^s \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{(-1)^{N-j-1}}{s-j}, \tag{4.1.18}$$

for all $s \in \mathbb{C}$.

Therefore, we have

$$\begin{aligned} \dim_B(A,\Omega) = D(\zeta_{A,\Omega}) = N-1 \\ \mathscr{P}(\zeta_{A,\Omega}) = \{0,1,\dots,N-1\} \quad \text{and} \quad \dim_{PC}(A,\Omega) = \{N-1\}. \end{aligned} \tag{4.1.19}$$

Furthermore,

$$\operatorname{res}(\zeta_{A,\Omega}, j) = (-1)^{N-j-1} N\omega_N \binom{N-1}{j} R^j \tag{4.1.20}$$

for $j = 0,1,\dots,N-1$. It is noteworthy that the set $\mathscr{P}(\zeta_{A,\Omega})$ of complex dimensions of (A,Ω) is not the same as the set $\mathscr{P}(\zeta_A)$ of complex dimensions of A; compare Equation (4.1.19) with Equation (2.2.58) of Example 2.2.21 on page 128. As a special case of (4.1.20), for $j = D := N-1$ we obtain that

$$\operatorname{res}(\zeta_{A,\Omega}, D) = N\omega_N R^{N-1} = \mathscr{M}^D(A,\Omega). \tag{4.1.21}$$

The last equality follows from a direct computation:

$$\mathscr{M}^D(A,\Omega) = \lim_{t\to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D}} = \lim_{t\to 0^+} \frac{\omega_N R^N - \omega_N (R-t)^N}{t} = N\omega_N R^{N-1}. \tag{4.1.22}$$

Furthermore, recall that $H^D(A) = H^{N-1}(\partial B_R(0)) = N\omega_N R^{N-1}$, where H^{N-1} is the $(N-1)$-dimensional Hausdorff measure. In particular, the relative fractal drum (A,Ω) is Minkowski measurable and

$$\mathscr{M}^D(A,\Omega) = H^D(A). \tag{4.1.23}$$

Equation (4.1.21) is a special case of Equation (4.5.13) in the case when $m := 0$ in Theorem 4.5.1 on page 353; see also Equation (4.5.1).

Proposition 4.1.20. (a) *For any relative fractal drum (A,Ω), with $|\Omega| < \infty$, we have*

$$\overline{\dim}_B(A,\Omega) = \overline{\dim}_B(\overline{A},\Omega),$$

and similarly for the relative lower box dimension.

(b) *The Cartesian product $(A_1 \times A_2, \Omega_1 \times \Omega_2)$ of two Minkowski nondegenerate RFDs (A_1,Ω_1) and (A_2,Ω_2), is also Minkowski nondegenerate. Furthermore,*

$$\dim_B(A_1 \times A_2, \Omega_1 \times \Omega_2) = \dim_B(A_1,\Omega_1) + \dim_B(A_2,\Omega_2).$$

Proof. Part (a) follows easily from the fact that $A_t = (\overline{A})_t$ for all $t > 0$, where $\overline{A}$ denotes the closure of A in $\mathbb{R}^N$. Part (b) follows from [Žu2, Proposition 4.3]. □

Some basic open questions about the relative upper box dimension can be found in Problem 6.2.31 of Section 6.2.2.

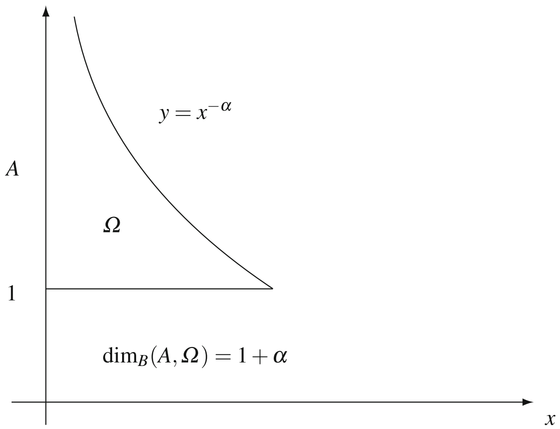

Fig. 4.1 A relative fractal drum (A,Ω) in the plane with relative box dimension $\dim_B(A,\Omega) = 1+\alpha \in (1,2)$, for $\alpha \in (0,1)$; see Example 4.1.21.

Example 4.1.21. Here, we deal with a situation where both of the sets A and Ω are unbounded. This example is based on [Žu2, Example 2.1]. Let $A = \{0\} \times (1,+\infty) \subseteq \mathbb{R}^2$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : x \in (0,1),\ 1 < y < x^{-\alpha}\}$, for some fixed $\alpha \in (0,1)$; see Figure 4.1. Note that Ω is unbounded, but has finite two-dimensional Lebesgue measure. The relative distance zeta function is then given by

$$\begin{aligned}\zeta_{A,\Omega}(s) &= \iint_\Omega d((x,y),A)^{s-2}\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1 x^{s-2}\mathrm{d}x\int_1^{x^{-\alpha}}\mathrm{d}y = \int_0^1 (x^{s-2-\alpha}-x^{s-2})\,\mathrm{d}x \qquad (4.1.24)\\ &= \frac{1}{s-1-\alpha}-\frac{1}{s-1} \sim \frac{1}{s-1-\alpha},\end{aligned}$$

where in the computation of the double integral, we have assumed that $\mathrm{Re}\,s > 1+\alpha$. It follows that $\sigma = 1+\alpha$ is the abscissa of convergence of the relative zeta function $\zeta_{A,\Omega}$: $D(\zeta_{A,\Omega}) = 1+\alpha$. Therefore (see (4.1.10)), the half-line A has a nontrivial relative box-dimension with respect to Ω, given by

$$\overline{\dim}_B(A,\Omega) = D(\zeta_{A,\Omega}) = 1+\alpha.$$

It is not difficult to show that a stronger result holds; namely, $\dim_B(A,\Omega)$ exists, $\dim_B(A,\Omega) = 1+\alpha$, and the relative fractal drum (A,Ω) is Minkowski measurable.

It follows from the above discussion that the set $\mathscr{P}_c(\zeta_{A,\Omega})$ of relative principal complex dimensions of the half-line A (with respect to the open unit square Ω) is given by

$$\mathscr{P}_c(\zeta_{A,\Omega}) = \{1+\alpha\}.$$

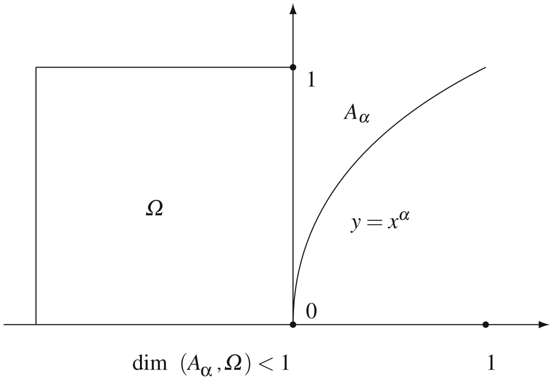

Fig. 4.2 A relative fractal drum (A_α, Ω) such that $\dim_B(A_\alpha, \Omega) = 1 - \alpha < 1$ (here, $0 < \alpha < 1$), whereas $\dim_B A_\alpha = 1$; see Example 4.1.23. This illustrates the *drop of dimension phenomenon* for relative Minkowski dimensions.

Actually, a more precise result holds. Indeed, note that according to the last equality of (4.1.24), $\zeta_{A,\Omega}$ has a meromorphic continuation to all of $\mathbb{C}$ (given by the right-hand side of the last equality of (4.1.24)), and furthermore, the set $\mathscr{P}(\zeta_{A,\Omega})$ of all relative complex dimensions of (A, Ω) is given by

$$\mathscr{P}(\zeta_{A,\Omega}) = \{1, 1+\alpha\} = \{1+\alpha\} \cup \{1\},$$

the union of $\{1+\alpha\}$, the set of scaling complex dimensions, and $\{1\}$, the set of positive integer dimensions (in the sense of [LapPe2–3] and [LapPeWi1], see also [Lap-vFr3, Section 13.1]). We point out, however, that the theory of [LapPe2, LapPeWi1] cannot be applied to the present example in order to also yield this result. Hence, the relative 'fractal drum' (A, Ω) is not fractal (in the sense of [Lap-vFr1–3]) since it does not have any nonreal principal complex dimensions, which is, of course, natural since both A and Ω are standard Euclidean geometric shapes.

Example 4.1.22. Let Ω be the same as in the preceding example, and define $A = \{(x,y) \in (-1,0) \times \mathbb{R} : y = |x|^{-\alpha}\}$, where $\alpha \in (0,1)$ is fixed. Here, we also have that $\dim_B(A, \Omega)$ exists and

$$\dim_B(A, \Omega) = D(\zeta_{A,\Omega}) = 1 + \alpha.$$

Note that now, the sets $\overline{A}$ and $\overline{\Omega}$ are disjoint.

It is clear that in the case of a bounded set A, we have $\overline{\dim}_B(A, \Omega) \le \overline{\dim}_B A$, and analogously for the lower box dimension. The following example shows that the inequality may be strict.

Example 4.1.23. We provide here an example showing that a smooth rectifiable curve (see part (a) of Remark 4.1.24 below) may have a relative box dimension strictly less than one, whereas its (ordinary) box dimension is equal to one. This example illustrates what we propose to refer to as *the drop of dimension phenomenon*, which is frequently encountered in the context of RFDs. For an even more dramatic example of this important and surprising phenomenon, see Corollary 4.1.38 and Remark 4.1.39 on pages 265–266.

Let $\Omega = (-1,0)\times(0,1)$ and let A_α be the graph of a Hölder continuous function $y = x^\alpha$, $0 < x < 1$, for a fixed $\alpha \in (0,1)$; see Figure 4.2. Then, the relative box dimension of the curve A_α (with respect to Ω) exists and is given by

$$\dim_B(A_\alpha,\Omega) = 1-\alpha.$$

Note that, in contrast, $\dim_B A_\alpha = 1$, independently of the value of $\alpha \in (0,1)$, since A_α is clearly rectifiable, i.e., of finite length. (See part (b) of Remark 4.1.24 below.) Also, it is worth noting that A_α and Ω are disjoint. The relative zeta function $\zeta_{A_\alpha,\Omega}(s)$ is holomorphic on the half-plane $\operatorname{Re} s > 1-\alpha$, and the bound is optimal:

$$D(\zeta_{A_\alpha,\Omega}) = \dim_B(A_\alpha,\Omega) = 1-\alpha.$$

Remark 4.1.24. (a) Note that A_α is a C^∞-curve, since it does not contain the origin. Furthermore, the curve $\overline{A}_\alpha$ is at least of class C^1 (more precisely, of class C^k with $k = \lfloor 1/\alpha \rfloor$), since it can be viewed as the graph of the function $x = y^{1/\alpha}$ for $y \in [0,1]$, where the exponent $1/\alpha$ is larger than 1 (and in particular, the function is Lipschitz continuous).

(b) The length of A_α is bounded by the sum of its projections onto the vertical and horizontal axes, that is, by 2. If a curve is rectifiable (i.e. of finite length), then its graph has box (i.e., Minkowski) dimension equal to 1; see, e.g., Federer [Fed2, Theorem 3.2.39] for a more general statement concerning k-rectifiable sets. Namely, the Minkowski (or box) dimension of a closed and *k-rectifiable set* (i.e., of the image in $\mathbb{R}^N$ under a Lipschitz map of a bounded set in $\mathbb{R}^k$) exists and does not exceed k, and, moreover, its k-dimensional Minkowski content exists and is finite. Here, we have $k = 1$, $N = 2$ and, clearly, the Minkowski dimension of a smooth curve is not smaller than 1.

Example 4.1.25. Slightly modifying the above example, let us set $A' = \{0\}\times(0,1)$ and consider the family of open sets $\Omega'_\alpha = \{(x,y) \in (0,1)^2 : y < x^\alpha\}$, where $\alpha \in (0,1)$. Then

$$D(\zeta_{A',\Omega'_\alpha}) = \dim_B(A',\Omega'_\alpha) = 1-\alpha.$$

This shows that the relative box dimension depends on the domain Ω'_α.

In the following proposition, we extend the well-known property of *finite stability* of the usual upper box dimension $\overline{\dim}_B A$ (see, e.g., [Fal1]) to the more general case of the relative upper box dimension $\overline{\dim}_B(A,\Omega)$; see Equation (4.1.25) in Proposition 4.1.26 below. The claim is not true for the relative lower box dimension $\underline{\dim}_B(A,\Omega)$; see the discussion immediately following Equation (6.1.8) in Subsection 6.1.2 of Chapter 6 below.

Proposition 4.1.26 (Finite stability of the relative upper box dimension). *Let (A,Ω) and (B,Ω) be two relative fractal drums in $\mathbb{R}^N$. Then $(A\cup B,\Omega)$ is an RFD as well, and the following property of* finite stability of the relative upper box dimension *holds*:

$$\overline{\dim}_B(A\cup B,\Omega)=\max\{\overline{\dim}_B(A,\Omega),\overline{\dim}_B(B,\Omega)\}. \tag{4.1.25}$$

Moreover, for any real number $s\in\mathbb{R}$, we have that

$$\max\{\mathscr{M}^{*s}(A,\Omega),\mathscr{M}^{*s}(B,\Omega)\}\le\mathscr{M}^{*s}(A\cup B,\Omega)\le\mathscr{M}^{*s}(A,\Omega)+\mathscr{M}^{*s}(B,\Omega). \tag{4.1.26}$$

Proof. Since (A,Ω) and (B,Ω) are RFDs, then $\Omega\subseteq A_\delta$ and $\Omega\subseteq B_\delta$ for some $\delta>0$; hence, $\Omega\subseteq A_\delta\cup B_\delta=(A\cup B)_\delta$. Therefore, $(A\cup B,\Omega)$ is an RFD as well.

Let us first prove the two inequalities appearing in (4.1.26). The first one follows immediately from the two inclusions $A\subseteq A\cup B$ and $B\subseteq A\cup B$, while the second one is an easy consequence of the fact that $(A\cup B)_t=A_t\cup B_t$, for all $t>0$:

$$\begin{aligned}\mathscr{M}^{*s}(A\cup B,\Omega)&=\limsup_{t\to0^+}\frac{|(A\cup B)_t\cap\Omega|}{t^{N-s}}=\limsup_{t\to0^+}\frac{|(A_t\cup B_t)\cap\Omega|}{t^{N-s}}\\&\le\limsup_{t\to0^+}\Big(\frac{|A_t\cap\Omega|}{t^{N-s}}+\frac{|B_t\cap\Omega|}{t^{N-s}}\Big)\\&\le\limsup_{t\to0^+}\frac{|A_t\cap\Omega|}{t^{N-s}}+\limsup_{t\to0^+}\frac{|B_t\cap\Omega|}{t^{N-s}}\\&=\mathscr{M}^{*s}(A,\Omega)+\mathscr{M}^{*s}(B,\Omega).\end{aligned} \tag{4.1.27}$$

Now, Equation (4.1.25), which we write as $L=R$, follows easily from Equation (4.1.26). Indeed, assume that (4.1.25) does not hold, i.e., that $L\neq R$. Let us consider the following two cases:

(a) If $L<R$ in (4.1.25), then by taking any real number $s\in(L,R)$, we have that $\mathscr{M}^{*s}(A\cup B,\Omega)=0$ and either $\mathscr{M}^{*s}(A,\Omega)=+\infty$ or $\mathscr{M}^{*s}(B,\Omega)=+\infty$. However, this is impossible, due to the first inequality in (4.1.26).

(b) If $L>R$, then by taking any real number $s\in(R,L)$, we obtain that $\mathscr{M}^{*s}(A\cup B,\Omega)=+\infty$ and $\mathscr{M}^{*s}(A,\Omega)=\mathscr{M}^{*s}(A,\Omega)=0$. This is also impossible, due to the second inequality in (4.1.26).

This completes the proof of Equation (4.1.25), as well as of the proposition. □

Remark 4.1.27. If (A,Ω_1) and (B,Ω_2) are two relative fractal drums in $\mathbb{R}^N$ such that for some $\varepsilon>0$, $A_\varepsilon\cap\Omega_2=\emptyset$ and $B_\varepsilon\cap\Omega_1=\emptyset$, then the property of *finite stability* holds in the following sense:

$$\overline{\dim}_B(A\cup B,\Omega_1\cup\Omega_2)=\max\{\overline{\dim}_B(A,\Omega_1),\overline{\dim}_B(B,\Omega_2)\}. \tag{4.1.28}$$

Moreover, for any real number $s \in \mathbb{R}$, we have that

$$\begin{aligned} \max\{\mathscr{M}^{*s}(A,\Omega_1),\mathscr{M}^{*s}(B,\Omega_2)\} &\le \mathscr{M}^{*s}(A\cup B,\Omega_1\cup\Omega_2) \\ \mathscr{M}^{*s}(A\cup B,\Omega_1\cup\Omega_2) &\le \mathscr{M}^{*s}(A,\Omega_1)+\mathscr{M}^{*s}(B,\Omega_2). \end{aligned} \tag{4.1.29}$$

Note, however, that Equations (4.1.28) and (4.1.29), jointly with the indicated assumptions, do not contain Proposition 4.1.25 as a special case.

In order to prove (4.1.29), it suffices to observe that for any $t \in (0,\varepsilon)$, we have that $(A\cup B)_t \cap(\Omega_1\cap\Omega_2) = (A_t\cap\Omega_1)\cup(B_t\cap\Omega_2)$, and then to proceed analogously as in the first part of the proof of Proposition 4.1.26. Equation (4.1.28) follows from (4.1.29) and the arguments from the second part of the proof of the proposition.

4.1.2 Cone Property and Flatness of Relative Fractal Drums

We introduce the cone property of a relative fractal drum (A,Ω) at a point, in order to ensure that the abscissa of convergence of the associated relative zeta function $\zeta_{A,\Omega}$ be nonnegative. The main result of this subsection is stated in Proposition 4.1.33. We also construct a simple class of RFDs for which the relative box dimension is negative; see Proposition 4.1.35.

Definition 4.1.28. Let $B_r(a)$ be a given ball in $\mathbb{R}^N$ of radius r. Let ∂B be the boundary of the ball, which is an $(N-1)$-dimensional sphere, and assume that G is a closed connected subset contained in a hemisphere of ∂B. [Intuitively, G is a disk-like subset ('calotte') of a hemisphere contained in the sphere ∂B.] We assume that G is open with respect to the relative topology of ∂B. The *cone* $K = K_r(a,G)$ *with vertex at* a, and of radius r, is defined as the interior of the convex hull of the union of $\{a\}$ and G.

Definition 4.1.29. Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. We say that a *relative fractal drum* (A,Ω) *has the cone property at a point* $a \in \overline{A}\cap\overline{\Omega}$ if there exists $r>0$ such that Ω contains a cone $K_r(a,G)$ with vertex at a (and of radius r).

Remark 4.1.30. If $a\in\overline{A}\cap\Omega$ (hence, a is an inner point of Ω), then the cone property of the relative fractal drum (A,Ω) is obviously satisfied at this point. So, the cone property is actually interesting only on the boundary of Ω, that is, at $a\in\overline{A}\cap\partial\Omega$.

Example 4.1.31. Given $\alpha>0$, let (A,Ω_α) be the relative fractal drum in $\mathbb{R}^2$ defined by $A=\{(0,0)\}$ and $\Omega_\alpha=\{(x,y)\in\mathbb{R}^2 : 0<y<x^\alpha,\ x\in(0,1)\}$. If $0<\alpha\le 1$, then the cone property of (A,Ω) is fulfilled at $a=(0,0)$, while it is not satisfied (at $a=(0,0)$), for $\alpha>1$. Using these domains, we can construct a one-parameter family of RFDs with negative relative box dimension; see Proposition 4.1.35 below.

Proposition 4.1.33 below is an extension of Lemma 2.1.52, which states that $D(\zeta_A)\ge 0$ for any bounded set A. We first need an auxiliary result.

Lemma 4.1.32. *Assume that $K = K_r(a,G)$ is an open cone in $\mathbb{R}^N$ with vertex at a (and of radius $r > 0$), and $f \in L^1(0,r)$ is a nonnegative function. Then there exists a positive integer m, depending only on N and on the opening angle of the cone, such that*

$$\int_{B_r(a)} f(|x-a|)\,\mathrm{d}x \le m \int_K f(|x-a|)\,\mathrm{d}x. \tag{4.1.30}$$

Proof. Since the sphere ∂B is compact, there exist finitely many calottes $G_1,\dots,G_m$ contained in the sphere, that are all congruent to G (that is, each G_i can be obtained from G by a rigid motion), and which cover ∂B. Let $K_i = K_r(a,G_i)$, $i = 1,\dots,m$, be the corresponding cones with vertex at a. It is clear that the value of

$$\int_{K_i} f(|x-a|)\,\mathrm{d}x \tag{4.1.31}$$

does not depend on i. Since $B_r(a) = \cup_{i=1}^m K_i$, we have

$$\int_{B_r(a)} f(|x-a|)\,\mathrm{d}x \le \sum_{i=1}^m \int_{K_i} f(|x-a|)\,\mathrm{d}x = m \int_K f(|x-a|)\,\mathrm{d}x. \tag{4.1.32}$$

□

Proposition 4.1.33. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Then:*

(*a*) *If the sets A and Ω are a positive distance apart (i.e., if $d(A,\Omega) > 0$), then $D(\zeta_{A,\Omega}) = -\infty$; that is, $\zeta_{A,\Omega}$ is an entire function. Furthermore,* $\dim_B(A,\Omega) = -\infty$.

(*b*) *Assume that there exists at least one point $a \in \overline{A} \cap \overline{\Omega}$ at which the relative fractal drum (A,Ω) satisfies the cone property. Then $D(\zeta_{A,\Omega}) \ge 0$.*

Proof. (*a*) For $r > 0$ small enough such that $r < d(A,\Omega)$, where $d(A,\Omega)$ is the distance between A and Ω, we have $A_r \cap \Omega = \emptyset$; so that $\zeta_{A,A_r\cap\Omega}(s) \equiv 0$ for all $s \in \mathbb{C}$. Therefore, $D(\zeta_{A,A_r\cap\Omega}) = -\infty$. Since $\zeta_{A,\Omega}(s) - \zeta_{A,A_r\cap\Omega}(s)$ is an entire function, we conclude that we also have that $D(\zeta_{A,\Omega}) = -\infty$. Since $|A_\varepsilon \cap \Omega| = 0$ for all sufficiently small $\varepsilon > 0$, we have $\mathscr{M}^r(A,\Omega) = 0$ for all $r \in \mathbb{R}$, and therefore, $\dim_B(A,\Omega) = -\infty$.

(*b*) Assume, by reasoning by contradiction, that $D(\zeta_{A,\Omega}) < 0$. In particular, $\zeta_{A,\Omega}(s)$ is continuous at $s = 0$ (because it must then be holomorphic at $s = 0$). By hypothesis, there exists an open cone $K = K_r(a,G)$, such that $K \subset \Omega$. Using the inequality $d(x,A) \le |x-a|$ (valid for all $x \in \mathbb{R}^N$ since $a \in \Omega$) and Lemma 4.1.32, we deduce that for any real number $s \in (0,N)$,

$$\begin{aligned}\zeta_{A,\Omega}(s) \ge \zeta_{A,K}(s) &= \int_K d(x,A)^{s-N}\mathrm{d}x \ge \int_K |x-a|^{s-N}\mathrm{d}x \\ &\ge \frac{1}{m}\int_{B_r(a)} |x-a|^{s-N}\mathrm{d}x = \frac{N\omega_N}{m} r^s s^{-1},\end{aligned}$$

where m is the positive constant appearing in Equation (4.1.30) of Lemma 4.1.32. This implies that $\zeta_{A,\Omega}(s) \to +\infty$ as $s \to 0^+$, $s \in \mathbb{R}$, which contradicts the holomorphicity (or simply, the continuity) of $\zeta_{A,\Omega}(s)$ at $s = 0$. □

The cone condition can be replaced by a much weaker condition, as we will now explain in the following proposition.

Proposition 4.1.34. *Let $(r_k)_{k\geq 0}$ be a decreasing sequence of positive real numbers, converging to zero. We define a subset of the cone $K_r(a,G)$ as follows*:

$$K_r(a,G,(r_k)_{k\geq 0}) = \Big\{x \in K_r(a,G) : |x-a| \in \bigcup_{k=0}^{\infty}(r_{2k}, r_{2k+1})\Big\}. \tag{4.1.33}$$

If we assume that the sequence $(r_k)_{k\geq 1}$ is such that

$$\sum_{k=0}^{\infty}(-1)^k r_k^s \to L > 0 \quad \text{as} \quad s \to 0^+, s \in \mathbb{R}, \tag{4.1.34}$$

then the conclusion of Proposition 4.1.33(b) still holds, with the cone condition involving $K := K(a,G)$ replaced by the above modified cone condition, involving the set $K' := K_r(a,G,(r_k)_{k\geq 0})$ contained in K.

Proof. In order to establish this claim, it suffices to use a procedure analogous to the one used in the proof of Proposition 4.1.33:

$$\begin{aligned}\zeta_{A,\Omega}(s) &\geq \int_{K'} |x-a|^{s-N}\mathrm{d}x \geq \frac{1}{m}\sum_{k=0}^{\infty}\int_{B_{r_{2k}}(a)\setminus B_{r_{2k+1}}(s)} |x-a|^{s-N}\mathrm{d}x \\ &= \frac{N\omega_N}{m}s^{-1}\sum_{k=0}^{\infty}(r_{2k}^s - r_{2k+1}^s) = \frac{N\omega_N}{m}s^{-1}\sum_{k=0}^{\infty}(-1)^k r_k^s.\end{aligned}$$

For example, if $r_k = 2^{-k}$, then condition (4.1.34) is fulfilled since

$$\sum_{k=0}^{\infty}(-1)^k r_k^s = \sum_{k=0}^{\infty}(-1)^k 2^{-ks} = \frac{1}{1+2^{-s}} \to \frac{1}{2} \quad \text{as} \quad s \to 0^+, s \in \mathbb{R}.$$

This concludes the proof of the proposition. □

The following proposition (building on Example 4.1.31 above) shows that the box dimension of a relative fractal drum can be negative. It also shows that the analog of Lemma 2.1.52 does not hold for arbitrary RFDs.

Proposition 4.1.35. *Let $A = \{(0,0)\}$ and*

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < y < x^{\alpha},\ x \in (0,1)\}, \tag{4.1.35}$$

where $\alpha > 1$; see Figure 4.3. Then the relative fractal drum (A,Ω) has a negative box dimension. More specifically, $\dim_B(A,\Omega)$ *exists, the relative fractal drum (A,Ω) is Minkowski measurable and*

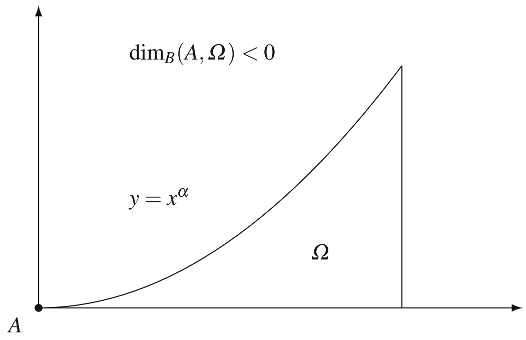

Fig. 4.3 A relative fractal drum (A,Ω) with negative box dimension $\dim_B(A,\Omega)=1-\alpha<0$ (here $\alpha>1$), due to the 'flatness' of the open set Ω at A; see Proposition 4.1.35. This provides a further illustration of *the drop in dimension phenomenon* (for relative box dimensions).

$$\begin{aligned} \dim_B(A,\Omega) &= D(\zeta_{A,\Omega}) = 1-\alpha<0,\\ \mathscr{M}^{1-\alpha}(A,\Omega) &= \frac{1}{1+\alpha},\\ D_{\rm mer}(\zeta_{A,\Omega}) &\le 3(1-\alpha). \end{aligned} \tag{4.1.36}$$

Furthermore, $s=1-\alpha$ *is a simple pole of* $\zeta_{A,\Omega}$.

Proof. First note that $A_\varepsilon=B_\varepsilon((0,0))$. Therefore, for every $\varepsilon>0$, we have

$$|A_\varepsilon\cap\Omega|\le\int_0^\varepsilon x^\alpha\mathrm{d}x=\frac{\varepsilon^{\alpha+1}}{\alpha+1}.$$

If we choose a point $(x(\varepsilon),y(\varepsilon))$ such that

$$(x(\varepsilon),y(\varepsilon))\in\partial(A_\varepsilon)\cap\{(x,y):y=x^\alpha,\ x\in(0,1)\},$$

then the following equation holds:

$$x(\varepsilon)^2+x(\varepsilon)^{2\alpha}=\varepsilon^2. \tag{4.1.37}$$

It is clear that

$$|A_\varepsilon\cap\Omega|\ge\int_0^{x(\varepsilon)} x^\alpha\mathrm{d}x=\frac{x(\varepsilon)^{\alpha+1}}{\alpha+1}.$$

Letting $D:=1-\alpha$, we conclude that

$$\frac{1}{\alpha+1}\Big(\frac{x(\varepsilon)}{\varepsilon}\Big)^{\alpha+1}\le\frac{|A_\varepsilon\cap\Omega|}{\varepsilon^{2-D}}\le\frac{1}{\alpha+1},\quad \text{for all}\quad \varepsilon>0. \tag{4.1.38}$$

We deduce from (4.1.37) that $x(\varepsilon) \sim \varepsilon$ as $\varepsilon \to 0^+$, since

$$\frac{x(\varepsilon)}{\varepsilon} = (1 + x(\varepsilon)^{2(\alpha-1)})^{-1/2} \to 1 \quad \text{as} \quad \varepsilon \to 0^+, \tag{4.1.39}$$

and therefore, (4.1.38) implies that $\dim_B(A,\Omega) = D$ and $\mathscr{M}^D(A,\Omega) = 1/(\alpha+1)$.

Using (4.1.38) again, we have that

$$0 \le f(\varepsilon) := \frac{1}{\alpha+1} - \frac{|A_\varepsilon \cap \Omega|}{\varepsilon^{2-D}} \le \frac{1}{\alpha+1}\Big(1 - \Big(\frac{x(\varepsilon)}{\varepsilon}\Big)^{\alpha+1}\Big). \tag{4.1.40}$$

Using (4.1.39) and the binomial expansion, we conclude that

$$\Big(\frac{x(\varepsilon)}{\varepsilon}\Big)^{\alpha+1} = 1 - \frac{\alpha+1}{2} x(\varepsilon)^{2\alpha-2} + o(x(\varepsilon)^{2\alpha-2}) \quad \text{as} \quad \varepsilon \to 0^+.$$

Therefore, we deduce from (4.1.40) that

$$f(\varepsilon) = O(x(\varepsilon^{2\alpha-2})) = O(\varepsilon^{2\alpha-2}) \quad \text{as} \quad \varepsilon \to 0^+.$$

Since $|A_\varepsilon \cap \Omega| = \varepsilon^{2-D}((\alpha+1)^{-1} + f(\varepsilon))$, by using Theorem 2.3.18 (adjusted to the case of RFDs, see Theorem 4.5.1), we then conclude that

$$D_{\mathrm{mer}}(\zeta_{A,\Omega}) \le D - (2\alpha - 2) = 3(1-\alpha).$$

Furthermore, $s = D$ is a simple pole.

Finally, we note that the equality $D(\zeta_{A,\Omega}) = D$ follows from (4.1.10). □

Example 4.1.36. Let (A,Ω) be the relative fractal drum in $\mathbb{R}^2$ defined by $A = \{(0,0)\}$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < y < x^2,\ x \in (0,1)\}$; see Figure 4.2, for $\alpha = 2$. This relative fractal drum does not satisfy the cone property (at any point). (Note that since $\overline{A} \cap \partial\Omega = \{(0,0)\}$, it suffices to check that (A,Ω) does not have the cone property at $a = (0,0)$, which is the case since $2 > 1$; see Remark 4.1.30 and Example 4.1.31.) According to Proposition 4.1.35, its relative box dimension is equal to -1. We will show directly that the relative distance zeta function $\zeta_{A,\Omega}(s)$ is well defined at $s = 0$, and equal to Catalan's constant. First, using polar coordinates (r,θ), we obtain that for every $s > 0$,

$$\begin{aligned}\zeta_{A,\Omega}(s) &= \int_\Omega d((x,y),A)^{s-2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{x^2} (\sqrt{x^2+y^2})^{s-2}\,\mathrm{d}y \\ &= \int_0^{\pi/4} \mathrm{d}\theta \int_{\tan\theta/\cos\theta}^{1/\cos\theta} r^{s-1}\,\mathrm{d}r = \frac{1}{s}\int_0^{\pi/4} \frac{1-\tan^s\theta}{\cos^s\theta}\,\mathrm{d}\theta.\end{aligned}$$

The function under the integral sign is dominated by a constant (independent of s), so we conclude from the Lebesgue dominated convergence theorem that the integral in the last expression above converges to zero. We can now apply l'Hospital's rule and differentiate under the integral sign in order to compute the limit at $s = 0$:

$$\begin{aligned}\lim_{s\to 0^+}\zeta_{A,\Omega}(s) &= \lim_{s\to 0^+}\int_0^{\pi/4}\frac{\partial}{\partial s}\left(\frac{1-\tan^s\theta}{\cos^s\theta}\right)\mathrm{d}\theta\\ &= \lim_{s\to 0^+}\int_0^{\pi/4}\left[\left(\frac{\tan\theta}{\cos\theta}\right)^s\log(\cot\theta)+\frac{\log(\cos\theta)}{\cos^s\theta}(\tan^s\theta-1)\right]\mathrm{d}\theta\\ &= \int_0^{\pi/4}\log(\cot\theta)\,\mathrm{d}\theta=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.\end{aligned}$$

The next-to-last equality again follows from an application of Lebesgue's dominated convergence theorem, while the last sum is *Catalan's constant*, which is approximately equal to 0.915.

In the following lemma, we show that for any $\delta>0$, the respective sets of principal complex dimensions corresponding to RFDs (A,Ω) and $(A,A_\delta\cap\Omega)$ coincide.

Lemma 4.1.37. *Assume that (A,Ω) is a relative fractal drum in $\mathbb{R}^N$. Then for any $\delta>0$ we have*

$$\zeta_{A,\Omega}(s)\sim\zeta_{A,A_\delta\cap\Omega}(s). \tag{4.1.41}$$

In particular,

$$\dim_{PC}(A,\Omega)=\dim_{PC}(A,A_\delta\cap\Omega) \tag{4.1.42}$$

and therefore,

$$\overline{\dim}_B(A,\Omega)=\overline{\dim}_B(A,A_\delta\cap\Omega). \tag{4.1.43}$$

Here, the δ-neighborhood of A can be taken with respect to any norm on $\mathbb{R}^N$.[3]

Proof. Recall that according to the definition of a relative fractal drum (A,Ω), there exists $\delta_1>0$ such that $d(x,A)<\delta_1$ for all $x\in\Omega$; see Definition 4.1.2. On the other hand, we have that $d(x,A)>\delta$ for all $x\in\Omega\setminus A_\delta$. Therefore, by using Theorem 2.1.45 with $\varphi(x):=d(x,A)$ and $\mathrm{d}\mu(x):=d(x,A)^{-N}\mathrm{d}x$, we conclude that the difference

$$\zeta_{A,\Omega}(s)-\zeta_{A,A_\delta\cap\Omega}(s)=\int_{\Omega\setminus A_\delta}d(x,A)^{s-N}\mathrm{d}x$$

defines an entire function. This proves the desired equivalence in (4.1.41). The remaining claims of the lemma follow immediately from this equivalence. Finally, the fact that any norm on $\mathbb{R}^N$ can be chosen to define A_δ follows from the equivalence of all the norms on $\mathbb{R}^N$. ☐

The following result provides an example of a nontrivial relative fractal drum (A,Ω) such that $\dim_B(A,\Omega)=-\infty$. It suffices to construct a domain Ω in $\mathbb{R}^2$ which is *flat* in a connected open neighborhood of one of its boundary points.

Corollary 4.1.38 (A maximally flat RFD). *Let $A=\{(0,0)\}$ and*[4]

$$\Omega'=\{(x,y)\in\mathbb{R}^2:0<y<\mathrm{e}^{-1/x},\ 0<x<1\}. \tag{4.1.44}$$

---

[3] This fact will be used in an essential manner in the proof of Corollary 4.1.38.

[4] The corresponding RFD (A,Ω') is very similar to the RFD (A,Ω) exhibited in Figure 4.2, but now with an extremely sharp spike at the origin.

Then $\dim_B(A,\Omega')$ *exists and*

$$\dim_B(A,\Omega') = D(\zeta_{A,\Omega'}) = -\infty. \tag{4.1.45}$$

Proof. Let us fix $\alpha > 1$. Then, by l'Hospital's rule,

$$\lim_{x\to 0^+} \frac{\mathrm{e}^{-1/x}}{x^\alpha} = \lim_{t\to+\infty} \frac{t^\alpha}{\mathrm{e}^t} = 0.$$

Hence, there exists $\delta = \delta(\alpha) > 0$ such that $0 < \mathrm{e}^{-1/x} < x^\alpha$ for all $x \in (0,\delta)$; that is,

$$\Omega'_{\delta(\alpha)} \subset \Omega_{\delta(\alpha)},$$

where

$$\Omega'_{\delta(\alpha)} := \{(x,y) \in \mathbb{R}^2 : 0 < y < \mathrm{e}^{-1/x},\ 0 < x < \delta(\alpha)\}$$

and

$$\Omega_{\delta(\alpha)} := \{(x,y) \in \mathbb{R}^2 : 0 < y < x^\alpha,\ 0 < x < \delta(\alpha)\}.$$

Using Lemma 4.1.37 (with Ω' instead of Ω and with the ∞-norm on $\mathbb{R}^2$ instead of the usual Euclidean norm)[5] and Proposition 4.1.35, we see that

$$\overline{\dim}_B(A,\Omega') = \overline{\dim}_B(A,\Omega'_{\delta(\alpha)}) \le \dim_B(A,\Omega_{\delta(\alpha)}) = 1-\alpha.$$

The claim follows by letting $\alpha \to +\infty$, since then, we have that

$$-\infty \le \underline{\dim}_B(A,\Omega') \le \overline{\dim}_B(A,\Omega') = -\infty.$$

We conclude, as desired, that $\dim_B(A,\Omega)$ exists and is equal to $-\infty$. ☐

Remark 4.1.39. (*Flatness and 'infinitely sharp blade'*). It is easy to see that Corollary 4.1.38 can be significantly generalized. For example, it suffices to assume that A is a point on the boundary of Ω such that Ω has the *flatness property of A relative to* Ω. This can even be formulated in terms of subsets A. We can imagine a bounded open set $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$, except on a subset $A \subset \partial\Omega$, which may be a line segment, near which Ω is flat. A simple construction of such a set is $\Omega = \Omega' \times (0,1)$, where Ω' is given as in Corollary 4.1.38, and $A = \{(0,0)\} \times (0,1)$; see Equation (4.1.44). Note that this domain is not Lipschitz near the points of A, and not even Hölderian; see Figure 4.4. The *flatness of a relative fractal drum* (A,Ω) can be defined by

$$\mathrm{fl}(A,\Omega) = \left(\overline{\dim}_B(A,\Omega)\right)^-,$$

where $(r)^- := \max\{0,-r\}$ is the negative part of a real number r. *We say that the flatness of* (A,Ω) *is nontrivial if* $\mathrm{fl}(A,\Omega) > 0$*, that is, if* $\overline{\dim}_B(A,\Omega) < 0$. In the example just mentioned above, we have a relative fractal drum (A,Ω) with infinite

[5] Note that $\Omega'_{\delta(\alpha)} = \Omega' \cap B_{\delta(\alpha)}(0)$, where $B_\delta(0) := \{(x,y) \in \mathbb{R}^2 : |(x,y)|_\infty < \delta\}$ and $|(x,y)|_\infty := \max\{|x|,|y|\}$.

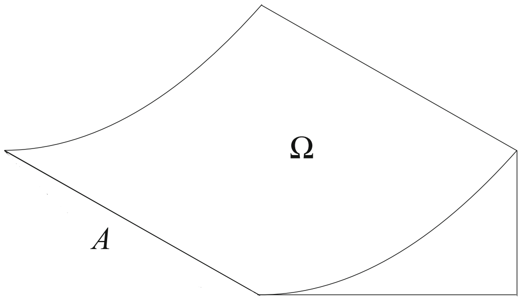

Fig. 4.4 A relative fractal drum (A,Ω) with infinite flatness, as described in Remark 4.1.39. In other words, Ω has infinite flatness near A; equivalently, $\dim_B(A,\Omega) = -\infty$, which provides an even more dramatic illustration of *the drop in dimension phenomenon* (for relative box dimensions).

flatness, i.e., with $\mathrm{fl}(A,\Omega) = +\infty$. Intuitively, it can be viewed as an 'ax' with an 'infinitely sharp' blade.

4.1.3 Scaling Property of Relative Zeta Functions

We start with the following result, which shows that if (A,Ω) is a given relative fractal drum, then for any $\lambda > 0$, the zeta function $\zeta_{\lambda A,\lambda\Omega}(s)$ of the scaled relative fractal drum $(\lambda A,\lambda\Omega)$ is equal to the zeta function $\zeta_{A,\Omega}(s)$ of (A,Ω) multiplied by λ^s. This result extends Proposition 2.1.77.

Theorem 4.1.40 (Scaling property of relative distance zeta functions). *Let $\zeta_{A,\Omega}(s)$ be the relative distance zeta function. Then, for any positive real number λ, we have that $D(\zeta_{\lambda A,\lambda\Omega}) = D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$ and*

$$\zeta_{\lambda A,\lambda\Omega}(s) = \lambda^s \zeta_{A,\Omega}(s), \tag{4.1.46}$$

for $\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)$ *and any* $\lambda > 0$. (*See also Corollary 4.1.42 below for a more general statement.*)

Proof. The claim is established by introducing a new variable $y = x/\lambda$, and by noting that $d(\lambda y, \lambda A) = \lambda\, d(y,A)$, for any $y \in \mathbb{R}^N$ (which is an easy consequence of the homogeneity of the Euclidean norm). Indeed, in light of Remark 4.1.8 or part (b) of Theorem 4.1.7, for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega) = D(\zeta_{A,\Omega})$, we have successively:

$$\begin{aligned}\zeta_{\lambda A,\lambda\Omega}(s) &= \int_{\lambda\Omega} d(x,\lambda A)^{s-N}\mathrm{d}x \\ &= \int_{\Omega} d(\lambda y,\lambda A)^{s-N}\lambda^N \mathrm{d}y \\ &= \lambda^s \int_{\Omega} d(y,A)^{s-N}\mathrm{d}y = \lambda^s \zeta_{A,\Omega}(s).\end{aligned}$$

It follows that (4.1.46) holds and $\zeta_{\lambda A,\lambda\Omega}(s)$ is holomorphic for $\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)$. Since $D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$ (by part (b) of Theorem 4.1.7 and by Remark 4.1.8, as was recalled above), we deduce that $D(\zeta_{\lambda A,\lambda\Omega}) \le D(\zeta_{A,\Omega})$, for every $\lambda > 0$. But then, replacing λ with its reciprocal λ^{-1} in this last inequality, we obtain the reverse inequality,[6] and hence, we conclude that

$$\overline{\dim}_B(A,\Omega) = D(\zeta_{A,\Omega}) = D(\zeta_{\lambda A,\lambda\Omega}),$$

for all $\lambda > 0$, as desired. □

If $\mathscr{L} = (\ell_j)_{j\ge 1}$ is a fractal string and λ is a positive constant, then for the scaled string $\lambda\mathscr{L} := (\lambda\ell_j)_{j\ge 1}$, the corresponding claim in Theorem 4.1.40 is trivial: $\zeta_{\lambda\mathscr{L}}(s) = \lambda^s\zeta_{\mathscr{L}}(s)$, for every $\lambda > 0$. Indeed, by definition of the geometric zeta function of a fractal string (see Equation (2.1.71) in Section 2.1.4), we have

$$\zeta_{\lambda\mathscr{L}}(s) = \sum_{j=1}^{\infty} (\lambda\ell_j)^s = \lambda^s \sum_{j=1}^{\infty} \ell_j^s = \lambda^s\zeta_{\mathscr{L}}(s),$$

for $\mathrm{Re}\, s > D(\zeta_{\mathscr{L}})$. (The exact same argument as above then shows that $D(\zeta_{\mathscr{L}}) = D(\zeta_{\lambda\mathscr{L}})$.) Then, by analytic (i.e., meromorphic) continuation, the same identity continues to hold in any domain to which $\zeta_{\mathscr{L}}$ can be meromorphically extended to the left of the critical line $\{\mathrm{Re}\, s = D(\zeta_{\mathscr{L}})\}$.

Remark 4.1.41. Let $\mathscr{L} := (A,\Omega)$ be a relative fractal drum in $\mathbb{R}^N$ and let $\lambda\mathscr{L} := (\lambda A,\lambda\Omega)$, where $\lambda > 0$. If we define $\zeta_{\mathscr{L}}(s) := \zeta_{A,\Omega}(s) = \int_\Omega d(x,A)^{s-N}\mathrm{d}x$, then we can reformulate Theorem 4.1.40 as follows: $D(\zeta_{\lambda\mathscr{L}}) = D(\zeta_{\mathscr{L}}) = \overline{\dim}_B\mathscr{L}$ and

$$\zeta_{\lambda\mathscr{L}}(s) = \lambda^s\zeta_{\mathscr{L}}(s), \quad \text{for} \quad \mathrm{Re}\, s > \overline{\dim}_B\mathscr{L} \quad \text{and} \quad \lambda > 0. \tag{4.1.47}$$

More explicitly,

$$\zeta_{\lambda A,\lambda\Omega}(s) = \lambda^s\zeta_{A,\Omega}(s), \quad \text{for} \quad \mathrm{Re}\, s > \overline{\dim}_B(A,\Omega) \quad \text{and} \quad \lambda > 0. \tag{4.1.48}$$

Clearly, in light of the principle of analytic continuation, the identities (4.1.47) and (4.1.48) continue to hold for all $s \in U$, where U is any domain of $\mathbb{C}$ to which $\zeta_{\mathscr{L}}$ can be meromorphically continued.

[6] More specifically, we replace (A,Ω) with $(\lambda^{-1}A,\lambda^{-1}\Omega)$ to deduce that for every $\lambda > 0$, $D(\zeta_{A,\Omega}) \le D(\zeta_{\lambda^{-1}A,\lambda^{-1}\Omega})$. We then substitute λ^{-1} for λ in this last inequality in order to obtain the desired reversed inequality: for every $\lambda > 0$, $D(\zeta_{A,\Omega}) \le D(\zeta_{\lambda A,\lambda\Omega})$.

The following result supplements Theorem 4.1.40 in several different and significant ways.

Corollary 4.1.42. *Fix* $\lambda > 0$. *Assume that* $\zeta_{A,\Omega}$ *admits a meromorphic continuation to some open connected neighborhood* U *of the open half-plane* $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$. *Then, so is the case for* $\zeta_{\lambda A,\lambda\Omega}$ *and the identity* (4.1.46) *continues to hold for every* $s \in U$ *which is not a pole of* $\zeta_{A,\Omega}$ *(and hence, of* $\zeta_{\lambda A,\lambda\Omega}$ *as well).*

Moreover, if we assume, for simplicity,[7] *that* ω *is a simple pole of* $\zeta_{A,\Omega}$ *(and hence also, of* $\zeta_{\lambda A,\lambda\Omega}$*), then the following identity holds:*[8]

$$\mathrm{res}(\zeta_{\lambda A,\lambda\Omega},\omega) = \lambda^{\omega}\,\mathrm{res}(\zeta_{A,\Omega},\omega). \tag{4.1.49}$$

Proof. The fact that $\zeta_{\lambda A,\lambda\Omega}$ is holomorphic at a given point $s \in U$ if and only if $\zeta_{A,\Omega}$ is holomorphic at s (i.e., if and only if $\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)$), follows from (4.1.46) and the equality $D(\zeta_{\lambda A,\lambda\Omega}) = D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$. An analogous statement is true if "holomorphic" is replaced with "meromorphic". More specifically, by analytic continuation of (4.1.46), $\zeta_{\lambda A,\lambda\Omega}$ is meromorphic in the domain U (containing the critical line $\{\mathrm{Re}\, s = \overline{\dim}_B(A,\Omega)\}$) if and only if $\zeta_{A,\Omega}$ is meromorphic in U, and then, clearly, identity (4.1.46) continues to hold for every $s \in U$ which is not a pole of $\zeta_{A,\Omega}$ (and hence also, of $\zeta_{\lambda A,\lambda\Omega}$). Therefore, the first part of the corollary is established.

Next, assume that ω is a simple pole of $\zeta_{A,\Omega}$. Then, in light of (4.1.46) and the discussion in the previous paragraph, we have that for all s in a punctured neighborhood of ω (contained in U but not containing any other pole of $\zeta_{A,\Omega}$),

$$(s-\omega)\zeta_{\lambda A,\lambda\Omega}(s) = \lambda^{s}\left((s-\omega)\zeta_{A,\Omega}(s)\right). \tag{4.1.50}$$

The fact that (4.1.49) holds now follows by letting $s \to \omega$, $s \neq \omega$ in (4.1.50). Indeed, we then have

$$\mathrm{res}(\zeta_{A,\Omega},\omega) = \lim_{s\to\omega}(s-\omega)\zeta_{A,\Omega}(s),$$

and similarly for $\mathrm{res}(\zeta_{\lambda A,\lambda\Omega},\omega)$. □

This important scaling property of distance zeta functions of RFDs, established in Theorem 4.1.40 and Corollary 4.1.42, is analogous to the well-known scaling property of Hausdorff measure in Euclidean space (see, e.g., [Fal2]), but note that in the spirit of the theory of complex fractal dimensions, it now holds for all *complex* values of s (rather than just for the Hausdorff fractal dimension in the case of Hausdorff measure). See, in addition, identity (4.1.49) of Corollary 4.1.42 where a corresponding scaling property also holds for the complex fractal dimensions themselves, at the level of the residues.

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[7] If s is a multiple pole, then an analogous scaling property holds for the principal parts (instead of the residues) of the zeta functions involved, as the reader can easily verify.

[8] If we use the notation $\mathscr{L} := (A,\Omega)$ and $\lambda\mathscr{L} := (\lambda A,\lambda\Omega)$ from Remark 4.1.41, Equation (4.1.49) can be written more compactly as $\mathrm{res}(\zeta_{\lambda\mathscr{L}},\omega) = \lambda^{\omega}\,\mathrm{res}(\zeta_{\mathscr{L}},\omega)$.

The scaling property of relative zeta functions (established in Theorem 4.1.40 and Corollary 4.1.42) motivates us to introduce the notion of relative fractal spray (Definition 4.2.1), which is very close to (but also subtly different from) the usual notion of fractal spray introduced by the first author and Carl Pomerance in [LapPo3] (see [Lap-vFr3] and the references therein). First, we define the operation of union of (disjoint) families of RFDs (Definition 4.1.43).

Definition 4.1.43. (*Union of relative fractal drums*). Let $(A_j,\Omega_j)_{j\geq 1}$ be a countable family of relative fractal drums in $\mathbb{R}^N$, such that the corresponding family of open sets $(\Omega_j)_{j\geq 1}$ is disjoint (i.e., $\Omega_j\cap\Omega_k=\emptyset$ for $j\neq k$), and the set $\Omega:=\cup_{j=1}^{\infty}\Omega_j$ is of finite N-dimensional Lebesgue measure (but may be unbounded). Then, the *union of the* (finite or countable) *family of relative fractal drums* (A_j,Ω_j) $(j\geq 1)$ is the relative fractal drum (A,Ω), where $A:=\cup_{j=1}^{\infty}A_j$ and $\Omega:=\cup_{j=1}^{\infty}\Omega_j$. We write

$$(A,\Omega)=\bigcup_{j=1}^{\infty}(A_j,\Omega_j). \tag{4.1.51}$$

It is easy to derive the following countable additivity property of the distance zeta functions.

Theorem 4.1.44. *Assume that* $(A_j,\Omega_j)_{j\geq 1}$ *is a finite or countable family of RFDs satisfying the conditions of Definition 4.1.43, and let* (A,Ω) *be its union* (*in the sense of Definition 4.1.43*). *Furthermore, assume that the following condition is fulfilled*:

$$\text{For any } j\in\mathbb{N} \text{ and } x\in\Omega_j, \text{ we have that } d(x,A)=d(x,A_j). \tag{4.1.52}$$

Then, for all $s\in\mathbb{C}$ *such that* $\operatorname{Re}s>\overline{\dim}_B(A,\Omega)$, *we have*

$$\zeta_{A,\Omega}(s)=\sum_{j=1}^{\infty}\zeta_{A_j,\Omega_j}(s). \tag{4.1.53}$$

Condition (4.1.52) *is satisfied, for example, if for every* $j\in\mathbb{N}$, A_j *is equal to the boundary of* Ω_j *in* $\mathbb{R}^N$*; that is,* $A_j=\partial\Omega_j$.

Proof. The claim follows from the following computation, which is valid for $\operatorname{Re}s>\overline{\dim}_B(A,\Omega)$:

$$\begin{aligned}\zeta_{A,\Omega}(s)&=\int_{\Omega}d(x,A)^{s-N}\mathrm{d}x=\sum_{j=1}^{\infty}\int_{\Omega_j}d(x,A)^{s-N}\mathrm{d}x\\&=\sum_{j=1}^{\infty}\int_{\Omega_j}d(x,A_j)^{s-N}\mathrm{d}x=\sum_{j=1}^{\infty}\zeta_{A_j,\Omega_j}(s).\end{aligned} \tag{4.1.54}$$

More specifically, clearly, (4.1.54) holds for s real such that $s>\overline{\dim}_B(A,\Omega)\geq D(\zeta_{A,\Omega})$. Therefore, for such a value of s,

$$\zeta_{A,\Omega_j}(s)=\int_{\Omega_j}d(x,A)^{s-N}\mathrm{d}x\leq\int_{\Omega}d(x,A)^{s-N}\mathrm{d}x=\zeta_{A,\Omega}(s)<\infty,$$

for every $j \geq 1$. Hence,

$$\sup_{j\geq 1}\{D(\zeta_{A,\Omega_j})\} \leq D(\zeta_{A,\Omega}) \leq \overline{\dim}_B(A,\Omega), \tag{4.1.55}$$

from which (4.1.54) now follows for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \overline{\dim}_B(A,\Omega)$, in light of the countable additivity of the complex Borel measure (and hence, bounded measure) on Ω, given by $\mathrm{d}\gamma(x) := d(x,A)^{s-N}\mathrm{d}x$. Note that according to the hypothesis of Definition 4.1.43, we have $|\Omega| < \infty$, so that $d\gamma$ is indeed a complex Borel measure; see, e.g., [Foll] or [Ru]. ☐

Remark 4.1.45. In the statement of Theorem 4.1.44, the numerical series on the right-hand side of (4.1.53) converges absolutely (and hence, converges also in $\mathbb{C}$) for $\operatorname{Re} s > \overline{\dim}_B(A,\Omega)$. In particular, for s real such that $s > \overline{\dim}_B(A,\Omega)$, it is a convergent series of positive terms (i.e., it has a finite sum). It remains to be investigated whether (and under which hypotheses) Equation (4.1.53) continues to hold for all $s \in \mathbb{C}$ in a common domain of meromorphicity of the zeta functions $\zeta_{A,\Omega}$ and ζ_{A_j,Ω_j} for $j \geq 1$ (away from the poles). At the poles, an analogous question could be raised for the corresponding residues (assuming, for simplicity, that the poles are simple). We will encounter a similar issue when discussing 'local distance zeta functions' in Appendix B.

Since, among other things, Theorem 4.1.44 gives a way to compute the distance zeta function of a given relative fractal drum if it can be appropriately subdivided into a disjoint union of relative fractal 'subdrums', we introduce the following important definition.

Definition 4.1.46. (*Disjoint union of relative fractal drums*). Let the conditions of Definition 4.1.43 be satisfied and also assume that condition (4.1.52) is satisfied (so that the conclusion of Theorem 4.1.44 holds). Then, we call the union given in (4.1.51) a *disjoint union of relative fractal drums* and write

$$(A,\Omega) = \bigsqcup_{j=1}^{\infty}(A_j,\Omega_j). \tag{4.1.56}$$

Furthermore, in the special case when for every $j \in \mathbb{N}$, we have that $(A_j,\Omega_j) = \lambda_j(A_0,\Omega_0)$ for some sequence of positive numbers $(\lambda_j)_{j\geq 1}$ and some given relative fractal drum (A_0,Ω_0), we will slightly abuse the notation and write

$$(A,\Omega) = \bigsqcup_{j=1}^{\infty}\lambda_j(A_0,\Omega_0), \tag{4.1.57}$$

in the sense that the scaled RFDs appearing in (4.1.57) are actually isometric images of $\lambda_j(A_0,\Omega_0)$ arranged in such a way that the union (4.1.57) is indeed a disjoint union of relative fractal drums.

4.1.4 Stalactites, Stalagmites and Caves Associated With Relative Fractal Drums

In this subsection, we extend the notions of stalactites, stalagmites and caves, introduced in Subsection 2.1.6, associated with fractal sets. Let (A,Ω) be a given relative fractal drum in $\mathbb{R}^N$. Assume that

$$\Omega\setminus\overline{A}=\bigcup_{k\in J}U_k,$$

where $\{U_k\}_{k\in J}$ is the disjoint family of connected components of the open set $\Omega\setminus\overline{A}$. It is clear that the index set is at most countable. Let r be a given nonzero real number, and let us define the following function:

$$f:\Omega\to[0,+\infty],\quad f(x):=d(x,A)^r.$$

(If $r<0$, we let $0^r=+\infty$.) For each $k\in J$, we also introduce the function $f_k:=f|_{U_k}$.

Definition 4.1.47. For each $k\in J$, the graph of the function f_k is called the k-th *stalactite corresponding to the relative fractal drum* (A,Ω) (and to r). The set $\mathrm{cave}(A,\Omega)=\mathrm{cave}(A,\Omega,r)$ defined by

$$\mathrm{cave}(A,\Omega):=\{(x,u)\in\Omega\times(0,+\infty):0<u<f(x)\}$$

and contained in $\mathbb{R}^{N+1}$, is called the (A,Ω)*-cave associated with the relative fractal drum* (A,Ω) (and corresponding to r).

Note that a connected component U_k of an unbounded open set $\Omega\setminus\overline{A}$ may be unbounded. However, when $r>0$, the corresponding function f_k is bounded, due to the assumption according to which there exists $\delta>0$ such that $\Omega\subset A_\delta$; see Definitions 4.1.1 and 4.1.2.

We could now proceed with further discussion and illustrative examples, in the spirit of Subsection 2.1.6. Instead, we will limit ourselves to stating the analog of Proposition 2.1.84.

Proposition 4.1.48. *If s is a real number and $s>\overline{\dim}_B(A,\Omega)$, then the volume of the (A,Ω)-cave, corresponding to the parameter $r=s-N$, is finite.*

Proof. This follows at once from Theorem 4.1.7. □

4.2 Relative Fractal Sprays With Principal Complex Dimensions of Arbitrary Orders

In this section, we consider a special type of RFDs, called *relative fractal sprays*, and consider their distance zeta functions. We then illustrate the results obtained by computing the complex dimensions of relative Sierpiński sprays. More specifically, we determine the complex dimensions of the relative Sierpiński gasket and of the relative Sierpiński carpet; we also calculate the associated residues.

4.2.1 Relative Fractal Sprays in $\mathbb{R}^N$

We now introduce the definition of relative fractal spray, which is very similar to (but more general than) the notion of fractal spray (see [LapPo3], [Lap-vFr3, Definition 13.2], [LapPe2–3] and [LapPeWi1–2]), itself a generalization of the notion of (ordinary) fractal string [LapPo1–2, Lap1–3, Lap-vFr3].

Definition 4.2.1. Let $(\partial\Omega_0,\Omega_0)$ be a fixed relative fractal drum in $\mathbb{R}^N$ (which we call the *base relative fractal drum*, or *generating relative fractal drum* or else, simply, the *generator*), $(\lambda_j)_{j\geq 0}$ a decreasing sequence of positive numbers (scaling factors), converging to zero, and $(b_j)_{j\geq 0}$ a given sequence of positive integers (multiplicities). The associated *relative fractal spray* is a relative fractal drum (A,Ω) obtained as the disjoint union of a sequence of RFDs $\mathscr{F} := \{(\partial\Omega_i,\Omega_i) : i\in\mathbb{N}_0\}$, where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, such that each Ω_i can be obtained from $\lambda_j\Omega_0$ by a rigid motion in $\mathbb{R}^N$, and for each $j\in\mathbb{N}_0$ there are precisely b_j RFDs in the family $\mathscr{F}$ that can be obtained from $\lambda_j\Omega_0$ by a rigid motion. Any relative fractal spray (A,Ω), generated by the base relative fractal drum (or 'basic shape') Ω_0 and the sequences of 'scales' $(\lambda_j)_{j\geq 0}$ with associated 'multiplicities' $(b_j)_{\geq 0}$, is denoted by

$$(A,\Omega) := \mathrm{Spray}(\Omega_0,(\lambda_j)_{j\geq 0},(b_j)_{j\geq 0}). \tag{4.2.1}$$

The family $\mathscr{F}$ is called the *skeleton of the spray*. The distance zeta function $\zeta_{A,\Omega}$ of the relative fractal spray is computed in Theorem 4.2.5 below.

If there exist $\lambda\in(0,1)$ and an integer $b\geq 2$ such that $\lambda_j=\lambda^j$ and $b_j=b^j$, for all $j\in\mathbb{N}_0$, then we simply write

$$(A,\Omega) = \mathrm{Spray}(\Omega_0,\lambda,b).$$

Here, it should be noted that there exist *nonsprayable* RFDs $(\partial\Omega_0,\Omega_0)$ in $\mathbb{R}^N$; see Example 4.2.13 below.

Definition 4.2.2. The relative fractal spray $(A,\Omega)=\mathrm{Spray}(\Omega_0,(\lambda_j)_{j\geq 0},(b_j)_{\geq 0})$ can be viewed as a relative fractal drum generated by $(\partial\Omega_0,\Omega_0)$ and a fractal string $\mathscr{L}=(\ell_j)_{j\geq 1}$, consisting of the decreasing sequence $(\lambda_j)_{j\geq 0}$ of positive real numbers, in which each λ_j has multiplicity b_j. Thus, we can write $(A,\Omega)=\mathrm{Spray}(\Omega_0,\mathscr{L})$. It is also convenient to view the construction of (A,Ω) in Definition 4.2.1 as the *tensor product* of the base relative fractal drum (A_0,Ω_0) and the fractal string $\mathscr{L}$:

$$(A,\Omega) = (\partial\Omega_0,\Omega_0)\otimes\mathscr{L}. \tag{4.2.2}$$

We can also define the *tensor product of two (possibly unbounded) fractal strings* $\mathscr{L}_1 = (\ell_{1j})_{j\geq 1}$ and $\mathscr{L}_2 = (\ell_{2k})_{k\geq 1}$ as the following fractal string (note that here, $\mathscr{L}_1$ and $\mathscr{L}_2$ are viewed as nonincreasing sequences of positive numbers tending to zero, but that we may have $\sum_{j=1}^{\infty} \ell_{1j} = +\infty$ or $\sum_{k=1}^{\infty} \ell_{2k} = +\infty$):

$$\mathscr{L}_1 \otimes \mathscr{L}_2 := (\ell_{1j}\ell_{2k})_{j,k\geq 1}. \tag{4.2.3}$$

The multiplicity of any $l \in \mathscr{L}_1 \otimes \mathscr{L}_2$ is equal to the number of ordered pairs of (ℓ_{1j}, ℓ_{2k}) in the Cartesian product $\mathscr{L}_1 \times \mathscr{L}_2$ of multisets such that $l = \ell_{1j}\ell_{2k}$.

We can easily modify the notion of relative fractal spray in Definition 4.2.1 in order to deal with a finite collection of K basic RFDs (or generating RFDs) $(\partial\Omega_{01}, \Omega_{01}),\ldots,(\partial\Omega_{0K}, \Omega_{0K})$, similarly as in [LapPo3], [Lap-vFr3, Definition 13.2] (and [LapPe2–3, LapPeWi1–2]). A slightly more general notion would consist in replacing $(\partial\Omega_0, \Omega_0)$ with any relative fractal drum (A_0, Ω_0); see Definition 4.2.9.

It is important to stress that, from our point of view, the sets Ω_i in the definition of a relative fractal spray (Definition 4.2.1) do not have to be 'densely packed'. In fact, in general, they cannot be 'densely packed', as indicated by Example 4.2.4(*c*) below. They can just be viewed as a union of the *disjoint family* $\{(\partial\Omega_i, \Omega_i)\}_{i\geq 0}$ *of RFDs* in $\mathbb{R}^N$, where the corresponding family of open sets $\{\Omega_i\}_{i\geq 1}$ is disjoint. Its union, $\cup_{i=0}^{\infty}\Omega_i$, can even be unbounded in $\mathbb{R}^N$, although it has to be of finite N-dimensional Lebesgue measure. As an example, we can consider the family of balls $\{\Omega_i := B_{r_i}(a_i)\}_{i\geq 0}$ in $\mathbb{R}^N$, such that $|a_i| \to +\infty$ as $i \to \infty$ and $\sum_{i=0}^{\infty} r_i^N < \infty$.

The following simple lemma provides necessary and sufficient conditions for a relative fractal spray (A, Ω) to be such that $|\Omega| < \infty$.

Lemma 4.2.3. *Assume that* $(A, \Omega) := \mathrm{Spray}(\Omega_0, (\lambda_j)_{j\geq 0}, (b_j)_{\geq 0})$ *in* $\mathbb{R}^N$ *is a relative fractal spray. Then* $|\Omega| < \infty$ *if and only if* $|\Omega_0| < \infty$ *and*

$$\sum_{j=0}^{\infty} b_j \lambda_j^N < \infty. \tag{4.2.4}$$

In that case, we have

$$|\Omega| = |\Omega_0| \sum_{j=0}^{\infty} b_j \lambda_j^N. \tag{4.2.5}$$

In particular, the relative fractal drum (A, Ω) *is well defined and* $\overline{\dim}_B(A, \Omega) \leq N$.

Proof. Let us prove the sufficiency part. For $\Omega_j = \lambda_j \Omega_0$ we have $|\Omega_j| = |\lambda_j \Omega_0| = \lambda_j^N |\Omega_0|$, and therefore,

$$|\Omega| = \sum_{j=0}^{\infty} |\Omega_j| = \sum_{j=0}^{\infty} b_j |\lambda_j \Omega_0| = |\Omega_0| \sum_{j=0}^{\infty} b_j \lambda_j^N.$$

The proof of the necessity part is also easy and is therefore omitted. □

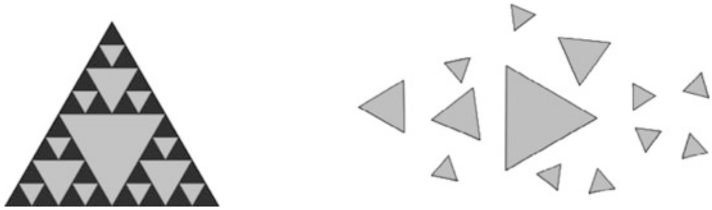

Fig. 4.5 *Left*: The Sierpiński gasket A, viewed as a relative fractal drum (A,Ω), with Ω being the countable disjoint union of open triangles contained in the unit triangle Ω_0. *Right*: An equivalent interpretation of the Sierpiński gasket drum (A,Ω). Here, Ω is a countable disjoint union of open equilateral triangles, and $A=\partial\Omega$. (There are 3^{j-1} triangles with sides 2^{-j} in the union, with $j\in\mathbb{N}$.) Both pictures depict the first three iterations of the construction. We can also view the standard Sierpiński gasket A as a relative fractal drum (A,Ω), in which Ω is just the open unit triangle in the left picture.

Example 4.2.4. Here, we provide a few simple examples of relative fractal sprays:

(a) The ternary Cantor set can be viewed as a relative fractal drum

$$(A,\Omega)=\mathrm{Spray}(\Omega_0,1/3,2)$$

(or the *Cantor relative fractal drum*, or the *relative Cantor fractal spray*), generated by

$$(\partial\Omega_0,\Omega_0)=(\{1/3,2/3\},(1/3,2/3))$$

as the base relative fractal drum, $\lambda=1/3$ and $b=2$. Its relative box dimension is given by $D=\log_3 2$. Of course, this is just an example of ordinary fractal string, namely, the well-known Cantor string.

(b) The Sierpiński gasket can be viewed as a relative fractal drum (or the *Sierpiński relative fractal drum*, or *Sierpiński relative fractal spray*), generated by $(\partial\Omega_0,\Omega_0)$ as the basic relative fractal drum, where Ω_0 is an open equilateral triangle of sides of length $1/2$, $\lambda=1/2$ and $b=3$. Its relative box dimension is given by $D=\log_2 3$.

(c) If Ω_0 is any bounded open set in $\mathbb{R}^2$ (say, an open disk), $\lambda=1/2$ and $b=3$, we obtain a fractal spray $(A,\Omega)=\mathrm{Spray}(\Omega_0,1/2,3)$, in the sense of Definition 4.2.1. In Theorem 4.2.5, we shall see that if Ω_0 has a Lipschitz boundary, then the set of poles of the relative zeta function of this fractal spray (which is a relative fractal drum), as well as the multiplicities of the poles, do not depend on the choice of Ω_0. In this sense, examples (b) and (c) are equivalent. In particular, the box dimension of the *generalized Sierpiński relative fractal drum* is constant, and equal to $D=\log_2 3$.

In other words, the Sierpiński gasket $(A,\Omega)=\mathrm{Spray}(\Omega_0,1/2,3)$, appearing in Example 4.2.4(b), can be viewed as *any* countable disjoint collection of open triangles in the plane (which can be even an unbounded collection) and their bounding triangles, of sizes $\lambda_j = 2^{-j}$ and multiplicities $b_j = 3^j$, $j \in \mathbb{N}_0$, and not just as the standard disjoint collection of open triangles, densely packed inside the unit open triangle. See Figure 4.5.

Using the scaling property stated in Theorem 4.1.40, it is easy to explicitly compute the distance zeta function of relative fractal sprays. Note that the zeta function involves the Dirichlet series $f(s) := \sum_{j=0}^{\infty} b_j \lambda_j^s$. Theorem 4.2.5 just below can be considered as an extension of Theorem 4.1.40.

Theorem 4.2.5 (Distance zeta function of relative fractal sprays). *Let*

$$(A,\Omega) = \operatorname{Spray}(\Omega_0,(\lambda_j)_{j\geq 0},(b_j)_{j\geq 0})$$

be a relative fractal spray in $\mathbb{R}^N$, in the sense of Definition 4.2.1, and such that $|\Omega_0| < \infty$. Assume that condition (4.2.4) of Lemma 4.2.3 is satisfied; that is, $|\Omega| < \infty$. Let Ω be the (countable, disjoint) union of all the open sets appearing in the skeleton, corresponding to the fractal spray. In other words, Ω is the disjoint union of the open sets Ω_j, each repeated with the multiplicity b_j for $j \in \mathbb{N}_0$. Let $f(s) := \sum_{j=0}^{\infty} b_j \lambda_j^s$.[9] *Then, for* $\operatorname{Re} s > \max\{\overline{\dim}_B(A,\Omega), D(f)\}$*, the distance zeta function of the relative fractal spray (A,Ω) is given by the* factorization formula

$$\zeta_{A,\Omega}(s) = \zeta_{\partial\Omega_0,\Omega_0}(s) \cdot f(s), \tag{4.2.6}$$

and

$$\overline{\dim}_B(A,\Omega) = \max\{\overline{\dim}_B(\partial\Omega_0,\Omega_0), D(f)\}. \tag{4.2.7}$$

Proof. Clearly, it follows from (4.2.4) that $f(N) < \infty$. Hence, $D(f) \leq N$; so that $\overline{\dim}_B(A,\Omega) \leq N$. Each open set of the skeleton of the relative fractal spray is obtained by a rigid motion of sets of the form $\lambda_j\Omega_0$, and for any fixed $j \in \mathbb{N}_0$, there are precisely b_j such sets. Identity (4.2.6) then follows immediately from Theorems 4.1.40 and 4.1.44. The remaining claims are easily derived by using this identity. ☐

Note that it follows from Definition 4.2.2 and relation (4.2.6) that the distance zeta function of the tensor product is equal to the product of the zeta functions of its components:

$$\zeta_{(\partial\Omega_0,\Omega_0)\otimes\mathscr{L}}(s) = \zeta_{\partial\Omega_0,\Omega_0}(s) \cdot \zeta_{\mathscr{L}}(s). \tag{4.2.8}$$

Equation (4.2.7) can therefore be written as follows:

$$\overline{\dim}_B((\partial\Omega_0,\Omega_0)\otimes\mathscr{L}) = \max\{\overline{\dim}(\partial\Omega_0,\Omega_0), \overline{\dim}_B\mathscr{L}\}. \tag{4.2.9}$$

Theorem 4.2.6. *Assume that a relative fractal spray* $(A,\Omega) = \operatorname{Spray}(\Omega_0,\lambda,b)$*, introduced at the end of Definition 4.2.1, is such that $|\Omega_0| < \infty$, $\lambda \in (0,1)$, $b \geq 2$ is an integer, and $b\lambda^N < 1$. Then, for* $\operatorname{Re} s > \max\{\overline{\dim}_B(\partial\Omega_0,\Omega_0), \log_{1/\lambda} b\}$*), we have*

[9] Note that according to (4.2.4), this Dirichlet series converges absolutely for $\operatorname{Re} s \geq N$; hence, $D(f) \leq N$.

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial\Omega_0,\Omega_0}(s)}{1-b\lambda^s}, \tag{4.2.10}$$

and the lower bound for Re s *is optimal. In particular, it is equal to* $D(\zeta_{A,\Omega})$*, and*

$$\overline{\dim}_B(A,\Omega) = D(\zeta_{A,\Omega}) = \max\{\overline{\dim}_B(\partial\Omega_0,\Omega_0), \log_{1/\lambda} b\}.$$

If, in addition, Ω_0 *is bounded and has a Lipschitz boundary* $\partial\Omega_0$ *which can be described by finitely many Lipschitz charts, then* $\dim_B(A,\Omega)$ *exists and*

$$\dim_B(A,\Omega) = \max\{N-1, \log_{1/\lambda} b\}. \tag{4.2.11}$$

If we assume that $\log_{1/\lambda} b \in (N-1,N)$*, then the set* $\dim_{PC}(A,\Omega) = \mathscr{P}_c(\zeta_{A,\Omega})$ *of principal complex dimensions of the relative fractal spray* (A,Ω) *is given by*

$$\dim_{PC}(A,\Omega) = \log_{1/\lambda} b + \frac{2\pi}{\log(1/\lambda)}\mathbb{i}\mathbb{Z}. \tag{4.2.12}$$

Proof. If $\lambda_j = \lambda$ and $b_j = b^j$ for all $j \in \mathbb{N}$, with $b\lambda^N < 1$, then $\sum_{j=0}^{\infty} b^j\lambda^{jN} = \frac{1}{1-b\lambda^N} < \infty$; so that $|\Omega| < \infty$. Identity (4.2.10) follows immediately from (4.2.6), by using the fact that for Ω_0 with a Lipschitz boundary satisfying the stated assumption, we have $\dim_B(\partial\Omega_0,\Omega_0) = \dim_B \partial\Omega_0 = N-1$ (this follows, for example, from [ŽuŽup2, Lemma 3]; see also [Lap1]), together with the property of finite stability of the upper box dimension; see, e.g., [Fal1, p. 44]. ☐

Example 4.2.7. Here, we construct a relative fractal spray

$$(A,\Omega) = \mathrm{Spray}(\Omega_0, (\lambda_j)_{j\geq 1}, (b_j)_{\geq 1})$$

in $\mathbb{R}^2$ such that $|\Omega_0| < \infty$, $b_j \equiv 1$, $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$ (hence, $|\Omega| < \infty$ by Lemma 4.2.4), and such that the base set Ω_0 is *unbounded*, as well as its boundary $\partial\Omega_0$. Let Ω_0 be any unbounded Borel set of finite 2-dimensional Lebesgue measure, such that both Ω_0 and $\partial\Omega_0$ are unbounded, and Ω_0 is contained in a horizontal strip

$$V_1 := \{(x,y) \in \mathbb{R}^2 : 0 < y < 1\}.$$

We can construct such a set explicitly as

$$\Omega_0 = \{(x,y) \in \mathbb{R}^2 : 0 < y < x^{-\alpha},\ x > 1\},$$

where $\alpha > 1$, so that $|\Omega_0| < \infty$.

Let $(V_j)_{j\geq 1}$ be a countable, disjoint sequence of horizontal strips in the plane, defined by $V_j = V_1 + (0,j)$ for each $j \in \mathbb{N}$. Let $(\lambda_j)_{j\geq 1}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$. It is clear that for any λ_j, $j \geq 2$, the set $\lambda_j\Omega_0$ is congruent (up to a rigid motion) to the subset $\Omega_j := \lambda_j\Omega_0 + (0,j)$ of V_j. Then, the fractal spray

$$(A,\Omega) = \bigcup_{j=1}^{\infty} (\partial\Omega_j, \Omega_j)$$

has the desired properties.

It is clear that the tensor product introduced in Definition 4.2.2 is associative, in the following sense:[10]

$$((A_0,\Omega_0)\otimes\mathscr{L}_1)\otimes\mathscr{L}_2=(A_0,\Omega_0)\otimes(\mathscr{L}_1\otimes\mathscr{L}_2). \tag{4.2.13}$$

This equation shows that the tensor product defines the *action* of bounded fractal strings $\mathscr{L}$ on the set of relative fractal drums. [Here, we consider the set of bounded fractal strings (with the trivial strings included and equipped with the tensor product $\otimes$) as a monoid, where the identity element is the trivial fractal string consisting of only one length $\ell=1$.] We can therefore extend Theorem 4.2.5 as follows.

Theorem 4.2.8. *Assume that (A_0,Ω_0) is a base relative fractal drum in $\mathbb{R}^N$, and let $(\mathscr{L}_k)_{k\geq 0}$ be a sequence of fractal strings. Let (A_k,Ω_k), $k\geq 1$, be a sequence of relative fractal sprays defined by*

$$(A_k,\Omega_k)=(A_{k-1},\Omega_{k-1})\otimes\mathscr{L}_{k-1}. \tag{4.2.14}$$

Then

$$(A_k,\Omega_k)=(A_0,\Omega_0)\otimes\left(\bigotimes_{j=0}^{k-1}\mathscr{L}_j\right). \tag{4.2.15}$$

Furthermore, for each $k\geq 1$, we have

$$\zeta_{A_k,\Omega_k}(s)=\zeta_{A_0,\Omega_0}(s)\cdot\prod_{j=0}^{k-1}\zeta_{\mathscr{L}_j}(s), \tag{4.2.16}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re}s>\max\{\overline{\dim}_B(A_0,\Omega_0),\overline{\dim}_B\mathscr{L}_0,\ldots,\overline{\dim}_B\mathscr{L}_{k-1}\}$, and

$$\overline{\dim}_B(A_k,\Omega_k)=\max\{\overline{\dim}_B(A_0,\Omega_0),\overline{\dim}_B\mathscr{L}_0,\ldots,\overline{\dim}_B\mathscr{L}_{k-1}\}. \tag{4.2.17}$$

Proof. Relation (4.2.15) follows easily by induction, using the associativity of the tensor product. The remaining claims then follow much as in the proof of Theorem 4.2.5. □

We close this subsection by providing the following generalization of the notion of fractal spray, which is quite natural in our context.

Definition 4.2.9. A *relative fractal spray* is defined exactly as a fractal spray in Definition 4.2.1, except that the generator of the spray is now allowed to be an arbitrary relative fractal drum (A_0,Ω_0), where $A_0\subseteq\mathbb{R}^N$ is arbitrary and $\Omega_0\subseteq\mathbb{R}^N$ is open, but not necessarily bounded; see Definition 4.1.2. (We assume that $\Omega_0\subseteq(A_0)_\delta$, for some $\delta>0$. In addition, we may also require that the total volume of the spray be finite: $|\Omega|<\infty$.) The corresponding relative fractal spray (A,Ω) is denoted by

$$(A,\Omega):=\operatorname{Spray}((A_0,\Omega_0),(\lambda_j)_{j\geq 0},(b_j)_{\geq 0}). \tag{4.2.18}$$

[10] This equality should be understood modulo isometric displacements of scaled copies of (A_0,Ω_0).

In the special case when $\lambda_j = \lambda^j$ and $b_j = b^j$, $j \geq 0$, where $\lambda \in (0,1)$ and an integer $b \geq 2$ are fixed, the corresponding relative fractal spray is denoted by $(A,\Omega) := \mathrm{Spray}((A_0,\Omega_0),\lambda,b)$.

For example, 'spraying' a given relative fractal spray $\mathrm{Spray}((A_0,\Omega_0),\lambda_0,b_0)$ is also possible:

$$(A_1,\Omega_1) = \mathrm{Spray}(\mathrm{Spray}((A_0,\Omega_0),\lambda_0,b_0),\lambda_1,b_1). \tag{4.2.19}$$

By continuing to spray as in Equation (4.2.19), we can define *iterated relative fractal sprays* (A_n,Ω_n) inductively by

$$(A_n,\Omega_n) = \mathrm{Spray}((A_{n-1},\Omega_{n-1}),\lambda_n,b_n), \quad \text{for each} \quad n \geq 1. \tag{4.2.20}$$

The notion of a relative fractal spray will be used in several places in the remainder of this chapter as well as in Chapters 5–6, most often without explicit mention. We leave it to the reader (or to future work) to further explore some of the additional properties of relative fractal sprays and their relative (distance or tube) zeta functions, defined as in Definition 4.1.1 and by using Theorem 4.1.40.

4.2.2 Principal Complex Dimensions of Arbitrary Multiplicities

The goal of this subsection is to show how one can effectively construct fractal sets (as well as fractal strings and even RFDs) which have poles along the critical line (i.e., principal complex dimensions) of any given order (i.e., multiplicity), and even infinitely many essential singularities (see Theorem 4.2.19 and Remark 4.2.21 below). Such fractal strings and more general RFDs are interesting examples of strongly hyperfractal RFDs, in the sense of part (*ii*) of Definition 4.6.23 in Subsection 4.6.3 below. The corresponding method for constructing these RFDs is explained and illustrated in the following example.

Example 4.2.10. (*Cantor sets of higher order*). We will provide here an example of a relative fractal drum of $\mathbb{R}$ such that, for any given $m \in \mathbb{N}$, its distance zeta function has an infinite set of poles of order m in arithmetic progression and located on the critical line. The construction is based on an 'iterated Cantor set', as we now explain.

Let C be the standard middle-third Cantor set contained in $[0,1]$ and let $\Omega := (0,1)$. Then, let (C,Ω) be our base relative fractal drum and let $\mathscr{L} := \mathscr{L}_{CS}$ be the Cantor string with total length 3; that is,

$$\mathscr{L} = (1, 3^{-1}, 3^{-1}, \underbrace{3^{-2},\dots,3^{-2}}_{4 \text{ times}}, \underbrace{3^{-3},\dots,3^{-3}}_{8 \text{ times}}, \dots).$$

We now define the relative fractal drum (C_2,Ω_2) as the tensor product $(C,\Omega)\otimes\mathscr{L}$; see Definitions 4.2.1, 4.2.2 and Figure 4.6. Furthermore, one can see clearly that

$$(C_2,\Omega_2) = (C,\Omega) \sqcup 3^{-1}(C_2,\Omega_2) \sqcup 3^{-1}(C_2,\Omega_2), \tag{4.2.21}$$

where $\sqcup$ denotes a disjoint union of isometric images of scaled copies of (C_2,Ω_2); see Definition 4.1.46. Then, by the scaling property of the relative distance zeta function (see Theorem 4.1.40), we have

$$\zeta_{C_2,\Omega_2}(s) = \zeta_{C,\Omega}(s) + 2\,\zeta_{3^{-1}C_2,3^{-1}\Omega_2}(s) = \zeta_{C,\Omega}(s) + 2\cdot 3^{-s}\,\zeta_{C_2,\Omega_2}(s),$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s$ is sufficiently large, or, in other words,

$$\zeta_{C_2,\Omega_2}(s) = \frac{3^s}{3^s-2}\zeta_{C,\Omega}(s) = \frac{2\cdot 3^s}{2^s s(3^s-2)^2}, \tag{4.2.22}$$

where, in the second equality, we have used the expression for $\zeta_C = \zeta_{C,\Omega}$ obtained in Example 2.1.82. In light of the principle of analytic continuation, it is clear that Equation (4.2.22) continues to hold for all $s \in \mathbb{C}$, and hence, ζ_{C_2,Ω_2} is meromorphic on all of $\mathbb{C}$ and

$$\mathscr{P}(\zeta_{C_2,\Omega_2}) = \{0\} \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right). \tag{4.2.23}$$

Furthermore, the poles $\omega_k := \log_3 2 + \frac{2\pi \mathbb{i} k}{\log 3}$ for $k \in \mathbb{Z}$ are all of second order (i.e., of multiplicity two). We conclude that $\overline{\dim}_B(C_2,\Omega_2) = \log_3 2$. More specifically, by Theorem 5.3.16 in Chapter 5 below, and in light of expression (4.2.22) for ζ_{C_2,Ω_2}, we obtain the following exact tube formula for the second order Cantor set, valid pointwise for all $t \in (0,1)$:

$$|(C_2)_t \cap \Omega_2| = t^{1-\log_3 2}\left(\log t^{-1} G(\log t^{-1}) + H(\log t^{-1})\right) + 2t, \tag{4.2.24}$$

where $G, H\colon \mathbb{R} \to \mathbb{R}$ are nonconstant, bounded periodic functions with minimal period $T := \log 3$. These functions can be computed explicitly in terms of their Fourier series but the algebraic expressions for their Fourier coefficients are too complicated to be given here in a concise manner. Furthermore, we conclude from the tube formula (4.2.24) that $\dim_B(C_2,\Omega_2)$ exists and $\dim_B(C_2,\Omega_2) = \log_3 2$, and moreover, that $\mathscr{M}^D(C_2,\Omega_2) = +\infty$.

We can now repeat the above process inductively; that is, for each integer $n \geq 2$, we define the relative fractal drum (C_n,Ω_n) as a relative fractal spray generated by (C_{n-1},Ω_{n-1}) and $\mathscr{L}$; that is, $(C_n,\Omega_n) := (C_{n-1},\Omega_{n-1}) \otimes \mathscr{L}$, for each integer $n \geq 2$. Much as before, we obtain that

$$\zeta_{C_n,\Omega_n}(s) = \frac{2\cdot 3^{(n-1)s}}{2^s s(3^s-2)^n}, \quad \text{for all} \quad s \in \mathbb{C}. \tag{4.2.25}$$

The set of complex dimensions of the RFD (C_n,Ω_n) is the same as in the case when $n = 2$ (see Equation (4.2.23) above), but except at $s := 0$ (which is simple), the corresponding multiplicities are not the same and depend on n. (Hence, the *multisets* $\mathscr{P}(\zeta_{C_n,\Omega_n})$ are different for each $n \in \mathbb{N}$.) More specifically, the poles of ζ_{C_n,Ω_n} at

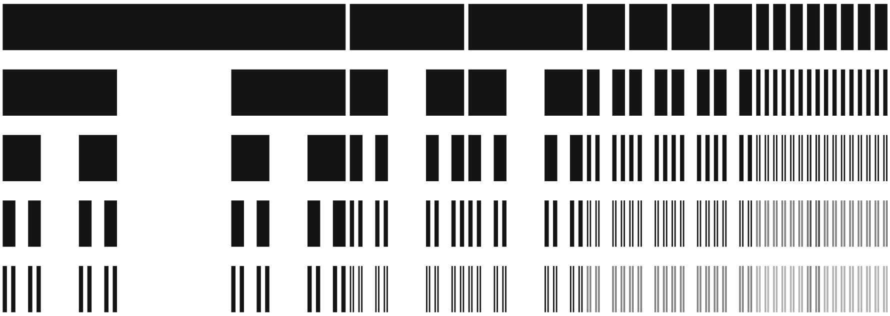

Fig. 4.6 The *second order Cantor set* from Example 4.2.10. Only the first four iterations are shown here. More precisely, from left to right, we have the middle-third Cantor set C in $[0,1]$, then two copies of C scaled by $1/3$, and then four copies of C scaled by $1/9$; and so on, ad infinitum.

$s := \omega_k = \log_3 2 + \frac{2\pi \mathrm{i} k}{\log 3}$ for each $k \in \mathbb{Z}$ are of order n and $D := \dim_B(C_2,\Omega_2) = \log_3 2$. Furthermore, again by Theorem 5.3.16, we have the following exact tube formula, valid pointwise for all $t \in (0,1)$:

$$|(C_n)_t \cap \Omega_n| = t^{1-\log_3 2} \sum_{i=1}^{n} (\log t^{-1})^{i-1} G_i(\log t^{-1}) + 2t, \tag{4.2.26}$$

where for $i = 1,\dots,n$, $G_i \colon \mathbb{R} \to \mathbb{R}$ is a nonconstant, bounded periodic function with minimal period $T := \log 3$. As in the case of the second order Cantor set, each of these functions can be computed explicitly in terms of its Fourier series.

Finally, we can now use the sequence of relative fractal drums (C_n,Ω_n), for $n \in \mathbb{N}$, in order to construct an RFD (A,Ω) which will have an infinite set of essential singularities on the critical line $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$. The construction is analogous to the one in the proof of Theorem 3.3.6 and in Example 3.3.7 in Section 3.3 above, dealing with Cantor strings of higher order. We let $(C_1,\Omega_1) := (C,\Omega)$, scale down every RFD (C_n,Ω_n) by the factor $3^{-n}/n!$ and define (A,Ω) as the disjoint union of copies of the resulting RFDs; that is,

$$(A,\Omega) := \bigsqcup_{n=1}^{\infty} \frac{3^{-n}}{n!} (C_n,\Omega_n). \tag{4.2.27}$$

(Here, we have used Definition 4.5.7 and Lemma 4.5.10 in Subsection 4.5.2 below.) We then have

$$\begin{aligned} \zeta_{A,\Omega}(s) &= \sum_{n=1}^{\infty} \zeta_{3^{-n}(n!)^{-1}(C_n,\Omega_n)}(s) = \sum_{n=1}^{\infty} \frac{3^{-ns}}{(n!)^s} \zeta_{C_n,\Omega_n}(s) \\ &= \frac{2}{6^s s} \sum_{n=1}^{\infty} \frac{1}{(n!)^s (3^s-2)^n}. \end{aligned} \tag{4.2.28}$$

By the Weierstrass M-test, $\zeta_{A,\Omega}(s)$ is holomorphic on $\{\mathrm{Re}\, s > 0\} \setminus \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right)$. More precisely, it has essential singularities at each point of the set $\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}$. Note that the critical line $\{\mathrm{Re}\, s = \log_3 2\}$ is clearly not a natural boundary for $\zeta_{A,\Omega}$ since $\zeta_{A,\Omega}$ given by Equation (4.2.28) can be holomorphically continued to the connected open set $\{\mathrm{Re}\, s > 0\} \setminus \dim_{PC}(A,\Omega)$.

In light of Theorem 5.3.16 of Chapter 5 below, we deduce that the tube formula of (A,Ω) has the following asymptotic expansion:

$$|A_t \cap \Omega| = t^{1-\log_3 2} \sum_{i=1}^{\infty} (\log t^{-1})^{i-1} G_i(\log t^{-1}) + O(t^{1-\alpha}) \quad \text{as} \quad t \to 0^+, \tag{4.2.29}$$

for any $\alpha > 0$, and where, similarly as before, the functions G_i for $i \in \mathbb{N}$ are nonconstant, bounded periodic functions with minimal period $T := \log 3$. Although in Chapter 5, we always assume that the corresponding fractal zeta function has a meromorphic extension to a suitable connected open neighborhood of the critical line, the results of Chapter 5 actually extend to functions having only isolated singularities in a suitable neighborhood of the critical line; that is, the corresponding fractal zeta functions may also have essential singularities. It is now easy to check that $\zeta_{A,\Omega}$ given by (4.2.28) satisfies the conditions of Theorem 5.3.16, with $\kappa_d := -1$; (see Definitions 5.1.3 and 5.3.9), and with the screen $\boldsymbol{S}$ taken as the vertical line $\{\mathrm{Re}\, s = \alpha\}$. The tube formula (4.2.29) now follows by calculating the residues $\mathrm{res}(t^{1-s}(1-s)^{-1}\zeta_{A,\Omega}(s), \omega_k)$, where $\omega_k \in \log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}$.

We close this discussion by observing that, as was alluded to earlier, the RFD (A,Ω) is a strongly hyperfractal RFD (in the sense of part (*ii*) of Definition 4.6.23 in Subsection 4.6.3 below and as strengthened in both parts of Remark 1.3.9), which is not maximally hyprefractal (in the sense of part (*iii*) of that same definition).

The above construction can be generalized verbatim for any (nontrivial) bounded fractal string $\mathscr{L}$ instead of the Cantor string $\mathscr{L}_{CS}$. *This suggests that the definition of complex dimensions should be extended to also include potential essential singularities* (*as well as algebraic and transcendental singularities*) *of the fractal zeta functions*, in the spirit of [Lap-vFr3, Subsection 13.4.3].

Let us now recall the definition of a self-similar spray or tiling (see [LapPe2–3], [LapPeWi1–2], [Lap-vFr3, Section 13.1]). More precisely, let us state this definition slightly more generally and in the context of relative fractal drums.

Definition 4.2.11. (*Self-similar spray or tiling*). Let G be a given open subset (*base set* or *generator*) of $\mathbb{R}^N$ of finite N-dimensional Lebesgue measure and let $\{r_1, r_2, \dots, r_J\}$ be a finite multiset (also called a *ratio list*) of positive real numbers (in $(0,1)$) such that $J \in \mathbb{N}$, $J \geq 2$ and

$$\sum_{j=1}^{J} r_j^N < 1. \tag{4.2.30}$$

Furthermore, let Λ be the multiset consisting of all the possible 'words' of multiples of the scaling factors $r_1, \dots, r_J$; that is, let

$$\Lambda := \{1, r_1, \dots, r_J, r_1 r_1, \dots, r_1 r_J, r_2 r_1, \dots, r_2 r_J, \dots, r_J r_1, \dots, r_J r_J, \\ r_1 r_1 r_1, \dots, r_1 r_1 r_J, \dots\} \tag{4.2.31}$$

and arrange all of the elements of the multiset Λ into a *scaling sequence* $(\lambda_i)_{i\geq 0}$, where $\lambda_0 := 1$. Note that $0 < \lambda_i < 1$, for every $i \geq 1$.

A *self-similar spray* (or *tiling*), generated by the base set G and the ratio list $\{r_1, r_2, \dots, r_J\}$ is an RFD $(\partial\Omega, \Omega)$ in $\mathbb{R}^N$, where Ω is a disjoint union of open sets G_i; i.e.,

$$\Omega := \bigsqcup_{i=0}^{\infty} G_i, \tag{4.2.32}$$

such that each G_i is congruent to $\lambda_i G$, for each $i \geq 0$. Here, the disjoint union $\sqcup$ can be understood as the disjoint union of RFDs given in Definition 4.1.46, with $(A_i, \Omega_i) := (\partial G_i, G_i)$ for every $i \geq 0$, in the notation of that definition.

Remark 4.2.12. Note that in the above definition, the scaling sequence $(\lambda_i)_{i\geq 0}$ consists of all the products of ratios $r_1, \dots, r_J$ appearing in the infinite sum

$$\sum_{n=0}^{\infty} \left(\sum_{j=1}^{J} r_j \right)^n, \tag{4.2.33}$$

after expanding the powers and counted with their multiplicities. More precisely, we have that for every multi-index $\alpha = (\alpha_1, \dots, \alpha_J) \in \mathbb{N}_0^J$, the multiplicity of $r_1^{\alpha_1} r_2^{\alpha_2} \dots r_J^{\alpha_J}$ in the multiset Λ is equal to the multinomial coefficient

$$\binom{|\alpha|}{\alpha_1, \alpha_2, \dots, \alpha_J} = \frac{|\alpha|!}{\alpha_1! \, \alpha_2! \cdots \alpha_J!}, \tag{4.2.34}$$

where $|\alpha| := \sum_{j=1}^{J} \alpha_j$. Of course, depending on the specific values of the ratios $r_1, \dots, r_J$, some of the numbers $r_1^{\alpha_1} r_2^{\alpha_2} \dots r_J^{\alpha_J}$ may be equal for different multi-indices $\alpha \in \mathbb{N}_0^J$.

Furthermore, the condition (4.2.30) ensures that the set $\Omega = \sqcup_{i\geq 0} G_i$ has finite N-dimensional Lebesgue measure. Indeed, we have

$$\begin{aligned} |\Omega| &= \sum_{i=0}^{\infty} |G_i| = \sum_{i=0}^{\infty} |\lambda_i G| = |G| \sum_{i=0}^{\infty} \lambda_i^N \\ &= |G| \sum_{n=0}^{\infty} \left(\sum_{j=1}^{J} r_j^N \right)^n = \frac{|G|}{1 - \sum_{j=1}^{J} r_j^N} < \infty, \end{aligned} \tag{4.2.35}$$

since (4.2.30) is satisfied. Note that the second to last equality above follows from the construction of the scaling sequence $(\lambda_i)_{i\geq 0}$.

In Definition 4.2.11, it is implicitly assumed that the generator G is such that it is indeed possible to construct the *disjoint* union appearing in (4.2.32), as given in Definition 4.1.46. This can always be achieved when G is bounded, which is the usual assumption made when dealing with self-similar sprays as, for instance, in [LapPe2–3], [LapPeWi1–2] and [Lap-vFr3, Section 13.1]. However, contrary to intuition, this does not have to be the case for a general open set G of finite N-dimensional Lebsgue measure, as is shown by the following example.

Example 4.2.13. Here, we construct an open set G in $\mathbb{R}^N$ of finite N-dimensional Lebesgue measure, and which is dense in $\mathbb{R}^N$. Therefore, *any* isometric image of a scaled copy of G has an intersection with G of positive N-dimensional Lebesgue measure. Let $A = \{a_k \in \mathbb{R}^N : k \in \mathbb{N}\}$ be a countable dense subset of $\mathbb{R}^N$ (for example, the set of points in $\mathbb{R}^N$ with rational coordinates). Let $(\rho_k)_{k\geq 1}$ be a sequence of positive real numbers such that $\sum_{k=1}^{\infty} \rho_k^N < \infty$, and consider the open set G defined as the (not necessarily disjoint) union of the open balls $B_{\rho_k}(a_k)$ of radius ρ_k and with centers at a_k, for $k \geq 1$:

$$G := \bigcup_{k=1}^{\infty} B_{\rho_k}(a_k). \tag{4.2.36}$$

Then, its N-dimensional volume is positive and finite since

$$0 < |G|_N \leq \sum_{k=1}^{\infty} |B_{\rho_k}(a_k)|_N = \omega_N \sum_{k=1}^{\infty} \rho_k^N < \infty, \tag{4.2.37}$$

where ω_N is the volume of the unit ball of $\mathbb{R}^N$. Since $\overline{A} = \mathbb{R}^N$, it follows that A (and hence, G as well) has a nonempty intersection with *any* nonempty open subset of $\mathbb{R}^N$.

We proceed by discussing some interesting properties of the RFD (A,G). Since $\overline{A} = \mathbb{R}^N$ and since $d(x,A) = d(x,\overline{A}) = 0$ for any $x \in \mathbb{R}^N$, we have that $A_t = \mathbb{R}^N$ for any $t > 0$; so that $A_t \cap G = G$, and therefore, $|A_t \cap G| = |G|$ for all $t > 0$. Hence, for any fixed real number s, we have

$$\frac{|A_t \cap G|}{t^{N-s}} = |G| t^{s-N} \sim t^{s-N} \quad \text{as} \quad t \to 0^+; \tag{4.2.38}$$

it follows that

$$\dim_B(A,G) = N. \tag{4.2.39}$$

Let us now compute the tube zeta function $\tilde{\zeta}_{A,G}$ of the RFD (A,G):

$$\tilde{\zeta}_{A,G}(s) := \int_0^{\delta} t^{s-N-1} |A_t \cap G| \, \mathrm{d}t = |G| \int_0^{\delta} t^{s-N-1} \mathrm{d}t = |G| \frac{\delta^{s-N}}{s-N}, \tag{4.2.40}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > N$. Therefore, $\tilde{\zeta}_{A,G}$ can be (uniquely) meromorphically extended to the whole complex plane by letting $\tilde{\zeta}_{A,G}(s) := |G| \frac{\delta^{s-N}}{s-N}$ for all $s \in \mathbb{C}$.

In order to compute the distance zeta function of the RFD (A,G), note (much as before) that for any $x \in \mathbb{R}^N$, we have

$$d(x,A) = d(x,\overline{A}) = d(x,\mathbb{R}^N) = 0. \tag{4.2.41}$$

Therefore, the distance zeta function $\zeta_{A,G}$ satisfies $\zeta_{A,G}(s) := \int_G d(x,A)^{s-N}\mathrm{d}x = 0$ for all $s \in \mathbb{C}$ such that $\mathrm{Re}\, s > N$. This function can be holomorphically extended to the whole complex plane by letting $\zeta_{A,G} \equiv 0$ on all of $\mathbb{C}$. We have thus constructed an RFD (A,G) in $\mathbb{R}^N$ such that

$$\begin{aligned} D(\zeta_{A,G}) = N, \quad D_{\mathrm{hol}}(\zeta_{A,G}) = D_{\mathrm{mer}}(\zeta_{A,G}) = -\infty, \\ D(\tilde{\zeta}_{A,G}) = D_{\mathrm{hol}}(\tilde{\zeta}_{A,G}) = N, \quad D_{\mathrm{mer}}(\tilde{\zeta}_{A,G}) = -\infty. \end{aligned} \tag{4.2.42}$$

Note that in the case of the relative distance zeta function $\zeta_{A,G}$, we have achieved the maximal possible gap between its abscissa of (absolute) convergence and its abscissa of holomorphic continuation, since for any RFD (A,G) in $\mathbb{R}^N$, we have $D(\zeta_{A,G}), D_{\mathrm{hol}}(\zeta_{A,G}) \in [-\infty, N]$.

It is also worth noting that the open set G has finite N-dimensional Lebesgue measure, while the N-dimensional Lebesgue measure of its boundary ∂G is infinite. Indeed, we have that

$$|\partial G|_N = |\overline{G} \setminus \Omega|_N = |\mathbb{R}^N \setminus G|_N = |\mathbb{R}^N|_N - |G|_N = +\infty - |G|_N = +\infty. \tag{4.2.43}$$

In light of the above example, it is natural to introduce the following definition.

Definition 4.2.14. We let $\mathrm{RFD}_\Lambda(\mathbb{R}^N)$ be the family of all relative fractal drums (A,Ω) in $\mathbb{R}^N$ such that for a given multiset $\Lambda = \Lambda(r_1,\dots,r_J)$ of scaling factors $\lambda \in (0,1)$ defined by (4.2.31), one can construct the *disjoint* union $\sqcup_{\lambda\in\Lambda}\lambda(A,\Omega)$, in the sense of Definition 4.1.46. We then say that (A,Ω) is Λ*-sprayable* in $\mathbb{R}^N$. Furthermore, we say that (A,Ω) is *universally sprayable* if it is sprayable with respect to any finite multiset of scaling factors Λ.

Example 4.2.15. We can provide two simple classes of RFDs $(\partial G, G)$ which are universally sprayable:

(a) Any $(\partial G, G)$, where G is a bounded subset of $\mathbb{R}^N$.

(b) Any $(\partial G, G)$, where G is a *strip-like* subset of $\mathbb{R}^N$; i.e., such that the set G is contained between two parallel hyperplanes in $\mathbb{R}^N$ (more precisely, there exists two real constants a and b and a nonzero vector $c \in \mathbb{R}^N$ such that $a \le c \cdot x \le b$ for all $x \in G$, where $\cdot$ denotes the inner product in $\mathbb{R}^N$).

Note that, according to this definition, each bounded set G is a strip-like set.

Consider now a self-similar spray as a relative fractal drum (A,Ω), which we refer to in the sequel as a *self-similar RFD* or as the *self-similar RFD associated with the self-similar spray* (A,Ω) (see Definition 4.2.20 on page 290 below, along with the corresponding footnote 13); that is, let $A := \partial\Omega$ and $\Omega := \sqcup_{i\ge 0} G_i$ (see Definition 4.2.11). The 'self-similarity' of (A,Ω) is nicely exhibited by the scaling relation (4.2.44) given in the following lemma.

Lemma 4.2.16. *Let (A,Ω) be a self-similar spray in $\mathbb{R}^N$, as in Definition 4.2.11. Then, the relative fractal drum (A,Ω) satisfies the following* 'self-similar identity':

$$(A,\Omega) = (\partial G, G) \sqcup (r_1 A, r_1 \Omega) \sqcup \cdots \sqcup (r_J A, r_J \Omega), \tag{4.2.44}$$

where (with the exception of the first term on the right-hand side of (4.2.44)) the symbol $\sqcup$ indicates that this represents a disjoint union of copies of (A,Ω) scaled by factors $r_1,\ldots,r_J$ and displaced by isometries of $\mathbb{R}^N$ (see Definition 4.1.46).

Proof. Let us reindex the scaling sequence $(\lambda_i)_{i\geq 0}$ in a way that keeps track of the actual construction of the numbers λ_i out of the scaling ratios $r_1,\ldots,r_J$; see Equation (4.2.31) above. We let

$$I := \{\emptyset\} \cup \bigcup_{m=1}^{\infty} \{1,\ldots,J\}^m \tag{4.2.45}$$

be the set of all finite sequences consisting of numbers $1,\ldots,J$ (or, equivalently, of all finite words based on the alphabet $\{1,\ldots,J\}$). Furthermore, for every $\alpha \in I$, define

$$\lambda_\alpha := \begin{cases} 1, & \alpha = \emptyset, \\ r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_m}, & \alpha \neq \emptyset. \end{cases} \tag{4.2.46}$$

We then deduce from the construction of (A,Ω) that

$$\begin{aligned}(A,\Omega) &= \bigsqcup_{i=0}^{\infty} (\partial G_i, G_i) = \bigsqcup_{i=0}^{\infty} \lambda_i (\partial G, G) \\ &= \bigsqcup_{\alpha\in I} \lambda_\alpha (\partial G, G) = (\partial G, G) \sqcup \bigsqcup_{\alpha \in I\setminus\{\emptyset\}} \lambda_\alpha(\partial G, G).\end{aligned}$$

Observe now that in the last disjoint union above, every $\alpha \in \{1,\ldots,J\}^m$ can be written as $\{j\} \times \{1,\ldots,J\}^{m-1}$, for some $j \in \{1,\ldots,J\}$, provided we identify $\{j\}$ with $\{j\} \times \{\emptyset\}$ when $m = 1$. Note that this identification is consistent with the definition of λ_α, in the sense that $\lambda_{\{j\}\times\beta} = r_j \lambda_\beta$ for all $j \in \{1\ldots,J\}$ and $\beta \in I$. In light of this, we can next partition the last union above with respect to which number $j \in \{1,\ldots,J\}$ the sequence α begins with:

$$\begin{aligned}(A,\Omega) &= (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} \bigsqcup_{\alpha\in\{j\}\times I} \lambda_\alpha (\partial G, G) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} \bigsqcup_{\beta\in I} r_j \lambda_\beta (\partial G, G) \\ &= (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} r_j \Big(\bigsqcup_{\beta\in I} \lambda_\beta (\partial G, G) \Big) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} r_j (A,\Omega).\end{aligned}$$

This completes the proof of the lemma. ☐

In light of (4.2.44) and the additivity of the distance zeta function, it is now clear that the distance zeta function of (A,Ω) satisfies the following functional equation:

$$\zeta_{A,\Omega}(s) = \zeta_{\partial G, G}(s) + \zeta_{r_1 A, r_1 \Omega}(s) + \cdots + \zeta_{r_J A, r_J \Omega}(s), \tag{4.2.47}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.[11] Furthermore, for such s, by using the scaling property of the relative distance zeta function (Theorem 4.1.40), we deduce that the above equation then becomes

$$\zeta_{A,\Omega}(s) = \zeta_{\partial G,G}(s) + r_1^s \zeta_{A,\Omega}(s) + \cdots + r_J^s \zeta_{A,\Omega}(s). \tag{4.2.48}$$

Finally, this last identity together with an application of the principle of analytic continuation now yields the following theorem. We note that the second equality in Equation (4.2.50) of Theorem 4.2.17 follows from Equation (4.2.17).

Theorem 4.2.17. *Let G be the generator of a self-similar spray in $\mathbb{R}^N$, and let $\{r_1, r_2, \ldots, r_J\}$, with $r_j > 0$ (for $j = 1, \ldots, J$, $J \geq 2$) and such that $\sum_{j=1}^J r_j^N < 1$, be its scaling ratios counted according to their multiplicities. Furthermore, let $(A,\Omega) := (\partial\Omega, \Omega)$ be the self-similar spray generated by G, as in Definition 4.2.11. Then, the distance zeta function of (A,Ω) is given by*

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial G,G}(s)}{1 - \sum_{j=1}^J r_j^s}, \tag{4.2.49}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. Furthermore,

$$D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega) = \max\{\overline{\dim}_B(\partial G, G), \sigma_0\}, \tag{4.2.50}$$

where $\sigma_0 > 0$ is the unique real solution s of the Moran equation $\sum_{j=1}^J r_j^s = 1$ (i.e., σ_0 is the similarity dimension *of the self-similar spray (A,Ω)).*[12]

More specifically, given a connected open neighborhood U of the critical line $\{\operatorname{Re} s = D\}$, where $D := \overline{\dim}_B(A,\Omega)$, $\zeta_{A,\Omega}$ has a meromorphic continuation to U if and only if $\zeta_{\partial G,G}$ does, and in that case, $\zeta_{A,\Omega}(s)$ is given by (4.2.49) for all $s \in U$. Consequently, the visible complex dimensions of (A,Ω) satisfy

$$\mathscr{P}(\zeta_{A,\Omega}, U) \subseteq (\mathfrak{D} \cap U) \cup \mathscr{P}(\zeta_{\partial G,G}, U), \tag{4.2.51}$$

where $\mathfrak{D}$ is the set of all the complex solutions s of the Moran equation $\sum_{j=1}^J r_j^s = 1$. Finally, if there are no zero-pole cancellations in (4.2.49), then we have an equality in (4.2.51).

We refer to [Lap-vFr3, Chapter 3, esp. Theorem 3.6] for detailed information about the structure of $\mathfrak{D}$; see also the brief discussion given before Corollary 5.4.23 and Problem 6.2.36 below.

Remark 4.2.18. There are two particularly interesting situations in which Theorem 4.2.17 can be applied:

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[11] For instance, it suffices to assume that $\operatorname{Re} s > N$ since, by Theorem 4.1.7, all of the zeta functions appearing in (4.2.47) are holomorphic on the right half-plane $\{\operatorname{Re} s > N\}$.

[12] Clearly, $\sigma_0 > 0$ since $J \geq 2 > 1$; furthermore, $\sigma_0 < N$ since $\sum_{j=1}^J r_j^N < 1$.

(i) The case when $U := \{\operatorname{Re} s > D_{\mathrm{mer}}(\zeta_{\partial G,G})\}$, the largest open right half-plane to which $\zeta_{\partial G,G}$ can be meromorphically extended.

(ii) The case when $U := \mathring{\boldsymbol{W}}$, where $\boldsymbol{W}$ is an arbitrary window for $\zeta_{\partial G,G}$ and hence also for $\zeta_{A,\Omega}$, either in the sense of Chapter 2 (see Subsection 2.1.5, page 95) or in the sense of Chapter 5 (see Definition 5.1.1 in Subsection 5.1.1). In that case, since the screen $\boldsymbol{S} = \partial \boldsymbol{W}$ associated with the window $\boldsymbol{W}$ does not contain any poles, the inclusion (4.2.51) can be equivalently written as follows:

$$\mathscr{P}(\zeta_{A,\Omega}, \boldsymbol{W}) \subseteq (\mathfrak{D} \cap \boldsymbol{W}) \cup \mathscr{P}(\zeta_{\partial G,G}, \boldsymbol{W}). \tag{4.2.52}$$

Furthermore, if there are no zero-pole cancellations for any $s \in \boldsymbol{W}$ in the right-hand side of (4.2.49), then we have an equality in (4.2.52).

The next theorem gives a general construction of complex dimensions of higher order generated by self-similar sprays. It is stated for RFDs in $\mathbb{R}^N$. For notational simplicity, in that theorem, we assume that $\zeta_{\partial G,G}$ admits a meromorphic continuation to all of $\mathbb{C}$ (which is very often the case, in practice), but the reader will easily be able to extend it to a more general situation, in the spirit of Theorem 4.2.17 and Remark 4.2.18 above.

Theorem 4.2.19. *Let $(A,\Omega) := (\partial\Omega,\Omega)$ be a self-similar spray in $\mathbb{R}^N$ (with $N \geq 1$) generated by an open set G and the set of scaling ratios $\{r_1, r_2, \ldots, r_J\}$, with $r_j > 0$ (for $j = 1, \ldots, J$, $J \geq 2$) and such that $\sum_{j=1}^J r_j^N < 1$; see Definition 4.2.11 above. Furthermore, assume that $\zeta_{\partial G,G}$ has a meromorphic continuation to all of $\mathbb{C}$ and that there are no zero-pole cancellations in (4.2.49); i.e., that $\mathfrak{D} \cap \mathscr{P}(\zeta_{\partial G,G}) = \emptyset$, where $\mathfrak{D}$ is the set of all the complex solutions s of the Moran equation $\sum_{j=1}^J r_j^s = 1$ (also called the* scaling complex dimensions *of the self-similar spray (A,Ω) in the sequel); see, e.g., Subsection 5.5.6 or Section 6.2.*

Then, given an arbitrary integer $n \in \mathbb{N}$, there is an explicitly constructible relative fractal drum (A_n, Ω_n) (in fact, a fractal spray also generated by G or, more precisely, with base RFD $(\partial G, G)$) which has exactly the same complex dimensions as (A,Ω), provided the corresponding multiplicities are not *taken into account, but with the orders (i.e., multiplicities) of the complex dimensions belonging to $\mathfrak{D}$ now being multiplied by n.*

Moreover, if we let $U := \{\operatorname{Re} s > 0\}$ and $\mathfrak{D}^+ := \mathfrak{D} \cap U$, then there is an explicitly constructible RFD $(A_\infty, \Omega_\infty)$ such that its complex dimensions visible through U are the same as the complex dimensions of (A,Ω) visible through U, provided the multiplicities are not *taken into account, but with the complex dimensions belonging to $\mathfrak{D}^+$ now being of infinite order, that is, being essential singularities of its distance zeta function $\zeta_{A_\infty,\Omega_\infty}$. In particular, we have that*

$$D_{\mathrm{mer}}(A_\infty, \Omega_\infty) = D(\zeta_{A_\infty,\Omega_\infty}) = \overline{\dim}_B(A_\infty, \Omega_\infty). \tag{4.2.53}$$

Proof. We use the RFD (A,Ω) as our new ‘generator’; that is, we define a new relative fractal drum (A_2, Ω_2) as a disjoint union of scaled copies of (A,Ω) by scaling factors λ_i, where $(\lambda_i)_{i\geq 0}$ is the scaling sequence of the self-similar spray (A,Ω). Much as in the proof of Lemma 4.2.16, this construction then implies that

$$(A_2,\Omega_2) = (A,\Omega) \sqcup \bigsqcup_{j=1}^{J} (r_j A_2, r_j \Omega_2). \tag{4.2.54}$$

Furthermore, much as before, by the scaling property of the relative distance zeta function (see Theorem 4.1.40) and in light of Theorem 4.2.17, we then obtain (after an application of the principle of analytic continuation) that

$$\zeta_{A_2,\Omega_2}(s) = \frac{\zeta_{A,\Omega}(s)}{1-\sum_{j=1}^{J} r_j^s} = \frac{\zeta_{\partial G,G}(s)}{\left(1-\sum_{j=1}^{J} r_j^s\right)^2}, \tag{4.2.55}$$

for all $s\in\mathbb{C}$, since, by hypothesis, $\zeta_{\partial G,G}$ admits a meromorphic continuation to all of $\mathbb{C}$. From the above identity (4.2.55), we now conclude that the relative fractal drum (A_2,Ω_2) has the same complex dimensions as (A,Ω), but with the orders of those belonging to $\mathfrak{D}$ being multiplied by two.

We can now proceed inductively by using (A_2,Ω_2) as a new 'generator'. Therefore, for each $n\in\mathbb{N}$, we obtain a relative fractal drum (A_n,Ω_n) (in fact, a fractal spray also generated by G or, more precisely, with base RFD $(\partial G, G)$) such that

$$\zeta_{A_n,\Omega_n}(s) = \frac{\zeta_{\partial G,G}(s)}{\left(1-\sum_{j=1}^{J} r_j^s\right)^n}, \tag{4.2.56}$$

for all $s\in\mathbb{C}$; that is, (A_n,Ω_n) has the same complex dimensions as (A,Ω), but the complex dimensions belonging to $\mathfrak{D}$ (i.e., the scaling complex dimensions of the fractal spray (A_n,Ω_n)) have their orders multiplied by n. By convention, we let $(A_1,\Omega_1) := (\partial G, G)$.

In order to generate essential singularities, we take a disjoint union of copies of the relative fractal drums (A_n,Ω_n) scaled by $(n!)^{-1}$. More specifically, we define (A_∞,Ω_∞) as follows:

$$(A_\infty,\Omega_\infty) := (A,\Omega) \sqcup \bigsqcup_{n=2}^{\infty} (n!)^{-1}(A_n,\Omega_n). \tag{4.2.57}$$

The construction of (A_∞,Ω_∞) and the scaling property of the relative distance zeta function (see Theorem 4.1.40) then imply that

$$\zeta_{A_\infty,\Omega_\infty}(s) = \zeta_{\partial G,G}(s) \sum_{n=1}^{\infty} \frac{1}{(n!)^s\left(1-\sum_{j=1}^{J} r_j^s\right)^n}, \tag{4.2.58}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. By the Weierstrass M-test, the sum in the above equation (4.2.58) defines a holomorphic function on $\{\operatorname{Re} s > 0\}\setminus\mathfrak{D}^+$ and, furthermore, it is easy to show that $\mathfrak{D}^+$ is the set of essential singularities of the function defined by this sum. This completes the proof of the theorem. □

Clearly, the relative fractal drums constructed in the proof of Theorem 4.2.19 also exhibit some kind of self-similarity. Indeed, we can introduce the notion of a *self-similar RFD*, which generalizes the notion of a self-similar spray used, in particular, in [Lap2–3], [LapPo3], [Lap-vFr1–3], [LapPe2–3] and [LapPeWi1–2]. Namely, we can give the following formal definition.

Definition 4.2.20. Take (A_0,Ω_0) to be a *base* or *generating* relative fractal drum and define the self-similar relative fractal drum (A,Ω) analogously as in Definition 4.2.11; that is, let

$$(A,\Omega) := \bigsqcup_{i=0}^{\infty} \lambda_i(A_0,\Omega_0), \tag{4.2.59}$$

where $(\lambda_i)_{i\geq 0}$ is the scaling sequence corresponding to the multiset Λ defined by (4.2.31). Of course, in this case, we implicitly assume that the base relative fractal drum (A_0,Ω_0) is such that the above *disjoint* union can be constructed (see Examples 4.2.13 and 4.2.15).

When this is the case (for example, when Ω_0 is bounded), (A,Ω) is called a *self-similar RFD*. More specifically, (A,Ω) is called "the" *self-similar RFD with base RFD* (*or generated by the RFD*) (A_0,Ω_0) and with the scaling ratios $r_1,\dots,r_J$.[13] Its *self-similarity dimension* σ_0 is the unique real number s such that $\sum_{j=1}^{J} r_j^s = 1$. Necessarily, we have that $0<\sigma_0<N$.

Note that in the terminology introduced in Definition 4.2.20, the self-similar spray $(\partial G,G)$ of Theorem 4.2.17 is also a self-similar RFD with base RFD (or generated by the RFD) $(\partial G,G)$.

Remark 4.2.21. The iterative construction given in the proof of Theorem 4.2.19 can also be applied in the more general situation where the relative fractal drum (A,Ω) is actually a relative fractal spray. Namely, we fix a fractal string $\mathscr{L}$ and define the sequence of fractal strings $(\mathscr{L}_k)_{k\geq 0}$ from Theorem 4.2.8 by $\mathscr{L}_k := \mathscr{L}$ for every $k\geq 0$. Then, under the assumption that the base relative fractal drum (A_0,Ω_0) is 'nice enough', for each given $k\in\mathbb{N}_0$, the set of complex dimensions of the relative fractal drum (A_k,Ω_k) from Theorem 4.2.19 will contain the set of complex dimensions of $\mathscr{L}$, but with their orders (i.e., multiplicities) now multiplied by k.

4.2.3 Relative Sierpiński Sprays and Their Complex Dimensions

In this subsection, we provide two examples (Example 4.2.24 and 4.2.29) of relative fractal sprays, dealing with the *inhomogeneous relative Sierpiński gasket RFD* and the *relative Sierpiński carpet*, respectively, viewed as RFDs. We also discuss higher-dimensional analogs of these classic examples of self-similar fractals, namely, the *inhomogeneous Sierpiński N-gasket RFD*, also called the *inhomogeneous N-gasket*

[13] Clearly, such an RFD is unique only up to multiple choices of isometries (or displacements) of $\mathbb{R}^N$, corresponding to the countably many copies of the base RFD (A_0,Ω_0) it is composed of.

RFD (Example 4.2.26) and associated with the so-called *inhomogeneous N-gasket*, along with the (relative) Sierpiński N-carpet, (Example 4.2.31), with $N \geq 3$. (In this notation, the $N = 2$ case corresponds to the above standard Sierpiński gasket and carpet RFDs, respectively.) In fact, far from being trivial generalizations to higher-dimensions, these families of examples reveal several interesting new phenomena, which will be discussed especially in Example 4.2.26 (the inhomogeneous N-gasket RFD, with $N \geq 3$) and whose consequences will be considered in several parts of Chapter 5, including Subsection 5.5.6 (particularly, part (c) of Remark 5.5.26), as well as in some of the open problems of Chapter 6 (especially, Problems 6.2.32, 6.2.35 and 6.2.36).

In order to avoid any possible confusion, we stress from the outset that for $N \geq 3$, the inhomogeneous Sierpiński N-gasket can be viewed as a 'self-similar RFD' (or a self-similar spray, called a relative Sierpiński spray in the title of this subsection) but is *not* associated with a self-similar set, in the usual sense of the term; see, e.g., [Hut] or [Fal1, Chapter 9]. Indeed it is not associatd with a 'homogeneous self-similar set', as in the aforementioned references and the classic literature on fractal geometry, but (still for $N \geq 3$) it is instead associated with (in a sense to be specified in Example 4.2.26 below) an 'inhomogeneous self-similar set', in the sense of Barnsley and Demko [BarDemk], a notion already briefly described in a specific example in Remark 2.1.87 above. (See also [Fra2], along with the relevant references therein, for more detailed information about this topic.) The same comment also applies to Examples 4.2.33, 4.2.34 and 4.2.35, except for the fact that the self-similar fractal nest from Example 4.2.35 is a self-similar set in an even more general sense, which will be described below.

We note that aspects of this subsection are closely related to Section 3.2. In the sequel, it will be useful to use the following definition.

Definition 4.2.22. We say that *two given relative fractal drums* (A_1, Ω_1) *and* (A_2, Ω_2) *in* $\mathbb{R}^N$ *are congruent* if there exists an isometry[14] $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $A_2 = f(A_1)$ and $\Omega_2 = f(\Omega_1)$.

It is easy to see that the congruence of RFDs is an equivalence relation.

The following lemma states, in particular, that any two congruent RFDs have the same distance zeta functions. We leave its proof as a simple exercise for the interested reader.

Lemma 4.2.23. *Let* (A_1, Ω_1) *and* (A_2, Ω_2) *be two congruent RFDs in* $\mathbb{R}^N$. *Then, for any* $r \in \mathbb{R}$, *we have*

$$\mathscr{M}_*^r(A_1, \Omega_1) = \mathscr{M}_*^r(A_2, \Omega_2), \quad \mathscr{M}^{*r}(A_1, \Omega_1) = \mathscr{M}^{*r}(A_2, \Omega_2) \tag{4.2.60}$$

and

$$\underline{\dim}_B(A_1, \Omega_1) = \underline{\dim}_B(A_2, \Omega_2), \quad \overline{\dim}_B(A_1, \Omega_1) = \overline{\dim}_B(A_2, \Omega_2) =: \overline{D}. \tag{4.2.61}$$

[14] Recall that, up to a translation, an *isometry* (or *displacement*) of $\mathbb{R}^N$ is necessarily linear, with determinant ± 1.

As a result, $\dim_B(A_1,\Omega_1)$ *exists if and only if* $\dim_B(A_2,\Omega_2)$ *exists and in that case, we have*

$$\dim_B(A_1,\Omega_1)=\dim_B(A_2,\Omega_2). \tag{4.2.62}$$

Furthermore,

$$\zeta_{A_1,\Omega_1}(s)=\zeta_{A_2,\Omega_2}(s), \tag{4.2.63}$$

for any $s\in\mathbb{C}$ *with* $\operatorname{Re}s>\overline{\dim}_B(A_1,\Omega_1)$.

More generally, the identity (4.2.63) *holds for all* $s\in U$*, where* U *is any connected open neighborhood of the common critical line* $\{\operatorname{Re}s=\overline{\dim}_B(A_1,\Omega_1)\}$ *to which* ζ_{A_1,Ω_1} *(or, equivalently,* ζ_{A_1,Ω_1}*) can be meromorphically continued.*

It follows from (4.2.63) that under the hypotheses of Lemma 4.2.23 and given a connected open set $U\subseteq\mathbb{C}$ chosen as in the last part of the theorem (containing the common critical line $\{\operatorname{Re}s=\overline{D}\}$ of the RFDs (A_1,Ω_1) and (A_2,Ω_2); see Equation (4.2.61)), ζ_{A_1,Ω_1} and ζ_{A_2,Ω_2} have the exact same mermomorphic continuation to U, and therefore the same poles in U and associated residues (or more generally, principal parts in the case of multiple poles). In particular, two congruent RFDs have the same (visible) complex dimensions.

Example 4.2.24. (*Relative Sierpiński gasket*). Let A be the Sierpiński gasket in $\mathbb{R}^2$, the outer boundary of which is an equilateral triangle with unit sides. Consider the countable family of all open triangles in the standard construction of the gasket. Namely, these are the open triangles which are deleted at each stage of the construction. If Ω is the largest open triangle (with unit sides), then the *relative Sierpiński gasket* is defined as the ordered pair (A,Ω). The distance zeta function $\zeta_{A,\Omega}$ of the relative Sierpiński gasket (A,Ω) can be computed as the distance zeta function of the following relative fractal spray (see Definition 4.2.1):

$$\operatorname{Spray}(\Omega_0,\lambda=1/2,b=3),$$

where Ω_0 is the first deleted open triangle with sides $1/2$. It suffices to apply Equation (4.2.10) from Theorem 4.2.6. Decomposing Ω_0 into the union of six congruent right triangles (determined by the heights of the triangle Ω_0, see Figure 4.7) with disjoint interiors, we have that

$$\begin{aligned}\zeta_{\partial\Omega_0,\Omega_0}(s)&=6\,\zeta_{A',\Omega'}(s)=6\int_{\Omega'}d((x,y),A')^{s-2}\mathrm{d}x\,\mathrm{d}y\\&=6\int_0^{1/4}\mathrm{d}x\int_0^{x/\sqrt{3}}y^{s-2}\mathrm{d}y=6\frac{(\sqrt{3})^{1-s}2^{-s}}{s(s-1)},\end{aligned} \tag{4.2.64}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re}s>1$. Using Equation (4.2.10) and appealing to Lemma 4.2.23, we deduce that the distance zeta function of the relative Sierpiński gasket (A,Ω) satisfies

$$\zeta_{A,\Omega}(s)=\frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(1-3\cdot 2^{-s})}\sim\frac{1}{1-3\cdot 2^{-s}}, \tag{4.2.65}$$

where the equality holds for all $s\in\mathbb{C}$ with $\operatorname{Re}s>\log_2 3$ and as usual, the equivalence $\sim$ holds in the sense of Definition 2.1.69. Therefore, by the principle of analytic

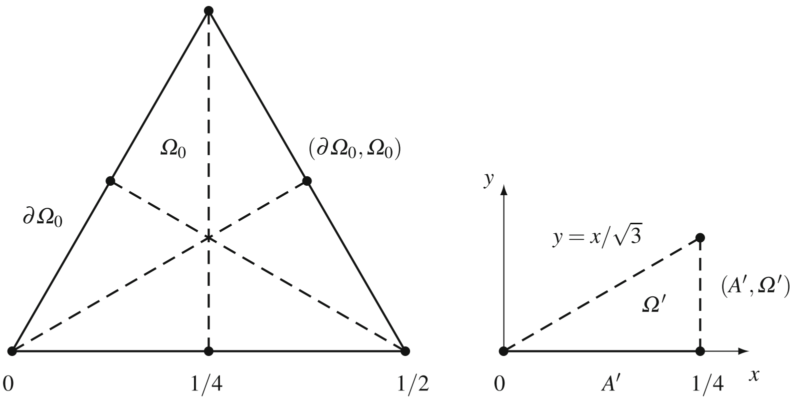

Fig. 4.7 On the left is depicted the base relative fractal drum $(\partial\Omega_0, \Omega_0)$ of the relative Sierpiński gasket, where Ω_0 is the associated (open) equilateral triangle with sides $1/2$. It can be viewed as the (disjoint) union of six RFDs, all of which are congruent to the relative fractal drum (A', Ω') on the right. This figure explains Equation (4.2.64) appearing in Example 4.2.24; see Lemma 4.2.23.

continuation, it follows that $\zeta_{A,\Omega}$ has a meromorphic extension to the entire complex plane, given by the same closed form as in Equation (4.2.65). More specifically,

$$\zeta_{A,\Omega}(s) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(1-3\cdot 2^{-s})}, \quad \text{for all } s \in \mathbb{C}. \tag{4.2.66}$$

Hence, the set of all of the complex dimensions (in $\mathbb{C}$) of the relative Sierpiński gasket is given by

$$\mathscr{P}(\zeta_{A,\Omega}) = \left(\log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right) \cup \{0,1\}. \tag{4.2.67}$$

Each of these complex dimensions is simple (i.e., is a simple pole of $\zeta_{A,\Omega}$). Note that here, $\{0,1\}$ can be interpreted as the set of *integer dimensions* of A, in the sense of [LapPe2–3] and [LapPeWi1] (see also [Lap-vFr3, Section 13.1]). In particular, we deduce from (4.2.67) that $D(\zeta_{A,\Omega}) = \log_2 3$, and we thus recover a well-known result. Namely, the set $\dim_{PC}(A,\Omega) := \mathscr{P}_c(\zeta_{A,\Omega})$ of principal complex dimensions of the relative Sierpiński gasket (A,Ω) is given by

$$\dim_{PC}(A,\Omega) = \log_2 3 + \mathbf{p}\mathbb{i}\mathbb{Z},$$

where $\mathbf{p} = 2\pi/\log 2$ is the oscillatory period of the Sierpiński gasket; see [Lap-vFr3, Subsection 6.6.1].

Note, however, that in [Lap-vFr1–3], the complex dimensions are obtained in a completely different manner (via an associated spectral zeta function corresponding to the Dirichlet Laplacian on the bounded open set Ω) and not geometrically. In addition, all of the complex dimensions of the Sierpiński gasket A are shown to be principal (that is, to be located on the vertical line $\operatorname{Re} s = \log_2 3 = \dim_B A$), a conclusion which is different from (4.2.67) above. We also refer to [ChrIvLap] and [LapSar], as well as to [LapPe2–3] and [LapPeWi1–2], for different approaches (via a spectral zeta function associated to a suitable geometric Dirac operator and via a self-similar tiling associated with A, respectively) leading to the same conclusion.

In light of (4.2.66), the residue of the distance zeta function $\zeta_{A,\Omega}$ of the relative Sierpiński gasket computed at any principal pole $s_k := \log_2 3 + \mathbf{p}k\mathbb{i}$, $k \in \mathbb{Z}$, is given by

$$\operatorname{res}(\zeta_{A,\Omega}, s_k) = \frac{6(\sqrt{3})^{1-s_k}}{2^{s_k}(\log 2)s_k(s_k-1)}.$$

In particular,

$$|\operatorname{res}(\zeta_{A,\Omega}, s_k)| \sim \frac{6(\sqrt{3})^{1-D}}{D\,2^D \log 2} k^{-2} \quad \text{as} \quad k \to \pm\infty,$$

where $D := \log_2 3$.

The following proposition shows that the relative Sierpiński gasket can be viewed as the relative fractal spray generated by the relative fractal drum (A', Ω') appearing in Figure 4.7 on the right.

Proposition 4.2.25 (Relative Sierpiński gasket). *Let (A', Ω') be the relative fractal drum defined in Figure 4.7 on the right. Let (A, Ω) be the relative fractal spray generated by the base relative fractal drum (A', Ω'), with scaling ratio $\lambda = 1/2$ and with multiplicities $m_k = 6 \cdot 3^{k-1}$, for any positive integer k:*

$$(A, \Omega) = \operatorname{Spray}((A', \Omega'), \lambda = 1/2, m_k = 6 \cdot 3^{k-1} \text{ for } k \in \mathbb{N}), \tag{4.2.68}$$

in the notation of Definition 4.2.1. (Observe that we assume here that the base relative fractal drum (A', Ω') has a multiplicity equal to 8.) Then, the relative distance zeta function of the relative fractal spray (A, Ω) coincides with the relative distance zeta function of the relative Sierpiński gasket; see Equation (4.2.66).

Example 4.2.26. (*Inhomogeneous Sierpiński N-gasket RFD*). The usual Sierpiński gasket is contained in the unit triangle in the plane. Its (inhomogeneous) analog in $\mathbb{R}^3$, which we call the *inhomogeneous Sierpiński 3-gasket* or *inhomogeneous tetrahedral gasket*, is obtained by deleting the middle open octahedron (from the initial compact, convex unit tetrahedron), defined as the interior of the convex hull of the midpoints of each of the six edges of the initial tetrahedron (thus, four sub-tetrahedra are left after the first step), and so on.

More generally, for $N \geq 2$, the *inhomogeneous Sierpiński N-gasket A_N*, contained in $\mathbb{R}^N$, can be defined as follows. (More briefly, A_N is also referred to as the *inhomogeneous N-gasket*.) Let $V_N := \{P_1, P_2, \ldots, P_{N+1}\}$ be a set of N points in $\mathbb{R}^N$ such that the mutual distance of any two points from the set is equal to 1.

The set V_N, where $N \geq 2$, with the indicated property, can be constructed inductively as follows. For $N = 2$, we take V_2 to be the set of vertices of any unit triangle in $\mathbb{R}^2$. We then reason by induction. Given $N \geq 2$, we assume that the set V_N of $N+1$

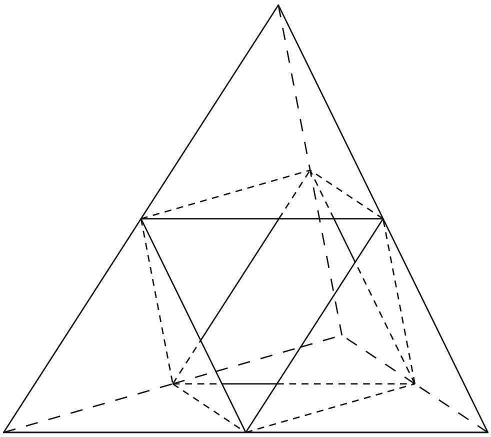

Fig. 4.8 The open octahedron $\Omega_{3,0}$ inscribed into the largest (compact) tetrahedron Ω_3, surrounded with 4 smaller (compact) tetrahedra scaled by the factor $1/2$. Each of them contains analogous scaled open octahedra, etc. The countable family of all open octahedra (viewed jointly with their boundaries) constitutes the tetrahedral gasket RFD or the Sierpiński 3-gasket RFD (A_3, Ω_3). The complement of the union of all open octahedra, with respect to the initial tetrahedron Ω_3, is called the *inhomogeneous Sierpiński* 3*-gasket RFD* (or the *relative Sierpiński* 3*-gasket*).

Unlike the classic Sierpiński 3-gasket (also known as the Sierpiński pyramid or tetrahedron) $S := S_3$, which is a (homogeneous or standard) self-similar set in $\mathbb{R}^3$ and satisfies the usual fixed point equation, $S = \cup_{j=1}^4 \Phi_j(S)$, where $\{\Phi_j\}_{j=1}^4$ are suitable contractive similitudes of $\mathbb{R}^3$ with respective fixed points $\{P_j\}_{j=1}^4$ and scaling ratios $\{r_j\}_{j=1}^4$ of common value $1/2$, the inhomogeneous Sierpiński 3-gasket RFD A_3 is *not* a self-similar set. Instead, it is an *inhomogeneous self-similar set* (in the sense of [BarDemk], see also [Fra2] and Remark 2.1.87 above). More specifically, $A := A_3$ satisfies the following *inhomogeneous* fixed point equation (of which it is the unique solution in the class of all nonempty compact subsets of $\mathbb{R}^3$), $A = \bigcup_{j=1}^4 \Phi_j(A) \cup B$, where B is the boundary $\partial\Omega_{3,0}$ of the first octahedron $\Omega_{3,0}$. In fact, B can simply be taken as the union of four middle triangles on the boundary of the outer tetrahedron Ω_3.

points in R^N has been constructed. Note that the set V_N is contained in a sphere, whose center is denoted by O. Let us consider the line of $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ through the point O and perpendicular to the hyperplane $\mathbb{R}^N = \mathbb{R}^N \times \{0\}$ in $\mathbb{R}^{N+1}$. There exists a unique point P_{N+2} in the half-plane $\{x_{N+1} > 0\}$ of $\mathbb{R}^{N+1}$, which is a unit distance from all of the N vertices of V_N. (Here, we identify V_N with $V_N \times \{0\} \subset \mathbb{R}^{N+1}$.) We then define V_{N+1} by $V_{N+1} := V_N \cup \{P_{N+2}\}$.

Let us define Ω_N as the convex hull of the set V_N. As usual, we call it the *N-simplex*. Let $\Omega_{N,0}$, called the *N-plex*, be the open set defined as the interior of the convex hull of the set of midpoints of all of the $\binom{N+1}{2}$ edges of the N-simplex Ω_N.[15] The set $\overline{\Omega}_N \setminus \Omega_{N,0}$ is equal to the union of $N+1$ congruent N-simplices with disjoint interiors, having all of their side lengths equal to $1/2$. This is the first step of the

[15] For example, for $N = 2$, the set $\Omega_{2,0}$ (that is, the 2-plex) is an open equilateral triangle in $\mathbb{R}^2$ of side lengths equal to $1/2$, while for $N = 3$, the set $\Omega_{3,0}$ (that is, the 3-plex) is an open octahedron in $\mathbb{R}^3$ of side lengths equal to $1/2$.

construction. We proceed analogously with each of the $N+1$ compact N-simplices. The compact set A_N obtained in this way is called the *inhomogeneous Sierpiński N-gasket* (or, more briefly, the *inhomogeneous N-gasket*). The corresponding relative fractal spray in $\mathbb{R}^N$, defined by

$$(A_N, \Omega_N) = \text{Spray}\left((\partial\Omega_{N,0}, \Omega_{N,0}), \lambda = 1/2, b = N+1\right), \tag{4.2.69}$$

is called the *inhomogeneous Sierpiński N-gasket RFD* (or, more briefly, the *inhomogeneous N-gasket RFD*). It is a self-similar spray RFD; see the end of Definition 4.2.1 or of Definition 4.2.20. According to Theorem 4.2.6, we have the following factorization formula:

$$\zeta_{A_N,\Omega_N}(s) = f(s)\cdot\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}(s), \tag{4.2.70}$$

where

$$f(s) = \sum_{k=0}^{\infty}(N+1)^k(2^{-k})^s = \frac{1}{1-(N+1)2^{-s}}.$$

Upon analytic continuation, we deduce that $f(s)$ has a meromorphic continuation to all of $\mathbb{C}$ given by

$$f(s) := \frac{1}{1-(N+1)2^{-s}}, \quad \text{for all } s\in\mathbb{C}. \tag{4.2.71}$$

Hence, the set of poles of the function f (which can be uniquely meromorphically extended to the whole complex plane), is given by

$$\mathscr{P}(f) = \log_2(N+1) + \frac{2\pi}{\log 2}\,\mathbb{i}\mathbb{Z}. \tag{4.2.72}$$

Furthermore, the set of poles of the distance zeta function of the *relative N-plex* $(\partial\Omega_{N,0}, \Omega_{N,0})$ is given by

$$\mathscr{P}(\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}) = \{0,1,\dots,N-1\}, \tag{4.2.73}$$

while $\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}(s) \neq 0$ for all $s\in\mathbb{C}\setminus\mathscr{P}(\zeta_{\partial\Omega_{N,0},\Omega_{N,0}})$. Both in (4.2.72) and (4.2.73), all of the poles are simple. Consequently, in light of (4.2.70), we conclude that the set of poles of $\zeta_{\partial\Omega_N,\Omega_N}$, i.e., the complex dimensions of the *inhomogeneous Sierpiński N-gasket* (A_N, Ω_N), is given by

$$\mathscr{P}(\zeta_{A_N,\Omega_N}) = \{0,1,\dots,N-1\} \cup \left\{\log_2(N+1) + \frac{2\pi}{\log 2}\,\mathbb{i}\mathbb{Z}\right\}, \tag{4.2.74}$$

with each nonreal complex dimension $\omega_k := \log_2(N+1) + \frac{2\pi}{\log 2}\mathbb{i}k$ (with $k \in \mathbb{Z} \setminus \{0\}$) being simple.[16] In particular, the set of principal complex dimensions of (A_N, Ω_N) is given by[17]

$$\dim_{PC}(A_N, \Omega_N) = \begin{cases} \log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}, & \text{for } N = 2, \\ 2 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}, & \text{for } N = 3, \\ \{N-1\}, & \text{for } N \geq 4, \end{cases} \tag{4.2.75}$$

while the (upper) box dimension of (A_N, Ω_N) is given by

$$\overline{\dim}_B(A_N, \Omega_N) = \max\left\{\log_2(N+1), N-1\right\} \tag{4.2.76}$$

and so

$$\overline{\dim}_B(A_N, \Omega_N) = \begin{cases} \log_2 3, & \text{for } N = 2, \\ N-1, & \text{for } N \geq 3, \end{cases} \tag{4.2.77}$$

which extends the well-known results for $N = 2$ and 3, corresponding to the usual Sierpiński gasket in $\mathbb{R}^2$ and the tetrahedral gasket in $\mathbb{R}^3$, respectively. Namely, their respective relative box dimensions are equal to $\log_2 3$ and 2.

It can be readily shown that in this case, $\dim_B(A_N, \Omega_N)$ and $\dim_B A_N$ exist and

$$\overline{\dim}_B(A_N, \Omega_N) = \dim_B(A_N, \Omega_N) = \dim_B A_N = \dim_H A_N, \tag{4.2.78}$$

where as before, $\dim_H(\cdot)$ denotes the Hausdorff dimension. More generally, it is easy to see that $\dim_{PC}(A_N, \Omega_N) = \dim_{PC} A_N$, where the equality holds between multisets, that is, counting multiplicities. See also Remark 4.2.27 below.

Moreover, it can also be easily checked (and is essentially known, at least for $N = 2$) that (A_N, Ω_N) is Minkowski nondegenerate if $N \neq 3$:[18]

$$0 < \mathscr{M}_*(A_N, \Omega_N) \leq \mathscr{M}^*(A_N, \Omega_N) < \infty. \tag{4.2.79}$$

In the special case when $N = 3$, due to the factorization formula (4.2.70), ζ_{A_3,Ω_3} has a double pole at $s = 2$ and it can be shown by some of the methods of Chapter 5 (see, especially, Theorem 5.3.21) that in this case, (A_3, Ω_3) is Minkowski degenerate with $\mathscr{M}(A_3, \Omega_3) = +\infty$, but that it is also *h-Minkowski measurable* where the gauge function h is given by $h(t) := \log t^{-1}$ for all $t \in (0,1)$. (For an introduction to gauge functions and gauge Minkowski contents, see the beginning of Subsection 4.5.1 and also Definition 6.1.4 below.) In particular, since $D = \dim_B(A_N, \Omega_N)$ exists and

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[16] Note that it could happen that $\omega_0 = \log_2(N+1)$ is equal to an integer $m \in \{0, 1, \ldots, N-1\}$, which occurs if and only if $N = 2^m - 1$ with $m \in \mathbb{N} \setminus \{1\}$ and $N \geq 3$ or if $N = 1$ (the trivial case of the unit interval discussed in Example 5.5.1 below). In that situation (when $N \geq 3$), $\omega_0 = \sigma_0$ (the similarity dimension of A_N and (A_N, Ω_N), to be discussed further on) is a *double* pole of ζ_{A_N,Ω_N}.

[17] Recall that, by definition, $\dim_{PC}(A_N, \Omega_N) := \mathscr{P}_c(\zeta_{A_N,\Omega_N})$, the set of principal complex dimensions of the RFD (A_N, Ω_N).

[18] The truth of this statement can also be deduced from the methods and results of Chapter 5 below, especially, in Sections 5.3–5.5, including Example 5.5.12 and Subsection 5.5.6.

$\mathscr{M}_*(A_N,\Omega_N)>0$, the hypothesis of part (c) of Theorem 4.1.7 (and hence, also of part (ii) of Corollary 4.1.10) are satisfied and so, in light of the factorization formula (4.2.70), a moment's reflection shows that

$$\begin{aligned} D := \dim_B(A_N,\Omega_N) &= \max\left\{\log_2(N+1), N-1\right\} \\ &= D(\zeta_{A_N,\Omega_N}) = D_{\mathrm{hol}}(\zeta_{A_N,\Omega_N}), \end{aligned} \tag{4.2.80}$$

as was claimed in Equation (4.2.76) above.

Note that in (4.2.76) and (4.2.80), $\log_2(N+1)$ stands for the *similarity dimension* σ_0 of the self-similar RFD or spray (A_N,Ω_N) (or, equivalently, of the *inhomogeneous* self-similar set A_N), while $N-1$ refers to the (inner) dimension of the boundary $\partial\Omega_{N,0}$ of the generator (the N-plex $\Omega_{N,0}$), i.e., of the RFD $(A_{N,0},\Omega_{N,0})$.

In the sequel, the function f appearing in Equations (4.2.70)–(4.2.72) will often be called the *scaling zeta function* of the RFD (A,Ω) and denoted by $\zeta_{\mathfrak{S}}$; see, e.g., Subsection 5.5.6 or Section 6.2. Therefore, for example, Equation (4.2.70) can be rewritten as follows (using the abbreviated notation ζ_{A_N,Ω_N} and $\zeta_{A_{N,0},\Omega_{N,0}}$):

$$\zeta_{A_N,\Omega_N}(s) = \zeta_{\mathfrak{S}}(s)\cdot\zeta_{A_{N,0},\Omega_{N,0}}(s). \tag{4.2.81}$$

Also, in Equation (4.2.74), and in agreement with the terminology of [LapPe2–3] and [LapPeWi1–2] (see also [Lap-vFr3, Section 13.1]), $\{0,1,\dots,N-1\}$ and $\mathscr{P}(\zeta_{\mathfrak{S}})=\left\{\log_2(N+1)+\frac{2\pi}{\log 2}\mathrm{i}\mathbb{Z}\right\}$ are called, respectively, the set of *integer dimensions* and the set of *scaling complex dimensions* of the self-similar RFD (A_N,Ω_N). Note that some points could be common to those two sets, for instance, when $N=3$, the point $s=2$, which is therefore a double pole of ζ_{A_N,Ω_N} or, equivalently, a complex dimension of (A_3,Ω_3) of multiplicity two.

Recall that the *classic Sierpiński N-gasket* S_N (used, for example, in [KiLap1], [Ki1], and the relevant references therein),[19] is a standard (or *homogeneous*) self-similar set. Hence, $S:=S_N$ satisfies the fixed point equation $S=\cup_{j=1}^{N+1}\Phi_j(S)$, where $\{\Phi_j\}_{j=1}^{N+1}$ are contractive similitudes of $\mathbb{R}^N$ with corresponding fixed points $\{V_j\}_{j=1}^{N+1}$ and scaling ratios $\{r_j\}_{j=1}^{N+1}$, of common value $1/2$: $r_1=\cdots=r_{N+1}=1/2$. In particular, S is the unique nonempty compact subset of $\mathbb{R}^N$ which is the solution of that equation. See, e.g., [Hut] or [Fal1, Chapter 9].

By contrast, the inhomogeneous Sierpiński N-gasket RFD (A_N,Ω_N) is a self-similar spray or RFD, but (for $N\geq 3$) A_N is *not* a (classic or homogeneous) self-similar set. Interestingly, however, A_N is an *inhomogeneous* self similar set (in the sense of [BarDemk], see also [Fra1] and Remark 2.1.87 above), as is explained in more detail when $N=3$ in the second part of the caption of Figure 4.8 above. In particular, when $N\geq 3$, $A:=A_N$ satisfies the following *inhomogeneous* fixed point equation:

$$A=\bigcup_{j=1}^{N+1}\Phi_j(A)\cup B, \tag{4.2.82}$$

[19] The classic Sierpiński N-gasket S_N has been used, for example, in [KiLap1], in a work dealing with the spectral analysis of Laplacians on self-similar fractals.

where B is a suitable (nonempty) compact subset of $\mathbb{R}^N$ (described in the caption of Figure 4.8 in the prototypical special case when $N = 3$); for example, B can be chosen to be the boundary of $\Omega_{N,0}$: we can let $B = \partial\Omega_{N,0} = A_{N,0}$. More specifically, A_N is the unique nonempty compact subset A of $\mathbb{R}^N$ satisfying the identity (4.2.82).

Nevertheless, since in the terminology of Definition 4.2.11, (A_N, Ω_N) is a self-similar spray with generator the N-plex $\Omega_{N,0}$ and ratio list $\{r_1 = \cdots = r_{N+1} = 1/2\}$, the self-similar set S_N (the classic Sierpiński N-gasket), the inhomogeneous self-similar set A_N (the inhomogeneous N-gasket) and the self-similar spray (or RFD) (A_N, Ω_N) have the same similarity dimension, σ_0, which is the unique real solution of the Moran equation $\sum_{j=1}^{N+1} r_j^s = 1$; that is, $(N+1)\cdot 2^{-s} = 1$, with $s \in \mathbb{R}$, or, equivalently,

$$\sigma_0 = \log_2(N+1). \tag{4.2.83}$$

Finally, we point out that Equations (4.2.78) and (4.2.80) imply that

$$\dim_B(A_N, \Omega_N) = \max\left\{\sigma_0, \dim_B(A_{N,0}, \Omega_{N,0})\right\}. \tag{4.2.84}$$

(See also Equation (4.2.50) in Theorem 4.2.17.) Therefore, $\dim_B(A_N, \Omega_N)$ is equal to $\sigma_0 = \dim_B A_N$ when $N \leq 3$ and is strictly greater than σ_0 when $N \geq 4$. (See also Remark 4.2.27 just below.) We will obtain a natural generalization and application of these results towards the end of Section 5.5.6; see, especially, part (c) of Remark 5.5.26.

It is noteworthy that when $N = 2$, we not only have that $A_2 = S_2$, the classic Sierpiński gasket, but it is also the case that $A_2 = S_2$ is both a (homogeneous or standard) self-similar set *and* an inhomogeneous self-similar set, with respect to the same iterated functions system (or IFS) $\{\Phi\}_{j=1}^3$, comprised of similarity transformations of $\mathbb{R}^2$. Indeed, in the inhomogeneous fixed point equation (4.2.82), with $A := A_2$ and $N = 2$, we can not only choose $B := \emptyset$ (the empty set), but we can also choose $B := \partial A_{2,0}$, the boundary of the unit triangle.

Remark 4.2.27. It is well known (see, e.g., [Hut], [Fal1, Theorem 9.3]), that if a higher-dimensional self-similar set satisfies the open set condition, as is the case (for every $N \geq 2$) of the standard Sierpiński N-gasket S_N but *not* (for any $N \geq 3$) of the inhomogeneous Sierpiński N-gasket A_N (see the discussion following Equation (4.2.80), along with the caption of Figure 4.8), then its Minkowski and Hausdorff dimensions coincide with its similarity dimension; moreover, $\dim_B S_N$ exists. Hence, this means that

$$\dim_B S_N = \dim_H S_N = \sigma_0 = \log_2(N+1), \tag{4.2.85}$$

where σ_0 is the common similarity dimension of the self-similar set S_N, the inhomogeneous N-gasket A_N, and the self-similar RFD (A_N, Ω_N). Hence, when $N \geq 4$, it follows that $\dim_B A_N > \sigma_0 = \dim_B S_N$. There is no contradiction, however, in light of Equations (4.2.76) and (4.2.77), along with the fact that A_N is not a self-similar set (only a nonhomogeneous self-similar set) for such values of N.[20]

[20] It is possible to construct simpler examples in $\mathbb{R}^2$ which also illustrate this situation; see Examples 4.2.33, 4.2.34 and 4.2.35 below.

What is true, in general, under the above hypotheses (see Theorem 4.2.17 and Definition 4.2.20 above) is that the Minkowski dimension of the self-similar RFD is equal to the maximum of the similarity dimension σ_0 and the (inner) Minkowski dimension D_G of the boundary of the generator G, which is assumed to be sufficiently nice (see Subsection 5.5.6 below);[21] here, $G := \Omega_{N,0}$ and hence, $D_G = \dim_B(A_{N,0}, \Omega_{N,0}) = N-1$. We will further discuss this issue in Remark 5.5.26 of Subsection 5.5.6, where we will see that the proper counterpart of this situation is case (iii) of part (c) of Remark 5.5.26, namely, when $D_G := \dim_B(\partial G, G) > \sigma_0$. This latter possibility cannot occur in the case of a (standard) self-similar set; see [Fal1, Theorem 9.3]. More specifically, as does not seem to have been observed before, this impossibility is a somewhat surprising consequence of the aforementioned result of Hutchinson in [Hut], as described in [Fal1, Theorem 9.3] and extending to any dimension $N \geq 1$ a corresponding one-dimensional result due to Moran in [Mora].

In closing this remark, we mention that such a problem does not occur when $N = 1$, which is the situation considered in the theory of the complex dimensions of geometric self-similar strings developed, in particular, in [Lap-vFr3, Chapters 2, 3 and Section 8.4]. Indeed, we then have that $D_G < \sigma_0$ since $D_G = 0$ (when $N = 1$) and $\sigma_0 > 0$ (always).

We next explain in more detail how to calculate the complex dimensions (and hence also the principal complex dimensions) of the relative inhomogeneous Sierpiński N-gasket (A_N, Ω_N) in the prototypical case when $N = 3$.

The relative distance zeta function $\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}$ of the N-plex RFD $(\partial\Omega_{N,0}, \Omega_{N,0})$ $= (A_{N,0}, \Omega_{N,0})$ can be explicitly computed as follows, in the case when $N = 3$. It is easy to see that the octahedral RFD $(\partial\Omega_{3,0}, \Omega_{3,0})$ can be identified with sixteen copies of disjoint RFDs, each of which is congruent to the pyramidal RFD (T, Ω') in $\mathbb{R}^3$, where Ω' is the open (irregular) pyramid with vertices at $O(0,0,0), A(1/4,0,0)$, $B(1/4,1/4,0)$ and $C(0,0,1/(2\sqrt{2}))$, while the triangle $T = \operatorname{conv}(A,B,C)$ is a face of the pyramid. Since for any $(x,y,z) \in \Omega'$, we have

$$d((x,y,z),T) = \frac{1}{\sqrt{3}}\left(\frac{1}{2\sqrt{2}} - \sqrt{2}x - z\right), \tag{4.2.86}$$

we deduce that (recall that $A_{3,0} := \partial\Omega_{3,0}$)

$$\begin{aligned}\zeta_{A_{3,0},\Omega_{3,0}}(s) &= 16\zeta_{T,\Omega'}(s)\\ &= 16\iiint_{\Omega'} d((x,y,z),T)^{s-3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &= 16\int_0^{1/4}\mathrm{d}x\int_0^x \mathrm{d}y\int_0^{\frac{1}{2\sqrt{2}}-\sqrt{2}x}\left(\frac{\frac{1}{2\sqrt{2}}-\sqrt{2}x-z}{\sqrt{3}}\right)^{s-3}\mathrm{d}z.\end{aligned} \tag{4.2.87}$$

[21] If we allow the boundary of G to be fractal, then new interesting phenomena may occur, as was illustrated in Subsection 4.2.2 above.

Evaluating the last integral in (4.2.87), we obtain via a direct computation that

$$\begin{aligned}\zeta_{A_{3,0},\Omega_{3,0}}(s) &= 16\frac{(\sqrt{3})^{3-s}}{s-2}\int_0^{1/4}\Big(\frac{1}{2\sqrt{2}}-\sqrt{2}x\Big)^{s-2}x\,\mathrm{d}x\\ &= 8\frac{(\sqrt{3})^{3-s}}{s-2}\int_0^{1/(2\sqrt{2})}u^{s-2}\Big(\frac{1}{2\sqrt{2}}-u\Big)\,\mathrm{d}u\\ &= \frac{8(\sqrt{3})^{3-s}(2\sqrt{2})^{-s}}{s(s-1)(s-2)},\end{aligned} \tag{4.2.88}$$

for any complex number s such that $\operatorname{Re}s > 2$. Therefore, we deduce from (4.2.70) that the distance zeta function of the thetrahedral RFD in $\mathbb{R}^3$ can be meromorphically extended to the whole complex plane and is given for all $s \in \mathbb{C}$ by

$$\zeta_{A_3,\Omega_3}(s) = \frac{8(\sqrt{3})^{3-s}(2\sqrt{2})^{-s}}{s(s-1)(s-2)(1-4\cdot 2^{-s})}. \tag{4.2.89}$$

It is worth noting that $s = 2$ is the only pole of ζ_{A_3,Ω_3} of order 2, since $s = 2$ is a simple pole of both $(s-2)^{-1}$ and $(2^s-4)^{-1}$. More specifically, since the derivative of $1-4\cdot 2^{-s}$ computed at $s = 2$ is nonzero (and, in fact, is equal to $4\log 2$), then $s = 2$ is a simple zero of $1-4\cdot 2^{-s}$; that is, it is a simple pole of $1/(1-4\cdot 2^{-s})$.

Moreover, it immediately follows from Equation (4.2.89) that

$$\zeta_{A_3,\Omega_3}(s) \sim \frac{1}{(s-2)(1-4\cdot 2^{-s})}. \tag{4.2.90}$$

In particular, as we have already seen in Equation (4.2.75) and recalling that $N := 3$ here, we have

$$\dim_{PC}(A_3,\Omega_3) = 2 + \frac{2\pi}{\log 2}\,\mathbb{i}\mathbb{Z}. \tag{4.2.91}$$

Since $D := \overline{\dim}_B(A_3,\Omega_3) = \dim_B(A_3,\Omega_3) = 2$ is a simple pole of both $1/(s-2)$ and $1/(2^s-4)$, we conclude that $D = 2$ is the only complex dimension of order two of the RFD (A_3,Ω_3). Consequently, the case of the relative Sierpiński 3-gasket (A_3,Ω_3) reveals a new phenomenon: its relative box dimension $D = 2$ is a complex dimension of order (i.e., multiplicity) two, while all the other complex dimensions of the relative Sierpiński 3-gasket (including the double sequence of nonreal complex dimensions on the critical line of convergence $\{\operatorname{Re}s = 2\}$) are simple. Since, as we have already observed earlier, we have that $\dim_{PC}(A_N,\Omega_N) = \dim_{PC}A_N$ for every $N \geq 2$, we deduce from (4.2.91) and the discussion following it that

$$\dim_{PC}A_3 = 2 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}, \tag{4.2.92}$$

with $s = \dim_B A_3 = 2$ being the only principal complex dimension of A_3 of order two, all the other complex dimensions being simple.

We challenge the interested reader to use similar arguments as in the case when $N=3$ in order to infer that for any $N\geq 3$, the distance zeta function of the relative N-plex $(\partial\Omega_{N,0},\Omega_{N,0})$ is of the form

$$\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}(s)=\frac{g(s)}{s(s-1)\cdots(s-(N-1))}, \tag{4.2.93}$$

where $g(s)$ is a nonvanishing entire function. (Note that, when $N=3$, this is in agreement with Equation (4.2.88) above.) Therefore, we conclude from Equations (4.2.70) and (4.2.71) above that

$$\zeta_{A_N,\Omega_N}(s)=\frac{g(s)}{s(s-1)\cdots(s-(N-1))(1-(N+1)2^{-s})}. \tag{4.2.94}$$

This extends Equation (4.2.89) to any $N\geq 3$ (really, of the base *RFD* $(A_{3,0},\Omega_{3,0})$ generating the self-similar RFD (A_3,Ω_3)).

In the case when $N\geq 4$, $D=N-1$ is the only principal complex dimension of the inhomogeneous Sierpiński N-gasket RFD. [Indeed, for $N\geq 4$, we have that $\log_2(N+1)<N-1$ (i.e., $N+1<2^{N-1}$), which can be easily proved, for example, by using mathematical induction on N.] Furthermore, we immediately deduce from Equation (4.2.94) that

$$\zeta_{A_N,\Omega_N}(s)\sim\frac{1}{s-(N-1)}. \tag{4.2.95}$$

Moreover, if $N\geq 4$ is of the form $N=2^q-1$ for some integer $q\geq 3$, then $q=\log_2(N+1)$ (note that it is smaller than $D=N-1$) is the only complex dimension of order two (since it is a simple pole of both $(s-q)^{-1}$ and $(1-(N+1)2^{-s})^{-1}$), while all of the other complex dimensions of (A_N,Ω_N) are simple.

On the other hand, if $N\geq 4$ is not of the form $N=2^q-1$ for any integer $q\geq 3$, then all of the complex dimensions of the inhomogeneous Sierpiński N-gasket RFD are simple.

Roughly speaking, in the case when when $N=3$, the fact that $s=2$ has multiplicity two can be explained geometrically as follows: firstly, $s=2$ is a simple pole of the scaling zeta function $\zeta_{\mathfrak{S}}(s)=1/(1-(N+1)2^{-s})=1/(1-4\cdot 2^{-s})$ of the RFD (A_3,Ω_3),[22] while at the same time, $s=2$ is the simple pole arising from the boundary of the first (deleted) octahedron, which is also 2-dimensional; more specifically, $s=2$ is also a simple pole of $\zeta_{A_{3,0},\Omega_{3,0}}$. Therefore, the double pole of ζ_{A_N,Ω_N} arises both from the (inhomogeneous) self-similarity of the RFD (A_N,Ω_N) (or, equivalently, of the set A_N) and from the special geometry of the boundary of the generator (really, of the base RFD $(A_{N,0},\Omega_{N,0})$ generating the self-similar RFD (A_N,Ω_N)) when $N=3$.

Remark 4.2.28. Since as was noted earlier, $\dim_{PC}A_N=\dim_{PC}(A_N,\Omega_N)$, where the equality holds between multisets, exactly the same comment as above holds about the *principal* complex dimensions of the inhomogeneous N-gasket A_N (instead of

[22] Indeed, the similarity dimension of the 3-gasket A_3 is equal to 2.

the complex dimensions of the inhomogeneous N-gasket RFD (A_N, Ω_N)). For example, if $N \geq 4$ is not of the form $2^q - 1$ for any integer $q \geq 3$, then all of the complex dimensions of A_N are simple, while otherwise (i.e., if $N = 2^q - 1$, for some $q \geq 3$), then $s = q$ is the only complex dimension of multiplicity 2 and all the other complex dimensions (including $D = \dim_B A_N = \dim_B(A_N, \Omega) = N - 1$) are simple. The multiplicity of $s = q$ arises both from the (inhomogeneous) self-similar structure of A_N and of the geometric structure of the boundary of the generator $\Omega_{N,0}$, $A_{N,0} := \partial\Omega_{N,0}$, exactly as in the case when $N = 3$.

In the case of the relative inhomogeneous Sierpiński 2-gasket (A_2, Ω_2), the value of $s = \log_2 3$ (which is the simple pole arising from the self-similarity of (A_2, Ω_2)) is strictly larger than the dimension $s = 1$ of the boundary of the deleted triangle (i.e., of the 2-plex $\Omega_{2,0}$). Moreover, the relative 2-Sierpiński gasket is Minkowski nondegenerate and Minkowski nonmeasurable, while the relative inhomogeneous 3-Sierpiński gasket (A_3, Ω_3) is Minkowski degenerate, with its 2-dimensional Minkowski content being equal to $+\infty$. Its gauge function (a notion introduced in Subsection 6.1.4 of Chapter 6.1.1) can be determined by methods involving the fractal tube formulas developed in Chapter 5.

More specifically, in Chapter 5, we will show that it is possible to find a gauge function (namely, $h(t) := \log t^{-1}$ for all $t \in (0,1)$) relative to which the relative inhomogeneous 3-gasket RFD (A_3, Ω_3) is Minkowski *nondegenerate* and moreover, Minkowski *measurable*; see Theorem 5.4.27. (The same is true for the inhomogeneous 3-gasket A_3.) This should be contrasted with the case of the ordinary (classical) Sierpiński N-gasket S_N, which is Minkowski nondegenerate and Minkowski nonmeasurable (in the usual sense, i.e., with respect to the trivial gauge function $h(t) \equiv 1$ correspoding to a standard power law).

On the other hand, when $N \geq 4$, the dimension $D_G = N - 1$ of the boundary of the N-plex $\Omega_{N,0}$ is larger than the similarity dimension $\sigma_0 = \log_2(N+1)$ arising from "fractality". Hence, $D_G = N - 1$. Since $D = N - 1$ is the only complex dimension on the critical line (and it is simple), we conclude that for $N \geq 4$, the RFD (A_N, Ω_N) is Minkowski measurable (see Theorem 5.4.20 in Chapter 5). Thus, the case when $N = 3$ is indeed very special among all of the inhomogeneous Sierpiński N-gasket RFDs. These issues will be clarified and revisited, as well as placed in a much broader framework, towards the end of Chapters 5 and 6; see, especially, part (c) of Remark 5.5.26 in Subsection 5.5.6, along with Problems 6.2.32, 6.2.35 and 6.2.36.

Example 4.2.29. (*Relative Sierpiński carpet*). Let A be the Sierpiński carpet contained in the unit square Ω. Let (A, Ω) be the corresponding *relative Sierpiński carpet* (or *Sierpiński carpet RFD*), with Ω being the unit square. (See Figure 2.1 on page 49 for a picture of the standard Sierpiński carpet.) Its distance zeta function $\zeta_{A,\Omega}$ coincides with the distance zeta function of the following relative fractal spray (see the end of Definition 4.2.1):

$$\mathrm{Spray}(\Omega_0, \lambda = 1/3, b = 8),$$

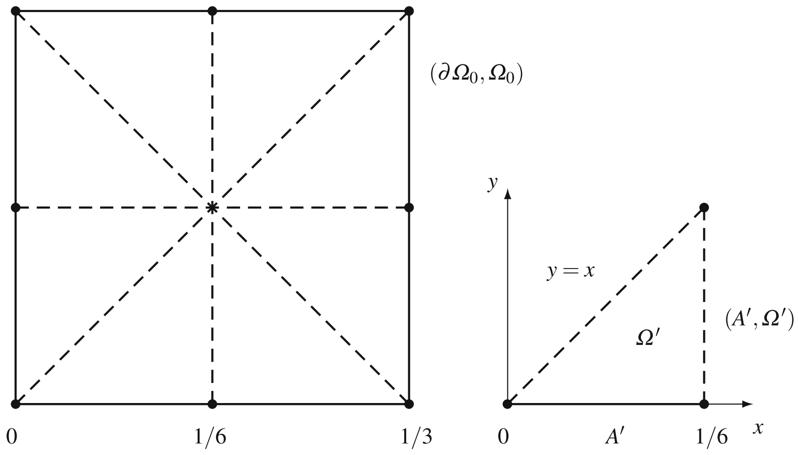

Fig. 4.9 On the left is the base relative fractal drum $(\partial\Omega_0,\Omega_0)$ of the relative Sierpiński carpet (A,Ω) described in Example 4.2.29, where Ω_0 is the associated (open) square with sides $1/3$. The base relative fractal drum $(\partial\Omega_0,\Omega_0)$ can be viewed as the (disjoint) union of eight RFDs, all of which are congruent to the relative fractal drum (A',Ω') depicted on the right. This figure explains Equation (4.2.97); see Lemma 4.2.23.

where Ω_0 is the first deleted open square with sides $1/3$. Similarly as in Example 4.2.24, by using Theorem 4.2.6 and Lemma 4.2.23, we obtain that $\zeta_{A,\Omega}$, the relative distance zeta functions of (A,Ω), has a meromorphic continuation to the entire complex plane given for all $s\in\mathbb{C}$ by

$$\zeta_{A,\Omega}(s)=\frac{8\cdot 6^{-s}}{s(s-1)(1-8\cdot 3^{-s})}. \tag{4.2.96}$$

Indeed, clearly, the base relative fractal drum $(\partial\Omega_0,\Omega_0)$ is the (disjoint) union of eight relative fractal drums, each of which is congruent to a relative fractal drum (A',Ω'), where Ω' is an appropriate isosceles right triangle; see Figure 4.9. We then deduce from Lemma 4.2.23 that

$$\begin{aligned}\zeta_{\partial\Omega_0,\Omega_0}(s)&=8\,\zeta_{A',\Omega'}(s)=8\int_{\Omega'}d((x,y),A')^{s-2}\mathrm{d}x\,\mathrm{d}y\\&=8\int_0^{1/6}\mathrm{d}x\int_0^x y^{s-2}\mathrm{d}y=\frac{8\cdot 6^{-s}}{s(s-1)},\end{aligned} \tag{4.2.97}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re}s>1$, and hence, in light of Theorem 4.2.6, that $\zeta_{A,\Omega}(s)$ is given by (4.2.96). Note that, after analytic continuation, we also have

$$\zeta_{\partial\Omega_0,\Omega_0}(s)=\frac{8\cdot 6^{-s}}{s(s-1)},\quad \text{for all } s\in\mathbb{C}. \tag{4.2.98}$$

Since by (4.2.96),

$$\zeta_{A,\Omega}(s) \sim \frac{1}{1-8\cdot 3^{-s}},$$

one deduces from this equivalence that the abscissa of convergence of $\zeta_{A,\Omega}$ is given by $D=\log_3 8=\dim_B(A,\Omega)$, where the equality follows from Theorem 4.1.7(b) and Remark 4.1.8.

Here, the relative box dimension of A coincides with its usual box dimension, namely, $\log_3 8$. Moreover, the set $\dim_{PC}(A,\Omega) := \mathscr{P}_c(\zeta_{A,\Omega})$ of the relative principal complex dimensions of the Sierpiński carpet A with respect to the unit square Ω is given by

$$\dim_{PC}(A,\Omega) = \log_3 8 + \mathbf{p}\mathbb{i}\mathbb{Z}, \tag{4.2.99}$$

where $\log_3 8 =: D$ is the Minkowski dimension and $\mathbf{p} := 2\pi/\log 3$ is the oscillatory period of the Sierpiński carpet RFD (A,Ω) (as well as of the ordinary Sierpiński carpet). Each principal complex dimension is simple (i.e., is a simple pole of $\zeta_{A,\Omega}$).

Observe that it follows immediately from (4.2.96) that the set $\mathscr{P}(\zeta_{A,\Omega})$ of all relative complex dimensions of the Sierpiński carpet A (with respect to the unit square Ω) is given by

$$\mathscr{P}(\zeta_{A,\Omega}) = \dim_{PC}(A,\Omega)\cup\{0,1\} = (\log_3 8 + \mathbf{p}\mathbb{i}\mathbb{Z})\cup\{0,1\}, \tag{4.2.100}$$

where $\dim_{PC}(A,\Omega) = \log_3 8 + \mathbf{p}\mathbb{i}\mathbb{Z}$ can be viewed as the set of 'scaling complex dimensions' of the self-similar RFD (A,Ω) and $\{0,1\}$ can be viewed as the set of 'integer dimensions' of (A,Ω) (in the sense of [LapPe2–3] and [LapPeWi1], see also [Lap-vFr3, Section 13.1]). Furthermore, each of these relative complex dimensions is simple (i.e., is a simple pole of $\zeta_{A,\Omega}$). Interestingly, these are exactly the complex dimensions which one would expect to be associated with A, according to the theory developed in [LapPe2–3] and [LapPeWi1–2] (as described in [Lap-vFr3, Section 13.1]) via self-similar tilings (or sprays) and associated tubular zeta functions.

Exactly the same results concerning the principal complex dimensions and the complex dimensions hold for the ordinary Sierpiński carpet A instead of the Sierpiński carpet RFD (A,Ω); in particular, the exact counterparts of (4.2.99) and (4.2.100) hold, with (A,Ω) replaced by A. See Proposition 3.2.1 in Subsection 3.2.1 above.

In light of (4.2.96), the residue of the distance zeta function of the relative Sierpiński carpet (A,Ω) computed at any principal pole $s_k := \log_3 8 + \mathbf{p}\mathbb{i}k$, $k\in\mathbb{Z}$, is given by

$$\operatorname{res}(\zeta_{A,\Omega}, s_k) = \frac{2^{-s_k}}{(\log 3)s_k(s_k-1)}.$$

In particular,

$$|\operatorname{res}(\zeta_{A,\Omega}, s_k)| \sim \frac{2^{-D}}{D\log 3}k^{-2} \quad \text{as} \quad k\to\pm\infty,$$

where $D := \log_3 8$.

Similarly as in the case of the relative Sierpiński gasket (see Proposition 4.2.25), the relative Sierpiński carpet can be viewed as a fractal spray generated by the base RFD appearing in Figure 4.9 on the right.

Proposition 4.2.30 (Relative Sierpiński carpet). *Let (A',Ω') be the RFD defined in Figure 4.9 on the right. Let (A,Ω) be the relative fractal spray generated by the base relative fractal drum (A',Ω'), with scaling ratio $\lambda = 1/3$ and with multiplicities $m_k = 8^k$ for any positive integer k:*

$$(A,\Omega) = \mathrm{Spray}((A',\Omega'),\ \lambda = 1/3,\ m_k = 8^k \text{ for } k \in \mathbb{N}). \tag{4.2.101}$$

(*Note that here we assume that the base relative fractal drum (A',Ω') has a multiplicity equal to* 8.) *Then, the relative distance zeta function of the relative fractal spray (A,Ω) coincides with the relative distance zeta function of the relative Sierpiński carpet.* (*See Equation* (4.2.96).)

Example 4.2.31. (*Sierpiński N-carpet*). It is easy to generalize the notion of a standard Sierpiński carpet (which is a compact subset of the unit square $[0,1]^2 \subset \mathbb{R}^2$), to the *Sierpiński N-carpet* (or *N-carpet*, for short), defined analogously as a compact subset A of the unit N-dimensional cube $[0,1]^N \subset \mathbb{R}^N$. More specifically, we divide $[0,1]^N$ into the union of 3^N congruent N-dimensional subcubes of length $1/3$ and with disjoint interiors and then remove the middle open subcube. The remaining compact set is denoted by F_1. We then remove the middle open N-dimensional cubes of length $1/3^2$ from the remaining $3^N - 1$ subcubes. The resulting compact subset is denoted by F_2. Proceeding analogously ad infinitum, we obtain a decreasing sequence of compact subsets F_k of $[0,1]^N$, $k \geq 1$. The Sierpiński N-carpet A is then defined by

$$A := \bigcap_{k=1}^{\infty} F_k. \tag{4.2.102}$$

Note that the Sierpiński 1-carpet coincides with the usual ternary Cantor set, while the 2-carpet coincides with the classic Sierpiński carpet; furthermore, the Sierpiński 3-carpet is discussed in [LapRaŽu5, Example 6.10].

It is clear that the *Sierpiński N-carpet RFD* (A,Ω), where A is the standard Sierpiński N-carpet and $\Omega := (0,1)^N$ is the open unit cube of $\mathbb{R}^N$, can be viewed as the following relative fractal spray; see the end of Definition 4.2.1:

$$(A,\Omega) = \mathrm{Spray}\left((\partial\Omega_0,\Omega_0),\lambda = 1/3, b = 3^N - 1\right). \tag{4.2.103}$$

Here, the cube $\Omega_0 = (0,1/3)^N$ is obtained by a suitable translation of the middle open subcube from the first step of the construction of the set A. According to Theorem 4.2.6, we then have that

$$\begin{aligned} \zeta_{A,\Omega}(s) &= f(s)\cdot\zeta_{\partial\Omega_0,\Omega_0}(s) \\ &= \frac{\zeta_{\partial\Omega_0,\Omega_0}(s)}{1-(3^N-1)3^{-s}} \sim \frac{1}{1-(3^N-1)3^{-s}}, \end{aligned} \tag{4.2.104}$$

where $f(s) = \zeta_{\mathfrak{S}}(s) = 1/(1-(3^N-1)3^{-s})$, for all $s \in \mathbb{C}$, is the scaling zeta function of the self-similar RFD (A,Ω). Since Ω_0 has a Lipschitz boundary and $\log_{1/\lambda} b = \log_3(3^N-1) \in (N-1,N)$, we deduce from (4.2.104) and from (4.2.12) in Theorem 4.2.6 that the set of principal complex dimensions of the relative Sierpiński N-carpet spray is given by

$$\dim_{PC}(A,\Omega) = \log_3(3^N-1) + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z} \tag{4.2.105}$$

and hence,

$$\dim_{PC}(A,\Omega) \subset \{\operatorname{Re} s = \log_3(3^N-1)\} \subset \{N-1 < \operatorname{Re} s < N\}.$$

In particular, according to Theorem 4.1.7(b), we have that

$$\overline{\dim}_B(A,\Omega) = \log_3(3^N-1). \tag{4.2.106}$$

Furthermore, it can be shown that in the present case of the Sierpiński N-carpet RFD, $\dim_B(A,\Omega)$ and $\dim_B A$ exist and

$$\dim_B(A,\Omega) = \dim_B A = \log_3(3^N-1). \tag{4.2.107}$$

It is easy to see that the set of principal complex dimensions $\dim_{PC} A$ of the Sierpiński N-carpet A in $\mathbb{R}^N$ coincides with the set $\dim_{PC}(A,\Omega)$ appearing in Equation (4.2.105) and that the multiplicities of the complex dimensions are the same (hence, all of the complex dimensions are simple). As simple special cases, for $N=1$ we obtain the set of principal complex dimensions of the ternary Cantor set appearing in Equation (2.1.114) on page 105, or of the usual Sierpiński carpet appearing in Equation (4.2.99), for $N=1$ or $N=2$, respectively.

Since the set of all complex dimensions of the RFD $(\partial\Omega_0,\Omega_0)$ is equal to $\{0,1,\dots,N-1\}$,[23] and hence, $\overline{\dim}_B(\partial\Omega_0,\Omega_0) = \dim_B(\partial\Omega_0,\Omega_0) = N-1$, it follows from Equation (4.2.104) that the set of all complex dimensions of the Sierpiński N-carpet relative fractal spray (A,Ω) is given by

$$\begin{aligned}\mathscr{P}(\zeta_{A,\Omega}) &= \dim_{PC}(A,\Omega) \cup \{0,1,\dots,N-1\}\\ &= \left(\log_3(3^N-1) + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \{0,1,\dots,N-1\}.\end{aligned} \tag{4.2.108}$$

This concludes for now our study of the relative fractal drum (A,Ω) naturally associated with the N-dimensional Sierpiński carpet.

We will return to this subject in Chapter 5 (Example 5.5.13) when obtaining a corresponding fractal tube formula in the case when $N=3$.

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[23] Note that the relative zeta function $\zeta_{A,\Omega}$ appearing in Equation (4.2.104) can be meromophically extended in a unique way to the whole complex plane $\mathbb{C}$ since the same can be done with $\zeta_{\partial\Omega_0,\Omega_0}$. See, for example, Equation (4.2.97) dealing with the case when $N=2$.

Remark 4.2.32. It is natural to wonder why the same new phenomena as for the relative inhomogeneous Sierpiński N-gasket (Example 4.2.26) do not occur in the case of the relative Sierpiński N-carpet (A,Ω). In particular, the Minkowski dimension $D := \dim_B(A,\Omega)$ and the similarity dimension σ_0 of the Sierpiński N-carpet RFD (A,Ω) coincide (in fact, $D = \sigma_0 = \log_3(3^N - 1)$, see Equation (4.2.107)). Furthermore, again in light of Equation (4.2.107), D also coincides with the Minkowski dimension of the classic Sierpiński N-carpet A. [Since A is a (homogeneous) self-similar set satisfying the open set condition, we must have that $\dim_B A$ exists and $\dim_B A = \sigma_0$, the common similarity dimension of A and of (A,Ω).] Moreover, $\dim_{PC} A := \mathscr{P}_c(\zeta_A)$ and $\dim_{PC}(A,\Omega) := \mathscr{P}_c(\zeta_{A,\Omega})$ coincide, as multisets. Finally, it is always the case that $\dim_B(A,\Omega) = \max\{\sigma_0, \dim_B(\partial\Omega_0,\Omega_0)\} = \sigma_0$, in agreement with (4.2.106).

All of these statements hold for every $N \geq 1$. The reason, of course, for all these simplifications (compared to the case of the N-gasket RFD in Example 4.2.26, when $N \geq 3$ and, especially, when $N \geq 4$) is that the first component, A, of the self-similar RFD (A,Ω) is precisely the classic Sierpiński N-carpet. Therefore, for every $N \geq 1$, A_N is a self-similar set, in the usual sense, and not just an inhomogeneous self-similar set (as was the case for every $N \geq 3$ of the first component, A_N, of the inhomogeneous self-similar RFD (A_N,Ω_N) in Example 4.2.26).

Example 4.2.33. (*The* $1/2$-*square fractal*). In this planar example, we will further investigate and illustrate the new interesting phenomenon which occurs in the case of the Sierpiński 3-gasket RFD discussed in Example 4.2.26. Namely, we start with the closed unit square $I = [0,1]^2$ in $\mathbb{R}^2$ and subdivide it into 4 smaller squares by taking the centerlines of its sides. We then remove the two diagonal open smaller squares, denoted by G_1 and G_2 in Figure 4.10; so that $G := G_1 \cup G_2$ is our generator in the sense of Definition 4.2.11. Next, we repeat this step with the remaining two closed smaller squares and continue this process, ad infinitum. The $1/2$-*square fractal* is then defined as the set A which remains at the end of the process; see Figure 4.10, where the first 6 iterations are shown. More precisely, the set A is the union of all of the boundaries of the disjoint family of open squares appearing in Figure 4.10 and packed in the unit square I. If we now let $\Omega := (0,1)^2$, we have that (A,Ω) is an example of a self-similar spray (or tiling), in the sense of Definition 4.2.11, with generator $G = G_1 \cup G_2$ and scaling ratios $r_1 = r_2 = 1/2$. Note, however, that A is not a (*homogeneous*) self-similar set in the usual sense (see, e.g., [Fal1, Hut]), defined via iterated function systems (IFSs), but is instead an *inhomogeneous* self-similar set.

More specifically, the set A is the unique nonempty compact subset K of $\mathbb{R}^2$ which is the solution of the inhomogeneous equation

$$K = \bigcup_{j=1}^{2} \Phi_j(K) \cup B, \tag{4.2.109}$$

where Φ_1 and Φ_2 are suitable contractive similitudes of $\mathbb{R}^2$ with fixed points located at the lower left vertex and the upper right vertex of the unit square, respectively, and with a common scaling ratio equal to $1/2$ (i.e., $r_1 = r_2 = 1/2$, where $\{r_j\}_{j=1}^2$ are the scaling ratios of the self-similar RFD (A,Ω)). Furthermore, the nonempty compact set B in Equation (4.2.109) is the union of the left and upper sides of the square G_1 and the right and lower sides of the square G_2; see Figure 4.10. We note that here, the corresponding (classic or homogeneous) self-similar set (i.e., the unique nonempty compact subset of $\mathbb{R}^2$ which is the solution of the homogeneous fixed point equation, $C = \cup_{j=1}^2 \Phi_j(C)$), is the diagonal C of the unit square connecting the lower left and the upper right vertices of the unit square.

Let us now compute the distance zeta function ζ_A of the $1/2$-square fractal. Without loss of generality, we may assume that $\delta > 1/4$; so that we have

$$\zeta_A(s) = \zeta_{A,\Omega}(s) + \zeta_I(s), \tag{4.2.110}$$

where, intuitively, ζ_I denotes the distance zeta function corresponding to the 'outer' δ-neighborhood of A. Clearly, ζ_I is equal to the distance zeta function of the unit square $I := [0,1]^2$; it is straightforward to compute it and show that it has a meromorphic extension to all of $\mathbb{C}$ given by[24]

$$\zeta_I(s) = \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s}, \tag{4.2.111}$$

for all $s \in \mathbb{C}$.

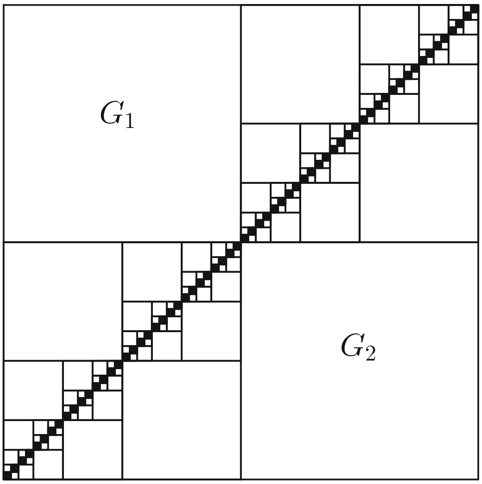

Fig. 4.10 The $1/2$-square fractal A from Example 4.2.33. The first 6 iterations are depicted. Here, $G := G_1 \cup G_2$ is the single generator of the corresponding self-similar spray or RFD (A,Ω), in the sense of Definition 4.2.11. The set A is equal to the complement of the union of the disjoint family of all open squares, with respect to $\Omega = (0,1)^2$. Equivalently, the set A coincides with the closure of the union of the boundaries of all the open squares.

[24] See also the proof of Proposition 3.2.1 where this computation was performed.

Furthermore, by using Theorem 4.2.17, we obtain that

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial G,G}(s)}{1-2\cdot 2^{-s}} = \frac{2^s \zeta_{\partial G,G}(s)}{2^s-2}, \tag{4.2.112}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. Next, we compute the distance zeta function of $(\partial G, G)$ by subdividing $G = G_1 \cup G_2$ into 16 congruent triangles (see Figures 4.9 and 4.10) and by using local Cartesian coordinates $(x,y) \in \mathbb{R}^2$ to deduce that

$$\zeta_{\partial G,G}(s) = 16 \int_0^{1/4} \mathrm{d}x \int_0^x y^{s-2}\,\mathrm{d}y = \frac{4^{-s}}{s(s-1)},$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Hence,

$$\zeta_{\partial G,G}(s) = \frac{4^{-s}}{s(s-1)}, \tag{4.2.113}$$

an identity valid initially for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$, and then, after meromorphic continuation, for all $s \in \mathbb{C}$. Finally, by combining Equations (4.2.110)–(4.2.113), we conclude that the distance zeta function ζ_A is meromorphic on all of $\mathbb{C}$ and is given by

$$\zeta_A(s) = \frac{2^{-s}}{s(s-1)(2^s-2)} + \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s}, \tag{4.2.114}$$

for all $s \in \mathbb{C}$.

Consequently (see just below), we have that $\dim_B A$ exists and

$$D(\zeta_A) = \dim_B A = 1,$$
$$\mathscr{P}(\zeta_A) := \mathscr{P}(\zeta_A, \mathbb{C}) = \{0\} \cup (1 + \mathbf{p}\mathbb{i}\mathbb{Z}) \tag{4.2.115}$$

and thus

$$\dim_{PC} A := \mathscr{P}_c(\zeta_A) = 1 + \mathbf{p}\mathbb{i}\mathbb{Z}, \tag{4.2.116}$$

where the oscillatory period $\mathbf{p}$ of A is given by $\mathbf{p} := \frac{2\pi}{\log 2}$. All of the complex dimensions in $\mathscr{P}(\zeta_A)$ are simple except for $\omega = 1$, which is a double pole of ζ_A.

We will revisit this example in Chapter 5 where we will use the distance zeta function of A given by (4.2.114) in order to derive a corresponding fractal tube formula (see Example 5.5.22 in Subsection 5.5.6). For now, we simply mention that it will follow from the results of Chapter 5 (see, especially, Theorem 5.4.30) that $\dim_B A$ exists, $\dim_B A = D(\zeta_A) = 1$ and that A is not Minkowski measurable because of the presence of the double pole of ζ_A at $\omega = 1$. On the other hand, we will show that A is h-Minkowski measurable, where the gauge function h is given by $h(t) := \log t^{-1}$ for all $t \in (0,1)$, and by Theorem 5.4.32, the corresponding h-Minkowski content is given by

$$\mathscr{M}^1(A,h) = \zeta_A[1]_{-2} = \frac{1}{4\log 2}, \tag{4.2.117}$$

where $\zeta_A[1]_{-2}$ is the (-2)-nd coefficient in the Laurent series expansion of ζ_A around $s=1$. Finally, we note that in light of Equation (4.2.115) (and hence, in light of the presence of nonreal complex dimensions), the set A is indeed *fractal*, according to our proposed definition of fractality given in Remark 4.6.24.

Example 4.2.34. (*The* $1/3$-*square fractal*). In the present planar example, we illustrate a situation which is similar to that of the inhomogeneous Sierpiński N-gasket RFD discussed in Example 4.2.26 for $N \geq 4$. Again, we start with the closed unit square $I=[0,1]^2$ in $\mathbb{R}^2$ and subdivide it into 9 smaller congruent squares (similarly as in the case of the Sierpiński carpet). Next, we remove 7 of those smaller squares; that is, we only leave the lower left and the upper right squares (see Figure 4.11). In other words, our generator G (in the sense of Definition 4.2.11) is the (nonconvex) open polygon depicted in Figure 4.11.

As usual, we proceed by iterating this procedure with the two remaining closed squares and then continue this process ad infinitum. The first 4 iterations are depicted in Figure 4.11. The $1/3$-*square fractal* is then defined as the set A which remains at the end of the process. We now let $\Omega := (0,1)^2$, which makes the RFD (A,Ω) a self-similar spray (or tiling), in the sense of Definition 4.2.11, with generator G and scaling ratios $\{r_j\}_{j=1}^2$ such that $r_1=r_2=1/3$. Again, the set A is not a homogeneous self-similar set, but is an inhomogeneous self-similar set.

More specifically, the set A is the unique nonempty compact subset K of $\mathbb{R}^2$ which is the solution of the inhomogeneous fixed point equation

$$K=\bigcup_{j=1}^{2}\Phi_j(K)\cup B, \tag{4.2.118}$$

where Φ_1 and Φ_2 are contractive similitudes of $\mathbb{R}^2$ with fixed points located at the lower left vertex and the upper right vertex of the unit square, respectively, and with a common scaling ratio equal to $1/3$ (i.e., $r_1=r_2=1/3$). Furthermore, the nonempty compact set B in Equation (4.2.118) is equal to the boundary of G without the part belonging to the boundary of the two smaller squares which are left behind in the first iteration; see Figure 4.11. We also observe that here, the corresponding (classic or homogeneous) self-similar set generated by the IFS consisting of Φ_1 and Φ_2, is the ternary Cantor set located along the diagonal of the unit square.

We now proceed by computing the distance zeta function ζ_A of the $1/3$-square fractal. Without loss of generality, we may assume that $\delta > 1/4$; so that we have

$$\zeta_A(s)=\zeta_{A,\Omega}(s)+\zeta_I(s), \tag{4.2.119}$$

where, as before in Example 4.2.33, ζ_I denotes the distance zeta function corresponding to the 'outer' δ-neighborhood of A and coincides with the distance zeta function of the unit square $I:=[0,1]^2$. Recall that ζ_I was computed in Example 4.2.33 and is given by Equation (4.2.111) for all $s\in\mathbb{C}$.

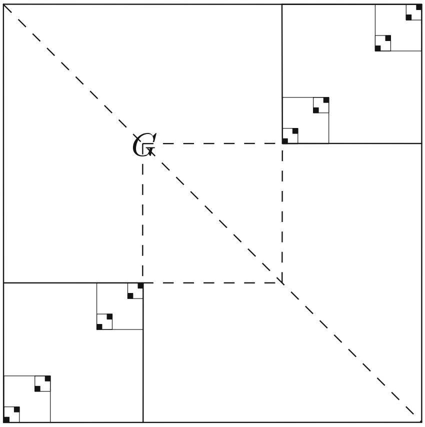

Fig. 4.11 The $1/3$-square fractal A from Example 4.2.34. The first 4 iterations are depicted. Here, G is the single generator of the corresponding self-similar spray or RFD (A,Ω), in the sense of Definition 4.2.11 or Definition 4.2.20. The set A is equal to the complement of the union of the disjoint family of all the open 8-gons, with respect to the open square $\Omega=(0,1)^2$. The largest 8-gon is equal to the union of two open squares indicated by dashed sides of length $2/3$, while each of the next two smaller 8-gons is obtained by scaling the first one by the factor $1/3$. Any of the 2^k 8-gons of the k-th generation is obtained by scaling the first one by the factor $1/3^{k-1}$, for any $k\in\mathbb{N}$. Equivalently, A coincides with the closure of the union of the boundaries of all the 8-gons.

Furthermore, by using Theorem 4.2.17, we obtain that

$$\zeta_{A,\Omega}(s)=\frac{\zeta_{\partial G,G}(s)}{1-2\cdot 3^{-s}}=\frac{3^s\zeta_{\partial G,G}(s)}{3^s-2}, \tag{4.2.120}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re}s$ sufficiently large.

Next, we compute the distance zeta function of $(\partial G,G)$ by subdividing G into 14 congruent triangles denoted by G_i, for $i=1,\dots,14$ (see Figure 4.11). Therefore, by symmetry, we obtain the following functional equation:

$$\zeta_{\partial G,G}(s)=12\zeta_{\partial G,G_1}(s)+2\zeta_{\partial G,G_{13}}, \tag{4.2.121}$$

valid initially for all $s\in\mathbb{C}$ such that $\operatorname{Re}s$ is sufficiently large.

We use local Cartesian coordinates $(x,y)\in\mathbb{R}^2$ to compute $\zeta_{\partial G,G_1}$ and obtain

$$\zeta_{\partial G,G_1}(s)=\int_0^{1/3}\mathrm{d}x\int_0^x y^{s-2}\,\mathrm{d}y=\frac{3^{-s}}{s(s-1)}.$$

Hence,

$$\zeta_{\partial G,G_1}(s)=\frac{3^{-s}}{s(s-1)}, \tag{4.2.122}$$

an identity valid initially for all $s\in\mathbb{C}$ such that $\operatorname{Re}s>1$, and then, after meromorphic continuation, for all $s\in\mathbb{C}$. In order to compute $\zeta_{\partial G,G_{13}}$, we use local polar coordinates (r,θ) and deduce that

$$\zeta_{\partial G, G_{13}}(s) = \int_0^{\pi/2} \mathrm{d}\theta \int_0^{3^{-1}(\sin\theta+\cos\theta)^{-1}} r^{s-1}\,\mathrm{d}r$$
$$= \frac{3^{-s}}{s}\int_0^{\pi/2}(\cos\theta+\sin\theta)^{-s}\,\mathrm{d}\theta, \tag{4.2.123}$$

valid initially for all $s \in \mathbb{C}$ such that $\mathrm{Re}\, s > 0$ and after meromorphic continuation, for all $s \in \mathbb{C}$. Note that by using Theorem 2.1.45 with $\varphi(\theta) := (\cos\theta+\sin\theta)^{-1}$ for $\theta \in (0, 2\pi)$, it is easy to check that

$$Z(s) := \int_0^{\pi/2}(\cos\theta+\sin\theta)^{-s}\,\mathrm{d}\theta \tag{4.2.124}$$

is an entire function, since it is of the form of the generalized DTI $f(s) := \int_E \varphi(\theta)^s \mathrm{d}\mu(\theta)$, where $E = [0, \pi/2]$, $\varphi(\theta) := (\cos\theta+\sin\theta)^{-1}$ for all $\theta \in E$ is uniformly bounded by positive constants both from above and below, and $\mathrm{d}\mu(\theta) := \mathrm{d}\theta$.

Finally, by combining Equation (4.2.111) and Equations (4.2.119)–(4.2.124), we obtain that ζ_A is given by

$$\zeta_A(s) = \frac{2}{s(3^s-2)}\left(\frac{6}{s-1}+Z(s)\right)+\frac{4\delta^{s-1}}{s-1}+\frac{2\pi\delta^s}{s}, \tag{4.2.125}$$

an identity valid initially for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$ and then, after meromorphic continuation, for all $s \in \mathbb{C}$.

Consequently, we deduce that

$$D(\zeta_A) = 1,$$
$$\mathscr{P}(\zeta_A) := \mathscr{P}(\zeta_A, \mathbb{C}) \subseteq \{0\} \cup (\log_3 2 + \mathbf{p}\mathbb{i}\mathbb{Z}) \cup \{1\} \tag{4.2.126}$$

and

$$\dim_{PC} A := \mathscr{P}_c(\zeta_A) = \{1\}, \tag{4.2.127}$$

where the oscillatory period $\mathbf{p}$ of A is given by $\mathbf{p} := \frac{2\pi}{\log 3}$. In Equation (4.2.126), we only have an inclusion since, in principle, some of the complex dimensions with real part $\log_3 2$ may be canceled by the zeros of $6/(s-1)+Z(s)$. However, it can be checked numerically that $\log_3 2 \in \mathscr{P}(\zeta_A)$ and that there also exist nonreal complex dimensions with real part $\log_3 2$ in $\mathscr{P}(\zeta_A)$. All of the complex dimensions in $\mathscr{P}(\zeta_A)$ are simple.

We will revisit this example in Subsection 5.5.6 (see Example 5.5.23) where we will obtain a fractal tube formula for the set A from Equation (4.2.125). For now, we simply mention that, $\dim_B A = 1$ and that, by Theorem 5.4.2, A is Minkowski measurable with Minkowski content given by

$$\mathscr{M}^1(A) = \mathrm{res}(\zeta_A, 1) = 16. \tag{4.2.128}$$

We also note that A is indeed *fractal*, according to our proposed definition of fractality (see Remark 4.6.24). More precisely, in light of Equation (4.2.126), it is *strictly subcritically fractal* and *fractal in dimension* $d := \log_3 2$, in the sense of case (ii) of Remark 5.5.15 below. In closing, we also mention that the set A is rectifiable and that its 'length' (i.e., its 1-dimensional Hausdorff measure) is given by

$$H^1(A) = \frac{\mathscr{M}^1(A)}{\omega_1} = 8, \tag{4.2.129}$$

which can, of course, be easily checked directly. Here, $\omega_1 = 2$ is the volume of the 1-dimensional ball of radius 1.

Example 4.2.35. (*A self-similar fractal nest*). In the final planar example of this subsection, we investigate the case of a 'self-similar fractal nest'.[25] The set A which we now define is an inhomogeneous self-similar set. Similarly as in Example 4.2.34, the set A will be *fractal* in the sense of our proposed definition of fractality given in Remark 4.6.24 and, moreover, will be *strictly subcritically fractal* in the sense of Remark 5.5.15.

Let $a \in (0,1)$ be a real parameter. We define the set A as the union of concentric circles with center at the origin and of radius a^k for $k \in \mathbb{N}_0$ (see Figure 4.12). Furthermore, let G be the open annulus such that ∂G consists of the circles of radius 1 and a, as depicted in Figure 4.12, and let $\Omega := B_1(0)$. We can now consider the RFD (A, Ω) as a self-similar spray with generator G, in the sense of Definition 4.2.11.

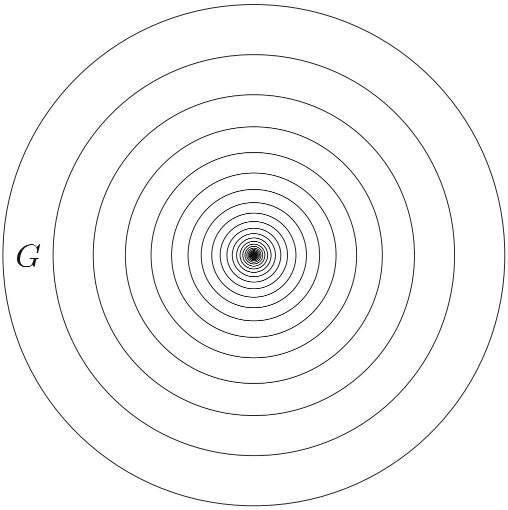

Fig. 4.12 The self-similar fractal nest from Example 4.2.35.

[25] As we shall see, throughout this example, the use of the adjective "self-similar" is somewhat abusive since only one similarity transformation is involved in order to define A.

We note that even though (A,Ω) is a fractal spray, with a single generator G, it is not (strictly speaking) self-similar in the traditional sense because it only has one scaling ratio $r = a$ (associated with a single contractive similitude). However, we will continue using this abuse of language throughout this example. Also, a moment's reflection reveals that this fact does not affect any of the conclusions relevant to the distance zeta function of such an RFD. Namely, we obviously have that

$$(A,\Omega) = (\partial G, G) \sqcup (aA, a\Omega); \tag{4.2.130}$$

so that

$$\zeta_{A,\Omega}(s) = \zeta_{\partial G,G}(s) + \zeta_{aA,a\Omega}(s), \tag{4.2.131}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s$ is sufficiently large. Furthermore, by using the scaling property of the relative distance zeta function (see Theorem 4.1.40), we conclude that

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial G,G}(s)}{1-a^s}, \tag{4.2.132}$$

again for all $s \in \mathbb{C}$ such that $\operatorname{Re} s$ is sufficiently large.

Next, we compute the distance zeta function of the generator by using polar coordinates (r,θ):

$$\begin{aligned}\zeta_{\partial G,G}(s) &= \int_0^{2\pi} \mathrm{d}\theta \int_a^{(1+a)/2} (r-a)^{s-2} r\,\mathrm{d}r \\ &\quad + \int_0^{2\pi} \mathrm{d}\theta \int_{(1+a)/2}^{1} (1-r)^{s-2} r\,\mathrm{d}r \\ &= \frac{2^{2-s}\pi(1+a)(1-a)^{s-1}}{s-1},\end{aligned} \tag{4.2.133}$$

an identity valid, after meromorphic continuation, for all $s \in \mathbb{C}$.

Equation (4.2.133) combined with Equation (4.2.132) now yields that $\zeta_{A,\Omega}$ is meromorphic on all of $\mathbb{C}$ and is given for all $s \in \mathbb{C}$ by

$$\zeta_{A,\Omega}(s) = \frac{2^{2-s}\pi(1+a)(1-a)^{s-1}}{(s-1)(1-a^s)}. \tag{4.2.134}$$

Finally, we fix an arbitrary $\delta > (1-a)/2$ and observe that for the distance zeta function of A, we have

$$\zeta_A(s) = \zeta_{A,\Omega}(s) + \zeta_{A,B_{1+\delta}(0)\setminus\Omega}(s), \tag{4.2.135}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. Furthermore, we have that

$$\zeta_{A,B_{1+\delta}(0)\setminus\Omega}(s) = \int_0^{2\pi} \mathrm{d}\theta \int_1^{1+\delta} (r-1)^{s-2} r\,\mathrm{d}r = \frac{2\pi\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s}, \tag{4.2.136}$$

where the last equality is valid, initially, for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$, and then, after meromorphic continuation, for all $s \in \mathbb{C}$.

Combining now the above equation with (4.2.135), we finally obtain that ζ_A is meromorphic on all of $\mathbb{C}$ and is given by

$$\zeta_A(s) = \frac{2^{2-s}\pi(1+a)(1-a)^{s-1}}{(s-1)(1-a^s)} + \frac{2\pi\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s}, \tag{4.2.137}$$

for all $s \in \mathbb{C}$.

Consequently (see also Theorem 5.4.30 below), we have that $\dim_B A$ exists and

$$\begin{aligned} D(\zeta_A) &= \dim_B A = 1, \\ \mathscr{P}(\zeta_A) &:= \mathscr{P}(\zeta_A, \mathbb{C}) = \mathbf{p}\mathrm{i}\mathbb{Z} \cup \{1\} \end{aligned} \tag{4.2.138}$$

and

$$\dim_{PC} A := \mathscr{P}_c(\zeta_A) = \{1\}, \tag{4.2.139}$$

where the oscillatory period $\mathbf{p}$ of A is given by $\mathbf{p} := \frac{2\pi}{\log a^{-1}}$ and all of the complex dimensions in $\mathscr{P}(\zeta_A)$ are simple. We will also revisit this example in Subsection 5.5.6 (see Example 5.5.24 below) where its fractal tube formula will be derived directly from Equation (4.2.137) and the results of Chapter 5. Here, we simply mention that $\dim_B A$ exists (which is also easy to check directly), $\dim_B A = 1$ and that, according to Theorem 5.4.2, A is Minkowski measurable, with Minkowski content given by

$$\mathscr{M}^1(A) = \operatorname{res}(\zeta_A, 1) = \frac{4\pi}{1-a}. \tag{4.2.140}$$

We also note that the set A is rectifiable and that its 'length' (really, its 1-dimensional Hausdorff measure) is given by

$$H^1(A) = \frac{\mathscr{M}^1(A)}{\omega_1} = \frac{2\pi}{1-a}, \tag{4.2.141}$$

where, as before, $\omega_1 = 2$ is the volume of the 1-dimensional ball of radius 1. Of course, formula (4.2.141) can also be easily recovered via a direct computation.

In closing, we mention that A is indeed *fractal* according to our proposed definition of fractality (see Remark 4.6.24). More specifically, in light of Equation (4.2.138), A is *strictly subcritically fractal* and *fractal in dimension* $d := 0$, in the sense of Remark 5.5.15 below.

The following example can be viewed as an analog of Example 4.2.35 (the self-similar fractal nest) in the one-dimensional Euclidean space $\mathbb{R}$.

Example 4.2.36. (*The geometric progression fractal string*). Fix $a \in (0,1)$, which will play the role of a parameter. Let $\mathscr{L} = (\ell_k)_{k\geq 0}$ be defined as the geometric sequence with progression a; i.e.,

$$\ell_k := a^k, \quad \text{for all} \quad k \geq 0. \tag{4.2.142}$$

The geometric zeta function of this fractal string is given by

$$\zeta_{\mathscr{L}}(s) = \sum_{k=0}^{\infty} (a^k)^s = \frac{1}{1-a^s}, \tag{4.2.143}$$

valid for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$. Upon meromorphic continuation, we see at once that $\zeta_{\mathscr{L}}$ is meromorphic on all of $\mathbb{C}$ and is given by

$$\zeta_{\mathscr{L}}(s) = \frac{1}{1-a^s}, \tag{4.2.144}$$

for all $s \in \mathbb{C}$.

Let $A_{\mathscr{L}}$ be the bounded subset of the real line generated by $\mathscr{L}$; i.e.,

$$A_{\mathscr{L}} := \left\{ a_k := \sum_{j \geq k} \ell_j : k \geq 0 \right\}. \tag{4.2.145}$$

Then, by means of Proposition 5.5.4 of Chapter 5 below (see also Example 2.1.58 and Equation (5.5.15)), we deduce that for any fixed $\delta > 1/2$, its distance zeta function $\zeta_{A_{\mathscr{L}}}$ is meromorphic on all of $\mathbb{C}$ and given by

$$\zeta_{A_{\mathscr{L}}}(s) = \frac{2^{1-s}}{s(1-a^s)} + \frac{2\delta^s}{s}, \tag{4.2.146}$$

for all $s \in \mathbb{C}$. Here, the term $2\delta^s/s$ corresponds to the 'outer' δ-neighborhood of $A_{\mathscr{L}}$, i.e., the left δ-neighborhood of the point 0 and the right δ-neighborhood of the point $a_0 = 1/(1-a)$.

We now see that the set of complex dimensions of $A_{\mathscr{L}}$ (or, equivalently, of $\mathscr{L}$) coincides with the set of principal complex dimensions of $A_{\mathscr{L}}$ (i.e., of $\mathscr{L}$); that is,

$$\mathscr{P}(\zeta_{A_{\mathscr{L}}}) = \dim_{PC} A_{\mathscr{L}} = \mathbf{p}\mathbb{i}\mathbb{Z}, \tag{4.2.147}$$

where $\mathbf{p} := 2\pi/\log a^{-1}$. Furthermore, all of the complex dimensions are simple, except for

$$D := D(\zeta_{A_{\mathscr{L}}}) = \dim_B A_{\mathscr{L}} = 0, \tag{4.2.148}$$

which has multiplicity two. See Remark 4.2.37 below for a justification of this claim.

In Example 5.5.25 of Chapter 5, we will use Equation (4.2.146) in order to obtain an exact fractal tube formula for the set $A_{\mathscr{L}} \subset \mathbb{R}$. For now, we mention that the presence of the double pole of $\zeta_{A_{\mathscr{L}}}(s)$ at $s = 0$ implies that the set $A_{\mathscr{L}}$ is not Minkowski measurable, since $\mathscr{M}^0(A_{\mathscr{L}}) = +\infty$, but that $A_{\mathscr{L}}$ is h-Minkowski measurable with respect to the gauge function h defined by $h(t) := \log t^{-1}$ for all $t \in (0,1)$, and that its h-Minkowski content is given by

$$\mathscr{M}^0(A_{\mathscr{L}}, h) = \frac{2}{\log a^{-1}}. \tag{4.2.149}$$

In closing this example, we point out that the geometric progression string $\mathscr{L}$ (or, equivalently, its canonical geometric realization $A_{\mathscr{L}}$, as well as any of its geometric realizations $\Omega \subset \mathbb{R}$ as bounded open sets of $\mathbb{R}$) is indeed an example of a *fractal* set, according to our proposed definition of fractality (see Remark 4.6.24), due to the presence of nonreal complex dimensions. More specifically, in light of Equation (4.2.147), $A_{\mathscr{L}}$ (or, equivalently, $\mathscr{L}$) is *critically fractal*; i.e., it is *fractal in dimension* $d := D = 0$, in the sense of Remark 5.5.15. Finally, we note that although $A_{\mathscr{L}}$ is h-Minkowski measurable, its intrinsic geometry still possesses geometric oscillations of order $O(t)$ in the fractal tube formula of $A_{\mathscr{L}}$, as will be shown in Example 5.5.25.

Remark 4.2.37. The fact that the complex dimension 0 of $A_{\mathscr{L}}$ has multiplicity two (and not one, as might naively be expected) follows from the following relation between $\zeta_{\mathscr{L}}$ and $\zeta_{A_{\mathscr{L}}}$ (see Equation (5.5.15) or, more generally, Equation (5.5.16) in Subsection 5.5.2 below):

$$\zeta_{A_{\mathscr{L}}}(s) = \frac{2^{1-s}}{s}\zeta_{\mathscr{L}}(s), \tag{4.2.150}$$

valid (in the present case of Example 4.2.36) for all $s \in \mathbb{C}$. Since 0 is a simple pole of $\zeta_{\mathscr{L}}$ (in light of Equation (4.2.144)), it is now apparent that 0 is a double pole of $\zeta_{A_{\mathscr{L}}}$, as claimed. For the same reason, the nonzero poles of $\zeta_{\mathscr{L}}$ and $\zeta_{A_{\mathscr{L}}}$ are the same, and have the same multiplicity.

Remark 4.2.38. When $a := p^{-1}$, where p is a prime number, Example 4.2.36 (the geometric progression string) reduces to the p-th Euler string $\mathscr{L}_p$, studied in [Lap-vFr3, esp., Subsection 4.2.1] (see also [HerLap1–5] and, in the p-adic setting, [LapLu2–3, LapLu-vFr1–2]) and whose geometric zeta function $\zeta_{\mathscr{L}_p}(s)$ coincides with the p-th local Euler factor $(1-p^{-s})^{-1}$, in agreement with (4.2.144) where we have set $a := p^{-1}$.

4.3 Spectral Zeta Functions of Fractal Drums and Their Meromorphic Extensions

We review here some of the known results concerning the spectral asymptotics of (relative) fractal drums, with emphasis on the leading asymptotic behavior of the spectral counting function (or, equivalently, of the eigenvalues), along with a corresponding sharp remainder estimate (obtained in [Lap1] and expressed in terms of the upper box dimension of the boundary).

We then apply these results, along with some results obtained in Section 2.3 of this monograph, in order to show that the spectral zeta functions of these fractal drums have a (nontrivial) meromorphic extension. This fact was already observed in [Lap2–3] by other means, but also by using the error estimates of [Lap1].

Moreover, we show the optimality (or sharpness) of the corresponding upper bound for the abscissa of meromorphic continuation of the spectral zeta function of the fractal drum. This latter result is new and makes use in an essential way of our results obtained later on in this chapter, especially in Sections 4.5 and 4.6.

4.3.1 Spectral Zeta Functions of Fractal Drums in $\mathbb{R}^N$

Let (A,Ω) be a given RFD in $\mathbb{R}^N$. In particular, this means that $|\Omega| < \infty$. We consider the corresponding *Dirichlet eigenvalue problem*, defined on the (possibly disconnected) open set $\Omega_A := \Omega \setminus \overline{A}$.[26] It consists in finding all ordered pairs $(\mu,u) \in \mathbb{C} \times H_0^1(\Omega_A)$ such that $u \neq 0$ and

$$\begin{cases} -\Delta u = \mu u, & \text{in } \Omega_A, \\ u = 0, & \text{on } \partial(\Omega_A), \end{cases} \tag{4.3.1}$$

in the variational sense (see, e.g., [LioMag], [Bre], along with [Lap1] and the relevant references therein). Here, $H_0^1(\Omega_A) := W_0^{1,2}(\Omega_A)$ is the standard *Sobolev space* (see, e.g., [Bre], [GilTru] or [MitŽu]), and $\Delta u = \sum_{k=1}^N \frac{\partial^2 u}{\partial x_k^2}$, where Δ is the Laplace operator. Recall that the Hilbert space $H_0^1(\Omega_A)$ is defined as the completion of $C_0^\infty(\Omega_A)$ (the space of infinitely differentiable complex-valued functions with compact support in Ω_A) under the Sobolev norm

$$\|u\| = \left(\int_{\Omega_A} |u(x)|^2 \,\mathrm{d}x + \int_{\Omega_A} |\nabla u(x)|^2 \,\mathrm{d}x \right)^{1/2} \tag{4.3.2}$$

and the associated inner product.

Equation (4.3.1) is, by definition, interpreted as follows: the scalar μ is an eigenvalue of $-\Delta$ if there exists $u \neq 0$, $u \in H_0^1(\Omega_A)$, such that

$$\int_{\Omega_A} \nabla u(x) \cdot \overline{\nabla \varphi(x)} \,\mathrm{d}x = \mu \int_{\Omega_A} u(x)\, \overline{\varphi(x)} \,\mathrm{d}x,$$

for all $\varphi \in C_0^\infty(\Omega_A)$ (or, equivalently, for all $\varphi \in H_0^1(\Omega_A)$).[27] This is the usual variational formulation of the Dirichlet eigenvalue problem on a bounded open set Ω_A with possibly nonsmooth (or even fractal) boundary. As it turns out, in order for (4.3.1) to be satisfied, μ must be real and even positive.

Throughout Section 4.3, we could assume equivalently that the relative fractal drum (A,Ω) is of the form of a standard fractal drum $(\partial\Omega_0, \Omega_0)$. Indeed, it suffices

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[26] For example, if Ω is the unit equilateral triangle and A is the Sierpiński gasket, then Ω_A is the union of a disjoint countable family of open triangles; see Figure 4.5 on page 275.

[27] In the case of Neumann boundary conditions, both $H_0^1(\Omega_A)$ and $C_0^\infty(\Omega_A)$ will be replaced by the Sobolev space $H^1(\Omega_A)$, as will be discussed further on. Also, we must then assume that Ω is a suitable bounded open subset of $\mathbb{R}^N$; see the discussion at the end of this section on pages 343–344.

to apply the quoted results (from [Lap1], for example), to the ordinary fractal drum $(\partial\Omega_0, \Omega_0)$, which is precisely what we will do, implicitly.

The following lemma describes the boundary of $\Omega_A := \Omega \setminus \overline{A}$. As we see, the subset $\overline{A} \setminus \Omega$ of $\overline{A}$ does not have any influence on Ω_A. In particular, if A and Ω are disjoint, then $\Omega_A = \Omega$. For example, $\Omega_{\partial\Omega} = \Omega$. Here and in the sequel of Section 4.3, in order to avoid trivial statements, we assume implicitly that all of the open sets $\Omega \subset \mathbb{R}^N$ are nonempty.

Lemma 4.3.1. *Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$. If the closure of A does not possess any interior points,*[28] *then $\partial(\Omega_A) = \partial\Omega \cup (\overline{A} \cap \overline{\Omega})$. In particular, if $A \subseteq \overline{\Omega}$, then $\partial(\Omega_A) = \partial\Omega \cup \overline{A}$.*

It is well known that the (eigenvalue) spectrum of the Dirichlet eigenvalue problem (4.3.1) is discrete and consists of an infinite and divergent sequence $(\mu_k)_{k\geq 1}$ of positive numbers (called eigenvalues), without accumulation point (except $+\infty$) and which can be written in nondecreasing order according to multiplicity as follows:

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \ldots, \quad \lim_{k\to\infty} \mu_k = +\infty.$$

Furthermore, each of the eigenvalues μ_k is of finite multiplicity. Moreover, if Ω_A is connected, then the first (or 'principal') eigenvalue μ_1 is of multiplicity one (i.e., $\mu_1 < \mu_2$); see [GilTru]. Because the Laplace operator is symmetric, the algebraic and geometric multiplicities of each of its eigenvalues coincide. We say for short that the sequence of eigenvalues $(\mu_k)_{k\geq 1}$ *corresponds* to the relative fractal drum (A, Ω).

Remark 4.3.2. (a) For the present Dirichlet problem (4.3.1), the discreteness of the spectrum, along with the finiteness of the multiplicity of each (necessarily positive) eigenvalue, follows from the fact that for any open subset Ω of $\mathbb{R}^N$ which is bounded (or, more generally, of finite volume), $H_0^1(\Omega)$ is compactly embedded into $H^1(\Omega)$ and hence, into the Lebesgue space $L^2(\Omega_A)$. (See, e.g., [EdmEv].) Recall that the Sobolev space $H^1(\Omega_A) := W^{1,2}(\Omega_A)$ (which is used to formulate the variational Neumann eigenvalue problem) is the space of all functions $u \in L^2(\Omega_A)$ with distributional (or 'weak') gradient $\nabla u \in [L^2(\Omega_A)]^N$. Like $H_0^1(\Omega_A)$, $H^1(\Omega_A)$ is a complex Hilbert space for the Sobolev norm $\|\cdot\|$ defined by (4.3.2) and the associated inner product.

(b) In contrast, for the Neumann problem, which will be briefly discussed towards the end of Subsection 4.3.2, even the discreteness of the spectrum does not always hold (for very rough boundaries) and even when it holds, the counterpart of Weyl's asymptotic formula (Equation (4.3.13) below) need not be verified. (See [Mét2–3].) This is why, following [Lap1], appropriate assumptions will be made on Ω in our discussion of the Neumann eigenvalue problem (or, more generally, of mixed Dirichlet-Neumann boundary conditions) towards the end of Subsection 4.3.2.

[28] It is easy to see that this condition is satisfied if $\overline{\dim}_B A < N$; see page 32.

Definition 4.3.3. The *spectrum of a relative fractal drum* (A,Ω) in $\mathbb{R}^N$, denoted by $\sigma(A,\Omega)$, is defined as the sequence of the square roots of the eigenvalues of the boundary value problem (4.3.1); that is,

$$\sigma(A,\Omega) := (\mu_k^{1/2})_{k\geq 1}. \tag{4.3.3}$$

Physically, the values of $\mu_k^{1/2}$, $k \in \mathbb{N}$, are interpreted as the (normalized)[29] *frequencies of the relative fractal drum*. The eigenvalues are scaled here with the exponent $1/2$, for technical (as well as physical) reasons (and because the Laplacian is a second order linear partial differential operator). See, for example, Lemma 4.3.6.

Definition 4.3.4. The *spectral zeta function* $\zeta^*_{A,\Omega}$ *of a relative fractal drum* (A,Ω) in $\mathbb{R}^N$ is given by

$$\zeta^*_{A,\Omega}(s) := \sum_{k=1}^{\infty} \mu_k^{-s/2}, \tag{4.3.4}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.

Example 4.3.5. The spectral zeta function of a fractal string $\mathscr{L} = (\ell_j)_{j\geq 1}$, where $\mathscr{L}$ is viewed as a relative fractal drum $(A_{\mathscr{L}}, \Omega_{\mathscr{L}})$, is given by

$$\zeta^*_{\mathscr{L}}(s) = \sum_{k,j=1}^{\infty} (k \cdot \ell_j^{-1})^{-s} = \zeta(s) \cdot \zeta_{\mathscr{L}}(s),$$

where $\zeta = \zeta_R$ is the Riemann zeta function and $\zeta_{\mathscr{L}}$ is the geometric zeta function of $\mathscr{L}$; see [Lap2–3], [LapMa2] and [Lap-vFr3, Section 1.3]. Hence, by analytic continuation (and since ζ is meromorphic on all of $\mathbb{C}$), we have

$$\zeta^*_{\mathscr{L}}(s) = \zeta(s) \cdot \zeta_{\mathscr{L}}(s), \tag{4.3.5}$$

in every domain $U \subset \mathbb{C}$ to which $\zeta_{\mathscr{L}}$ can be meromorphically continued.

The above definition of the spectrum $\sigma(A,\Omega)$ and of the spectral zeta function $\zeta^*_{A,\Omega}$ of a relative fractal drum is in agreement with the definition of the spectrum of a bounded fractal string $\mathscr{L} = (\ell_j)_{j\geq 1}$ given in [Lap-vFr3, p. 2] or more generally, of a fractal drum (see, e.g., [Lap1–3]). (See also [Lap-vFr3, Equation (1.45), p. 29] and [Lap-vFr3, Appendix B], along with the relevant references therein, including [Gilk] and [See1].) Note that the sequence

$$\mathscr{L}(A,\Omega) := (\mu_k^{-1/2})_{k\geq 1}, \tag{4.3.6}$$

which consists of the reciprocal frequencies in $\sigma(A,\Omega)$, is also a fractal string (possibly unbounded, i.e., $\sum_{k=1}^{\infty} \mu_k^{-1/2} = +\infty$). As we see, the spectral zeta function of a relative fractal drum (A,Ω) is by definition equal to the geometric zeta function of

[29] When $N = 1$, see [Lap-vFr3, footnote 1 on page 2].

the fractal string $\mathscr{L}(A,\Omega)$. It is clear that $D(\zeta^*_{A,\Omega})\geq 0$. Furthermore, since $\zeta^*_{A,\Omega}$ is a Dirichlet series with positive coefficients (and the spectrum of (A,Ω) is infinite), we also have $D(\zeta^*_{A,\Omega})=D_{\rm hol}(\zeta^*_{A,\Omega})$; see Subsection 2.1.3.

We note that the usual definition of the spectrum involves the sequence of eigenvalues $(\mu_k)_{k\geq1}$ rather than the sequence of their square roots $(\mu_k^{1/2})_{k\geq1}$, as in Equation (4.3.3). We prefer the definition of the spectrum $\sigma(A,\Omega)$ given in Equation (4.3.3) and hence, the use of the exponent $-s/2$ (rather than of $-s$) in the definition of the spectral zeta function $\zeta^*_{A,\Omega}$ in Equation (4.3.4) since, in this case, Lemma 4.3.6, Proposition 4.3.10 and Theorem 4.3.17 below take a more elegant form. See [Lap2–3] and [Lap-vFr3, p. 29 and Appendix B], and compare, for example, with [Gilk] and [See1].

The spectrum of a relative fractal drum has an interesting (but elementary) scaling property, which we now state.

Lemma 4.3.6. *Let $\sigma(A,\Omega)$ be the spectrum of a relative fractal drum (A,Ω) in $\mathbb{R}^N$. If λ is any fixed positive real number, then*

$$\sigma(\lambda A,\lambda\Omega)=\lambda^{-1}\sigma(A,\Omega); \tag{4.3.7}$$

that is, $\sigma(\lambda A,\lambda\Omega)=(\lambda^{-1}\mu_k^{1/2})_{k\geq1}$, where $(\mu_k)_{k\geq1}$ is the sequence of eigenvalues of problem (4.3.1) on Ω_A. Equivalently, $\mathscr{L}(\lambda A,\lambda\Omega)=\lambda\,\mathscr{L}(A,\Omega)$; see Equation (4.3.6).

Proof. It is easy to see that if μ_k is an eigenvalue corresponding to $-\Delta$, with respect to the domain $\Omega_A=\Omega\setminus\overline{A}$, generated by the relative fractal drum (A,Ω), then $\lambda^{-2}\mu_k$ is an eigenvalue corresponding to the operator $-\Delta$ with respect to the domain $(\lambda\Omega)_{\lambda A}$, generated by $(\lambda A,\lambda\Omega)$. Indeed, if $u_k\in H^1_0(\Omega_A)$ is such that $-\Delta u_k=\mu_k u_k$, $u_k\neq 0$, then for $v_k(y):=u_k(x/\lambda)$, where $y\in(\lambda\Omega)_{\lambda A}$, we have

$$-\Delta v_k(y)=\frac{\mu_k}{\lambda^2}v_k(y).$$

In other words, the sequence of eigenvalues of $-\Delta$ on $(\lambda\Omega)_{\lambda A}$ is equal to $(\mu_k\lambda^{-2})_{k\geq1}$. (This claim can also be checked directly by using the aforementioned variational formulation of the eigenvalue problem (4.3.1).) Therefore, by Definition 4.3.3,

$$\sigma(\lambda A,\lambda\Omega)=(\lambda^{-1}\mu_k^{1/2})_{k\geq1}=\lambda^{-1}\sigma(A,\Omega).$$

This completes the proof of the lemma. □

An immediate consequence of Lemma 4.3.6 is the following scaling result for the spectral zeta functions of RFDs.

Proposition 4.3.7 (Scaling property of spectral zeta functions). *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Then for any $\lambda>0$, and for all $s\in\mathbb{C}$ such that* $\mathrm{Re}\,s>\overline{\dim}_B(A,\Omega)$*, we have*

$$\zeta^*_{\lambda A,\lambda\Omega}(s) = \lambda^s \zeta^*_{A,\Omega}(s).$$

The following result represents a partial extension in the present context of Example 4.3.5 (see also [Lap-vFr3, Theorem 2.1]), in the special case of fractal strings, or of the corresponding result for fractal sprays in [Lap2–3] and [LapPo3]. Its proof is similar to that of Theorem 4.2.5.

Theorem 4.3.8. *Let (A_0,Ω_0) be a base RFD in $\mathbb{R}^N$, and let $\mathscr{L} = (\lambda_j)_{j\geq 1}$ be a nonincreasing sequence of positive numbers tending to zero (and repeated according to multiplicities), i.e., a (not necessarily bounded) fractal string. Assume that (A_j,Ω_j), $j \geq 1$, is a disjoint sequence of RFDs, each of which is obtained by a rigid motion of $\lambda_j(A_0,\Omega_0) = (\lambda_j A_0, \lambda_j \Omega_0)$. Let $(A,\Omega) = \bigcup_{j\geq 1}(A_j,\Omega_j)$ be the corresponding relative fractal spray, generated by (A_0,Ω_0) and $\mathscr{L}$; that is, $(A,\Omega) = (A_0,\Omega_0)\otimes\mathscr{L}$. Then, assuming that $s \in \mathbb{C}$ is such that* $\mathrm{Re}\, s > \max\{D(\zeta^*_{A_0,\Omega_0}), \overline{\dim}_B \mathscr{L}\}$, *we have*

$$\zeta^*_{A,\Omega}(s) = \zeta^*_{A_0,\Omega_0}(s)\cdot\zeta_{\mathscr{L}}(s), \tag{4.3.8}$$

where $\zeta_{\mathscr{L}}$ is the geometric zeta function of $\mathscr{L}$ (see Equation (2.1.71) *of Subsection 2.1.4). In particular, for all $s \in \mathbb{C}$ with* $\mathrm{Re}\, s$ *sufficiently large, we have*

$$\zeta^*_{A,\Omega}(s) = \sum_{k=1}^{\infty} (\mu_k^{(0)})^{-s/2} \sum_{j=1}^{\infty} \lambda_j^s, \tag{4.3.9}$$

where $(\mu_k^{(0)})_{k\geq 1}$ is the sequence of eigenvalues corresponding to the relative fractal drum (A_0,Ω_0). Furthermore, by the principle of analytic continuation, Equation (4.3.8) *continues to hold on any domain to which $\zeta_{\mathscr{L}}$ and $\zeta^*_{A_0,\Omega_0}$ can both be meromorphically continued. (A similar comment applies to Equation* (4.3.11) *below.)*

Moreover,

$$D(\zeta^*_{A,\Omega}) = \max\{D(\zeta^*_{A_0,\Omega_0}), \overline{\dim}_B \mathscr{L}\}. \tag{4.3.10}$$

In particular, if $\lambda_j = \lambda^j$ for some fixed $\lambda \in (0,1)$, and each λ^j is of multiplicity b^j, where $b \in \mathbb{N}$, $b \geq 2$, then for $\mathrm{Re}\, s > D(\zeta^*_{A_0,\Omega_0})$

$$\zeta^*_{A,\Omega}(s) = \frac{b\lambda^s}{1-b\lambda^s}\sum_{k=1}^{\infty}(\mu_k^{(0)})^{-s/2} = \frac{b\lambda^s}{1-b\lambda^s}\zeta^*_{A_0,\Omega_0}(s), \tag{4.3.11}$$

and

$$D(\zeta^*_{A,\Omega}) = \max\{D(\zeta^*_{A_0,\Omega_0}), \log_{1/\lambda} b\}.$$

Remark 4.3.9. In the case of fractal sprays (and of fractal strings, in particular), the factorization formula (4.3.8) was first observed in [Lap2–3]. In the special case of fractal strings, it has proved to be very useful; see, especially, [Lap2–3, LapPo1–3, LapMa1–2, HeLap, Lap-vFr1–3, Tep1–2, LalLap1–2, HerLap1–5]. See also, e.g., [Lap-vFr3, Sections 1.4 and 1.5] and [Lap-vFr3, Chapters 6, 9, 10 and 11], both for the case of fractal strings and (possibly generalized or even virtual) fractal sprays.

4.3.2 Meromorphic Extensions of Spectral Zeta Functions of Fractal Drums

It is well known that if Ω_0 is any (nonempty) bounded open subset of $\mathbb{R}^N$, and $\sigma(\partial\Omega_0,\Omega_0) = \big((\mu_k^{(0)})^{1/2}\big)_{k\geq 1}$ (that is, $(\mu_k^{(0)})_{k\geq 1}$ is the sequence of eigenvalues of $-\Delta$ with zero (or Dirichlet) boundary data on $\partial\Omega_0$, counting the multiplicities of the eigenvalues), then the following classical asymptotic result holds, known as *Weyl's law* [Wey1–2]:

$$\mu_k^{(0)} \sim \frac{4\pi^2}{(\omega_N|\Omega_0|)^{2/N}} \cdot k^{2/N} \quad \text{as} \quad k \to \infty, \tag{4.3.12}$$

where $\omega_N = \pi^{N/2}/(N/2)!$ is the volume of the unit ball in $\mathbb{R}^N$.[30] We recall that here, consistent with the notation introduced on page 41, the symbol $\sim$ means that the ratio of the left and right sides of (4.3.12) tends to 1 as $k\to\infty$.

The main result of this subsection is stated in Theorem 4.3.17. Its proof is based on the asymptotic result due to the first author, stated in Theorem 4.3.11, combined with Proposition 4.3.10.

The asymptotic result stated in Equation (4.3.12) was obtained by Hermann Weyl in 1912 for piecewise smooth boundaries, in [Wey1–2]. It has since then been extended to a variety of settings (for example, to smooth, compact Riemannian manifolds with or without boundary, various boundary conditions, broader classes of elliptic operators, fractal boundaries, etc.). See, for example, the well-known treatises by Courant and Hilbert [CouHil, Section VI.4] and by Reed and Simon [ReeSim1], along with [Hö3] and the introduction of [Lap1], as well as [Lap2–3] and [Lap-vFr3, Section 12.5 and Appendix B]. It has been extended by G. Métivier in [Mét1–3] during the 1970s to arbitrary bounded subsets of $\mathbb{R}^N$ (in the present case of Dirichlet boundary conditions). Independently and at about the same time, this latter result was also obtained by M. Sh. Birman and M. Z. Solomyak in [BiSo]. Furthermore, in this general setting (for example), sharp error estimates, expressed in terms of the upper Minkowski (or box) dimension of the boundary of Ω_0, were obtained by the first author in the early 1990s in [Lap1]; see Theorem 4.3.11 below, along with the comments following Theorem 4.3.17 and Remark 4.3.23 for further extensions about other boundary conditions and higher-order elliptic operators, with possibly variable coefficients.

In the following result, we consider a class of bounded open subsets Ω_0 of $\mathbb{R}^N$ such that the corresponding sequence of eigenvalues $(\mu_k^{(0)})_{k\geq 1}$ satisfies an asymptotic condition involving the error term as well:

$$\mu_k^{(0)} = \frac{4\pi^2}{(\omega_N|\Omega_0|)^{2/N}} \cdot k^{2/N} + O(k^{\gamma}) \quad \text{as} \quad k \to \infty. \tag{4.3.13}$$

[30] For odd N, we have $(N/2)! = \frac{N}{2}(\frac{N}{2}-1)\cdots\frac{1}{2}$, since $(N/2)! := \Gamma(\frac{N}{2}+1)$, where Γ is the classic gamma function.

Here, we assume that $\gamma \in (-\infty, 2/N)$. It will also be convenient to use the following short-hand notation: $\zeta^*_{\Omega_0} = \zeta^*_{\partial\Omega_0,\Omega_0}$, and more generally, $\zeta^*_{\Omega_0} = \zeta^*_{A_0,\Omega_0}$, provided A_0 and Ω_0 are disjoint. We say for brevity that $\zeta^*_{\Omega_0}$ is the *spectral zeta function of the bounded open subset* Ω_0 of $\mathbb{R}^N$.

Proposition 4.3.10. *Assume that Ω_0 is an arbitrary bounded open subset of $\mathbb{R}^N$ such that the corresponding sequence of eigenvalues of $-\Delta$, with zero (or Dirichlet) boundary data on $\partial\Omega_0$, counting the multiplicities of the eigenvalues, satisfies the asymptotic condition (4.3.13), where $\gamma < 2/N$. Then the spectral zeta function*

$$\zeta^*_{\Omega_0}(s) = \sum_{k=1}^{\infty} (\mu_k^{(0)})^{-s/2} \tag{4.3.14}$$

possesses a (necessarily unique) meromorphic extension (at least) to the open half-plane

$$\{\operatorname{Re} s > N - (2 - \gamma N)\}. \tag{4.3.15}$$

In other words, $D_{\mathrm{mer}}(\zeta^*_{\Omega_0}) \leq N - (2 - \gamma N)$. As we see, the meromorphic extension vertical strip, to the left of the vertical line $\{\operatorname{Re} s = N\}$, is of width at least $2 - \gamma N$.

The only pole of $\zeta^*_{\Omega_0}$ in this half-plane is $s = N$, and in particular, $D(\zeta^*_{\Omega_0}) = N$. Furthermore, it is simple and

$$\operatorname{res}(\zeta^*_{\Omega_0}, N) = \frac{N\omega_N}{(2\pi)^N}\,|\Omega_0|. \tag{4.3.16}$$

Proof. Letting $C := 4\pi^2(\omega_N|\Omega_0|)^{-2/N}$, we have that $\mu_k^{(0)} = C \cdot k^{2/N} + d_k$, where $d_k = O(k^\gamma)$ as $k \to \infty$, and hence,

$$\zeta^*_{\Omega_0}(s) = \sum_{k=1}^{\infty} (C \cdot k^{2/N} + d_k)^{-s/2}.$$

To prove the proposition, it suffices to apply Theorem 2.3.12 with $a = 2/N$, $\gamma < a$ and $s_1 = s/2$. Indeed, we obtain that $\zeta^*_{\Omega_0}(s)$ possesses a unique meromorphic extension (at least) to the open half-plane $\{\operatorname{Re} \frac{s}{2} > \frac{N}{2} - (1 - \frac{\gamma}{a})\}$, or, equivalently, to the open half-plane $\{\operatorname{Re} s > N - (2 - \gamma N)\}$, as claimed in (4.3.15). Furthermore, according to the same theorem, the residue of $\zeta^*_{\Omega_0}(2s) = \sum_{k=1}^{\infty}(\mu_k^{(0)})^{-s}$ at $s = a = 2/N$ is equal to $(1/a)C^{-1/a} = (N/2)C^{-N/2}$. Hence, the residue of $\zeta^*_{\Omega_0}(s)$ at $s = N$ can be obtained as follows:

$$\begin{aligned}\operatorname{res}(\zeta^*_{\Omega_0}, N) &= \lim_{s\to N}(s-N)\zeta^*_{\Omega_0}(s) = \lim_{2s\to N}(2s-N)\zeta^*_{\Omega_0}(2s)\\ &= 2\lim_{s\to N/2}\Big(s - \frac{N}{2}\Big)\zeta^*_{\Omega_0}(2s) = 2\frac{N}{2}C^{-N/2} = \frac{N\omega_N}{(2\pi)^N}\,|\Omega_0|,\end{aligned}$$

where in the next-to-last equality, we have used Equation (2.3.18) from Theorem 2.3.12. This completes the proof of Proposition 4.3.10. ☐

In practice, in light of the remainder estimates of [Lap1] recalled in Theorem 4.3.11 and in Corollary 4.3.14 below, we will apply Proposition 4.3.10 under the assumption that $\tilde{D} := \overline{\dim}_B(\partial\Omega_0, \Omega_0) < N$ and $\gamma \in [(2+\tilde{D}-N)/N, 2/N)$.

In light of Equation (4.3.16), under the hypotheses of Proposition 4.3.10, the residue of the spectral zeta function $\zeta^*_{\Omega_0}$ computed at $s = N$ is proportional to the N-dimensional Lebesgue measure (volume) of Ω_0; see (4.3.16). As we see, this result is of a similar nature as Equation (2.2.4) in Theorem 2.2.3. Moreover, the volume of Ω_0 can be explicitly computed by using the spectral zeta function:

$$|\Omega_0| = \frac{(2\pi)^N}{N\omega_N}\operatorname{res}(\zeta^*_{\Omega_0}, N). \tag{4.3.17}$$

Theorem 4.3.8, combined with Proposition 4.3.10, generalizes [Lap-vFr3, Theorem 1.19] to the N-dimensional case. See also Theorem 4.3.17 below, which relies on Theorem 4.3.11 (or, equivalently, on Corollary 4.3.14) and provides explicit conditions under which Equation (4.3.13) holds, and hence Proposition 4.3.10 can be applied.

It is clear that the claim of Proposition 4.3.10 is true if in (4.3.13) we replace $O(k^\gamma)$ by $O(k^{(\gamma)}$ as $k \to \infty$. For example, we may have $O(k^\gamma \log k)$ as $k \to \infty$ in (4.3.13).

Let $(\mu_k^{(0)})_{k\geq 1}$ be the sequence of eigenvalues of $-\Delta$, where Δ is the Dirichlet Laplacian, associated with a given bounded open subset Ω_0 of $\mathbb{R}^N$. In what follows, we denote by

$$N_\nu(\mu) := \#\{k \in \mathbb{N} : \mu_k^{(0)} \leq \mu\}, \qquad \text{for} \quad \mu > 0, \tag{4.3.18}$$

the *eigenvalue counting function of the fractal drum*, taking into account the multiplicities. It is also called the *spectral counting function* in the literature; see, e.g., [Lap1–5], [Lap-vFr1–3] and the relevant references therein.

In the proof of Theorem 4.3.17 below, we shall need the following significant result (see [Lap1, Equation (1.8), Theorems 1.1 and 2.3]), which provides a partial resolution of the *modified Weyl–Berry conjecture*. See [Lap1, Corollary 2.1], as well as [Lap1, Theorems 2.1 and 2.3], along with the comments following Theorem 4.3.17 and Remark 4.3.23, for a more general statement involving positive uniformly elliptic linear differential operators (with variable and possibly nonsmooth coefficients) and mixed Dirichlet–Neumann boundary conditions.

Theorem 4.3.11 (Lapidus, [Lap1]). *Let Ω_0 be an arbitrary (nonempty) bounded open subset of $\mathbb{R}^N$. Let $\tilde{D} := \overline{\dim}_B(\partial\Omega_0, \Omega_0)$ denote the upper relative Minkowski (or box) dimension of Ω_0, with respect to $\partial\Omega_0$. Then we have the following remainder estimates:*

(*i*) *If $\tilde{D} \in (N-1, N]$, then for any $d > \tilde{D}$,*

$$N_\nu(\mu) = (2\pi)^{-N}\omega_N|\Omega_0|\cdot\mu^{N/2} + O(\mu^{d/2}) \quad \text{as} \quad \mu \to +\infty. \tag{4.3.19}$$

(ii) If $\tilde{D} = N-1$, then for any $d > \tilde{D}$,

$$N_\nu(\mu) = (2\pi)^{-N}\omega_N|\Omega_0| \cdot \mu^{N/2} + O(\mu^{d/2}\log\mu) \quad \text{as} \quad \mu \to +\infty. \tag{4.3.20}$$

Moreover, in both cases (i) and (ii), the choice $d = \tilde{D}$ is allowed, provided

$$\mathscr{M}^{*\tilde{D}}(\partial\Omega_0, \Omega_0) < \infty;$$

that is, Ω_0 has finite upper Minkowski content, relative to $\partial\Omega_0$ (i.e., it has finite inner Minkowski content).

Remark 4.3.12. (*a*) For Dirichlet boundary conditions, Theorem 4.3.11 just above (and hence also, Corollary 4.3.14 below) remains valid without change for an arbitrary (and possibly unbounded as well as disconnected) nonempty open set Ω_0 with finite volume: $|\Omega_0| < \infty$.

(*b*) Note that in [Lap1–3], $\tilde{D}$ is referred to as the *inner Minkowski dimension* of Ω_0. It is known that since Ω_0 is a (nonempty) bounded open set, we have $N-1 \le \tilde{D} \le N$; see [Lap1, Section 3]. Also, in [Lap1], the case when $\tilde{D} = N-1$ is referred to as the '*nonfractal case*' (or the least fractal case), and the case when $\tilde{D} \in (N-1, N]$ is referred to as the '*fractal case*'. Finally, note that in the *most fractal case* when $\tilde{D} = N$, the error estimate (4.3.19) is still valid, but is uninformative; indeed, even when $d := \tilde{D} = N$, the 'error term' is then of the same order as the 'leading term' in (4.3.19).

(*c*) According to the notation introduced in Remark 2.3.4, this condition can be written more succinctly in the following form:

$$N_\nu(\mu) = (2\pi)^{-N}\omega_N|\Omega_0| \cdot \mu^{N/2} + O(\mu^{(\tilde{D}/2)}) \quad \text{as} \quad \mu \to \infty. \tag{4.3.21}$$

A similar comment applies to the error estimate (4.3.20).

Various aspects of the study of the (possibly modified) Weyl–Berry conjecture are discussed in the introduction of [Lap1], in [Lap3] and, more recently, in a brief survey given in [Lap-vFr3, Section 12.5.1]. See also [Berr1–2], [BroCar], [Lap1–3], [LapPo1–3], [Cae], [vBGilk], [HamLap], [FlVa], [Ger], [GerSc], [MolVai] and the references therein. The result stated in case (*ii*) of Theorem 4.3.11, that is, in the nonfractal case when $\tilde{D} = N-1$, and under the additional assumption that $\mathscr{M}^{*(N-1)}(\partial\Omega)$ is finite, was already obtained in Métivier's work [Mét3, Theorem 6.1 on page 191]; see also [Mét1–2]. Métivier stated his result without the explicit use of box (that is, Minkowski) dimension or Minkowski content. See [Lap-vFr3, Section 12.5] for a more complete list of references. Results concerning the partition function (the trace of the heat semigroup) of the Dirichlet Laplacian have been obtained by Brossard and Carmona [BroCar]. The main estimate in [BroCar] is now a consequence of the results of [Lap1] stated in Theorem 4.3.9, but the converse is not true. Indeed, as is well known, beyond the leading term, the spectral asymptotics for the trace of the heat semigroup do not imply corresponding asymptotics for the eigenvalue counting function (or, equivalently, for the eigenvalues themselves). In fact, when they hold, the pointwise estimates for the eigenvalue counting function are considerably more difficult to prove.

Remark 4.3.13. (a) As was mentioned earlier, the first general result concerning the leading term of the asymptotic expansion of the eigenvalues is due to Hermann Weyl in [Wey1–2] towards the beginning of the 20th century, in the case of a sufficiently smooth boundary. Eventually, this result was extended in the 1970s by Guy Métivier in [Mét1–3] (see also [BiSo]) for an arbitrary bounded open set (and for the Dirichlet Laplacian or more general elliptic operators and boundary conditions). The error estimate (4.3.20) is due to Courant in the case of a piecewise smooth boundary (and hence, $\tilde{D} = N-1$), a very special case of (ii) in Theorem 4.3.11. An elementary concrete example of that situation can be found in the monograph by Courant and Hilbert [CouHil, p. 431], where in the case when Ω_0 is a rectangle in the plane, with sides a and b, it is shown that the counting function of the associated sequence of eigenvalues of the Dirichlet Laplacian $-\Delta$ satisfies

$$N_\nu(\mu) = \frac{ab}{4\pi}\cdot\mu + O(\sqrt{\mu}) \quad \text{as} \quad \mu \to +\infty.$$

In fact, an equivalent number-theoretic formulation of this result was already known to Gauss in 1834; see [Gau].

(b) In case (ii) of Theorem 4.3.11, the remainder estimate (4.3.20) is known to hold without the logarithmic term (i.e., Equation (4.3.19) holds with $d = N-1$ as well as $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0,\Omega_0) < \infty$) if the boundary of Ω_0 is (sufficiently) smooth, or more generally, for sufficiently smooth compact Riemannian manifolds with or without boundary. (By "smooth" here, we mean C^r, that is, r times continuously differentiable, with the positive integer $r \geq 2$ large enough.) See [Hö2–3], [Lap-vFr3, Appendix B] and the introduction of [Lap1] as well as the many references therein, describing, in particular, the results of Hörmander [Hö1], Seeley [See2–3] and Pham The Lai [Ph]. See also [Lap-vFr3, Remark B.1 of Appendix B].

From Theorem 4.3.11 it is possible to derive a result, also due to the first author, regarding the error term for the leading asymptotics of the eigenvalues of the Dirichlet Laplacian $-\Delta$, associated with bounded open sets in $\mathbb{R}^N$. As is well known by the experts in spectral theory, the statement of Corollary 4.3.14 is equivalent to that of Theorem 4.3.11 (by means of a standard Abelian/Tauberian argument, for example); furthermore, Corollary 4.3.14 can be deduced from Theorem 4.3.11 by means of the converse of a Tauberian theorem, called an Abelian theorem in [Sim], for example; see [Lap1, Appendix A] and [Sim] for a closely related situation. In order to keep this part of the exposition essentially self-contained, we provide (at least in a special case) a different proof, based on the elementary Lemma 4.3.15 below.

Corollary 4.3.14 ([Lap1])**.** *Let Ω_0 be an arbitrary (nonempty) bounded open subset of $\mathbb{R}^N$. As before, we let $\tilde{D} := \overline{\dim}_B(\partial\Omega_0,\Omega_0)$, and let $(\mu_k^{(0)})_{k\geq1}$ be the sequence of* *eigenvalues of* $-\Delta$*, where* Δ *is the Dirichlet Laplacian on* Ω_0*. Then the following conclusions hold*:

(i) *If* $\tilde{D} \in (N-1, N]$*, then for any* $d > \tilde{D}$*,*

$$\mu_k^{(0)} = \frac{4\pi^2}{(\omega_N|\Omega_0|)^{2/N}} \cdot k^{2/N} + O(k^{(2+d-N)/N}) \quad \text{as} \quad k \to \infty. \tag{4.3.22}$$

(ii) *If* $\tilde{D} = N-1$*, then for any* $d > \tilde{D}$*,*

$$\mu_k^{(0)} = \frac{4\pi^2}{(\omega_N|\Omega_0|)^{2/N}} \cdot k^{2/N} + O(k^{(2+d-N)/N} \log k) \quad \text{as} \quad k \to \infty. \tag{4.3.23}$$

Moreover, in each of the cases (i) *and* (ii)*, the choice of* $d = \tilde{D}$ *is allowed, provided* $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0, \Omega_0) < \infty$*.*

More succinctly, according to the notation introduced in Remark 2.3.4, we can rewrite (4.3.22) in the following equivalent manner:

$$\mu_k^{(0)} = \frac{4\pi^2}{(\omega_N|\Omega_0|)^{2/N}} \cdot k^{2/N} + O\big(k^{\left(\frac{2+\tilde{D}-N}{N}\right)}\big) \quad \text{as} \quad k \to \infty. \tag{4.3.24}$$

A similar comment applies to the remainder term in (4.3.23).

Postponing the proof of Corollary 4.3.14 for a while, we first state and prove an auxilliary technical result.

Lemma 4.3.15. *Let* $c > 0$*,* $m > 0$ *and* $\alpha \in (-\infty, m)$ *be given real numbers. Assume that* $(\mu_k)_{k\geq 1}$ *is a sequence of positive real numbers satisfying the following condition*:[31]

$$c \cdot \mu_k^m + O(\mu_k^\alpha) = k \quad \text{as} \quad k \to \infty. \tag{4.3.25}$$

Then

$$\mu_k = c^{-1/m} \cdot k^{1/m} + O(k^{\frac{\alpha+1}{m}-1}) \quad \text{as} \quad k \to \infty. \tag{4.3.26}$$

Proof. Step 1: Let us first prove the lemma for $m = 1$. Note that in this case, we have $\alpha < 1$. Without loss of generality, we may assume that $c = 1$; otherwise, we introduce a new sequence $\mu_k' = c\mu_k$. In this case, by the assumption made in the lemma, there exists a positive real number C such that $|k - \mu_k| \leq C\mu_k^\alpha$ for all positive integers k. Since this implies that $k \leq \mu_k + C\mu_k^\alpha$ for all $k \geq 1$, then, clearly,

$$\lim_{k\to\infty} \mu_k = +\infty.$$

Therefore, from

$$\left|\frac{k}{\mu_k} - 1\right| \leq C\mu_k^{\alpha-1}, \tag{4.3.27}$$

[31] Here, we write μ_k^m instead of $(\mu_k)^m$, for example; see also, (4.3.50) below, for instance.

and using $\alpha - 1 < 0$, we conclude that $\lim_{k\to\infty} \frac{k}{\mu_k} = 1$. In particular, there exists a positive constant C_1 such that $\mu_k \le C_1 k$ for all positive integers k. Hence,

$$|k - \mu_k| \le C\mu_k^\alpha \le CC_1^\alpha k^\alpha;$$

that is, $\mu_k = k + O(k^\alpha)$ as $k \to \infty$, which proves the lemma for $m = 1$.

Step 2: We now consider the case when $c \cdot \mu_k^m + O(\mu_k^\alpha) = k$ as $k \to \infty$, with $m > 0$ and $m \neq 1$. Letting $\lambda_k := \mu_k^m$ for every $k \ge 1$, we obtain $c \cdot \lambda_k + O(\lambda_k^{\alpha/m}) = k$ as $k \to \infty$. By Step 1, we then conclude that

$$\lambda_k = \frac{1}{c} \cdot k + O(k^{\alpha/m}) \quad \text{as} \quad k \to \infty.$$

Therefore,

$$\begin{aligned}\mu_k = \left(\frac{1}{c} \cdot k + O(k^{\alpha/m})\right)^{1/m} &= c^{-1/m} k^{1/m} \left(1 + O(k^{\frac{\alpha}{m}-1})\right)^{1/m} \\ &= c^{-1/m} k^{1/m} \left(1 + O(k^{\frac{\alpha}{m}-1})\right) \quad \text{as} \quad k \to \infty,\end{aligned}$$

where in the last equality we have used the fact that $\alpha < m$. This concludes the proof of the lemma. □

Remark 4.3.16. Lemma 4.3.15 permits a slight generalization. If instead of condition (4.3.25), we assume that

$$c \cdot \mu_k^m + O(\mu_k^\alpha) = k + O(k^\beta) \quad \text{as} \quad k \to \infty, \tag{4.3.28}$$

where $\beta < 1$, then (retaining the remaining conditions in the lemma) we have that:

$$\mu_k = c^{-1/m} \cdot k^{1/m} + O(k^{\frac{1}{m} + \max\{\frac{\alpha}{m}, \beta\} - 1}) \quad \text{as} \quad k \to \infty. \tag{4.3.29}$$

This conclusion is obtained by an easy modification of the proof of Lemma 4.3.15.

We are now ready to prove Corollary 4.3.14 (in a special case).

Proof of Corollary 4.3.14. Let us first assume that $\tilde{D} \in (N-1, N]$; that is, let us assume that we are in case (*i*) of the corollary.

For simplicity, we assume that $\tilde{D} \in (N-1, N)$ and that the eigenvalues all have multiplicity one. (The case when $\tilde{D} = N$ is of no interest while the case when $\tilde{D} = N-1$ can be dealt with similarly.) For the general case when the eigenvalues may have multiplicities larger than one, it would be best to work directly with the eigenvalue counting function (and, hence, to use Theorem 4.3.11 instead of Corollary 4.3.14), as is standard and done in [Lap2–3]. See the comment following the proof of Theorem 4.3.17 below.

From the definition of the counting function, we obviously have that $N_\nu(\mu_k^{(0)}) = k$, for all $k \geq 1$. By using Theorem 4.3.11(i), we obtain that

$$(2\pi)^{-N}\omega_N|\Omega_0|\cdot\mu_k^{N/2} + O(\mu_k^{d/2}) = k \quad \text{as} \quad k \to \infty.$$

Now, if we set $m = N/2$ and $\alpha = d/2$, Lemma 4.3.15 immediately implies claim (i) in the corollary. The proof of case (ii) is similar. □

Combining Corollary 4.3.14 with Proposition 4.3.10, we deduce the main result of this section, already obtained by the first author in [Lap2–3]. As in [Lap2–3], it makes an essential use (via Corollary 4.3.14) of the key remainder estimate obtained in [Lap1]. Furthermore, it is stated a little bit more precisely than in [Lap2–3] and makes use of the notation introduced in Subsection 2.1.5. It shows that $D_{\rm mer}(\zeta_{\Omega_0}^*)$, the abscissa of meromorphic continuation of $\zeta_{\Omega_0}^*$, does not exceed the upper box dimension of the boundary $\partial\Omega_0$ relative to Ω_0, denoted (as above) $\overline{\dim}_B(\partial\Omega_0,\Omega_0)$ and called the *inner Minkowski dimension of* $\partial\Omega_0$ in [Lap1].

Theorem 4.3.17 (Lapidus, [Lap2–3]). *Let Ω_0 be an arbitrary (nonempty) bounded open subset of $\mathbb{R}^N$ such that $\overline{\dim}_B(\partial\Omega_0,\Omega_0)\} < N$. Then the spectral zeta function $\zeta_{\Omega_0}^*$ of Ω_0 is holomorphic in the open half-plane $\{{\rm Re}\, s > N\}$ and $D_{\rm hol}(\zeta_{\Omega_0}^*) = N$. Furthermore, $\zeta_{\Omega_0}^*$ can be (uniquely) meromorphically extended from $\{{\rm Re}\, s > N\}$ to (at least) $\{{\rm Re}\, s > \overline{\dim}_B(\partial\Omega_0,\Omega_0)\}$. In other words,*[32]

$$D_{\rm mer}(\zeta_{\Omega_0}^*) \leq \overline{\dim}_B(\partial\Omega_0,\Omega_0). \tag{4.3.30}$$

Moreover, $s = N$ is the only pole of $\zeta_{\Omega_0}^*$ in the half-plane $\{{\rm Re}\, s > \overline{\dim}_B(\partial\Omega_0,\Omega_0)\}$; it is a simple pole and

$$\operatorname{res}(\zeta_{\Omega_0}^*, N) = \frac{N\omega_N}{(2\pi)^N}|\Omega_0|. \tag{4.3.31}$$

Proof. Let us prove the statement regarding the meromorphicity of $\zeta_{\Omega_0}^*$. [The proof of the statement regarding the holomorphicity of $\zeta_{\Omega_0}^*$ in $\{{\rm Re}\, s > N\}$ is left as an easy exercise for the interested reader. (Actually, as will be explained further below, it follows from the known properties of generalized Dirichlet series with positive coefficients recalled in Subsection 2.1.3.)] Let us set $\gamma = \frac{2+d-N}{N}$. Since we then have $N-(2-\gamma N) = d$, using Proposition 4.3.10 applied to the sequence $(\mu_k^{(0)})_{k\geq 1}$ in Corollary 4.3.14, we conclude that $\zeta_{\Omega_0}^*$ can be meromorphically extended in a unique way to the half-plane $\{{\rm Re}\, s > d\}$. This property holds for any $d > \tilde{D} := \overline{\dim}_B(\partial\Omega_0,\Omega_0)$; hence, the function $\zeta_{\Omega_0}^*$ can be meromophically extended to the half-plane $\{{\rm Re}\, s > \tilde{D}\}$.

Finally, assume that $\tilde{D} \in (N-1,N)$, for simplicity. Then, since $\tilde{D} < N$, we see that the meromorphic continuation of $\zeta_{\Omega_0}^*$ must have a (simple) pole at $s = N$. Indeed, since $\zeta_{\Omega_0}^*$ is initially given by a (generalized) Dirichlet series with positive

[32] Recall that, by definition, $\{{\rm Re}\, s > D_{\rm mer}(\zeta_{\Omega_0}^*)\}$ is the largest open right half-plane to which $\zeta_{\Omega_0}^*$ can be meromorphically extended.

coefficients, $\zeta^*_{\Omega_0}$ must have a singularity at $s = N$; but since $\zeta^*_{\Omega_0}$ can be meromorphically continued to a connected open neighborhood of $s = N$ (and in light of either (4.3.19) or (4.3.22)), this singularity must be a simple pole of $\zeta^*_{\Omega_0}$. The value of the residue given in (4.3.31) follows from (4.3.16) in Proposition 4.3.10. This concludes the proof of the theorem. □

Alternatively, Theorem 4.3.17 follows easily from Theorem 4.3.11 (via standard arguments, well known to the experts in spectral theory) by proceeeding as follows, which is the method used in [Lap2–3]. First, observe that, as is well known, the spectral zeta function (essentially) coincides with the Mellin transform of the spectral counting function, at least for $\operatorname{Re} s > \tilde{D}$, where $\tilde{D} := \overline{\dim}_B(\partial\Omega_0, \Omega_0)$. Then, use the remainder estimate for the eigenvalue counting function (see Theorem 4.3.11 above, from [Lap1]), along with a suitable (and standard) Abelian theorem[33] (the converse of a Tauberian theorem, in the terminology of [Sim]) or simply, a direct analysis of the corresponding integral (essentially, the Mellin transform of the spectral counting function N_ν) in order to deduce that $\zeta^*_{\Omega_0}$ admits a meromorphic extension to the open half-plane $\{\operatorname{Re} s > \tilde{D}\}$, as desired. More specifically, one can use Theorem 2.1.47 about the holomorphicity of integrals depending analytically on a parameter, along with Theorem 4.3.11, to deduce that the spectral zeta function $\zeta^*_{\Omega_0}$ admits a meromorphic continuation to $\{\operatorname{Re} s > \tilde{D}\}$, with a single, simple pole located at $s = N$ (thus, the meromorphic continuation is holomorphic for $\operatorname{Re} s > \tilde{D}$ except at $s = N$).

The proof of Theorem 4.3.17 provided above, just after the statement of Theorem 4.3.17, presents the advantage of being elementary (assuming, of course, the results of Theorem 4.3.11, which are not at all elementary). However, at least for now, it is only valid under special assumptions on the multiplicities of the eigenvalues (see Lemma 4.3.15 and Remark 4.3.16 on pages 329 and 330), whereas the aforementioned proof (from [Lap3]) is valid in full generality since it directly relies on Theorem 4.3.11 rather than on Corollary 4.3.14.

We next state an easy but useful consequence of Theorem 4.3.17. At this stage, the reader may wish to review some of the relevant notation introduced in Section 2.1.

Corollary 4.3.18. *Under the same hypotheses as in Theorem 4.3.17, we have* (*with the notation introduced in Section 2.1*)

$$D(\zeta^*_{\Omega_0}) = D_{\mathrm{hol}}(\zeta^*_{\Omega_0}) = N \tag{4.3.32}$$

and so

$$\Pi(\zeta^*_{\Omega_0}) = \mathscr{H}(\zeta^*_{\Omega_0}) = \{\operatorname{Re} s > N\},\text{[34]} \tag{4.3.33}$$

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[33] See, e.g., [Sim] or [Lap1, Theorem A in Appendix A] for the case of the Laplace transform instead of the Mellin transform. Of course, a simple change of variable of the form $x = e^t$ then converts the (additive) Laplace transform to the (multiplicative) Mellin transform.

[34] That is, $\Pi(\zeta^*_{\Omega_0})$, the half-plane of (absolute) convergence of $\zeta^*_{\Omega_0}$, coincides with $\mathscr{H}(\zeta^*_{\Omega_0})$, the half-plane of holomorphic continuation of $\zeta^*_{\Omega_0}$.

whereas

$$D_{\rm mer}(\zeta_{\Omega_0}^*) < D_{\rm hol}(\zeta_{\Omega_0}^*). \tag{4.3.34}$$

Proof. The second equality in (4.3.32), $D_{\rm hol}(\zeta_{\Omega_0}^*) = N$, holds because (by the second part of Theorem 4.3.17 and since $\overline{\dim}_B(\partial\Omega_0, \Omega_0) < N$) the meromorphic continuation of $\zeta_{\Omega_0}^*$ has a pole at $s = N$, so that $\{\mathrm{Re}\, s > N\}$ is the largest open right half-plane on which $\zeta_{\Omega_0}^*$ is holomorphic: $\mathscr{H}(\zeta_{\Omega_0}^*) = \{\mathrm{Re}\, s > N\}$. Furthermore, the first equality in (4.3.32), $D(\zeta_{\Omega_0}^*) = D_{\rm hol}(\zeta_{\Omega_0}^*)$, holds because $\zeta_{\Omega_0}^*$ is initially given by a (generalized) Dirichlet series with positive coefficients; see Equation (4.3.4) of Definition 4.3.4 above, along with Subsection 2.1.3.1. This proves Equation (4.3.32) and hence also Equation (4.3.33).

Finally, we note that clearly, in light of the second equality in (4.3.32) and of the inequality (4.3.30) in Theorem 4.3.17, the claimed strict inequality (4.3.34) holds since, by hypothesis, we have that $\overline{\dim}_B(\partial\Omega_0, \Omega_0) < N$. This concludes the proof of the corollary. ☐

For the sake of brevity, let $\tilde{D} := \overline{\dim}_B(\partial\Omega_0, \Omega_0)$, in the sequel. It is noteworthy that the estimates obtained in [Lap1] (and recalled, in particular, in Equations (4.3.19) and (4.3.22)) are best possible (i.e., sharp), in general, in the most important case of a fractal drum for which $N > \tilde{D} > N-1$ and $d = \tilde{D}$, with $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0, \Omega_0) < \infty$;[35] that is, in case (*i*) of Theorem 4.3.11 and of Corollary 4.3.14 (as well as for an open set Ω_0 satisfying $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0, \Omega_0) < \infty$), respectively, the estimates (4.3.19) and (4.3.22) are sharp. See [Lap1, Examples 5.1 and 5.1'] for a one-parameter family of examples $\{\Omega_{0,\alpha}\}_{\alpha>0}$ (based on the α-string, often used in the present book) for which $\tilde{D}$ takes all possible values (as α varies in $(0, +\infty)$) in the allowed open interval $(N-1, N)$ and the error estimates (4.3.19) and (4.3.22) are sharp, with $d := \tilde{D}$; furthermore, each open set $\Omega_{0,\alpha}$ is Minkowski measurable and, in particular, is Minkowski nondegenerate (hence, the condition $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0, \Omega_0) < \infty$ is satisfied).

More specifically, for $\alpha > 0$, let $V_\alpha := \bigcup_{j=1}^{\infty}((j+1)^{-\alpha}, j^{-\alpha})$ denote the α-string. Then, given $N \geq 2$, let $\Omega_{0,\alpha} := V_\alpha \times (0,1)^{N-1}$; so that the bounded open set $\Omega_{0,\alpha}$ is the 'fractal comb' obtained as the disjoint union of the 'teeth' $((j+1)^{-\alpha}, j^{-\alpha}) \times (0,1)^{N-1}$. According to the results of [Lap1, Examples 5.1 and 5.1'] along with [Lap1, Appendix C], for each $\alpha > 0$,

$$\tilde{D} := \dim_B(\partial\Omega_{0,\alpha}, \Omega_{0,\alpha}) = (N-1) + (\alpha+1)^{-1} \tag{4.3.35}$$

exists, and the relative fractal drum $(\partial\Omega_{0,\alpha}, \Omega_{0,\alpha})$ is Minkowski measurable with Minkowski content

$$\mathscr{M}^{\tilde{D}}(\partial\Omega_{0,\alpha}, \Omega_{0,\alpha}) = \frac{2^{1-\tilde{D}}\alpha^{\tilde{D}}}{1-\tilde{D}}.$$

[35] Recall from [Lap1, Corollary 3.2] that (since Ω_0 is a nonempty, bounded and open subset of $\mathbb{R}^N$) $\tilde{D} = \overline{\dim}_B(\partial\Omega_0, \Omega_0)$ always satisfies the following inequality: $N-1 \leq \tilde{D} \leq N$.

Clearly, in light of (4.3.35), $\tilde{D}$ ranges through all of $(N-1,N)$ as α ranges through $(0,+\infty)$. Furthermore, it can be shown by a direct computation (see [Lap1], *loc. cit.*) that the error estimates (4.3.19) and (4.3.22) hold with $d=\tilde{D}$ and cannot be improved. Actually, much more is true in this case, although it is not necessary to know about it for the present argument. Indeed, in light of later results obtained in [LapPo1–2] about the spectral asymptotics of Minkowski measurable fractal strings, one can even show that the error term in (4.3.19) can be replaced by an explicitly computable monotonic (asymptotic) second term, proportional to $\mathscr{M}^{\tilde{D}}(\partial\Omega_0,\Omega_0)\,\mu^{\tilde{D}/2}$ and with the implied constant of proportionality involving the positive number $-\zeta(\tilde{d})$, where $\tilde{d}:=1/(\alpha+1)\in(0,1)$ and ζ denotes the Riemann zeta function; and analogously for (4.3.22). (See [LapPo2–3] and [Lap-vFr3, Subsections 6.5.1 and 8.1.2].)

We note that the open sets $\Omega_{0,\alpha}$ constructed in [Lap1] are not connected. However, much as in [BroCar] and [FlVa], one can open appropriately small gates in each of the ‘teeth’ of the ‘fractal combs’ $\Omega_{0,\alpha}$ in order to obtain a one-parameter family $\{\Omega'_{0,\alpha}\}_{\alpha>0}$ of *connected* (and even *simply connected*) open subsets of $\mathbb{R}^N$ (with $N\geq 2$ arbitrary) having the same properties as the family $\{\Omega_{0,\alpha}\}_{\alpha>0}$. More specifically, each domain $\Omega'_{0,\alpha}$ is Minkowski measurable, with

$$\dim_B\Omega'_{0,\alpha}=\dim_B\Omega_{0,\alpha}=(N-1)+(\alpha+1)^{-1} \tag{4.3.36}$$

taking all possible values in $(N-1,N)$, as α varies in the interval $(0,+\infty)$, and for the Dirichlet Laplacian on $\Omega'_{0,\alpha}$, both of the remainder estimates (4.3.19) and (4.3.22) are best possible (with $d:=\tilde{D}$ and $\Omega_0=\Omega_{0,\alpha}$ or $\Omega'_0=\Omega'_{0,\alpha}$, respectively).

We leave it to the interested reader to verify that the exact same conclusion as above can be reached (for the same two families of examples) in the case of Neumann (instead of the Dirichlet) Laplacian. In this case, we must replace $\tilde{D}$ by $D:=\overline{\dim}_B(\partial\Omega_0)$ (as was done in [Lap1] when dealing with Neumann boundary conditions), and, of course, exclude the zero eigenvalue in the original definition (4.3.14) of the corresponding spectral zeta function (which we continue to denote by $\zeta^*_{\Omega_0}$, for simplicity). Observe that for these examples, it is easy to check that $\dim_B(\partial\Omega_0)$ exists and $\tilde{D}=D=\dim_B(\partial\Omega_0)$.

We could naturally be tempted to use the same one-parameter families $\{\Omega_{0,\alpha}\}_{\alpha>0}$ and $\{\Omega'_{0,\alpha}\}_{\alpha>0}$ of open sets and simply connected domains, respectively, along with some of the results of [LapPo2] concerning the modified Weyl–Berry conjecture (in dimension one) to solve the following open problem (Problem 4.3.20), to which we will provide a partial answer in Theorem 4.3.21 below. However, this is not possible, as will be explained in the next remark in the case of this first family.

Remark 4.3.19. To see why the one-parameter family $\{\Omega_{0,\alpha}\}_{\alpha>0}$ cannot be used to resolve part (i) of Problem 4.3.20 below, one can reason as follows (in the case of the Dirichlet Laplacian). First of all, since $\Omega_{0,\alpha}=V_\alpha\times(0,1)^{N-1}$, where $V_\alpha=\cup_{j=1}^{\infty}((j+1)^{-\alpha},j^{-\alpha})$ is the α-string, we have that the eigenvalues of $\Omega_{0,\alpha}$ are the sums of the eigenvalues of V_α and of those of $(0,1)^{N-1}$; therefore, similarly, the poles of $\zeta^*_{\Omega_0}$ are the sums of the poles of $\zeta^*_{V_\alpha}$ and those of $\zeta^*_{(0,1)^{N-1}}$.

Also, the spectral zeta function of a cube ($(0,1)^{N-1}$, in this case) can be expressed as a linear combination of Epstein zeta functions; it therefore admits a meromorphic extension to all of $\mathbb{C}$, with poles which are all simple and located on the real axis at $\{1,2,\dots,N-1\}$. Furthermore, according to the classic formula for the spectal zeta function of a fractal string ([Lap2–3], [Lap-vFr3, Theorem 1.10]), we have

$$\zeta^*_{V_\alpha}(s) = \zeta(s)\cdot\zeta_{V_\alpha}(s), \tag{4.3.37}$$

where $\zeta = \zeta_R$ is the Riemann zeta function and ζ_{V_α} is the geometric zeta function of the α-string. Now, by [Lap-vFr3, Theorem 6.21], ζ_{V_α} has a meromorphic extension to all of $\mathbb{C}$ (with simple poles located at $\tilde{d}$ and in (a subset of) $\{-\tilde{d},-2\tilde{d},-3\tilde{d},\dots\}$), where $\tilde{d} := \dim_B(\partial V_\alpha, V_\alpha) = 1/(\alpha+1)$. Hence, in light of (4.3.37), $\zeta^*_{V_\alpha}$ is meromorphic in all of $\mathbb{C}$ (with one more pole than ζ_{V_α}, namely, the simple pole of ζ_{V_α} at 1). Therefore, $\zeta^*_{0,\alpha}$ can also be meromorphically extended to all of $\mathbb{C}$ (with poles which are all simple, with the exception of $s=1$, which is double, and located on the real axis). We conclude that $D_{\rm mer}(\zeta^*_{0,\alpha}) = -\infty$ for every $\alpha > 0$, whereas $\tilde{D} := \dim_B(\partial\Omega_{0,\alpha},\Omega_{0,\alpha}) = N-1+(\alpha+1)^{-1}$ sweeps out the interval $(N-1,N)$ as α ranges through $(0,+\infty)$. Therefore, inequality (4.3.30) is strict, in this case, and is in fact, as far as possible from being an equality.

We expect that the following open problem has a positive answer in every dimension $N \geq 1$. (We will show in Theorem 4.3.21 and the ensuing comment, Remark 4.3.23, that this is so both for the Dirichlet and Neumann Laplacians.) In the sequel, we assume implicitly that $\tilde{D} := \dim_B(\partial\Omega_0,\Omega_0) < N$.

Problem 4.3.20. (*i*) *Determine whether the inequality* (4.3.30) *in Theorem 4.3.17 is sharp; that is, find a bounded open set $\Omega_0 \subset \mathbb{R}^N$ for which*

$$D_{\rm mer}(\zeta^*_{\Omega_0}) = \overline{\dim}_B(\partial\Omega_0,\Omega_0)$$

for the Dirichlet Laplacian on Ω_0.

(*ii*) *More generally, address the exact counterpart of this problem for higher order elliptic operators* (*see inequality* (4.3.56) *below*) *and/or for Neumann* (*or, more generally, for mixed Dirichlet–Neumann*) *boundary conditions instead of for Dirichlet boundary conditions* (*see the comment following the statement of this problem*)*; that is, find a bounded open set $\Omega_0 \subset \mathbb{R}^N$ for which $D_{\rm mer}(\zeta^*_{\Omega_0}) = \overline{\dim}_B(\partial\Omega_0,\Omega_0)$.*

(*iii*) *Either in the setting of* (*i*)*, or, more generally, in the setting of* (*ii*)*, find a one-parameter family of bounded open sets solving* (*i*) (*or, more generally,* (*ii*)) *in the affirmative and for which the dimension $\tilde{D} := \overline{\dim}_B(\partial\Omega_0,\Omega_0)$ takes all the possible values in $(N-1,N)$, as the parameter of the family varies. Furthermore, when $N \geq 2$, find such a family consisting of connected* (*or even simply connected*) *open sets.*

As before, in the case of Neumann (or, more generally, mixed Dirichlet–Neumann) boundary conditions, we must assume that Ω is a suitable bounded open subset of $\mathbb{R}^N$ (see pages 343–344 at the very end of this section) and replace $\tilde{D} := \overline{\dim}_B(\partial\Omega_0, \Omega_0)$ by $D := \overline{\dim}_B(\partial\Omega_0)$. (We have $\tilde{D} \leq D$ and so $N-1 \leq D \leq N$.) Furthermore, we then let $N(\mu)$ denote the number of (strictly) positive eigenvalues which do not exceed μ (since 0 is always an eigenvalue of the Neumann problem) and similarly exclude the eigenvalue 0 in the original definition of $\zeta^*_{\Omega_0}$, given in Equation (4.3.14), for example (or, more generally, when $m \geq 1$, by (4.3.50) below).

The next theorem is new and provides a partial solution to Problem 4.3.20. It actually answers part (*i*) of the problem in the affirmative. We expect (but do not want to claim) that a suitable modification and/or extension of the construction can be used to solve part (*iii*) as well, at least for the Dirichlet and Neumann Laplacians. It is noteworthy that our construction makes an essential use of aspects of classic fractal string theory and of the theory developed in this book (especially in Sections 4.4–4.6 of the present chapter).

Theorem 4.3.21. *There is an explicitly constructible bounded open subset of $\mathbb{R}^N$ solving part* (*i*) *of Problem 4.3.20 in the affirmative* (*and for which $N-1 < \tilde{D} < N$*). *Actually, this open set has a maximally hyperfractal* (*and transcendentally ∞-quasiperiodic*) *boundary, in the sense of Section 4.6* (*specifically, of Definition 4.6.23*(*iii*) *and Definition 4.6.7*(*a*)) *below. Equivalently, the associated relative fractal drum* $(\partial\Omega_0, \Omega_0)$ *is maximally hyperfractal* (*and transcendentally ∞-quasiperiodic*).

Proof. The proof parallels in part the reasoning outlined in Remark 4.3.19 above. It relies, however, in an essential way on the concepts introduced and the results obtained in Section 4.6 below.

More specifically, assume for now that $N = 1$ and let $\mathscr{L}$ be the (effectively constructible) bounded fractal string obtained in Corollary 4.6.17 of Section 4.6 below. Here, $\mathscr{L}$ is viewed as a relative fractal drum $(\partial V_0, V_0)$, with V_0 a bounded open subset of $\mathbb{R}$. By construction, we have that $(\partial V_0, V_0)$ is transcendentally ∞-quasiperiodic (see Definition 4.6.7(*a*)) and maximally hyperfractal (see Definition 4.6.23(iii)); so that (with $\tilde{d} := \overline{\dim}(\partial V_0, V_0)$) all of the points of the critical line $\{\mathrm{Re}\, s = \tilde{d}\}$ are nonisolated singularities of the geometric zeta function $\zeta_{\mathscr{L}}$ of $\mathscr{L}$. Furthermore, we have

$$\tilde{d} := \overline{\dim}(\partial V_0, V_0) = D(\zeta_{\mathscr{L}}) = D_{\mathrm{mer}}(\zeta_{\mathscr{L}}). \tag{4.3.38}$$

(See part (*a*) of Corollary 4.6.17 below.) Then, in light of the (the counterpart for V_0) of the factorization formula (4.3.37) above, the spectral zeta function $\zeta^*_{V_0}$ also satisfies

$$D_{\mathrm{mer}}(\zeta^*_{V_0}) = \tilde{d} = \overline{\dim}_B(\partial V_0, V_0). \tag{4.3.39}$$

Indeed, the critical line $\{\mathrm{Re}\, s = \tilde{d}\}$ consists entirely of nonisolated singularities of $\zeta^*_{V_0}$. (Also, $\zeta^*_{V_0}$ has a single, simple pole at $s = 1$.) This takes care of the $N = 1$ case.

Next, given a fixed integer $N \geq 2$, let $\Omega_0 := V_0 \times (0,1)^{N-1}$, viewed as a bounded open subset of $\mathbb{R}^N$ (or rather, as the relative fractal drum $(\partial\Omega_0, \Omega_0)$ of $\mathbb{R}^N$). Then, just as in Remark 4.3.19, note that the principal poles / singularities of $\zeta^*_{\Omega_0}$ are the sums of the principal poles / singularities of $\zeta^*_{V_0}$ and the principal pole of $\zeta_{(0,1)^{N-1}}$, which is equal to $N-1$. (Note that the poles of $\zeta_{(0,1)^{N-1}}$ are all simple and located on the real axis, at $\{1,\dots,N-1\}$.) Therefore, we deduce that $\zeta^*_{\Omega_0}$ has a single (simple) pole at $s=N$ $(=(N-1)+1)$ and that the critical line $\{\mathrm{Re}\, s=\tilde{D}\}$ consists entirely of singularities of $\zeta^*_{\Omega_0}$. Here,

$$\tilde{D}=(N-1)+\tilde{d}=\overline{\dim}_B(\partial\Omega_0,\Omega_0). \tag{4.3.40}$$

It follows that (much as in the $N=1$ case above) we must have

$$\tilde{D}=D_{\mathrm{mer}}(\zeta^*_{\Omega_0}), \tag{4.3.41}$$

as desired. In light of Equations (4.3.40) and (4.3.41), this completes the proof of the theorem. □

We deduce from the above discussion and known properties of the quantities involved that for the bounded open set Ω_0 of Theorem 4.3.21, the abscissa of meromorphic continuation of the spectral zeta function $\zeta^*_{\Omega_0}$ does not only coincide with $\tilde{D}:=\overline{\dim}_B(\partial\Omega_0,\Omega_0)$ (as is stated in Theorem 4.3.21 above), but also coincides with the abscissae of meromorphic, holomorphic and (absolute) convergence of the fractal (i.e., distance and tube) zeta functions of Ω_0, as is stated in the next result. Note that it follows from Theorem 4.3.21 that $D_{\mathrm{hol}}(\zeta^*_{\Omega_0})=D(\zeta^*_{\Omega_0})=N>\tilde{D}$.

Corollary 4.3.22. *For the example discussed in Theorem 4.3.21, we have*

$$\begin{aligned} D_{\mathrm{mer}}(\zeta^*_{\Omega_0}) &= \overline{\dim}_B(\partial\Omega_0,\Omega_0)=:\tilde{D} \\ &= D_{\mathrm{mer}}(f)=D_{\mathrm{hol}}(f)=D(f), \end{aligned} \tag{4.3.42}$$

for all $f\in\{\zeta_{\partial\Omega_0,\Omega_0},\tilde{\zeta}_{\partial\Omega_0,\Omega_0}\}$.

Proof. In light of Theorem 4.3.21, all we have to prove are the last three equalities of (4.3.42) and the equality $D(f)=\tilde{D}$. Now, these inequalities follow by combining the relevant result of Subsections 2.1.2, 2.1.3 and 4.1.1 (see part (*b*) of Theorem 2.1.11, Proposition 2.2.19 and part (*b*) of Theorem 4.1.7). □

Remark 4.3.23. The use of the same geometric example as before in the proof of Theorem 4.3.21, $\Omega_0=V_0\times(0,1)^{N-1}$, and an entirely similar (but slightly simpler) argument, show that the exact counterpart of Theorem 4.3.21 and Corollary 4.3.22 holds for the Neumann (instead of the Dirichlet) Laplacian. Recall that in that case, we must exclude the eigenvalue 0 in the original definition (4.3.14) of the spectral zeta function. We leave the easy verification as an exercise for the interested reader.

Thus far, in connection with the remainder estimates for the leading spectral asymptotics (see Theorems 4.3.11 and 4.3.17 along with Corollaries 4.3.14 and 4.3.18), we have restricted ourselves to discussing the Dirichlet Laplacian $-\Delta$, although the Neumann Laplacian can also be discussed, as well as general positive uniformly elliptic linear differential operators (with variable and possibly nonsmooth coefficients) of order $2m$ (with $m\geq 1$) and of the form

$$\mathscr{A} = \sum_{|\alpha|\le m, |\beta|\le m} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x) D^{\beta}), \tag{4.3.43}$$

described in [Lap1, Section 2.2]. We use here the standard multi-index notation: for example, $\alpha := (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$, $|\alpha| := \alpha_1 + \cdots + \alpha_N$ and

$$D^{\alpha} := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}.$$

All of these extensions are obtained in [Lap1]; see Theorem 2.1 and its corollaries in [Lap1]. In the latter case, the assumed assymptotic expansion of the eigenvalues of $\mathscr{A}$ in the corresponding version of Proposition 4.3.10, and implied (or, actually, equivalent to) by [Lap1, Theorem 2.1], should be replaced by

$$\mu_k^{(0)} = (\mu'_{\mathscr{A}}(\Omega_0))^{-2m/N} \cdot k^{2m/N} + O(k^{\gamma}), \quad \text{as} \quad k \to \infty, \tag{4.3.44}$$

where $\mu'_{\mathscr{A}}(\Omega_0)$ is the "Browder–Gårding measure" of Ω_0 defined, for example, in [Hö3] or in [Lap1, Equation (2.18a) in Section 2.2] in terms of the (positive definite, unbounded) quadratic form associated with $\mathscr{A}$ and

$$\gamma := \frac{2m + d - N}{N}, \tag{4.3.45}$$

with $d > \tilde{D}$ arbitrary and (as before) $\tilde{D} := \overline{\dim}_B(\partial\Omega_0, \Omega_0)$, the upper Minkowski dimension of the relative fractal drum $(\partial\Omega_0, \Omega_0)$. Furthermore, we may also take $d = \tilde{D}$ provided $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0, \Omega_0) < \infty$. We note that the remainder estimate (4.3.44) actually holds in the above form in the 'fractal case' when $\tilde{D} > N - 1$ (or, equivalently, when $\tilde{D} \in (N-1, N]$, since we always have $\tilde{D} \in [N-1, N]$). Furthermore, in the nonfractal case when $\tilde{D} = N - 1$, we must replace $O(k^{\gamma})$ by $O(k^{\gamma} \log k)$ on the right-hand side of (4.3.44). Here and in the sequel, and as was mentioned earlier, we should replace $\tilde{D}$ by D, where $D := \overline{\dim}_B(\partial\Omega_0)$, the upper Minkowski dimension of the boundary $\partial\Omega_0$, in the case of Neumann (or, more generally, mixed Dirichlet–Neumann) boundary conditions. We should also assume that Ω is a suitable bounded open subset of $\mathbb{R}^N$; see the discussion on pages 343–344 at the very end of this section.

Remark 4.3.24. For Neumann boundary conditions, and for example, for the Neumann Laplacian, one must also use the weak (or variational) formulation of the classic eigenvalue problem $-\Delta u = \mu u$ in Ω_0, with $\partial u/\partial n = 0$ on $\partial\Omega_0$, where $\partial u/\partial n$ stands for the normal derivative of u along $\partial\Omega_0$. However, since $\partial\Omega_0$ is irregular (and hence, $\partial u/\partial n$ is not defined, in general), one must now use the Sobolev space $H^1(\Omega_0) := W^{1,2}(\Omega_0)$ instead of $H_0^1(\Omega_0) := W_0^{1,2}(\Omega_0)$, which was used to formulate the Dirichlet eigenvalue problem; see the discussion following Equation (4.3.1) in Subsection 4.3.1, along with references [LioMag], [Bre] and [Lap1]. (*Neumann boundary conditions* are sometimes referred to as *natural boundary conditions* in the physics and applied mathematics literature, because they are automatically satisfied once the problem has been written in variational form.) An entirely analogous comment applies to general, uniformly elliptic, positive self-adjoint operators of the form (4.3.43); see, e.g., the aforementioned references.

Recall that the *Browder–Gårding measure* $\mu'_{\mathscr{A}}(\mathrm{d}x) := \mu'_{\mathscr{A}}(x)\,\mathrm{d}x$ is the absolutely continuous measure on $\mathbb{R}^N$ (with respect to the Lebesgue measure on $\mathbb{R}^N$), with density $\mu'_{\mathscr{A}}(x)$ given (for a.e. $x \in \Omega_0$) as follows (with $|\cdot| = |\cdot|_N$ denoting the N-dimensional volume or measure, as usual):

$$\mu'_{\mathscr{A}}(x) := (2\pi)^{-N}|\{\xi \in \mathbb{R}^N : a'(x,\xi) < 1\}|, \tag{4.3.46}$$

where $a'(x,\xi)$ denotes the *leading symbol of the quadratic form a* associated with the opeator $\mathscr{A}$ given by (4.3.43):

$$a'(x,\xi) := \sum_{|\alpha|=m,\,|\beta|=m} a_{\alpha\beta}(x)\,\xi^{\alpha+\beta}, \tag{4.3.47}$$

with $\xi^{\kappa} := \xi_1^{\kappa_1}\cdots\xi_N^{\kappa_N}$ for $\kappa = (\kappa_1,\ldots,\kappa_N) \in (\mathbb{N}\cup\{0\})^N$ and $\xi = (\xi_1,\ldots,\xi_N) \in \mathbb{R}^N$ (as well as with $x \in \Omega_0$). So that

$$\mu'_{\mathscr{A}}(\Omega_0) = \int_{\Omega_0} \mu'_{\mathscr{A}}(x)\,\mathrm{d}x, \tag{4.3.48}$$

with $\mu'_{\mathscr{A}}(x)$ given by (4.3.46) and (4.3.47) just above.

Physically, and in light of (4.3.46)–(4.3.48), $\mu'_{\mathscr{A}}(\Omega_0)$ can be interpreted as an integral in the phase space $\mathbb{R}^{2N}$. In fact, it is well known that in the special case when $\mathscr{A}$ is a Schrödinger-type operator, the corresponding Weyl term (namely, the leading term in Equation (4.3.49) below) can be viewed as a volume in phase space (with the eigenvalue parameter μ being thought of as an energy), in agreement with the semiclassical limit of quantum mechanics (see, e.g., [ReeSim1] and [Sim], along with the relevant references therein).

We have just stated, in the remainder estimate (4.3.44), the analog (obtained in [Lap1]) of case (*i*) of Corollary 4.3.14 above. (Observe that when $m = 1$ and in light of (4.3.45), estimate (4.3.44) does reduce to estimate (4.3.22) of Corollary 4.3.14.) Now, in the nonfractal case (or 'least fractal case', still following the terminology of [Lap1]) where $\tilde{D} = N-1$, the exact analog of part (ii) of Corollary 4.3.14 also holds. More specifically, still according to [Lap1, Theorem 2.1 and its corollaries], the precise counterpart of estimate (4.3.44) holds, with $O(k^{\gamma})$ replaced by $O(k^{\gamma}\log k)$, exactly as in estimate (4.3.23) of part (*ii*) of Corollary 4.3.14 (which corresponds to the case when $m = 1$).

Observe that if $N(\mu)$ denotes the eigenvalue counting function of the operator $\mathscr{A}$, the asymptotic remainder estimate (4.3.44) can be written equivalently as follows:

$$N(\mu) = \mu'_{\mathscr{A}}(\Omega_0)\,\mu^{N/2m} + R(\mu), \tag{4.3.49}$$

where the error term $R(\mu)$ is given by $R(\mu) := O(\mu^{d/2m})$ in the fractal case when $\tilde{D} > N-1$ and $R(\mu) := O(\mu^{d/2m}\log\mu)$ in the nonfractal case when $\tilde{D} = N-1$. Here, $d \in (\tilde{D}, N]$ is arbitrary and if $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0,\Omega_0) < \infty$, we may choose $d = \tilde{D}$ as well. [And, similarly, with $D = \overline{\dim}_B(\partial\Omega_0)$ instead of $\tilde{D} = \overline{\dim}_B(\partial\Omega_0, \Omega_0)$ and with $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0)$ instead of $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0, \Omega_0)$, for Neumann or, more generally, for mixed Dirichlet–Neumann (instead of Dirichlet) boundary conditions.] As was observed before, when $\tilde{D} = N-1$, then $O(k^\gamma)$ must be replaced by $O(k^\gamma \log k)$ on the right-hand side of (4.3.44).

Note that the value (4.3.45) of the exponent γ, appearing in (4.3.44), corresponds to letting $m' := N/2m$ and $\alpha' := d/2m$ (instead of m and α, respectively) in (4.3.25) of Lemma 4.3.15. See Equation (4.3.49) (which we cited from [Lap1, Theorem 2.1]) and recall that $N(\mu_k^{(0)}) = k$ for all $k \geq 1$.

Next, we consider the consequences of the above error estimates ((4.3.44) or, equivalently, (4.3.49)) for the spectral zeta function $\zeta^*_{\Omega_0} := \zeta^*_{\mathscr{A},\Omega_0}$ of the uniformly elliptic operator $\mathscr{A}$ of order $2m$, defined (for $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large) by

$$\zeta^*_{\Omega_0}(s) := \sum_{k=1}^{\infty} (\mu_k^{(0)})^{-s/2m}. \tag{4.3.50}$$

Observe that since $\mathscr{A}$ is of order $2m$, the (normalized) 'frequencies' of the corresponding drum are given by $\nu_k := (\mu_k^{(0)})^{-1/2m}$, so that $\zeta^*_{\Omega_0}(s) := \sum_{k=1}^{\infty}(\nu_k)^{-s}$, exactly as was done in Definition 4.3.4 when $m = 1$; see Equations (4.3.3) and (4.3.4). Indeed, note that for $m = 1$, Equation (4.3.50) reduces to (4.3.14).

The following result generalizes Theorem 4.3.17 to the present context. We point out that thanks to our definition of $\zeta^*_{\Omega_0}$ in Equation (4.3.50) just above, Theorem 4.3.25 and its consequences (stated, in particular, in Equation (4.3.57) below) take a form which is essentially identical to their counterpart in Theorem 4.3.21 (and in Corollary 4.3.22), for which $m = 1$ and $\mathscr{A}$ is the Laplace operator.

Theorem 4.3.25. *Assume that Ω_0 is a bounded open subset of $\mathbb{R}^N$ such that*

$$\overline{\dim}_B(\partial\Omega_0, \Omega_0) < N.$$

Let $\mathscr{A}$ be a positive uniformly elliptic self-adjoint operator of order $2m$, as described in [Lap1, Section 2.2]. *Then the corresponding spectral zeta function $\zeta^*_{\Omega_0} := \zeta^*_{\mathscr{A},\Omega_0}$, defined by* (4.3.50)*, possesses a (necessarily unique) meromorphic extension (at least) to the open half-plane*

$$\{\operatorname{Re} s > \overline{\dim}_B(\partial\Omega_0, \Omega_0)\}.$$

In other words,

$$D_{\mathrm{mer}}(\zeta^*_{\Omega_0}) \leq \overline{\dim}_B(\partial\Omega_0, \Omega_0). \tag{4.3.51}$$

The only pole of $\zeta^*_{\Omega_0}$ in the above half-plane is $s = N$, and hence, in particular, $D(\zeta^*_{\Omega_0}) = N$. Furthermore, it is a simple pole and

$$\operatorname{res}(\zeta^*_{\Omega_0}, N) = N\mu'_{\mathscr{A}}(\Omega_0). \tag{4.3.52}$$

In other words, the residue of the spectral zeta function of the operator $\mathscr{A}$*, computed at* $s = N$*, is proportional to the Browder–Gårding measure of* Ω_0.

Proof. Let $\tilde{D} := \overline{\dim}_B(\partial\Omega_0, \Omega_0) < N$. According to [Lap1, Theorem 2.1, case (*i*)], assuming $\tilde{D} > N-1$ the sequence of eigenvalues corresponding to the operator $\mathscr{A}$ satisfies (4.3.49), or, equivalently, condition (4.3.44) with γ defined by (4.3.45) for any $d > \tilde{D}$. Let us fix an arbitrary number $d \in (\tilde{D}, N)$.

First of all, applying Theorem 2.3.12, with $a = 2m/N$, to the sequence of eigenvalues of $\mathscr{A}$ satisfying (4.3.44), we immediately obtain (much as in the proof of Proposition 4.3.10) that $s = N$ is a simple pole. (Note that, since $d < N$, then $\gamma < a$, as required in Theorem 2.3.12.) Furthermore, using (2.3.17) from Theorem 2.3.12, we see that $\zeta^*_{\Omega_0}$ can be meromophically extended at least to the open set of all complex numbers s such that

$$\operatorname{Re}\frac{s}{2m} > \frac{N}{2m} - \left(1 - \frac{\gamma N}{2m}\right) = \frac{d}{2m},$$

that is, to the open half-plane $\{\operatorname{Re} s > d\}$. Since $d > \tilde{D}$ can be chosen arbitrarily close to $\tilde{D}$, we deduce that $D_{\mathrm{mer}}(\zeta^*_{\Omega_0}) \le \tilde{D}$.

Finally, the residue of the spectral zeta function $\zeta^*_{\Omega_0}$ at $s = N$ can then be computed as follows (much in the same way as in the proof of Proposition 4.3.10):

$$\begin{aligned}\operatorname{res}(\zeta^*_{\Omega_0}, N) &= \lim_{s\to N}(s-N)\zeta^*_{\Omega_0}(s) = \lim_{2ms\to N}(2ms-N)\zeta^*_{\Omega_0}(2ms)\\ &= 2m\lim_{s\to N/2m}\left(s - \frac{N}{2m}\right)\zeta^*_{\Omega_0}(2ms) = 2m\frac{N}{2m}\cdot C^{-N/2m}\\ &= N\mu'_{\mathscr{A}}(\Omega_0),\end{aligned} \tag{4.3.53}$$

where in the next-to-last equality, we have used Equation (2.3.18) from Theorem 2.3.12 with $C := \mu'_{\mathscr{A}}(\Omega_0)^{-2m/N}$ and $a := 2m/N$.

In the case when $\tilde{D} = N-1$, we use [Lap1, Theorem 2.1, case (*ii*)], which can be stated equivalently as follows (using, for example, Lemma 4.3.15):

$$\mu_k^{(0)} = (\mu'_{\mathscr{A}}(\Omega_0))^{-2m/N}\cdot k^{2m/N} + O(k^\gamma \log k), \quad \text{as} \quad k \to \infty.$$

Now, we can proceed analogously as in the above case when $\tilde{D} > N-1$. This completes the proof of the theorem. $\square$

As we see, assuming that the hypotheses of Theorem 4.3.25 are satisfied, the Browder–Gårding measure of Ω_0 can be recovered by using the spectral zeta function $\zeta^*_{\Omega_0}$ in the following manner:

$$\mu'_{\mathscr{A}}(\Omega_0) := \frac{1}{N}\operatorname{res}(\zeta^*_{\Omega_0}, N). \tag{4.3.54}$$

Let us now assume that $\tilde{D} < N$ in order for the analog of Weyl's asymptotic estimate to hold (in light of (4.3.44), or, equivalently, (4.3.49)); that is, in order for the error term to be negligible compared to the leading term in (4.3.44) and (4.3.49). It then follows from the above discussion (that is, from estimate (4.3.44) or (4.3.49) when $\tilde{D} > N-1$ or from its counterpart when $\tilde{D} = N-1$) that $\zeta^*_{\Omega_0}$ is holomorphic in the open half-plane $\{\operatorname{Re} s > N\}$ and can be (uniquely) meromorphically extended to the (strictly) larger open half-plane $\{\operatorname{Re} s > \tilde{D}\}$, with a single (simple) pole at $s = N$ in that half-plane. (This statement is true for any value of $\tilde{D}$ in $[N-1, N)$, whether or not $\mathscr{M}^{*\tilde{D}}(\partial\Omega_0, \Omega_0)$ is finite.) Consequently, we deduce that the abscissa of (absolute) convergence of $\zeta^*_{\Omega_0}$, defined by (4.3.50), satisfies the following identity:

$$D_{\text{hol}}(\zeta^*_{\Omega_0}) = D(\zeta^*_{\Omega_0}) = N, \tag{4.3.55}$$

whereas the abscissa of meromorphic continuation of $\zeta^*_{\Omega_0}$ satisfies the inequality

$$D_{\text{mer}}(\zeta^*_{\Omega_0}) \le \tilde{D}. \tag{4.3.56}$$

(Observe that when $m = 1$, inequality (4.3.56) formally looks exactly like inequality (4.3.30) of Theorem 4.3.17.) In particular, (since $\tilde{D} < N$, by assumption) we have that

$$D_{\text{mer}}(\zeta^*_{\Omega_0}) < D_{\text{hol}}(\zeta^*_{\Omega_0}).$$

As is noted in [Lap2–3], this latter result (in inequality (4.3.56)) follows from the analog of Theorem 4.3.11 (and Corollary 4.3.14) corresponding to uniformly elliptic differential operators $\mathscr{A}$ of order $2m$, which is obtained in [Lap1, Theorem 2.1 and Corollary 2.2]. See the precise definition of the spectrum and the domain of the operator $\mathscr{A}$ given in [Lap1, Section 2.2]; see also [LioMag] or [Mét1].

In addition, much as in Corollary 4.3.22 (where $m = 1$), we have the following identity (concerning not only the spectral zeta function but also the fractal zeta functions of $(\partial\Omega_0, \Omega_0)$):

$$\begin{aligned} D_{\text{mer}}(\zeta^*_{\Omega_0}) &= \overline{\dim}_B(\partial\Omega_0, \Omega_0) =: \tilde{D} \\ &= D_{\text{mer}}(f) = D_{\text{hol}}(f) = D(f), \end{aligned} \tag{4.3.57}$$

for all $f \in \{\zeta_{\partial\Omega_0,\Omega_0}, \tilde{\zeta}_{\partial\Omega_0,\Omega_0}\}$. And analogously for Neumann or, more generally, mixed Dirichlet–Neumann boundary conditions, except with $D := \overline{\dim}_B(\partial\Omega_0)$ instead of $\tilde{D} := \overline{\dim}_B(\partial\Omega_0, \Omega_0)$ and with $\partial\Omega_0$ instead of the relative fractal drum $(\partial\Omega_0, \Omega_0)$.

Recall that the sharpness of inequality (4.3.56) is addressed in Problem 4.3.20, and that for the Dirichlet Laplacian and in the most important case when $\tilde{D} \in (N-1, N)$, it is established in Theorem 4.3.21 and Corollary 4.3.22 above (which relies in an essential way on the results of Sections 4.5–4.6 below). In light of Remark 4.3.23, the counterpart of the latter statement is also true for the Neumann Laplacian (with $\tilde{D}$ replaced by D, as usual).

In the case of Neumann, or more generally, of mixed Dirichlet–Neumann boundary conditions, it follows from the results of [Lap1] (and [Lap2–3]) that Theorem 4.3.11, Corollary 4.3.14, and hence also Theorem 4.3.17 still hold (along with their more general counterparts for positive uniformly elliptic operators of order $2m$) provided that Ω_0 is assumed to be a bounded open set of $\mathbb{R}^N$ satisfying the *extension property* (explicited in the next paragraph) and $\tilde{D} = \overline{\dim}_B(\partial\Omega_0, \Omega_0)$ (the upper, inner Minkowski dimension of $\partial\Omega_0$) is replaced by $D = \overline{\dim}_B(\partial\Omega_0)$, the upper Minkowski (or box) dimension of $\partial\Omega_0$ in the statement of Theorem 4.3.11, Corollary 4.3.14 and Theorem 4.3.17, as well as Theorem 4.3.21 and Corollary 4.3.22 (which rely on results of Section 4.6 below and Theorem 4.3.25 along with Equation (4.3.56)).[36] See, in particular, [Lap1, Theorem 2.3 and Corollary 2.2].

Recall that the open set $\Omega_0 \subseteq \mathbb{R}^N$ is said to satisfy the *extension property* if every function in the Sobolev space $H^1(\Omega_0) := W^{1,2}(\Omega_0)$ can be extended to a function in $H^1(\mathbb{R}^N) := W^{1,2}(\mathbb{R}^N)$, and the resulting extension operator is a bounded linear operator. For example, a bounded domain Ω_0 in $\mathbb{R}^N$ satisfies the extension property if its boundary $\partial\Omega_0$ is of class C^1; see, e.g., [Bre, Théorème IX.7]. Note that, in this latter case, $\dim_B(\partial\Omega_0, \Omega_0) = N - 1$ and $\mathscr{M}^{*D}(\partial\Omega_0, \Omega_0) < \infty$.

Alternatively, the aforementioned results of [Lap1] imply that (still for Neumann or mixed Dirichlet–Neumann boundary conditions) instead of satisfying the extension property, Ω_0 can be assumed to satisfy the so-called (C')-*condition* [Lap1, Definition 2.2] (which is satisfied, for example, if Ω_0 is locally Lipschitz, or satisfies either a 'segment condition', a 'cone condition', or else is an open set with cusp; see [Mét2–3] or [Lap1, Examples 2.1 and 2.2]), in which case we are necessarily in case (*ii*) of the counterparts of Theorem 4.3.11 and Corollary 4.3.14 (see, especially, Equation (4.3.49) and the text following it), with D $(:= \overline{\dim}_B(\partial\Omega_0, \Omega_0)) = N - 1$ and $\mathscr{M}^{*D}(\partial\Omega_0, \Omega_0) < \infty$.

Recall that (as is proved by Jones in [Jon] and discussed in [Lap1, Example 4.2]; see also [Maz]) in two dimensions (i.e., when $N = 2$), a simply connected domain Ω_0 satisfies the *extension property* (or is an *extension domain*) if and only if it is a *quasidisk* (i.e., a Jordan curve which is the quasiconformal image of the unit disk in $\mathbb{R}^2$). The boundary $\partial\Omega_0$ of a quasidisk is called a *quasicircle*, and the property of being a quasicircle can be characterized geometrically by a *chord-arc condition*. Furthermore, a quasicircle can have any Hausdorff dimension between 1 and 2. See [Maz] and [Pom], along with the relevant references therein, for a detailed discussion of quasidiscs, quasicircles and extension domains. The class of quasicircles includes the classic Koch snowflake curve and its natural generalizations, as well as the Julia sets associated with the quadratic maps $z \mapsto z^2 + c$ $(z \in \mathbb{C})$, provided the parameter $c \in \mathbb{C}$ is sufficiently small. Therefore, the Koch snowflake domain (and

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[36] It is clear from the definitions that $\tilde{D} \le D$, and it can also be shown (since Ω_0 is open and bounded) that $N - 1 \le \tilde{D} \le D \le N$; see [Lap1, Corollary 3.2]. Furthermore, there are natural examples of planar domains for which $\tilde{D} < D$; see [Lap1, Note added in proof, p. 525] and the relevant reference therein, [Tri2].

its generalizations) and the bounded simply connected domains having for boundary the aforementioned Julia sets, are natural examples of quasidisks and hence, of extension domains.

In higher dimensions, *extension domains* (i.e., domains of $\mathbb{R}^N$ satisfying the (Sobolev) extension property) are more difficult to characterize. However, it has been shown by Hajlasz, Koskela and Tuominen in [HajKosTu1–2] that a bounded domain $\Omega_0 \subset \mathbb{R}^N$ is an extension domain if and only if it satisfies a certain functional analytic condition and the following *measure density condition*; see [HajKosTu1, Theorem 5]. The set $\Omega_0 \subseteq \mathbb{R}^N$ is said to satisfy the *measure density condition* (or to be a *lower Ahlfors regular N-set*) if there exists a positive constant M such that

$$|\Omega_0 \cap B_r(x)| \geq Mr^N,$$

for all $x \in \Omega_0$ and all $0 < r \leq 1$, where $B_r(x)$ denotes the open ball of center x and radius r in $\mathbb{R}^N$; see [HajKosTu1].

Finally, we note that for Neumann boundary conditions, the above results concerning spectral asymptotics and spectral zeta functions also extend to higher order uniformly elliptic self-adjoint operators (with variable coefficients), under the analogous hypotheses and with the same changes as those indicated above; see [Lap1].

Remark 4.3.26. It is noteworthy that when the extension property (or else the (C')-condition) is not satisfied, the continuous embedding of $H^1(\Omega_0)$ into $L^2(\Omega_0)$ need not be compact and, hence, the spectrum of the Neumann Laplacian may not be discrete. Actually, even when this spectrum is discrete, there are explicit examples of bounded open sets for which the leading spectral asymptotics of the Neumann Laplacian does not satisfy Weyl's classic law (4.3.12), and hence, let alone the corresponding remainder estimate (4.3.22) (or, equivalently, (4.3.19)). See, e.g., [Mét1–3], [Lap1] and the relevant references therein.

A similar comment can be made about more general uniformly elliptic operators of order $2m$, with Neumann (or, more generally, Dirichlet-Neumann) boundary conditions and with $H^m(\Omega_0)$ instead of $H^1(\Omega_0)$.

4.4 Further Examples of Relative Distance Zeta Functions

The aim of this section is to introduce several classes of RFDs and to study their associated fractal zeta functions. We will focus here on the distance zeta functions, although the corresponding tube zeta functions could be studied as well, either directly or by using the functional equation (4.5.2) below. Of special interest are the unbounded geometric chirps, associated with the standard geometric chirps occurring, for example, in the oscillation theory of differential equations. We also compute the relative distance function of the Cartesian product of fractal strings.

4.4.1 Relative Distance Zeta Functions of Unbounded Geometric Chirps

The following example and result (namely, Example 4.4.1 and Proposition 4.4.3) deal with unbounded geometric chirps; see Figures 4.13, 4.14 and 4.15. Also, refer to Section 3.6 for the case of bounded geometric chirps.

Example 4.4.1. Let A be an (α,β)-geometric chirp, for $\alpha\in(-1,0)$ and $\beta>0$; i.e., A is a union of vertical segments at $x=k^{-1/\beta}$ of length $k^{-\alpha/\beta}$ for $k\in\mathbb{N}$; see (3.6.1). For Ω we take the union of open rectangles R_k for $k\in\mathbb{N}$, where R_k has a base of length $k^{-1/\beta}-(k+1)^{-1/\beta}$ and height $k^{-\alpha/\beta}$; see Figure 4.15. The associated unbounded geometric chirp RFD (A,Ω) approximates the graph of the function

$$x\mapsto x^{\alpha}\sin(\pi x^{-\beta}),\quad \text{for all } x\in(0,1).$$

The relative distance zeta function of (A,Ω) is then given by

$$\zeta_{A,\Omega}(s)=\frac{1}{2^{s-2}(s-1)}\sum_{k=1}^{\infty}k^{-\alpha/\beta}\left(k^{-1/\beta}-(k+1)^{-1/\beta}\right)^{s-1}. \tag{4.4.1}$$

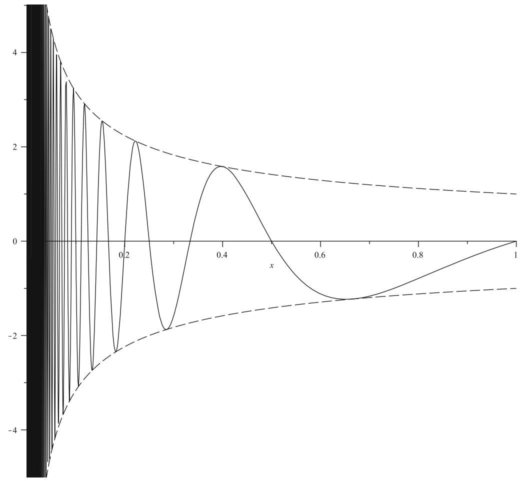

Fig. 4.13 The unbounded $(-1/2,1)$-chirp; the graph of $f(x)=x^{-1/2}\sin(\pi x^{-1})$, $0<x<1$, is fractal near $x=0$. We expect that $\overline{\dim}_B(A,\Omega)=7/4$, as for the related geometric chirp in Proposition 4.4.3(a) and depicted in Figure 4.15 on page 347.

It can be shown that it has a singularity at $s=2-\frac{1+\alpha}{1+\beta}$ and is holomorphic in the open right half-plane $\left\{\operatorname{Re}s>2-\frac{1+\alpha}{1+\beta}\right\}$. (See Remark 4.4.2 just below for a justification of this claim.) We conclude from Theorem 4.1.7 that $\overline{\dim}_B(A,\Omega)=2-\frac{1+\alpha}{1+\beta}$, which is Tricot's formula in the case when α is negative and β positive. We note that the original Tricot formula was obtained for $0 < \alpha < \beta$ and can be found in [Tri3, p. 122].

Remark 4.4.2. We provide here a short heuristic proof of the above claim (compare with the proof of Equation (3.6.3) in Subsection 3.6.1 above). Using the Lagrange mean value theorem, we approximate the difference $k^{-1/\beta} - (k+1)^{-1/\beta}$ (where $k \in \mathbb{N}$) by $k^{-\frac{1}{\beta}-1}$. The Dirichlet series on the right-hand side of Equation (4.4.1) then becomes

$$\sum_{k=1}^{\infty} k^{-\alpha/\beta} \left(k^{-\frac{1}{\beta}-1}\right)^{s-1} = \sum_{k=1}^{\infty} k^{-\left(\frac{\alpha}{\beta}+(\frac{1}{\beta}+1)(s-1)\right)}.$$

It converges absolutely if and only if $\frac{\alpha}{\beta} + (\frac{1}{\beta}+1)(\operatorname{Re} s - 1) > 1$; that is, when $\operatorname{Re} s > 2 - \frac{1+\alpha}{1+\beta}$. This heuristic proof can be easily made precise using Cahen's classical result stated in Theorem 2.1.27. We leave the details as a simple exercise for the interested reader. (A different proof of the claim as well as additional information can be found in Example 5.5.19 in Subsection 5.5.5 below.)

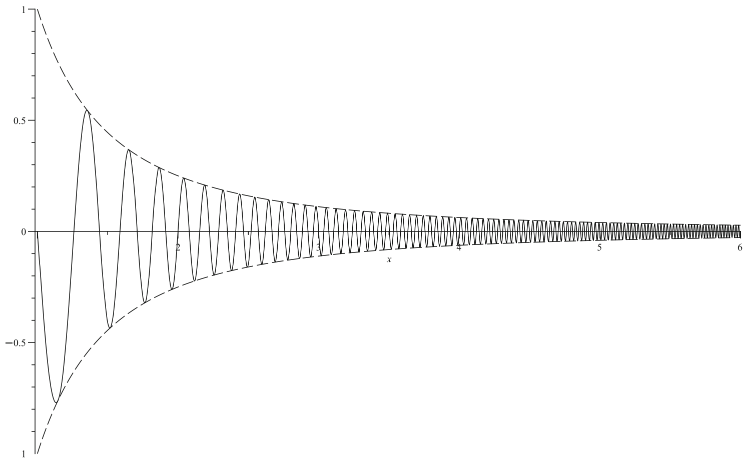

Fig. 4.14 The unbounded $(-2,-3)$-chirp; the graph of $f(x) = x^{-2}\sin(\pi x^3)$, $x > 1$, is fractal near $x = \infty$. We expect that $\overline{\dim}_B(A,\Omega) = 3/2$, as for the related geometric chirp in Proposition 4.4.3(b).

Proposition 4.4.3. *Let A be an (α,β)-geometric chirp defined by (3.6.1), and assume that one of the following conditions holds*:

(a) $-1 < \alpha < 0 < \beta$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : x \in (0,1),\ 0 < x^\alpha < y\}$,

(b) $\beta < \alpha < -1$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : x \in (1,+\infty),\ 0 < x^\alpha < y\}$.

Then

$$\overline{\dim}_B(A,\Omega) = 2 - \frac{1+\alpha}{1+\beta},$$

and moreover, this value coincides with $\dim_{PC}(A,\Omega)$.

Proof. Computing the relative distance zeta function of the (α,β)-geometric chirp from Example 4.4.1 with respect to the 'outer' rectangles and using Lemma 4.1.15, we obtain the result in case (a). We can use the same technique in case (b), due to the fact that $\beta < -1$. □

We note that in Example 4.4.1 and Proposition 4.4.3, we can replace $\overline{\dim}_B(A,\Omega)$ by $d = \dim_B(A,\Omega)$. This can be seen by direct computation: indeed, there exist positive constants c_1 and c_2 such that $c_1\delta^{2-d} \le |A_\delta \cap \Omega| \le c_2\delta^{2-d}$. Therefore, $c_1 \le \mathscr{M}_*^d(A,\Omega) \le \mathscr{M}^{*d}(A,\Omega) \le c_2$.

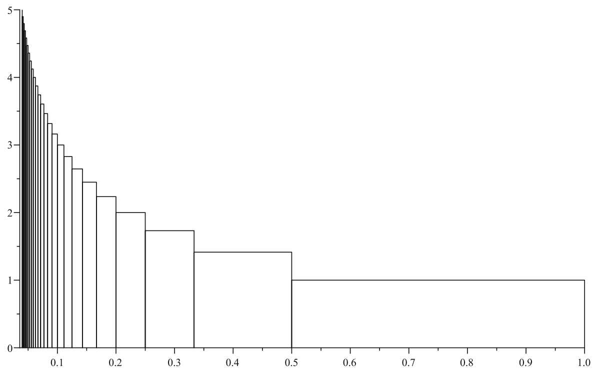

Fig. 4.15 Approximation of the unbounded chirp in Figure 4.13 on page 345, using rectangles. Here, $\alpha = -1/2$ and $\beta = 1$; hence, $\overline{\dim}_B A = 7/4$. In the corresponding RFD (A,Ω), the set A is defined as the union of the vertical segments while the open set Ω is defined as the union of the open rectangles.

In the case when $-1 < \alpha < 0$ and $\beta < 0$, we have $\dim_B(A,\Omega) = 1$, which complements Proposition 4.4.3(a). Analogously, in the case when $\alpha < -1$ and $\alpha \le \beta$, $\beta \ne 0$, we have $\dim_B(A,\Omega) = 1$, which complements Proposition 4.4.3(b).

In the work of the second and third authors with V. Županović [RaŽuŽup], a different approach to the study of the fractal properties of unbounded sets at infinity in $\mathbb{R}^N$ has been undertaken, instead of using the relative box dimensions. If A is an unbounded set which does not possess the origin as its accumulation point, then it is natural to define the *box dimension of* A *at infinity* as the usual box dimension of $A^{-1} = \{x/|x|^2 : x \in A\}$. Here, A^{-1} is the geometric inversion of A, which under the stated condition is clearly a bounded set. This tool has been applied to the study of the Hopf bifurcation of several polynomial dynamical systems at infinity. In his thesis [Ra1] and in [Ra2], the second author has significantly expanded these ideas. In particular, in [Ra1–2], the notions of Minkowski contents and box dimensions of unbounded open sets with respect to infinity have been introduced and studied, as well as the associated classes of fractal zeta functions, thereby extending to (suitable) unbounded sets $A \subseteq \mathbb{R}^N$ the theory developed in this book and in [LapRaŽu1–8].

4.4.2 Relative Zeta Functions of Cartesian Products of Fractal Strings

In Theorem 3.6.5, we have computed a representative of the zeta function of E relative to E_δ, where E is the boundary of the Cartesian product of two fractal strings $\mathscr{L} = (\ell_j)_{j\geq 1}$ and $\mathscr{M} = (m_k)_{k\geq 1}$. If we consider the zeta function of E relative to the rectangle $\Omega = (0,a_1)\times(0,b_1)$, then we deduce from the proof of Theorem 3.6.5 that $D(s) \equiv 0$ in (3.6.29), which yields the following explicit result.

Theorem 4.4.4. *Assume that the hypotheses of Theorem 3.6.5 are satisfied. Then, for E given as in Theorem 3.6.5 and for $\Omega = (0,a_1)\times(0,b_1)$, we have*

$$\zeta_{E,\Omega}(s) = \frac{2^{2-s}}{s-1}\sum_{j,k=1}^{\infty}\left[|\ell_j - m_k|\min\{\ell_j,m_k\}^{s-1} + \frac{2}{s}\min\{\ell_j,m_k\}^s\right].$$

Furthermore,

$$\overline{\dim}_B(E,\Omega) = \overline{\dim}_B E = 1+\max\{\overline{\dim}_B\mathscr{L},\overline{\dim}_B\mathscr{M}\} \tag{4.4.2}$$

is the abscissa of convergence of $\zeta_{E,\Omega}(s)$, and $\zeta_{E,\Omega}(s)\to+\infty$ as $\mathbb{R}\ni s\to\overline{\dim}_B E$ from the right.

Let us consider the product of three fractal strings. Assume that $\mathscr{L} = (\ell_j)_{j\geq 1}$, $\mathscr{M} = (m_k)_{k\geq 1}$, and $\mathscr{N} = (n_r)_{r\geq 1}$, with $a_1 := \sum_j \ell_j$, $b_1 := \sum_k m_k$ and $c_1 := \sum_r n_r$. We identify the strings with three standard families of open intervals, $\mathscr{L} = (I_j)_{j\geq 1}$, $\mathscr{M} = (J_k)_{k\geq 1}$ and $\mathscr{N} = (K_r)_{r\geq 1}$. Furthermore, for any ordered triple $L = (\ell_j, m_j, n_r)$ in $\mathscr{L}\times\mathscr{M}\times\mathscr{N}$, we define its nondecreasing permutation $(M_1(L),M_2(L),M_3(L))$ by $\{\ell_j,m_j,n_r\} = \{M_1(L),M_2(L),M_3(L)\}$ and $M_1(L)\leq M_2(L)\leq M_3(L)$. Note that then, $M_1(L) = \min L$ and $M_3(L) = \max L$. (Further, observe that we are really working here with a Cartesian product of multisets, rather than of ordinary sets. Recall that a multiset is simply a set with multiplicities. For example, $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{N}$ are multisets since, for instance, scales ℓ_j in the sequence $\mathscr{L}$ may have finite multiplicities.) In this context, we obtain the following result.

Theorem 4.4.5. *Let $\mathscr{L} = (I_j)_{j\geq 1}$, $\mathscr{M} = (J_k)_{k\geq 1}$ and $\mathscr{N} = (n_r)_{r\geq 1}$ be three fractal strings, and define the set $E = \partial(\mathscr{L}\times\mathscr{M}\times\mathscr{N})$. Let $\Omega = (0,a_1)\times(0,b_1)\times(0,c_1)$. Then, with the notation introduced just above, we have*

$$\zeta_{E,\Omega}(s) = \frac{2^{3-s}}{s-2}\sum_L \Big[(M_3(L)-M_1(L))(M_2(L)-M_1(L))(M_1(L))^{s-2} + \frac{2}{s-1}(M_3(L)+M_2(L)-2M_1(L))(M_1(L))^{s-1} + \frac{4}{s(s-1)}(M_1(L))^s\Big], \tag{4.4.3}$$

where the summation runs over all ordered triples $L=(\ell_j,m_j,n_r)$ from $\mathscr{L}\times\mathscr{M}\times\mathscr{N}$. Furthermore, the abscissa of convergence $D(\zeta_{E,\Omega})$ of $\zeta_{E,\Omega}$ is given by

$$\overline{\dim}_B(E,\Omega) = \overline{\dim}_B E = 2+\max\{\overline{\dim}_B\mathscr{L},\overline{\dim}_B\mathscr{M},\overline{\dim}_B\mathscr{N}\}, \tag{4.4.4}$$

and $\zeta_{E,\Omega}(s)\to+\infty$ as $\mathbb{R}\ni s\to\overline{\dim}_B E$ from the right.

Proof. The set Ω is a countable disjoint union of open parallelepipeds. Let R be a typical parallelepiped with sides $n\le m\le \ell$. We split R into 16 prisms (8 of them being pairwise isometric and having width $m-n$, and the rest with side $\ell-n$), 32 isometric tetrahedra, and two isometric parallelepipeds with sides $n/2$, $m-n$, $\ell-n$, placed at the center of R. We have to integrate the function $d(x,\partial R)^{s-3}$ over these sets. The integral over each prism of width $m-n$ is equal to

$$\frac{2^{1-s}}{(s-1)(s-2)}(m-n)n^{s-1},$$

(and analogously for the prism of width $\ell-n$). The integral over each tetrahedron is equal to

$$\frac{2^{-s}}{s(s-1)(s-2)}n^s,$$

while the integral over each parallelepiped is equal to

$$\frac{2^{2-s}}{s-2}(m-n)(\ell-n)n^{s-2}.$$

From this, the claim follows easily. We omit the details. The dimension result is an immediate consequence of the finite stability of the upper box dimension and the fact that

$$E = (\overline{A}\times[0,b_1]\times[0,c_1])\cup([0,a_1]\times\overline{B}\times[0,c_1])\cup([0,a_1]\times[0,b_1]\times\overline{C}),$$

where $A=(a_j)_{j\ge1}$ with $a_j:=\sum_{k\ge j}\ell_k$, and analogously for $B=(b_k)_{k\ge1}$ and $C=(c_r)_{r\ge1}$. $\square$

Note that the relative distance zeta function of the set $E=\partial(\mathscr{L}\times\mathscr{M})$ generated by two fractal strings is represented by a double sum (see Theorem 4.4.4), while the relative distance zeta function of $E=\partial(\mathscr{L}\times\mathscr{M}\times\mathscr{N})$ is equal to a triple sum (see Theorem 4.4.5) taken over the indices (j,k,r), since $L=(\ell_j,m_k,n_r)$. Analogously, the relative distance zeta function of the set $E = \partial(\mathscr{L}_1 \times \cdots \times \mathscr{L}_N)$ generated by N fractal strings $\mathscr{L}_i$, $i = 1, \ldots, N$, can then be computed, and it is equal to an N-tuple sum. Furthermore, we then have

$$\overline{\dim}_B(E,\Omega) = \overline{\dim}_B E = N - 1 + \max\{\overline{\dim}_B \mathscr{L}_i : i = 1, \ldots, N\}.$$

4.5 Meromorphic Extensions of Relative Zeta Functions and Applications

If we consider a class of RFDs with a prescribed value D for the abscissa of convergence of the associated distance relative zeta functions, it is of interest to know the corresponding values of the abscissa of meromorphic continuation $D_{\mathrm{mer}} = D - \alpha$. Clearly, we have $\alpha \geq 0$. Intuitively, the smaller α, the more complex the (fractal) nature of the relative fractal drum. This can be considered even as a definition for comparing the complexity of different RFDs in the class. The most complex, then, is the subclass of relative fractal drums for which the abscissa of meromorphic continuation is equal to D; that is, $\alpha = 0$. And among these, the most complex are the relative fractal drums for which the set of nonisolated singularities is equal to the *whole* critical line $\{\mathrm{Re}\, s = D\}$. Indeed, there cannot be more complexity than that, at least from the present point of view of the higher-dimensional theory of complex fractal dimensions. We call them *maximally hyperfractal drums*; see Definition 4.6.23.

This section is organized as follows. We first study the problem of determining an upper bound for the abscissa of meromorphic extension of the distance (or tube) zeta function for a class of RFDs; see Theorems 4.5.1 and 4.5.2. Furthermore, we construct a class of RFDs for which the abscissa of meromorphic continuation can be explicitly computed. We also construct an explicit class of maximally hyperfractal drums (A,Ω); see Theorem 4.5.8. As a consequence, we then construct (in Section 4.6) a class of maximally hyperfractal strings $\mathscr{L}$, which in turn generate maximally hyperfractal sets $A = A_{\mathscr{L}}$ on the real line; see Corollary 4.6.17.

4.5.1 Meromorphic Extensions of Zeta Functions of Relative Fractal Drums

By analogy with (2.2.8), we introduce the *relative tube zeta function* associated with the relative fractal drum (A,Ω) in $\mathbb{R}^N$. It is defined by

$$\tilde{\zeta}_{A,\Omega}(s) = \int_0^\delta t^{s-N-1} |A_t \cap \Omega| \, \mathrm{d}t, \tag{4.5.1}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s$ sufficiently large, where $\delta > 0$ is fixed. As we see, it involves the *relative tube function* $t \mapsto |A_t \cap \Omega|$. Its abscissa of convergence is given by $D(\tilde{\zeta}_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$. This follows from the following fundamental identity or *functional equation*, which connects the relative tube zeta function $\tilde{\zeta}_{A,\Omega}$ and the relative distance zeta function $\zeta_{A,\Omega}$, defined by (4.1.1):

$$\zeta_{A,A_\delta\cap\Omega}(s) = \delta^{s-N}|A_\delta\cap\Omega| + (N-s)\tilde{\zeta}_{A,\Omega}(s). \tag{4.5.2}$$

This identity is analogous to (2.2.1) and (2.2.23). Its proof is analogous to that of Theorem 2.2.1, using the known identity

$$\int_{A_\delta\cap\Omega} d(x,A)^{-\gamma}\,\mathrm{d}x = \delta^{-\gamma}|A_\delta\cap\Omega| + \gamma\int_0^\delta t^{-\gamma-1}|A_t\cap\Omega|\,\mathrm{d}t, \tag{4.5.3}$$

where $\gamma > 0$; see Lemma 2.1.4, [Žu2, Theorem 2.9(a)], or a more general form provided in [Žu4, Lemma 3.1].

It follows from the identity (4.5.2) that the analog of Proposition 2.2.19 and of Equation (2.2.50) holds in the present more general context. More specifically, provided that $\overline{\dim}_B(A,\Omega) < N$, the relative tube zeta function $\tilde{\zeta}_{A,\Omega}$ and the relative distance zeta function $\zeta_{A,A_\delta\cap\Omega}$ can be (uniquely) meromorphically extended to exactly the same domain $U \subseteq \mathbb{C}$ (chosen to be a connected open neighborhood of a given window $\boldsymbol{W}$, say), when it is possible. Furthermore, the relative fractal drum (A,Ω) has exactly the same visible complex dimensions (and with the same multiplicities), as measured from the point of view of either of these two fractal zeta functions:

$$\mathscr{P}(\tilde{\zeta}_{A,\Omega},\boldsymbol{W}) = \mathscr{P}(\zeta_{A,A_\delta\cap\Omega},\boldsymbol{W}). \tag{4.5.4}$$

In particular, we have

$$\overline{\dim}_B(A,\Omega) = D(\tilde{\zeta}_{A,\Omega}) = D(\zeta_{A,A_\delta\cap\Omega}), \tag{4.5.5}$$

$$\tilde{\zeta}_{A,\Omega} \sim \zeta_{A,A_\delta\cap\Omega}, \tag{4.5.6}$$

and so

$$\dim_{PC}(A,\Omega) = \mathscr{P}_c(\tilde{\zeta}_{A,\Omega}) = \mathscr{P}_c(\zeta_{A,A_\delta\cap\Omega}). \tag{4.5.7}$$

Finally, if $\omega \in U$ is a simple pole of $\tilde{\zeta}_{A,\Omega}$, then it is also a simple pole of $\zeta_{A,A_\delta\cap\Omega}$ and we have

$$\operatorname{res}(\tilde{\zeta}_{A,\Omega},\omega) = \frac{1}{N-\omega}\operatorname{res}(\zeta_{A,A_\delta\cap\Omega},\omega); \tag{4.5.8}$$

that is, the counterpart of Equation (2.2.50) holds in this context. Note that as was the case for ordinary fractal sets A, these residues are independent of the choice of $\delta > 0$.

We next consider a class of RFDs (A,Ω) such that both $D = \dim_B(A,\Omega)$ and $\mathscr{M}^D(A,\Omega)$ exist, but the relative Minkowski content is *degenerate* in the sense that $\mathscr{M}^D(A,\Omega) = +\infty$. In general, we may also have $\mathscr{M}^D(A,\Omega) = 0$, but we do not treat this case here.

We shall treat these two cases by using the following assumption on the asymptotics of the relative tube function $t \mapsto |A_t\cap\Omega|$:

$$|A_t \cap \Omega| = t^{N-D} h(t)(\mathscr{M} + O(t^\alpha)) \quad \text{as} \quad t \to 0^+, \tag{4.5.9}$$

where $\mathscr{M} > 0$, $\alpha > 0$ and $D \leq N$ are given in advance. Here, we assume that the function $h(t)$ has a sufficiently slow growth near the origin, in the sense that for any $c > 0$, $h(t) = O(t^c)$ as $t \to 0^+$. Typical examples of such functions are $h(t) = (\log t^{-1})^m$, $m \geq 1$, or more generally,

$$h(t) = \big(\underbrace{\log \ldots \log}_{n}(t^{-1})\big)^m$$

(the m-th power of the n-th iterated logarithm of t^{-1}, $n \geq 1$), and in these cases we obviously have $\mathscr{M}^D(A,\Omega) = +\infty$. For this and other examples, see [HeLap]. The function $t \mapsto t^D h(t)^{-1}$ is usually called the *gauge function*, but for the sake of simplicity, we shall rather use this name for the function $h(t)$ only.

Assuming that a relative fractal drum (A,Ω) in $\mathbb{R}^N$ is such that $D = \dim_B(A,\Omega)$ exists, and $\mathscr{M}_*^D(A,\Omega) = 0$ or $+\infty$ (or $\mathscr{M}^{*D}(A,\Omega) = 0$ or $+\infty$), it is natural to define as follows a new class of relative lower and upper Minkowski contents of (A,Ω), associated with a suitably chosen gauge function $h(t)$:

$$\begin{aligned} \mathscr{M}_*^D(A,\Omega,h) &= \liminf_{t \to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D} h(t)}, \\ \mathscr{M}^{*D}(A,\Omega,h) &= \limsup_{t \to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D} h(t)}. \end{aligned} \tag{4.5.10}$$

The aim is to find an *explicit* gauge function so that these two contents are in $(0,+\infty)$, and the functions $r \mapsto \mathscr{M}_*^r(A,\Omega,h)$ and $r \mapsto \mathscr{M}^{*r}(A,\Omega,h)$, $r \in \mathbb{R}$, defined exactly as in (4.5.10), except for D replaced with r, have a jump from $+\infty$ to 0 when r crosses the value of D. In this generality, the above *gauge relative Minkowski contents* have been introduced in [Žu4], motivated by [HeLap].

In Equation (4.5.10) above, $\mathscr{M}_*^D(A,\Omega,h)$ (resp., $\mathscr{M}^{*D}(A,\Omega,h)$) is called the *lower* (resp., *upper*) h-Minkowski content of (A,Ω). Furthermore, much as in the usual case when $h \equiv 1$, the RFD (A,Ω) is said to be *h-Minkowski nondegenerate* if

$$0 < \mathscr{M}_*^D(A,\Omega,h) \leq \mathscr{M}^{*D}(A,\Omega,h) < \infty.$$

If for some gauge function h, say, we have that $\mathscr{M}^D(A,\Omega,h) \in (0,+\infty)$ (which means, as usual, that $\mathscr{M}_*^D(A,\Omega,h) = \mathscr{M}^{*D}(A,\Omega,h)$ and that this common value, denoted by $\mathscr{M}^D(A,\Omega,h)$, lies in $(0,+\infty)$), we say (as in [HeLap]) that the fractal drum (A,Ω) is *h-Minkowski measurable*, with *h-Minkowski content* $\mathscr{M}^D(A,\Omega,h)$.

It is easy to see that the counterparts of Theorems 2.3.18 and 2.3.25 also hold in the present context of relative fractal drums. It suffices to replace $|A_t|$ by $|A_t \cap \Omega|$, $\tilde{\zeta}_A(s)$ by $\tilde{\zeta}_{A,\Omega}(s)$, and $\dim_B A$ by $\dim_B(A,\Omega)$. Below, we state and prove two results dealing with RFDs with associated gauge functions; see Theorems 4.5.1 and 4.5.2. In both of these theorems, we have $\mathscr{M}^D(A,\Omega) = +\infty$. As we shall see, certain gauge functions generate higher-order poles of fractal zeta functions. The presence of a nonstandard gauge function may also force the tube (or distance) zeta function to have a partial natural boundary along the critical line $\{\mathrm{Re}\, s = D\}$ (i.e., not to have a meromorphic continuation beyond the open half-plane $\{\mathrm{Re}\, s > D\}$ of holomorphicity). One could then try to view the fractal zeta function as an analytic function on an appropriate Riemann surface. However, we will not investigate this interesting topic here.

In what follows, we denote the Laurent expansion of a meromorphic extension (assumed to exist) of the relative tube zeta function $\tilde{\zeta}_{A,\Omega}$ to a connected open neighborhood of $s = D$ (more specifically, an open punctured disk centered at $s = D$) by

$$\tilde{\zeta}_{A,\Omega}(s) = \sum_{j=-\infty}^{\infty} c_j (s-D)^j, \tag{4.5.11}$$

where, of course, $c_j = 0$ for all $j \ll 0$ (that is, there exists $j_0 \in \mathbb{Z}$ such that $c_j = 0$ for all $j < j_0$).

The following theorem shows that, in order to obtain a meromorphic extension of the zeta function to the left of the abscissa of convergence, it is important to have some information about the second term in the asymptotic expansion of the relative tube function $t \mapsto |A_t \cap \Omega|$ near $t = 0$. We will provide two proofs of this result, because they each highlight different aspects of the issues involved. See Theorem 5.4.29 in Chapter 5 below (as well as Theorem 5.4.30) for a partial converse of Theorem 4.5.1.

Theorem 4.5.1 (Minkowski measurable RFDs). *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$, such that (4.5.9) holds for some $D \le N$, $\mathscr{M} > 0$, $\alpha > 0$ and with $h(t) := (\log t^{-1})^m$ for all $t \in (0,1)$, where m is a nonnegative integer. Then the relative fractal drum (A,Ω) is h-Minkowski measurable, $\dim_B(A,\Omega) = D$, and $\mathscr{M}^D(A,\Omega,h) = \mathscr{M}$. Furthermore, the relative tube zeta function $\tilde{\zeta}_{A,\Omega}(s)$ has for abscissa of convergence $D(\tilde{\zeta}_{A,\Omega}) = D$, and it possesses a (necessarily unique) meromorphic extension (at least) to the half-plane $\{\mathrm{Re}\, s > D - \alpha\}$; that is,*

$$D_{\mathrm{mer}}(\tilde{\zeta}_{A,\Omega}) \le D - a. \tag{4.5.12}$$

Moreover, $s = D$ is the unique pole in this half-plane, and it is of order $m+1$. In addition, the coefficients of the Laurent series expansion (4.5.11) corresponding to the principal part of $\tilde{\zeta}_{A,\Omega}(s)$ at $s = D$ are given by

$$\begin{aligned} c_{-m-1} &= m!\mathscr{M}, \\ c_{-m} = \cdots &= c_{-1} = 0 \quad (\text{provided } m \ge 1.) \end{aligned} \tag{4.5.13}$$

If $m = 0$, then D is a simple pole of $\tilde{\zeta}_{A,\Omega}$ and we have that

$$\mathrm{res}(\tilde{\zeta}_{A,\Omega}, D) = \mathscr{M}. \tag{4.5.14}$$

Proof. (First proof of Theorem 4.5.1.) Let us set

$$\zeta_1(s) = \mathscr{M} z_m(s), \quad z_m(s) = \int_0^\delta t^{s-D-1}(\log t^{-1})^m \mathrm{d}t,$$
$$\zeta_2(s) = \int_0^\delta t^{s-N-1}(\log t^{-1})^m O(t^{N-D+\alpha})\,\mathrm{d}t. \tag{4.5.15}$$

Since $\tilde{\zeta}_{A,\Omega}(s) = \zeta_1(s) + \zeta_2(s)$, we can proceed as follows. It is easy to see that for each $\varepsilon > 0$, we have $(\log t^{-1})^m = O(t^{-\varepsilon})$ as $t \to 0^+$; hence,

$$|\zeta_2(s)| \leq \int_0^\delta O(t^{\mathrm{Re}\, s-1-D+(\alpha-\varepsilon)})\,\mathrm{d}t.$$

Then, since the integral is well defined for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > D-(\alpha-\varepsilon)$, in the same way as in the proof of Theorem 2.3.18, we deduce that $D(\zeta_2) \leq D-(\alpha-\varepsilon)$. Letting $\varepsilon \to 0^+$, we obtain the following desired inequality: $D(\zeta_2) \leq D-\alpha$.

By means of the change of variable $\tau := \log t^{-1}$ (for $0 < t \leq \delta$), it is easy to see that

$$z_m(s) = \int_{\log \delta^{-1}}^{+\infty} \mathrm{e}^{-\tau(s-D)} \tau^m \mathrm{d}\tau. \tag{4.5.16}$$

Integration by parts yields the following recursion equation, where we have to assume (at first) that $\mathrm{Re}\, s > D$:

$$z_m(s) = \frac{1}{s-D}\left((\log \delta^{-1})^m \delta^{s-D} + m z_{m-1}(s)\right), \quad m \geq 1, \tag{4.5.17}$$

and $z_0(s) := (s-D)^{-1}\delta^{s-D}$. Since $D(\zeta_2) \leq D-\alpha$, it is clear that the coefficients c_j, $j < 0$, of the Laurent series expansion (4.5.11) of $\tilde{\zeta}_{A,\Omega}(s) = \zeta_1(s) + \zeta_2(s)$ in a neighborhood of $s = D$ do not depend on $\delta > 0$. Indeed, changing the value of $\delta > 0$ to $\delta_1 > 0$ in (4.5.1) amounts to adding $\int_\delta^{\delta_1} t^{s-N-1}|A_t \cap \Omega|\,\mathrm{d}t$, which is an entire function of s. Therefore, without loss of generality, we may take $\delta = 1$ in (4.5.17):

$$z_m(s) = \frac{m}{s-D} z_{m-1}(s) = \cdots = \frac{m!}{(s-D)^m} z_0(s) = \frac{m!}{(s-D)^{m+1}}. \tag{4.5.18}$$

In this way, we obtain that

$$\zeta_1(s) = \frac{m!}{(s-D)^{m+1}} \mathscr{M}, \tag{4.5.19}$$

and we can meromorphically extend the definition of ζ_1 from the half-plane $\{\mathrm{Re}\, s > D\}$ to the entire complex plane. The claim then follows from Lemma 2.3.5. $\square$

Proof. (Second proof of Theorem 4.5.1.) Let us define z_0 by

$$z_0(s) = \int_0^\delta t^{s-N-1} \frac{|A_t \cap \Omega|}{(\log t^{-1})^m} \, \mathrm{d}t, \tag{4.5.20}$$

where $\mathrm{Re}\, s > N - D$. As we see, $z_0(s) = \zeta_1(s)$, where $\zeta_1(s)$ is defined as in the proof of Theorem 2.3.25, except with $|A_t|$ replaced by $|A_t \cap \Omega|$, and $\zeta_0(s)$ has all the properties stated in this theorem for $\zeta_A(s)$. It is easy to see that

$$z_0'(s) = -\int_0^\delta t^{s-N-1} \frac{|A_t \cap \Omega|}{(\log t^{-1})^{m-1}} \, \mathrm{d}t.$$

Therefore, proceeding inductively, we obtain that (still for $\mathrm{Re}\, s > N - D$)

$$z_0^{(m)}(s) = (-1)^m \int_0^\delta t^{s-N-1} |A_t \cap \Omega| \, \mathrm{d}t = (-1)^m \zeta_{A,\Omega}(s). \tag{4.5.21}$$

We conclude that $\zeta_{A,\Omega}(s)$ and $z_0(s)$ possess the same meromorphic extensions. By using Theorem 2.3.18, we see that $z_0(s) = \zeta_1(s)$ can be meromorphically extended to $\{\mathrm{Re}\, s > D - \alpha\}$, and therefore the same holds for $\tilde{\zeta}_{A,\Omega}(s)$. The remaining claims follow from the fact that $\tilde{\zeta}_{A,\Omega}(s) = (-1)^m z_0^{(m)}(s)$. □

Next, we consider a class of Minkowski nonmeasurable RFDs with associated gauge functions. The following result is a partial generalization of Theorem 2.3.25, which corresponds to the case when $m = 0$.

Theorem 4.5.2 (Minkowski nonmeasurable RFDs). *Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$, such that there exist $D \le N$, a nonconstant periodic function $G : \mathbb{R} \to \mathbb{R}$ with minimal period $T > 0$, and a nonnegative integer m, satisfying*

$$|A_t \cap \Omega| = t^{N-D} (\log t^{-1})^m \left(G(\log t^{-1}) + O(t^\alpha) \right) \quad \text{as} \quad t \to 0^+. \tag{4.5.22}$$

Then $\dim_B(A, \Omega)$ exists and $\dim_B(A, \Omega) = D$, G is continuous, and

$$\mathscr{M}_*^D(A, \Omega, h) = \min G, \quad \mathscr{M}^{*D}(A, \Omega, h) = \max G,$$

where $h(t) := (\log t^{-1})^m$ for all $t \in (0, 1)$. Furthermore, the tube zeta function $\tilde{\zeta}_{A,\Omega}$ has for abscissa of convergence $D(\tilde{\zeta}_{A,\Omega}) = D$, and it possesses a (necessarily unique) meromorphic extension (at least) to the half-plane $\{\mathrm{Re}\, s > D - \alpha\}$; that is,

$$D_{\mathrm{mer}}(\tilde{\zeta}_{A,\Omega}) \le D - \alpha. \tag{4.5.23}$$

Moreover, all of its poles located in this half-plane are of order $m + 1$, and the set of poles $\mathscr{P}(\tilde{\zeta}_{A,\Omega})$ is contained in the vertical line $\{\mathrm{Re}\, s = D\}$. More precisely,

$$\begin{aligned} \mathscr{P}(\tilde{\zeta}_{A,\Omega}) &= \mathscr{P}_c(\tilde{\zeta}_{A,\Omega}) \\ &= \left\{ s_k = D + \frac{2\pi}{T} k \mathbb{i} \in \mathbb{C} : \hat{G}_0\left(\frac{k}{T}\right) \neq 0, \, k \in \mathbb{Z} \right\}, \end{aligned} \tag{4.5.24}$$

where $s_0 = D \in \mathscr{P}(\tilde{\zeta}_{A,\Omega})$ *and* $\hat{G}_0$ *is the Fourier transform of* G_0 (*as given by* (2.3.29)). *The poles come in complex conjugate pairs; that is, if* s_k *is a pole, then* s_{-k} *is a pole as well* (*reality principle, see Remark 2.3.28*).

In addition, if $\tilde{\zeta}_{A,\Omega}(s) = \sum_{j=-\infty}^{\infty} c_j^{(k)}(s-s_k)^j$ *is the Laurent expansion of the tube zeta function in a neighborhood of* $s = s_k$*, for a given* $k \in \mathbb{Z}$*, then*

$$\begin{aligned} c_j^{(k)} &= 0 \quad \textit{for} \quad j<0 \textit{ and } j \neq -m-1 \\ c_{-m-1}^{(k)} &= \frac{m!}{T}\hat{G}_0\left(\frac{k}{T}\right), \end{aligned} \tag{4.5.25}$$

where G_0 *is the restriction of* G *to the interval* $[0,T]$*, and* $\hat{G}_0$ *is given by* (2.3.29)*, as above. Also,*

$$|c_{-m-1}^{(k)}| \le \frac{m!}{T}\int_0^T G(\tau)\,\mathrm{d}\tau, \quad \lim_{k\to\infty} c_{-m-1}^{(k)} = 0. \tag{4.5.26}$$

In particular, for $k = 0$*, that is, for* $s_0 = D$*, we have*

$$\begin{aligned} c_{-m-1}^{(0)} &= \frac{m!}{T}\int_0^T G(\tau)\,\mathrm{d}\tau \\ m!\,\mathscr{M}_*^D(A,\Omega,h) &< c_{-m-1}^{(0)} < m!\,\mathscr{M}^{*D}(A,\Omega,h). \end{aligned} \tag{4.5.27}$$

If $m = 0$ (*i.e.,* $h(t) = 1$ *for all* $t \in (0,1)$)*, then* D *is a simple pole of* $\tilde{\zeta}_{A,\Omega}$ *and we have that*

$$\operatorname{res}(\tilde{\zeta}_{A,\Omega}, D) = \frac{1}{T}\int_0^T G(\tau)\,\mathrm{d}\tau = \widetilde{\mathscr{M}} \tag{4.5.28}$$

and

$$\mathscr{M}_*^D(A,\Omega) < \operatorname{res}(\tilde{\zeta}_{A,\Omega}, D) < \mathscr{M}^{*D}(A,\Omega), \tag{4.5.29}$$

where $\widetilde{\mathscr{M}} = \widetilde{\mathscr{M}}^D(A,\Omega)$ *denotes the average Minkowski content of* (A,Ω). (*See Remark 4.5.3 below.*)

Proof. For $m \in \mathbb{N}_0$, let us define z_m by

$$z_m(s) = \int_0^\delta t^{s-D-1}(\log t^{-1})^m G(\log t^{-1})\,\mathrm{d}t.$$

The function $z_0(s)$ is the exact counterpart of $\zeta_1(s)$ from the proof of Theorem 2.3.25, with $|A_t|$ replaced by $|A_t \cap \Omega|$ and where, much as in that proof, $\tilde{\zeta}_{A,\Omega} = \zeta_1 + \zeta_2$ and ζ_2 is an entire function. It is easy to see that $z_m(s) = (-1)^m z_0^{(m)}(s)$, therefore, the functions $z_m(s)$ and $z_0(s)$ have the same meromorphic extension, and the same sets of poles. This proves that $\tilde{\zeta}_{A,\Omega}(s)$ can be meromorphically extended from $\{\operatorname{Re} s > D\}$ to the half-plane $\{\operatorname{Re} s > D - \alpha\}$. The set of poles (complex dimensions of (A,Ω)) of the relative zeta function of (A,Ω), contained in this half-plane, is given by

$$\mathscr{P}(\tilde{\zeta}_{A,\Omega}) = \mathscr{P}(z_m) = \mathscr{P}(z_0)$$
$$= \left\{ s_k = D + \frac{2\pi}{T} k\mathbb{i} \in \mathbb{C} : \hat{G}_0\left(\frac{k}{T}\right) \neq 0,\ k \in \mathbb{Z} \right\}.$$

Each of these poles is simple. Furthermore, if

$$z_0(s) = \sum_{j=-1}^{\infty} a_j^{(k)} (s - s_k)^j$$

is the Laurent series of $z_0(s)$ in a neighborhood of $s = s_k$, then

$$z_0^{(m)}(s) = (-1)^m m!\, a_{-1}^{(k)} (s - s_k)^{-m-1} + \sum_{j=0}^{\infty} \frac{(m+j)!}{j!} a_{m+j}^{(k)} (s - s_k)^j.$$

Hence,

$$c_{-m-1}^{(k)} = m! a_{-1}^{(k)} = m! \frac{1}{T} \hat{G}_0\left(\frac{k}{T}\right),$$

where, in the last equality, we have used (2.3.33). The remaining claims are easily deduced from the corresponding ones in Theorem 2.3.25. □

Remark 4.5.3. In Equation (4.5.28), $\widetilde{\mathscr{M}} = \widetilde{\mathscr{M}}^D(A,\Omega)$, the *average Minkowski content* of (A,Ω), is defined as the multiplicative Cesàro average of $t^{-(N-D)}|A_t \cap \Omega|$:

$$\widetilde{\mathscr{M}}^D(A,\Omega) := \lim_{\tau \to +\infty} \frac{1}{\log \tau} \int_{1/\tau}^{1} \frac{|A_t \cap \Omega|}{t^{N-D}} \frac{\mathrm{d}t}{t}, \tag{4.5.30}$$

provided the limit exists in $[0,+\infty]$. See Section 2.4, Equation (4.5.9) and compare with [Lap-vFr3, Definition 8.29, Equation (8.55)].

Remark 2.3.19 also applies to Theorems 4.5.1 and 4.5.2. This means that in the statements of these theorems, we may have more general functions, which are $O(t^{\alpha)})$ (instead of $O(t^{\alpha})$) as $t \to 0^+$, like $t^{\alpha} \log(1/t)$.

Remark 4.5.4. In light of the functional equation (4.5.2) connecting $\zeta_{A,\Omega}$ and $\tilde{\zeta}_{A,\Omega}$, Theorems 4.5.1 and 4.5.2 also hold for relative *distance* zeta functions (instead of relative tube zeta functions), provided $D < N$, and in that case, all of the expressions for the residues and the Laurent coefficients are multiplied by $N - D$.

Example 4.5.5. (*Torus relative fractal drum*). Let Ω be an open torus in $\mathbb{R}^3$ defined by two radii r and R, where $0 < r < R < \infty$, and let $A := \partial\Omega$ be its topological boundary. In order to compute the tube zeta function of the *torus RFD* (A,Ω), we first compute its tube function. Let $\delta \in (0,r)$ be fixed. Using Cavalieri's principle, we obtain that

$$|A_t \cap \Omega|_3 = 2\pi R\left(r^2 - (r-t)^2\right) = 2\pi R(2rt - t^2), \tag{4.5.31}$$

where $t \in (0,\delta)$, from which it follows that

$$\tilde{\zeta}_{A,\Omega}(s) := \int_0^\delta t^{s-4}|A_t \cap \Omega|_3\,\mathrm{d}t = 2\pi R\Big(2r\frac{\delta^{s-2}}{s-2} - \frac{\delta^{s-1}}{s-1}\Big) \tag{4.5.32}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > 2$. The right-hand side defines a meromorphic function on the entire complex plane; so that, by the principle of analytic continuation, $\tilde{\zeta}_{A,\Omega}$ can be (uniquely) meromorphically extended to the whole of $\mathbb{C}$. In particular, we see that the multiset of complex dimensions of the torus RFD (A,Ω) is given by $\mathscr{P}(A,\Omega) = \{1,2\}$. Each of the complex dimensions 1 and 2 is simple. In particular, we have that

$$\dim_{PC}(A,\Omega) = \{2\} \quad \text{and} \quad \operatorname{res}(\tilde{\zeta}_{A,\Omega},2) = 4\pi Rr. \tag{4.5.33}$$

Also, $\overline{\dim}_B(A,\Omega) = D(\tilde{\zeta}_A) = 2$. From Equation (4.5.14) appearing in Theorem 4.5.1 below, we conclude that the 2-dimensional Minkowski content of the torus RFD (A,Ω) is given by

$$\mathscr{M}^2(A,\Omega) = 4\pi Rr. \tag{4.5.34}$$

Since $|A_t|_3 = 2\pi R\left((r+t)^2 - (r-t)^2\right)$, we can also easily compute the 'ordinary' tube zeta function $\tilde{\zeta}_A$ of the torus surface A in $\mathbb{R}^3$:

$$\tilde{\zeta}_A(s) = 8\pi Rr\frac{\delta^{s-2}}{s-2} \tag{4.5.35}$$

for all $s \in \mathbb{C}$. In particular, $\operatorname{res}(\tilde{\zeta}_A,2) = 8\pi Rr$. Using Equations (4.5.2) and (2.2.1), from (4.5.35) we obtain the corresponding distance zeta functions for all $s \in \mathbb{C}$:

$$\zeta_{A,\Omega}(s) = 2\pi R\Big(2r\frac{\delta^{s-2}}{s-2} - \frac{2\delta^{s-1}}{s-1}\Big), \quad \zeta_A(s) = 8\pi Rr\frac{\delta^{s-2}}{s-2}. \tag{4.5.36}$$

Also,

$$\mathscr{P}(\zeta_{A,\Omega}) = \mathscr{P}(\tilde{\zeta}_{A,\Omega}) = \{1,2\}$$

and

$$\mathscr{P}_c(\zeta_{A,\Omega}) = \mathscr{P}_c(\tilde{\zeta}_{A,\Omega}) = \{2\}$$

(with each pole 1 and 2 being simple) and

$$\overline{\dim}_B(A,\Omega) = D(\zeta_{A,\Omega}) = D(\tilde{\zeta}_{A,\Omega}) = 2.$$

Furthermore, we see that $\operatorname{res}(\zeta_{A,\Omega},2) = 4\pi Rr$ and $\operatorname{res}(\zeta_A,2) = 8\pi Rr$, in agreement with Equation (4.5.8), while

$$\mathscr{P}(\zeta_A) = \mathscr{P}(\tilde{\zeta}_A) = \mathscr{P}_c(\zeta_A) = \mathscr{P}_c(\tilde{\zeta}_A) = \{2\}.$$

One can easily extend the example of the 2-torus to any (smooth) closed, compact submanifold of $\mathbb{R}^N$ (and, in particular, of course, to the n-torus, with $n \geq 2$).

This can be done by using Federer's tube formula [Fed1] for sets of positive reach, which extends and unifies Weyl's tube formula [Wey3] for smooth compact submanifolds of $\mathbb{R}^N$ and Steiner's formula (obtained by Steiner and his successors [Stein]) for compact convex subsets of $\mathbb{R}^N$. The global form of Federer's tube formula expresses the volume of t-neighborhoods of a (compact) set of positive reach $A \subset \mathbb{R}^N$ as a polynomial of degree at most N in t, whose coefficients are (essentially) the so-called *Federer's curvatures* and which generalize Weyl's curvatures in [Wey3] (see [BergGos] for an exposition) and Steiner's curvatures in [Stein] (see [Schn2, Chapter 4] for a detailed exposition) in the case of compact submanifolds of $\mathbb{R}^N$ and compact convex sets, respectively.

Recall from [Fed1] that a closed subset A of $\mathbb{R}^N$ is said to be of *positive reach* if there exists $\delta_0 > 0$ such that every point $x \in \mathbb{R}^N$ within a distance less than δ_0 from A has a unique metric projection onto A; see [Fed1] and, e.g., [Schn2]. The *reach* of A, denoted by $\operatorname{reach}(A)$, is then defined as the supremum of all such positive numbers δ_0. Clearly, every closed convex subset of $\mathbb{R}^N$ is of infinite (and hence, positive) reach. Furthermore, if $A \subset \mathbb{R}^2$ is an arc of a circle of radius r, then the reach of A is equal to r.

In the present context, for a compact set of positive reach $A \subset \mathbb{R}^N$, it is easy to deduce from the tube formula in [Fed1] an explicit expression for $\tilde{\zeta}_A$.[37]

Theorem 4.5.6. *Let A be a (nonempty) compact set of positive reach in $\mathbb{R}^N$. Then, for any δ such that $0 < \delta < \operatorname{reach}(A)$, we have that*

$$\tilde{\zeta}_A(s) := \tilde{\zeta}_A(s;\delta) = \sum_{k=0}^{N} c_k \frac{\delta^{s-k}}{s-k}, \tag{4.5.37}$$

where $|A_t| = \sum_{k=0}^{N} c_k t^{N-k}$ for all $t \in (0,\delta)$ and the coefficients c_k are the (normalized) Federer curvatures. (From the functional equation (4.5.2) above, one then deduces at once a corresponding explicit expression for $\zeta_A(s) := \zeta_A(s;\delta)$.)

Hence, $\dim_B A$ *exists and*

$$D := D(\tilde{\zeta}_A) = D(\zeta_A) = \dim_B A = \max\{k \in \{0,1,\dots,N\} : c_k \neq 0\} \tag{4.5.38}$$

and[38]

$$\mathscr{P} := \mathscr{P}(\tilde{\zeta}_A) = \mathscr{P}(\zeta_A) \subseteq \{0,1,\dots,N\}. \tag{4.5.39}$$

In fact,

$$\mathscr{P} = \big\{k \in \{0,1,\dots,N\} : c_k \neq 0\big\} \subseteq \{k_0,\dots,D\}, \tag{4.5.40}$$

where $k_0 := \min\big\{k \in \{0,1,\dots,D\} : c_k \neq 0\big\}$. Furthermore, each of the complex dimensions of A is simple.

[37] Relative versions of Theorem 4.5.6 are also possible, but we will not consider them here.

[38] More precisely, the second equality in Equation (4.5.39) holds only if $D < N$.

Finally, if A is such that its affine hull is all of $\mathbb{R}^N$ (which is the case when the interior of A is nonempty and, in particular, if A is a convex body), then $D = N$, while if A is a (smooth) compact d-dimensional submanifold (with $0 \le d \le N$), then $D = d$.

For the 2-torus $A \subset \mathbb{R}^3$, we have $N = 3$, $D = 2$ (since the Euler characteristic of A is equal to zero), $c_2 \neq 0$,[39] $c_1 = 0$, and hence, $c_0 = 0$, $k_0 = 2$ and $\mathscr{P} = \{2\}$, as was also found in the last displayed equation of Example 4.5.5 via a direct computation.

We note that much more general tube formulas, called (as in [Lap-vFr2–3] and [LapPeWi1–2]) "fractal tube formulas", are obtained in [LapRaŽu5] (as announced in [LapRaŽu4]), as well as in Chapter 5 below, for arbitrary bounded sets (and even more generally, RFDs) in $\mathbb{R}^N$, under mild growth assumptions on the associated fractal zeta functions.

4.5.2 Precise Meromorphic Extensions of Zeta Functions of Countable Unions of Relative Fractal Drums

In Theorem 4.5.8, we construct a class of RFDs in $\mathbb{R}$, with prescribed values of the abscissa of meromorphic continuation of the corresponding zeta functions. This will enable us to construct a class of bounded sets A, with prescribed values of the abscissa of meromorphic continuation of the associated distance or tube zeta functions; see Theorem 4.5.20. The construction makes use of the generalized Cantor sets $C^{(a)}$ introduced in Example 2.2.6.

Definition 4.5.7. Let (A_j, Ω_j), $j \ge 1$, be a given sequence of RFDs in $\mathbb{R}^N$, where $(\Omega_j)_{j\ge 1}$ is a disjoint sequence of open subsets of $\mathbb{R}^N$. We define the *union of the relative fractal drums*

$$(A, \Omega) = \bigcup_{j=1}^{\infty} (A_j, \Omega_j),$$

by $A := \cup_{j=1}^{\infty} A_j$ and $\Omega := \cup_{j=1}^{\infty} \Omega_j$, assuming that there exists $\delta > 0$ such that $\Omega \subset A_\delta$ and $|\Omega| < \infty$.

Theorem 4.5.8. *Let $D \in (0,1)$ and $\alpha \in (0, D)$ be prescribed. Let (A, Ω) be a relative fractal drum, defined by $(A, \Omega) = \cup_{j\ge 1}(A_j, \Omega_j)$, where $(\Omega_j)_{j\ge 1}$ is a family of disjoint open intervals in $\mathbb{R}$, $|\Omega_j| = 2^{-j}$, $A^{(j)} = 2^{-j} C^{(a_j)} + \inf \Omega_j$, and $C^{(a_j)}$ are generalized Cantor sets described in Example 2.2.6, with $a_j \in (0, 1/2)$, $j \ge 1$. Assume that $a_1 = 2^{-1/D}$, and let $(a_j)_{j\ge 2}$ be an increasing sequence of positive numbers converging to $2^{-1/(D-\alpha)}$ as $j \to \infty$.*

[39] Note that c_2 is just proportional to the area of the 2-torus, with the proportionality constant being a standard positive constant.

Then, for the relative tube zeta function $\tilde{\zeta}_{A,\Omega}$ *of* (A,Ω), *we have*:

$$D(\tilde{\zeta}_{A,\Omega}) = D, \quad D_{\mathrm{mer}}(\tilde{\zeta}_{A,\Omega}) = D - \alpha. \tag{4.5.41}$$

(*See Definition 2.1.53.*) *Analogous result holds for the distance zeta function*:

$$D(\zeta_{A,\Omega}) = D, \quad D_{\mathrm{mer}}(\zeta_{A,\Omega}) = D - \alpha. \tag{4.5.42}$$

The set of poles of these zeta functions, contained in $\{\mathrm{Re}\, s > D - \alpha\}$, *coincides with the set* $\dim_{PC}(A,\Omega) = \mathscr{P}_c(\zeta_{A,\Omega})$ *of principal complex dimensions of the relative fractal drum* (A,Ω):

$$\dim_{PC}(A,\Omega) = D + \frac{2\pi}{\log(1/a_1)}\mathbb{i}\mathbb{Z}. \tag{4.5.43}$$

In particular, the oscillatory period of the RFD (A,Ω) *is given by* $\mathbf{p} = 2\pi D/\log 2$.[40]

In order to prove Theorem 4.5.8, we shall need the following technical lemma.

Lemma 4.5.9. *Let* $(A_j,\Omega_j)_{j\geq 1}$ *be a sequence of RFDs in* $\mathbb{R}^N$ *such that the family of open sets* $(\Omega_j)_{j\geq 1}$ *is disjoint. Consider their union* $(A,\Omega) = \cup_{j=1}^{\infty}(A_j,\Omega_j)$, *as introduced in Definition 4.5.7, and assume that* $|\Omega| < \infty$. *If*

$$\partial\Omega_j \subseteq A_j \quad \textit{for all} \quad j \in \mathbb{N}, \tag{4.5.44}$$

then

$$|A_t \cap \Omega| = \sum_{j=1}^{\infty} |(A_j)_t \cap \Omega_j|. \tag{4.5.45}$$

In particular,

$$\tilde{\zeta}_{A,\Omega}(s) = \sum_{j=1}^{\infty} \tilde{\zeta}_{A_j,\Omega_j}(s) \tag{4.5.46}$$

for all $s \in \mathbb{C}$ *such that* $\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)$.

Proof. For any $j \neq k$ and $a \in A_j$, since $a \notin \Omega_j$, we obviously have $B_t(a) \cap \Omega_k \subset (\partial A_k)_t \cap \Omega_k$. Taking the union over all $a \in A_j$, we obtain

$$(A_j)_t \cap \Omega_k \subseteq (\partial\Omega_k)_t \cap \Omega_k.$$

Using (4.5.44), we see that $(A_j)_t \cap \Omega_k \subseteq (A_k)_t \cap \Omega_k$, and hence,

$$\begin{aligned} A_t \cap \Omega &= \Big(\bigcup_{j=1}^{\infty}(A_j)_t\Big) \cap \Big(\bigcup_{k=1}^{\infty}\Omega_k\Big) \\ &= \bigcup_{j,k=1}^{\infty}\big((A_j)_t) \cap \Omega_k\big) = \bigcup_{k=1}^{\infty}\big(A_k)_t \cap (\Omega_k)\big). \end{aligned}$$

[40] Compare with Equation (2.2.17) on page 117. It is interesting to note that $\mathbf{p} \to 0^+$ as $D \to 0^+$; see Figure 4.16.

Since the family $(\Omega_k)_{k\geq 1}$ is disjoint, this implies (4.5.45). From this we conclude that for any positive real number s such that $s > \overline{\dim}_B(A,\Omega)$,

$$\tilde{\zeta}_{A,\Omega}(s) = \int_0^\delta t^{s-N-1}|A_t \cap \Omega|\,\mathrm{d}t = \int_0^\delta t^{s-N-1}\Big(\sum_{k=1}^\infty |(A_k)_t \cap \Omega_k|\Big)\,\mathrm{d}t$$
$$= \sum_{k=1}^\infty \int_0^\delta t^{s-N-1}|(A_k)_t \cap \Omega_k|\,\mathrm{d}t = \sum_{k=1}^\infty \tilde{\zeta}_{A_k,\Omega_k}(s).$$

Hence, (4.5.46) holds for s real such that $s > \overline{\dim}_B(A,\Omega)$. But now, using the principle of analytic continuation, we can extend this identity to the open half-plane $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$, as desired. ☐

We shall also need the following technical lemma.

Lemma 4.5.10. *Let $D \in (0,1)$ and $\alpha \in (0,D)$ be given. Assume that $(T_j)_{j\geq 1}$ is a decreasing sequence of positive real numbers, converging to a limit which is less than $\pi/(D-\alpha)$. Let $(D_j)_{j\geq 1}$ be a bounded sequence of positive real numbers, and $(G_j)_{j\geq 1}$ a bounded sequence of periodic functions (with G_j being T_j-periodic, for each $j \geq 1$).*

Then, the sequence of functions $(z_j)_{j\geq 1}$, defined by (see (2.3.42) *and* (2.3.43)*)*

$$z_j(s) = \frac{\mathrm{e}^{T_j(s-D_j)}}{\mathrm{e}^{T_j(s-D_j)}-1}\int_{\log\delta^{-1}}^{\log\delta^{-1}+T_j} \mathrm{e}^{-\tau(s-D_j)}G_j(\tau)\,\mathrm{d}\tau \quad \text{for} \quad j \geq 1, \tag{4.5.47}$$

is locally uniformly bounded on $\{0 < \mathrm{Re}\, s < D-\alpha+\varepsilon\} \setminus \overline{\mathscr{S}}$, where

$$\mathscr{S} := \bigcup_{j=1}^\infty \Big(D_j + \frac{2\pi}{T_j}\mathrm{i}\mathbb{Z}\Big) \tag{4.5.48}$$

and ε is a sufficiently small positive real number; that is, for each s_0 in the connected open set $\{0 < \mathrm{Re}\, s < D-\alpha+\varepsilon\} \setminus \overline{\mathscr{S}}$, there exists $M > 0$ and a neighborhood $N = N(s_0)$ of s_0 such that $|z_j(s)| \leq M$ for all $j \in \mathbb{N}$ and $s \in N(s_0)$.[41]

Proof. There exists $k_0 \in \mathbb{N}$ such that $T_j < \pi/(D-\alpha)$ for all $j \geq k_0$. Therefore, we can assume without loss of generality that $k_0 = 1$; that is, $T_j < \pi/(D-\alpha)$ for all $j \geq 1$. The sequences of real numbers $(T_j)_{j\geq 1}$ and $(D_j)_{j\geq 1}$ are bounded, as well as the sequence of functions $(G_j)_{j\geq 1}$. In light of (4.5.47), it suffices to prove that for any fixed complex number s_0 there exist a neighborhood $N(s_0)$ of s_0, and a positive number c, such that

$$|\mathrm{e}^{T_j(s-D_j)}-1| \geq c, \quad \text{for all } j \in \mathbb{N} \text{ and } s \in N(s_0). \tag{4.5.49}$$

Let us first fix $s_0 \in \{0 < \mathrm{Re}\, s < D-\alpha+\varepsilon\} \setminus \mathscr{S}$. Furthermore, let us choose $s_{jk} = s_{jk}(s_0) \in \mathscr{P}_j$ which is closest to s_0. Let $R := d(s_0,S) = d(s_0,s_{jk})$. Then

[41] See also Figure 4.16 and the discussion surrounding Equations (4.5.58)–(4.5.60) below.

$$\begin{aligned}|\mathrm{e}^{T_j(s_0-D_j)}-1| &= |\mathrm{e}^{T_j(s_0-D_j)}-\mathrm{e}^{T_j(s_{jk}-D_j)}|\\ &= |\mathrm{e}^{T_j(s_{jk}-D_j)}|\,|\mathrm{e}^{T_j(s_0-s_{jk})}-1|\\ &= |\mathrm{e}^{T_j(s_0-s_{jk})}-1|.\end{aligned}$$

Let us write $s_0-s_{jk}=R\mathrm{e}^{\mathrm{i}\varphi}$, and

$$\begin{aligned}w_j &:= \mathrm{e}^{T_j(s_0-s_{jk})}=\exp(T_jR\mathrm{e}^{\mathrm{i}\varphi})\\ &= \mathrm{e}^{T_jR\cos\varphi}\mathrm{e}^{\mathrm{i}T_jR\sin\varphi}=:r_j\mathrm{e}^{\mathrm{i}\psi_j},\end{aligned}$$

where we have set

$$r_j=\mathrm{e}^{T_jR\cos\varphi}\qquad \psi_j=T_jR\sin\varphi.$$

We assume without loss of generality that $R<D-\alpha+\varepsilon$, since it suffices to consider $0<\operatorname{Re}s_0<D-\alpha+\varepsilon$. We would like to estimate the value of $|w_j-1|$ from below. Let us fix $\varphi_0\in(0,\pi/2)$, and consider the following two cases:

Case 1: Assume that

$$\varphi\in(-\pi,\pi]\setminus\{(\frac{\pi}{2}-\varphi_0,\frac{\pi}{2}+\varphi_0)\cup(-\frac{\pi}{2}-\varphi_0,-\frac{\pi}{2}+\varphi_0)\}.$$

We consider the following two subcases:

(a) Assume that $\varphi\in[-\frac{\pi}{2}+\varphi_0,\frac{\pi}{2}-\varphi_0]$. Then

$$r_j=\mathrm{e}^{T_jR\cos\varphi}\geq\mathrm{e}^{T_jR\cos(\frac{\pi}{2}-\varphi_0)}=\mathrm{e}^{T_jR\sin\varphi_0}>1.$$

Hence,

$$|w_j-1|\geq|w_j|-1=r_j-1=\mathrm{e}^{T_jR\sin\varphi_0}-1\geq\mathrm{e}^{T_0R\sin\varphi_0}-1>0. \tag{4.5.50}$$

(b) Assume that $\varphi\in(-\pi,-\frac{\pi}{2}+\varphi_0]\cup[\frac{\pi}{2}+\varphi_0,\pi]$. Then

$$r_j=\mathrm{e}^{T_jR\cos\varphi}\leq\mathrm{e}^{T_jR\cos(\frac{\pi}{2}+\varphi_0)}=\mathrm{e}^{-T_jR\sin\varphi_0}\leq\mathrm{e}^{-T_0R\sin\varphi_0}<1.$$

Hence,

$$|w_j-1|\geq1-|w_j|=1-r_j\geq1-\mathrm{e}^{-T_0R\sin\varphi_0}>0. \tag{4.5.51}$$

Case 2: Assume that $\varphi\in(\frac{\pi}{2}-\varphi_0,\frac{\pi}{2}+\varphi_0)\cup(-\frac{\pi}{2}-\varphi_0,-\frac{\pi}{2}+\varphi_0)$. Then we have

$$\psi_j=T_jR\sin\varphi\geq T_jR\sin(\frac{\pi}{2}-\varphi_0)=T_jR\cos\varphi_0\geq T_0R\cos\varphi_0>0.$$

Since $T_0(D-\alpha+\varepsilon)<\pi$ for $\varepsilon>0$ small enough, then for any j large enough we have

$$0<\psi_j=T_jR\sin\varphi\leq T_jR\leq T_0(D-\alpha+\varepsilon)<\pi;$$

that is,

$$\psi_j \in [T_0 R\cos\varphi_0, T_0(D-\alpha+\varepsilon)] \subset (0,\pi),$$

and therefore,

$$\sin\psi_j \geq \min\{\sin(T_0 R\cos\varphi_0), \sin(T_0(D-\alpha+\varepsilon))\} = \sin(T_0 R\cos\varphi_0) > 0,$$

since $R < D-\alpha+\varepsilon$, for $\varepsilon > 0$ small enough.

If we consider a triangle with vertices at the points 0, 1 and w_j with respect to the (r_j,ψ_j)-polar system (the origin 0 of which is the point s_{jk}), it is clear that the length of the side of the triangle joining 1 with w_j is not smaller than the length of the height of the triangle drawn from 1 to the opposite side connecting 0 and w_j; that is,

$$|w_j - 1| \geq \sin\psi_j > 0.$$

Therefore,

$$|w_j - 1| \geq \sin(T_0 R\cos\varphi_0) > 0. \tag{4.5.52}$$

Making use of (4.5.50), (4.5.51) and (4.5.52), we obtain that

$$|\mathrm{e}^{T_j(s_0-D_j)} - 1| \geq g(s_0),$$

where

$$g(s_0) = \min\{\mathrm{e}^{T_0 d(s_0,S)\sin\varphi_0} - 1, 1 - \mathrm{e}^{-T_0 d(s_0,S)\sin\varphi_0}, \sin(T_0 d(s_0,S)\cos\varphi_0)\}.$$

If we take s in a sufficiently small neighborhood $N(s_0)$ of s_0, such that $d(s,S) \geq d_0 > 0$ for some positive constant d_0, then the same type of inequality holds for all $s \in N(s_0)$:

$$|\mathrm{e}^{T_j(s-D_j)} - 1| \geq g(s),$$

where $\varphi_0 \in (0,\pi/2)$ is a fixed angle. The desired constant c is obtained as the infimum of the right-hand side over $s \in N(s_0)$:

$$|\mathrm{e}^{T_j(s-D_j)} - 1| \geq c := \inf_{s\in N(s_0)} g(s) \quad \text{for all } j \in \mathbb{N} \text{ and } s \in N(s_0). \tag{4.5.53}$$

More explicitly, if we let $d_0 := d(N(s_0),S) = \inf_{s\in N(s_0)} d(s,S) > 0$, then we may take

$$c := \min\{\mathrm{e}^{T_0 d_0\sin\varphi_0} - 1, 1 - \mathrm{e}^{-T_0 d_0\sin\varphi_0}, \sin(T_0 d_0\cos\varphi_0)\}. \tag{4.5.54}$$

This concludes the proof of Lemma 4.5.10. □

Proof of Theorem 4.5.8. Note that $|\Omega| = \sum_{j=1}^{\infty} 2^{-j} < \infty$, and $\Omega \subset A_\delta$ for any $\delta > 1/2$. The first equality in (4.5.41) follows from Theorem 4.1.7.

In order to prove the second equality in (4.5.41), we first find a periodic function $G(\tau)$ and $f(t) = O(t^\alpha)$ as $t \to 0^+$, such that

$$|A_t \cap \Omega| = t^{1-D}\Big(G\Big(\log\frac{1}{t}\Big) + f(t)\Big).$$

Since $(A_j)_t \subset \overline{\Omega}_j$, where $(A_j)_t$ denotes the t-neighborhood of A_j, we have

$$|(A_j)_t \cap \Omega_j| = 2^{-j} t^{1-D_j}\Big(G_j\Big(\log\frac{1}{t}\Big) - 2t^{D_j}\Big). \tag{4.5.55}$$

We note that this identity (called a *fractal tube formula* in [Lap-vFr3, Chapter 8]) is obtained in the same manner as in [Lap-vFr3, Equation (1.11)]; therefore, we will not repeat its proof. Here, $D_j = \dim_B(A_j, \Omega_j) = \log_{1/a_j} 2$, each function G_j is T_j-periodic, where $T_j := \log(1/a_j)$, and $G_j(\tau) \in [\mathscr{M}_*^{D_j}(A_j), \mathscr{M}^{*D_j}(A_j)]$ for every $\tau \in [0, T_j]$ (or equivalently, for all $\tau \in \mathbb{R}$), and the values of the Minkowski contents are given in (2.2.12). Let $D_1 := D$, and note that the sequence $(D_j)_{j\ge 2}$ is monotonically increasing in $(0, D-\alpha)$, and converging to $D-\alpha$.

Using Lemma 4.5.9 and (4.5.55), we obtain

$$\begin{aligned} |A_t \cap \Omega| &= \sum_{j=1}^{\infty} 2^{-j} t^{1-D_j}\Big(G_j\Big(\log\frac{1}{t}\Big) - 2t^{D_j}\Big) \\ &= t^{1-D}\Big(2^{-1} G_1\Big(\log\frac{1}{t}\Big) + f(t)\Big), \end{aligned} \tag{4.5.56}$$

where

$$f(t) := -t^D + \sum_{j=2}^{\infty} 2^{-j} t^{D-D_j}\Big(G_j\Big(\log\frac{1}{t}\Big) - 2t^{D_j}\Big).$$

Since $D - D_j > \alpha$ and $t < 1$, we have

$$|f(t)| \le t^D + \sum_{j\ge 2} 2^{-j} t^{\alpha}(M+2) = (t^{D-\alpha} + M + 2)t^{\alpha} \le (M+3)t^{\alpha},$$

where for every $\tau \in [0, T_j]$ (i.e., for every $\tau \in \mathbb{R}$),

$$\begin{aligned} 0 < G_j(\tau) \le M &:= \sup_{j\ge 2} \mathscr{M}^{*D_j}(A^{(j)}) = \sup_{j\ge 2} 2(1-a_j)\Big(\frac{1}{2} - a_j\Big)^{D_j - 1} \\ &< \ 2(1-a_2)\Big(\frac{1}{2} - 2^{-1/(D-\alpha)}\Big)^{D_2 - 1}, \end{aligned}$$

since both $(a_j)_{j\ge 2}$ and $(D_j)_{j\ge 2}$ are increasing sequences; see (2.2.12). Therefore, $f(t) = O(t^{\alpha})$ as $t \to 0^+$, and we conclude from Theorem 4.5.2 that $D_{\rm mer}(\tilde{\zeta}_{A,\Omega}) \le D - \alpha$.

To show the equality, it suffices to prove that $s = D - \alpha$ is a singularity which is not a pole of a meromorphic extension of $\tilde{\zeta}_{A,\Omega}$. More precisely, we show that $D - \alpha$ is a nonisolated singularity of a meromorphic extension of $\tilde{\zeta}_{A,\Omega}$, to which a sequence of distinct poles $(D_j)_{j\geq 2}$ converges from the left. Using the first equality in (4.5.56), we obtain the following identity valid on $\{\mathrm{Re}\, s > D\}$:

$$\begin{aligned}\tilde{\zeta}_{A,\Omega}(s) &= \int_0^\delta t^{s-2}|A_t\cap\Omega|\,\mathrm{d}t\\ &= \sum_{j\geq 1} 2^{-j}\int_0^\delta t^{s-D_j-1}G_j\big(\log t^{-1}\big)\,\mathrm{d}t - 2\sum_{j\geq 1}2^{-j}\frac{\delta^s}{s}\\ &= \sum_{j\geq 1}2^{-s}z_j(s) - 2\frac{\delta^s}{s}.\end{aligned} \tag{4.5.57}$$

The functions $z_j(s)$ have meromorphic extensions to the entire complex plane; see the proof of Theorem 4.5.2. Furthermore, since

$$T_0 := \lim_{j\to\infty} T_j = \lim_{j\to\infty}\log(1/a_j) = \frac{\log 2}{D-\alpha} < \frac{\pi}{D-\alpha},$$

Lemma 4.5.10 shows us that the last series appearing in (4.5.57) converges to a function which is holomorphic on the connected open set

$$\{0<\mathrm{Re}\, s<D-\alpha+\varepsilon\}\setminus\mathscr{S}_1,$$

for arbitrarily small $\varepsilon>0$, where $\mathscr{S}_1$ is the set of singularities of $\tilde{\zeta}_{A,\Omega}(s)$ contained in the open right half-plane $\{\mathrm{Re}\, s>0\}$. More specifically, $\mathscr{S}_1=\overline{\mathscr{S}}$ (the closure of $\mathscr{S}$ in $\mathbb{C}$) is the closed subset of $\mathbb{C}$ given by

$$\mathscr{S}_1 = \mathscr{S}\cup\mathscr{A}, \tag{4.5.58}$$

where

$$\mathscr{S} := \bigcup_{j=2}^{\infty}\Big(D_j + \frac{2\pi}{T_j}\mathbb{i}\mathbb{Z}\Big) \tag{4.5.59}$$

and

$$\mathscr{A} := D-\alpha+\mathbf{p}\mathbb{i}\mathbb{Z}. \tag{4.5.60}$$

Here, $\mathscr{S}$ is the set of poles of $\tilde{\zeta}_{A,\Omega}(s)$ in $\{0<\mathrm{Re}\, s<D-\alpha+\varepsilon\}$,[42] $\mathscr{A}$ is the set of nonisolated singular points (the accumulation points of $\mathscr{S}$) and $\mathbf{p}:=2\pi/T_0$. Therefore, the function $\tilde{\zeta}_{A,\Omega}(s)$, defined by the last expression in (4.5.57), possesses a holomorphic extension to $G=\{0<\mathrm{Re}\, s<D-\alpha+\varepsilon\}\setminus\mathscr{S}_1$ (note that, as was stated earlier, G is a domain, that is, an open and connected subset of $\mathbb{C}$). In particular, $D-\alpha$ is a singularity of $\tilde{\zeta}_{A,\Omega}(s)$ which is not a pole. This proves that

$$D_{\mathrm{mer}}(\tilde{\zeta}_{A,\Omega}) = D-\alpha.$$

The analogous claims (made in the statement of the theorem) for relative distance zeta functions follow from (4.5.2). ☐

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[42] The set $\mathscr{S} := \cup_{j\geq 2}(D_j+\frac{2\pi}{T_j}\mathbb{i}\mathbb{Z})$ in Equation (4.5.59) corresponds to the set $\mathscr{S}$ in Equation (4.5.48) of Lemma 4.5.10.

In connection with Equations (4.5.58)–(4.5.60), we point out that in (4.5.58), if we let $\mathscr{P}_j := D_j + \frac{2\pi}{T_j}\mathbb{i}\mathbb{Z}$ for each $j \geq 1$, then we have that $\mathscr{P}_j \to \mathscr{A}$ in the Hausdorff metric, as $j \to \infty$; see Figure 4.16. Note that the sequence $(T_j)_{j\geq 2}$ is decreasing, since a_j is increasing, and hence, the sequence $\mathbf{p}_j := \frac{2\pi}{T_j}$ of *oscillatory quasiperiods* of (A,Ω) is increasing. Also,

$$\mathbf{p}_j \to \mathbf{p} = \frac{2\pi}{T_0} \quad \text{as} \quad j \to \infty,$$

where $T_0 := \frac{\log 2}{D-\alpha}$. It is also interesting to note that, although $\mathrm{Mer}(\tilde{\zeta}_{A,\Omega}) = \{\mathrm{Re}\, s > D-\alpha\}$, the tube zeta function $\tilde{\zeta}_{A,\Omega}$ is meromorphic on $\{\mathrm{Re}\, s > 0\} \setminus \mathscr{A}$. Here, the set $\mathrm{Mer}\, \tilde{\zeta}_{A,\Omega}$ is the half-plane of meromorphic continuation introduced in Definition 2.1.53.

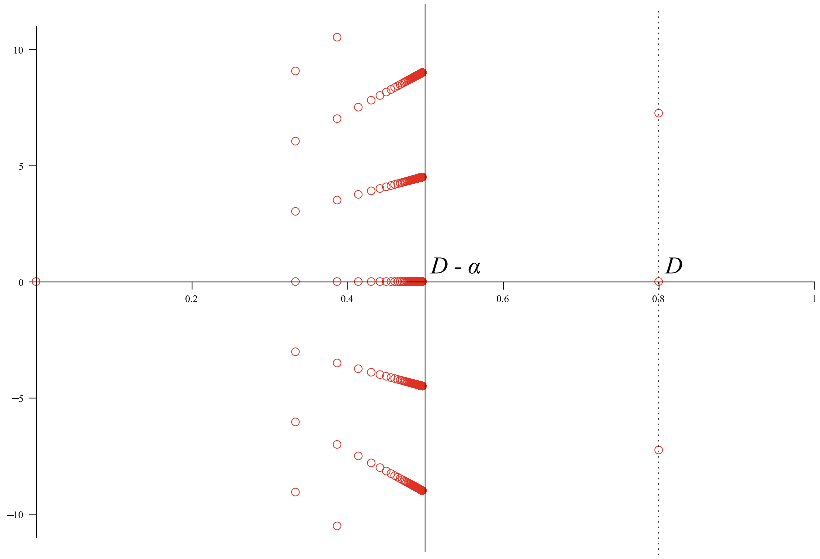

Fig. 4.16 An interesting set of complex dimensions: a sketch of the set $\mathscr{S} := \cup_{j\geq 2}(D_j + \frac{2\pi}{T_j}\mathbb{i}\mathbb{Z})$ of the poles of the tube zeta function (or "complex dimensions") of the relative fractal drum (A,Ω) from Theorem 4.5.8, with parameters $D = 4/5$, $\alpha = 3/10$ and the sequence $(a_j)_{j\geq 2}$ defined as $a_j = j/(4(j+1))$ for $j \geq 2$. Here, $D(\tilde{\zeta}_{A,\Omega}) = 4/5$, $D_{\mathrm{mer}}(\tilde{\zeta}_{A,\Omega}) = D - \alpha = 1/2$ and $\mathscr{A} = 2^{-1} + 4\pi(\log 2)\mathbb{i}\mathbb{Z}$ is the set of nonisolated singularities of $\tilde{\zeta}_{A,\Omega}$. (See Equations (4.5.58)–(4.5.60) and the discussion surrounding it.) It is easy to check that the set $\mathscr{S}$ appearing in Equations (4.5.58) and (4.5.59) of the proof of Theorem 4.5.8 is contained in a union of countably many rays emanating from the origin. The dotted vertical line is the critical line $\{\mathrm{Re}\, s = D\}$ of $\tilde{\zeta}_{A,\Omega}$, and to the left of it, the solid vertical line $\{\mathrm{Re}\, s = D - \alpha\}$ is the meromorphy critical line of $\tilde{\zeta}_{A,\Omega}$. It is worth pointing out that in the light of some of the results obtained in Chapters 4 and 5, we will suggest to extend the notion of "complex dimensions" from poles to nonremovable singularities of the associated fractal zeta function (here, $\tilde{\zeta}_{A,\Omega}$).

4.5.3 Precise Meromorphic Extensions of Zeta Functions of Countable Unions of Fractal Strings

In the sequel, an important role is played by the notion of countable union of a sequence of fractal strings, which we now introduce. It extends Definition 3.1.19, in which we have defined the union of two fractal strings.

Definition 4.5.11. Let $\mathscr{L}_j = (\ell_{jk})_{k\geq 1}$, $j \geq 1$, be a sequence of fractal strings in $\mathbb{R}$. The (disjoint) *union of fractal strings*, denoted by

$$\mathscr{L} = \bigsqcup_{j=1}^{\infty} \mathscr{L}_j, \tag{4.5.61}$$

is a new fractal string, defined as the multiset consisting of all $l \in \cup_{j=1}^{\infty} \mathscr{L}_j$, with the multiplicity of l equal to the sum of its multiplicities in all $\mathscr{L}_j$, $j \in \mathbb{N}$. Here, we assume that each $l \in \mathscr{L}$ belongs to at most finitely many fractal strings $\mathscr{L}_j$, and that $\mathscr{L}$ is a sequence of positive numbers converging to zero. Without these assumptions, the union of fractal strings is not well defined. Furthermore, we say that *the union of fractal strings is disjoint*, if for any two indices $j, j' \in \mathbb{N}$, the assumption that $j < j'$ implies that $\mathscr{L}_j \cap \mathscr{L}_{j'} = \emptyset$, where $\mathscr{L}_j$ and $\mathscr{L}_{j'}$ are viewed as ordinary sets.

The following lemma provides a simple construction of well defined countable unions of fractal strings.

Lemma 4.5.12. *Let $\mathscr{L}_j = (\ell_{jk})_{k\geq 1}$, $j \geq 1$, be a sequence of bounded fractal strings. If the sequence of the first elements of the fractal strings converges to zero (that is, if $\ell_{j1} \to 0^+$ as $j \to \infty$), then $\mathscr{L} := \sqcup_{j=1}^{\infty} \mathscr{L}_j$ is a well-defined fractal string.*

Proof. To show that any given element $\ell = \ell_{jk} \in \mathscr{L}$ is of finite multiplicity, it suffices to take $j_0 \in \mathbb{N}$ large enough, $j_0 = j_0(j,k)$, such that $\ell_{jk} > \ell_{j_0 1}$ (this is possible since $\ell_{j_0 1} \to 0^+$ as $j_0 \to \infty$). Then we have

$$\ell_{jk} \in \mathscr{L}_1 \sqcup \cdots \sqcup \mathscr{L}_{j_0-1}, \quad \ell_{jk} \notin \bigsqcup_{n=j_0}^{\infty} \mathscr{L}_n, \tag{4.5.62}$$

and hence, the multiplicity of ℓ_{jk} in $\mathscr{L}$ is equal to the sum of the multiplicities of this element in finitely many fractal strings, namely, $\mathscr{L}_1, \ldots, \mathscr{L}_{j_0-1}$.

We now show that $\mathscr{L}$ can be ordered as a nonincreasing sequence of positive real numbers $(\ell_m)_{m\geq 1}$, converging to zero. To see this, consider the following sequence of sets

$$\triangle\mathscr{L}_j := \{\ell_{j'k'} : \ell_{j+1,1} \leq \ell_{j'k'} < \ell_{j1}\}. \tag{4.5.63}$$

Here, we assume without loss of generality that the sequence $(\ell_{j1})_{j\geq 1}$ is nonincreasing. Therefore, the sets $\triangle\mathscr{L}_j$ are finite and pairwise disjoint. Furthermore, since ℓ_j converges to zero as $j \to \infty$, we have that

$$\mathscr{L} = \bigsqcup_{j=1}^{\infty} \triangle\mathscr{L}_j. \tag{4.5.64}$$

Here, the union of finite multisets is defined similarly to the union of fractal strings in Definition 4.5.11. Also note that $\min \triangle\mathscr{L}_j = \ell_{j+1,1} > \max \triangle\mathscr{L}_{j+1}$. The desired sequence $\mathscr{L} = (\ell_m)_{m\geq 1}$ is then obtained so that we first order $\triangle\mathscr{L}_1$ as a nonincreasing finite sequence, then continue with $\triangle\mathscr{L}_2$, and so on. ☐

Lemma 4.5.13. *Let $\mathscr{L}_j$, $j \geq 1$, be a sequence of fractal drums associated with (generalized) Cantor RFDs (A_j, Ω_j) in $\mathbb{R}$, where $(\Omega_j)_{\geq 1}$ is a disjoint family of unit intervals in $\mathbb{R}$, $|\Omega_j| = 2^{-j}$, $A_j = 2^{-j}C^{(a_j)} + \inf \Omega_j$, and $a_j \in (0,1/2)$ for each $j \geq 1$. Then $\mathscr{L} := \sqcup_{j=1}^{\infty}\mathscr{L}_j$ is a well-defined fractal string.*

Proof. Recall that $\mathscr{L}_j = (\ell_{jk})_{k\geq 1}$ is defined by $\ell_{jk} = |I_{jk}|$, where $(I_{jk})_{k\geq 1}$ is the family of connected components (open intervals) of $\overline{\Omega}_j \setminus A_j = \cup_{k\geq 1} I_{jk}$. We have

$$\ell_{j1} = 2^{-j}(1-2a_j) < 2^{-j}, \tag{4.5.65}$$

and hence, $\ell_{j1} \to 0^+$ as $j \to \infty$. The claim now follows from Lemma 4.5.12. ☐

In the following lemma, we construct a disjoint union of fractal strings, in the sense of Definition 4.5.11. It admits many variations, which we do not discuss here. By $(p_j)_{j\geq 1}$ we denote the usual sequence of prime numbers: $(2,3,5,7,11,\dots)$. We construct a disjoint sequence of fractal drums $\mathscr{L}_j = (\ell_{jk})_{k\geq 1}$, $j \in \mathbb{N}$, associated with generalized Cantor sets $C^{(a_j)}$ (see Example 2.2.6), with a suitable choice of the numbers $a_j \in (0,1/2)$.

Lemma 4.5.14. *Let $\mathscr{L}_j = (\ell_{jk})_{k\geq 1}$, $j \geq 2$, be a sequence of (scaled) Cantor strings, generated by relative fractal drums (A_j, Ω_j), $j \geq 2$, where $(\Omega_j)_{j\geq 1}$ is a disjoint family of intervals in $\mathbb{R}$, $A_j = 2^{-j}C^{(a_j)} + \inf \Omega_j$, and Ω_j is an open interval such that $|\Omega_j| = 2^{-j}$ for each $j \geq 2$. Assume that*

$$a_j = \frac{n_j}{p_j} \quad \textit{for} \quad j \geq 2, \tag{4.5.66}$$

where p_j is the j-th prime number,[43] *and $n_j \in \mathbb{N}$ is such that $n_j < \frac{1}{2}p_j$. Then the fractal string $\mathscr{L} := \sqcup_{j=2}^{\infty}\mathscr{L}_j$ is well defined, and moreover, the union of fractal strings is disjoint. In other words, for every $j,k \geq 1$, each value $\ell_{jk} \in \mathscr{L}$ occurs with the multiplicity 2^{k-1} in $\mathscr{L}$, the same multiplicity as in $\mathscr{L}_j$.*

Proof. From the construction of the Cantor string $\mathscr{L}_j$, we know that

$$\ell_{jk} = 2^{-j}a_j^{k-1}(1-2a_j). \tag{4.5.67}$$

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[43] Here, the sequence of prime numbers is written in increasinig order: $p_1 < p_2 < \cdots < p_j < \dots$, with $p_j \to \infty$ as $j \to \infty$.

Assume, contrary to the claim, that there exists a pair of indices $j < j'$, such that $\mathscr{L}_j \cap \mathscr{L}_{j'} \neq \emptyset$. In other words, $\ell_{jk} = \ell_{j'k'}$ for some $k, k' \in \mathbb{N}$; that is,

$$2^{-j} a_j^{k-1}(1-2a_j) = 2^{-j'} a_{j'}^{k'-1}(1-2a_{j'}).$$

Using $a_j = n_j/p_j$ and $a_{j'} = n_{j'}/p_{j'}$, we obtain

$$2^{j'-j} n_j^{k-1}(p_j - 2n_j) p_{j'}^{k'} = n_{j'}^{k'-1}(p_{j'} - 2n_{j'}) p_j^k.$$

However, this is impossible since the prime number $p_{j'}$ divides the left-hand side, but not the right-hand side. Indeed, $p_{j'}$ divides neither $n_{j'}$, nor $p_{j'} - 2n_{j'}$, nor p_j. $\square$

Lemma 4.5.15. *Assume that the union* $\mathscr{L} = \sqcup_{j=1}^{\infty} \mathscr{L}_j$ *of a sequence of fractal strings* $(\mathscr{L}_j)_{j\geq 1}$ *is well defined, and that it is bounded. Then*

$$\zeta_{\mathscr{L}}(s) = \sum_{j=1}^{\infty} \zeta_{\mathscr{L}_j}(s) \tag{4.5.68}$$

for all $s \in \mathbb{C}$ *such that* $\operatorname{Re} s > D(\zeta_{\mathscr{L}})$. *Furthermore,* $D(\zeta_{\mathscr{L}}) \geq \sup_{j\geq 1} D(\zeta_{\mathscr{L}_j})$.

Proof. If $\mathscr{L}_j = (\ell_{jk})_{k\geq 1}$, then clearly, $\mathscr{L} = (\ell_{jk})_{j,k\geq 1}$. We have

$$\begin{aligned} \zeta_{\mathscr{L}}(s) &= \sum_{j,k\geq 1} \ell_{jk}^s \quad \text{on} \quad \{\operatorname{Re} s > D(\zeta_{\mathscr{L}})\}, \\ \zeta_{\mathscr{L}_j}(s) &= \sum_{k=1}^{\infty} \ell_{jk}^s \quad \text{on} \quad \{\operatorname{Re} s > D(\zeta_{\mathscr{L}_j})\}, \end{aligned} \tag{4.5.69}$$

for all $j \in \mathbb{N}$. The identity (4.5.68) now follows, as we now explain. Indeed, the two series appearing in (4.5.69) are absolutely convergent, and $D(\zeta_{\mathscr{L}_j}) \leq D(\zeta_{\mathscr{L}})$, for all $j \geq 1$, since $(A_j, \Omega_j) \subseteq (A, \Omega)$ implies that

$$D(\zeta_{\mathscr{L}_j}) = \overline{\dim}_B(A_j, \Omega_j) \leq \overline{\dim}_B(A, \Omega) = D(\zeta_{\mathscr{L}}).$$

$\square$

Definition 4.5.16. Assume that Ω is a bounded interval in $\mathbb{R}$, and $A \subseteq \overline{\Omega}$ is such that A is closed (in $\mathbb{R}$), $|A| = 0$ and $\partial\Omega \subseteq A$. We say that a *fractal string* $\mathscr{L}$ *is associated with a given relative fractal drum* (A, Ω) in $\mathbb{R}$ if $\mathscr{L} = (\ell_k)_{k\geq 1}$, where $\ell_k := |J_k|$ for each $k \geq 1$, and $(J_k)_{k\geq 1}$ is the disjoint family of all the connected components (i.e., open intervals) of the open set $\Omega \setminus A \subseteq \mathbb{R}$.

Proposition 4.5.17. *Let* (A_j, Ω_j) *be a sequence of RFDs in* $\mathbb{R}$ *such that* $(\Omega_j)_{j\geq 1}$ *is a family of disjoint open intervals, and* $\partial\Omega_j \subset A_j \subset \overline{\Omega}_j$*,* A_j *is closed (in* $\mathbb{R}$*) and* $|A_j| = 0$ *for each* $j \in \mathbb{N}$. *Let* $(A, \Omega) = \cup_{j=1}^{\infty}(A_j, \Omega_j)$*,* $|\Omega| < \infty$*, and let* $\mathscr{L}_j$ *be the fractal strings associated with the RFDs* (A_j, Ω_j)*,* $j \in \mathbb{N}$. *Assume that the sequence* $\mathscr{L}_j = (\ell_{jk})_{k\geq 1}$ *is nonincreasing for each* $j \geq 1$*, and is such that the union* $\mathscr{L} :=$

$\sqcup_{j=1}^{\infty}\mathscr{L}_j$ *of fractal strings is well defined* (*see Definition 4.5.11*). *If* $\delta > \frac{1}{2}\sup_{j\geq 1}\ell_{j1}$, *then*

$$\zeta_{\mathscr{L}}(s) = s(2\delta)^{s-1}|\Omega| + s(1-s)2^{s-1}\tilde{\zeta}_{A,\Omega}(s), \tag{4.5.70}$$

for every complex number s in the open half-plane $\{\mathrm{Re}\, s > D(\zeta_{A,\Omega})\}$.

Proof. It suffices to prove (4.5.70) on $\{\mathrm{Re}\, s > D(\zeta_{A,\Omega})\}$. For any such s, we have that

$$\zeta_{A_j,\Omega_j}(s) = \frac{2^{1-s}}{s}\zeta_{\mathscr{L}_j}(s). \tag{4.5.71}$$

This follows from (2.1.84), dropping the second term on the right-hand side, since we deal here with relative zeta functions. Furthermore, using (4.5.2) with $N = 1$, we have

$$\zeta_{A_j,\Omega_j}(s) = \delta^{s-1}|(A_j)_\delta \cap \Omega_j| + (1-s)\tilde{\zeta}_{A_j,\Omega_j}(s). \tag{4.5.72}$$

Note that since $\delta > \frac{1}{2}\ell_{j1}$ for all $j \in \mathbb{N}$, we have $(A_j)_\delta \cap \Omega_j = \Omega_j$. Therefore, we conclude from (4.5.71) and (4.5.72) that for each $j \geq 1$,

$$\zeta_{\mathscr{L}_j}(s) = s(2\delta)^{s-1}|\Omega_j| + s(1-s)2^{s-1}\tilde{\zeta}_{A_j,\Omega_j}(s).$$

The claim now follows by summing up over $j \geq 1$, and using Lemma 4.5.15. ☐

From previous considerations, it is easy to deduce the following result.

Corollary 4.5.18. *Let* (A_j,Ω_j) *be a sequence of RFDs in* $\mathbb{R}$, *such that* $(\Omega_j)_{j\geq 1}$ *is a disjoint family of open intervals, and* $|\Omega_j| = 2^{-j}$, $A_j = 2^{-j}C^{(a_j)} + \inf\Omega_j$. *Let* $\mathscr{L}_j$, $j \geq 1$, *be a sequence of fractal strings associated with RFDs* (A_j,Ω_j). *Then the fractal string* $\mathscr{L} := \sqcup_{j=1}^{\infty}\mathscr{L}_j$ *is well defined, and*

(*a*) *for every complex number s in* $\{\mathrm{Re}\, s > D(\zeta_{\mathscr{L}})\}$,

$$\zeta_{\mathscr{L}}(s) = \sum_{j=1}^{\infty} 2^{-js}\frac{(1-2a_j)^s}{1-2a_j^s}; \tag{4.5.73}$$

(*b*) *the distance zeta function of the relative fractal drum* $(A,\Omega) = \cup_{j=1}^{\infty}(A_j,\Omega_j)$ *is given by*

$$\zeta_{A,\Omega}(s) = \frac{2^{1-s}}{s}\sum_{j=1}^{\infty} 2^{-js}\frac{(1-2a_j)^s}{1-2a_j^s}, \tag{4.5.74}$$

on $\{\mathrm{Re}\, s > D(\zeta_{A,\Omega})\}$.

Proof. The fractal string $\mathscr{L}$ is well defined due to Lemma 4.5.12. The claim in part (a) follows from

$$\zeta_{\mathscr{L}_j}(s) = \sum_{k=1}^{\infty} 2^{k-1}(2^{-j}a_j^{k-1}(1-2a))^{js} = 2^{-js}\frac{(1-2a_j)^s}{1-2a_j^s},$$

by using Lemma 4.5.15. In order to prove part (b), it suffices to use part (a), along with (4.5.71). ☐

We next state the main result of this section, in which we construct a set A with a given specified value of the abscissa of meromorphic continuation of A. In order to do so, it will be convenient to use the following definition.

Definition 4.5.19. Let $\mathscr{L} = (\ell_{jk})_{j,k\geq 1}$ be a fractal string; that is, $\mathscr{L}$ is representable in the form $\mathscr{L} = (m_i)_{i\geq 1}$, where the sequence $(m_i)_{i\geq 1}$ is a nonincreasing reordering of $(\ell_{jk})_{j,k\geq 1}$. Then, the sequence $A = (a_j)_{j\geq 1}$ of positive real numbers is said to be *associated with the fractal string* $\mathscr{L}$ if $a_j := \sum_{i\geq j} m_i$ for each $j \geq 1$.

Theorem 4.5.20. *Let $D \in (0,1)$ and $\alpha \in (0,D)$ be given. Let $\mathscr{L} = (\ell_{jk})_{j,k=1}^{\infty}$ be a bounded fractal string defined as follows. For $j = 1$, we let $\ell_{1k} := a_1^{k-1}(1-2a_1)$, $k \geq 1$, where $a_1 := 2^{-1/D}$. For $j \geq 2$ and $k \geq 1$, we let*

$$\ell_{jk} := 2^{-j} a_j^{k-1}(1-2a_j). \tag{4.5.75}$$

Assume that the sequence $(a_j)_{j\geq 1}$ is increasing and converges to $2^{-1/(D-\alpha)}$ as $j \to \infty$. Then

$$D(\zeta_{\mathscr{L}}) = D, \quad D_{\mathrm{mer}}(\zeta_{\mathscr{L}}) = D - \alpha. \tag{4.5.76}$$

Furthermore, if $A = A_{\mathscr{L}} := \{a_j : j \geq 1\} \subseteq (0,+\infty)$ is the bounded subset of $\mathbb{R}$ associated with the fractal string $\mathscr{L}$, then the same conclusion holds for the distance and tube zeta functions of A:

$$\begin{aligned} D(\zeta_A) = D(\tilde{\zeta}_A) = D, \\ D_{\mathrm{mer}}(\zeta_A) = D_{\mathrm{mer}}(\tilde{\zeta}_A) = D - \alpha. \end{aligned} \tag{4.5.77}$$

Moreover, $\dim_{PC} A = D + \frac{2\pi}{T_1}\mathrm{i}\mathbb{Z}$.

Proof. For $j = 1$, the associated fractal string $\mathscr{L}_1 = (\ell_{1k})_{k\geq 1}$ is the Cantor string generated by $A_1 = C^{(a_1)}$. We have $a_1 = 2^{-1/D} < 1/2$, so that the Cantor set $C^{(a_1)}$ is well defined. Furthermore, the box dimension of $C^{(a_1)}$ is given by $\log_{1/a_1} 2 = D$.

For $j \geq 2$, we have $a_j < 2^{-1/(D-\alpha)} < 2^{-1/(1-\alpha)} < 1/2$, so that the (scaled) Cantor sets $A_j = 2^{-j} C^{(a_j)} + \inf \Omega_j$, where $(\Omega_j)_{j\geq 1}$ is a family of disjoint open intervals in $\mathbb{R}$, $|\Omega_j| = 2^{-j}$, are also well defined. Lemma 4.5.13 then implies that the union of fractal strings $\mathscr{L} := \sqcup_{j=1}^{\infty} \mathscr{L}_j$ is well defined, where $\mathscr{L}_j$ are fractal strings associated with (A_j, Ω_j).

We have that

$$T_0 := \lim_j \log(1/a_j) = \log 2^{1/(D-\alpha)} = \frac{\log 2}{D-\alpha} < \frac{\pi}{D-\alpha},$$

so that Lemma 4.5.10 applies. The claim (4.5.76) follows from Theorem 4.5.8. The claims in (4.5.77) follow from Proposition 4.5.17 and (4.1.1), connecting the zeta function of a fractal string $\mathscr{L}$, the distance zeta function of the associated set $A = A_{\mathscr{L}}$, and the tube zeta function of A. ☐

The set A in Theorem 4.5.20 can be effectively constructed as a set associated with the fractal string $\mathscr{L} = \sqcup_{j=1}^{\infty}\mathscr{L}_j$, where each $\mathscr{L}_j$ is associated with a relative Cantor drum (A_j, Ω_j), described in the proof.

Theorem 4.5.20 shows, in particular, that our main results on the meromorphic extension of distance and tube zeta functions, obtained in Section 2.3, are in general optimal. We plan to study other applications and examples of relative zeta functions in a later work.

4.6 Transcendentally ∞-Quasiperiodic Relative Fractal Drums

One of the new notions explored and used in a key manner in this section is that of 'transcendentally quasiperiodic relative fractal drums', for which the corresponding quasiperiods are algebraically independent; see Section 4.6.1. It enables us, in particular, to construct bounded sets, fractal strings and RFDs that are 'maximally hyperfractal' (in the sense of the new Definition 4.6.23 below); that is, for which the corresponding fractal zeta function has nonisolated singularities at every point of the critical line $\{\mathrm{Re}\, s = D\}$ —and hence, for which the critical line is a (meromorphic) natural boundary (in the sense of part (*ii*) of Definition 1.3.8 in Subsection 1.3.2). The complexity or 'fractality' of the resulting geometric objects is therefore most extreme.

4.6.1 Quasiperiodic Relative Fractal Drums With Infinitely Many Algebraically Independent Quasiperiods

Here, we describe a general construction of quasiperiodic fractal drums possessing infinitely many algebraically incommensurable periods. It is based on properties of generalized Cantor sets, and on Baker's Theorem 3.1.14 from transcendental number theory; see [Ba, Theorem 2.1].

Let $m \geq 2$ be a given integer and $D \in (0,1)$ a given real number. Then for $a > 0$ defined by $a = m^{-1/D}$, we have $am = m^{1-1/D} < 1$, and hence, the generalized Cantor set $A = C^{(m,a)}$ is well defined (see Definition 3.1.1), and $\dim_B A = \log_{1/a} m = D$.

Definition 4.6.1. A finite set of real numbers is said to be *rationally* (*resp., algebraically*) *linearly independent* or simply, *rationally* (*resp., algebraically*) *independent*, if it is linearly independent over the field of rational (resp., algebraic) real numbers.

Definition 4.6.2. A sequence $(T_i)_{i\geq 1}$ of real numbers is said to be *rationally* (*resp., algebraically*) *linearly independent*, if any of its finite subsets is rationally (resp., algebraically) independent.

Definition 4.6.3. Let $m \geq 2$ be a positive integer. Let $\mathbf{p} = (p_i)_{i\geq 1}$ be the sequence of all prime numbers, arranged in increasing order; that is,

$$\mathbf{p} = (2, 3, 5, 7, 11, \dots).$$

We then define the *exponent sequence* $\mathbf{e} = \mathbf{e}(m) := (\alpha_i)_{i \geq 1}$ *associated with* m, where $\alpha_i \geq 0$ is the multiplicity of p_i in the factorization of m. We also let

$$\mathbf{p}^{\mathbf{e}} := \prod_{\{i \geq 1 : \alpha_i > 0\}} p_i^{\alpha_i}. \tag{4.6.1}$$

The set of all sequences $\mathbf{e}$ with components in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, such that all but at most finitely many components are equal to zero, is denoted by $(\mathbb{N}_0)_c^\infty$.

With this definition, for any integer $m \geq 2$, we obviously have $m = \mathbf{p}^{\mathbf{e}(m)}$. Conversely, any $\mathbf{e} \in (\mathbb{N}_0)_c^\infty$ defines a unique integer $m \geq 2$ such that $m = \mathbf{p}^{\mathbf{e}}$.

Definition 4.6.4. Given an exponent vector $\mathbf{e} = (\alpha_i)_{i \geq 1} \in (\mathbb{N}_0)_c^\infty$, we define the *support of* $\mathbf{e}$ as the set of all indices $i \in \mathbb{N}$ for which $\alpha_i > 0$, and we write

$$S(\mathbf{e}) = \operatorname{supp}(\mathbf{e}) = \{i \geq 1 : \alpha_i > 0\}. \tag{4.6.2}$$

The *support of an integer* $m \geq 2$ is denoted by $\operatorname{supp} m$ and defined by $\operatorname{supp} m := \operatorname{supp} \mathbf{e}(m)$.

The following definition is motivated by Theorem 3.1.15.

Definition 4.6.5. We say that a set $\{\mathbf{e}_i : i \geq 1\}$ of exponent vectors is *rationally linearly independent* if any of its finite subsets is linearly independent over $\mathbb{Q}$. We then say for short that the exponent vectors are rationally independent.

The following two definitions, Definition 4.6.6 and Definition 4.6.7, refine and extend the definition of n-quasiperiodic function and set (Definition 3.1.9 and Definition 3.1.11, respectively).

Definition 4.6.6. We say that a function $G : \mathbb{R} \to \mathbb{R}$ is ∞-*quasiperiodic*, if it is of the form

$$G(\tau) = H(\tau, \tau, \dots),$$

where $H : \ell^\infty(\mathbb{R}) \to \mathbb{R}$,[44] $H = H(\tau_1, \tau_2, \dots)$ is a function which is T_j-periodic in its j-th component, for each $j \in \mathbb{N}$, with $T_j > 0$ as minimal periods, and such that the set of periods

$$\{T_j : j \geq 1\} \tag{4.6.3}$$

is *rationally* independent. We say that the *order of quasiperiodicity* of the function G is equal to infinity (or that the function G is ∞-*quasiperiodic*).

In addition, we say that G is

(a) *transcendentally quasiperiodic of infinite order* (or *transcendentally* ∞-*quasiperiodic*) if the periods in (4.6.3) are *algebraically* independent;

[44] Here, $\ell^\infty(\mathbb{R})$ stands for the usual Banach space of bounded sequences $(\tau_j)_{j \geq 1}$ of real numbers, endowed with the norm $\|(\tau_j)_{j \geq 1}\|_\infty := \sup_{j \geq 1} |\tau_j|$.

(b) *algebraically quasiperiodic of infinite order* (or *algebraically ∞-quasiperiodic*) of infinite order if the periods in (4.6.3) are *rationally* independent and algebraically dependent.

We say that a sequence $(T_i)_{i\geq 1}$ of real numbers is *algebraically dependent* of infinite order if there exists a finite subset J of $\mathbb{N}$ such that $(T_i)_{i\in J}$ is algebraically dependent (that is, linearly dependent over the field of algebraic numbers). Recall that a finite set of real numbers $\{T_1,\dots,T_k\}$ is said to be *algebraically dependent* if there exist k algebraic real numbers $\lambda_1,\dots,\lambda_k$, not all of them equal to zero, such that $\lambda_1 T_1+\cdots+\lambda_k T_k=0$.

Definition 4.6.7. Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ having the following tube formula:

$$|A_t\cap\Omega| = t^{N-D}(G(\log t^{-1})+o(1)) \quad \text{as} \quad t\to 0^+, \tag{4.6.4}$$

where $D\leq N$,[45] and G is a nonnegative function such that

$$0<\liminf_{\tau\to+\infty} G(\tau)\leq \limsup_{\tau\to+\infty} G(\tau)<\infty.$$

(Note that it then follows that $\dim_B(A,\Omega)$ exists and is equal to D. Moreover, $\mathscr{M}_*^D(A,\Omega)=\liminf_{\tau\to+\infty}G(\tau)$ and $\mathscr{M}^{*D}(A,\Omega)=\limsup_{\tau\to+\infty}G(\tau)$.)

We then say that the *relative fractal drum* (A,Ω) *in* $\mathbb{R}^N$ *is quasiperiodic* and of *infinite order of quasiperiodicity* (or, in short, *∞-quasiperiodic*) if the function $G=G(\tau)$ is ∞-quasiperiodic; see Definition 4.6.6.

In addition, (A,Ω) is said to be

(a) a *transcendentally ∞-quasiperiodic relative fractal drum* if the corresponding function G is transcendentally ∞-quasiperiodic;

(b) an *algebraically ∞-quasiperiodic relative fractal drum* if the corresponding function G is algebraically ∞-quasiperiodic.

Definition 4.6.8. We say that a relative fractal drum (A,Ω) is *n-quasiperiodic*, where $n\geq 2$, if the function G appearing in Definition 4.6.7 is n-quasiperiodic; see Definition 3.1.9. Likewise, one can define *transcendentally n-quasiperiodic relative fractal drums* and *algebraically n-quasiperiodic relative fractal drums*.

In light of Definitions 4.6.7 and 4.6.8, we see that each n-quasiperiodic relative fractal drum, where $n\in(\mathbb{N}\setminus\{1\})\cup\{\infty\}$, is either transcendentally n-quasiperiodic or algebraically n-quasiperiodic. In other words, the family $\mathscr{D}_{\rm qp}(n)$ of n-quasiperiodic RFDs is equal to the disjoint union of the family $\mathscr{D}_{\rm tqp}(n)$ of transcendentally n-quasiperiodic RFDs and the family $\mathscr{D}_{\rm aqp}(n)$ of algebraically n-quasiperiodic RFDs:

$$\mathscr{D}_{\rm qp}(n)=\mathscr{D}_{\rm tqp}(n)\cup\mathscr{D}_{\rm aqp}(n), \quad \text{for} \quad n\in(\mathbb{N}\setminus\{1\})\cup\{\infty\}.$$

[45] Here, D may be negative as well; see Proposition 4.1.35.

Letting

$$\mathscr{D}_{\rm qp} := \bigcup_{n\geq 2} \mathscr{D}_{\rm qp}(n), \quad \mathscr{D}_{\rm tqp} := \bigcup_{n\geq 2} \mathscr{D}_{\rm tqp}(n), \quad \mathscr{D}_{\rm aqp} := \bigcup_{n\geq 2} \mathscr{D}_{\rm aqp}(n)$$

and

$$\overline{\mathscr{D}}_{\rm qp} := \mathscr{D}_{\rm qp} \cup \mathscr{D}_{\rm qp}(\infty), \quad \overline{\mathscr{D}}_{\rm tqp} := \mathscr{D}_{\rm tqp} \cup \mathscr{D}_{\rm tqp}(\infty), \quad \overline{\mathscr{D}}_{\rm aqp} := \mathscr{D}_{\rm aqp} \cup \mathscr{D}_{\rm aqp}(\infty),$$

we have that

$$\overline{\mathscr{D}}_{\rm qp} = \overline{\mathscr{D}}_{\rm tqp} \cup \overline{\mathscr{D}}_{\rm aqp}.$$

Theorem 4.6.9 below shows that the family $\mathscr{D}_{\rm tqp}(\infty)$ is nonempty. Moreover, the family $\mathscr{D}_{\rm aqp}(n)$ of algebraically n-quasiperiodic RFDs is nonempty for any $n \geq 2$, as well as for $n = \infty$, as shown by Radunović in [Ra1].

As we know, the family of bounded fractal strings can be naturally embedded into the family of bounded subsets of $\mathbb{R}$, while the family of bounded subsets of $\mathbb{R}^N$ can be naturally embedded into the family of RFDs. Therefore, we have the following natural embeddings

$$\mathscr{L}_{\rm qp}(n) \subset \mathscr{S}_{\rm qp}(n) \subset \mathscr{D}_{\rm qp}(n). \tag{4.6.5}$$

It is clear that we can define the families $\mathscr{L}_{\rm qp}(\infty)$ and $\mathscr{S}_{\rm qp}(\infty)$, much as we have defined $\mathscr{D}_{\rm qp}(\infty)$ above. In light of the embedding (4.6.5), we then have

$$\mathscr{L}_{\rm qp}(\infty) \subset \mathscr{S}_{\rm qp}(\infty) \subset \mathscr{D}_{\rm qp}(\infty),$$

and analogously

$$\mathscr{L}_{\rm tqp}(\infty) \subset \mathscr{S}_{\rm tqp}(\infty) \subset \mathscr{D}_{\rm tqp}(\infty),$$

$$\mathscr{L}_{\rm aqp}(\infty) \subset \mathscr{S}_{\rm aqp}(\infty) \subset \mathscr{D}_{atp}(\infty).$$

Theorem 4.6.9 below shows that the family $\mathscr{L}_{\rm tqp}(\infty)$ is nonempty. Therefore, the families $\mathscr{S}_{\rm tqp}(\infty)$ and $\mathscr{D}_{\rm tqp}(\infty)$ are nonempyt as well.

The following result can be considered as a fractal set-theoretic interpretation of Baker's theorem [Ba, Theorem 2.1], i.e., of Theorem 2.11, from transcendental number theory. It provides a construction of a transcendentally ∞-quasiperiodic relative fractal drum. In particular, this drum possesses infinitely many algebraically incommensurable quasiperiods T_i. In our construction, we use the two-parameter family of generalized Cantor sets $C^{(m,a)}$ described in Definition 3.1.1 and whose basic properties are described in Proposition 3.1.2.

Theorem 4.6.9. *Let $D \in (0,1)$ be a given real number, and let $(m_i)_{i\geq 1}$ be a sequence of integers, $m_i \geq 2$. For each $i \geq 1$, define $a_i = m_i^{-1/D}$, and let $C^{(m_i,a_i)}$ be the corresponding generalized Cantor set (see Definition 3.1.1). Assume that $(\Omega_i)_{i\geq 1}$ is a family of disjoint open intervals on the real line such that $|\Omega_i| \leq C_1 m_i^{1-1/D} c_i^{1/D}$ for each $i \geq 1$, where the sequence $(c_i)_{i\geq 1}$ of positive real numbers is summable, and $C_1 > 0$. Let*

$$(A,\Omega) := \bigcup_{i\geq 1}(A_i,\Omega_i), \quad \text{where} \quad A_i := |\Omega_i| C^{(m_i,a_i)} + \inf \Omega_i, \quad \text{for all} \quad i \geq 1.$$[46]

Assume that the sequence of real numbers

$$(\log m_1, \ldots, \log m_n, \ldots) \text{ is rationally independent.} \tag{4.6.6}$$

Then the sequence of real numbers

$$\left(\frac{1}{D}, T_1, T_2, \ldots\right) \tag{4.6.7}$$

is algebraically independent (*that is, lineary independent over the field of algebraic numbers*)*. In other words, the relative fractal drum* (A,Ω) *is transcendentally quasiperiodic with infinite order of quasiperiodicity. More specifically, its sequence* $(T_i)_{i\geq 1}$ *of quasiperiods is given by* $T_i := \log(1/a_i) = (\log m_i)/D$*, for every* $i \geq 1$*. Furthermore,*

$$D(\zeta_{A,\Omega}) = D_{\text{mer}}(\zeta_{A,\Omega}), \tag{4.6.8}$$

and moreover, all of the points on the critical line $\{\operatorname{Re} s = D\}$ *are nonisolated singularities of* $\zeta_{A,\Omega}$*; in other words, the relative fractal drum* (A,Ω) *is also maximally hyperfractal* (*in the sense of Definition 4.6.23*(*iii*) *below and the comment following it*)*.*

Finally, the relative fractal drum (A,Ω) *is Minkowski nondegenerate, in the sense that*

$$0 < \mathscr{M}_*^D(A,\Omega) \leq \mathscr{M}^{*D}(A,\Omega) < \infty.$$

Theorem 4.6.9 admits a partial extension. If instead of condition (4.6.6) we assume that $m_i \to \infty$ as $i \to \infty$, then (4.6.8) still holds, and, moreover, all the points of the critical line are nonisolated singularities of ζ_A. Furthermore, the fractal drum (A,Ω) is Minkowski nondegenerate.

We shall need the following lemma, which states a simple scaling property of the tube functions and Minkowski contents of RFDs. We note that the identity (4.6.10) below yields a partial extension of [Žu4, Proposition 4.4.]. Compare also with the scaling property of the corresponding distance zeta function $\zeta_{A,\Omega}$, obtained in Theorem 4.1.40.

Lemma 4.6.10. (*a*) *Let* (A,Ω) *be a relative fractal drum in* $\mathbb{R}^N$*. Then for any fixed* $\lambda > 0$*, and for all* $t > 0$*, we have that*

$$(\lambda A)_t \cap \lambda\Omega = \lambda(A_{t/\lambda} \cap \Omega), \quad |(\lambda A)_t \cap \lambda\Omega| = \lambda^N |A_{t/\lambda} \cap \Omega|. \tag{4.6.9}$$

Furthermore, for any real parameter $r \in \mathbb{R}$*, we have the following scaling* (*or homogeneity*) *properties of relative Minkowski contents*:

$$\mathscr{M}^{*r}(\lambda A, \lambda\Omega) = \lambda^r \mathscr{M}^{*r}(A,\Omega), \quad \mathscr{M}_*^r(\lambda A, \lambda\Omega) = \lambda^r \mathscr{M}_*^r(A,\Omega). \tag{4.6.10}$$

---

[46] Note that here, $|\Omega_i|$ plays the role of the scaling factor of the generalized Cantor set $C^{(m_i,a_i)}$.

(b) *If A is a generalized Cantor set $C^{(m,a)}$ (see Proposition 3.1.2), then*

$$|(\lambda C^{(m,a)})_t \cap (0,\lambda)| = t^{1-D}(G_\lambda(\log t^{-1}) - 2t^D),$$

where

$$G_\lambda(\tau) := \lambda^D G(\tau + \log\lambda)$$

and G is the T-periodic function defined in Equation (3.1.3) *of Proposition 3.1.2.*

Proof. We shall establish parts (a) and (b) separately.

(a) Scaling the set $A_t \cap \Omega$ by the factor λ, we obtain $\lambda(A_t \cap \Omega)$. On the other hand, the same result is then obtained as the intersection of the scaled sets $(\lambda A)_{\lambda t}$ and $\lambda\Omega$; that is,

$$\lambda(A_t \cap \Omega) = (\lambda A)_{\lambda t} \cap \lambda\Omega.$$

The first equality in (4.6.9) now follows by replacing t with t/λ. The second one is an immediate consequence of the first one. We also have

$$\begin{aligned}\mathscr{M}^{*r}(\lambda A, \lambda\Omega) &= \limsup_{t\to 0^+} \frac{|(\lambda A)_t \cap \lambda\Omega|}{t^{N-r}} = \lambda^N \limsup_{t\to 0^+} \frac{|(A)_{t/\lambda} \cap \Omega|}{t^{N-r}} \\ &= \lambda^N \limsup_{\tau\to 0^+} \frac{|(A)_\tau \cap \Omega|}{(\lambda\tau)^{N-r}} = \lambda^r \mathscr{M}^{*r}(A,\Omega).\end{aligned}$$

The second equality in (4.6.10) is proved in the same way, but by now using the lower limit instead of the upper limit.

(b) In the case of the generalized Cantor set, we use (4.6.9) with $N := 1$ along with Proposition 3.1.2:

$$\begin{aligned}|(\lambda C^{(m,a)})_t \cap (0,\lambda) &= \lambda |C^{(m,a)}_{t/\lambda} \cap (0,1)| = \lambda \left(\frac{t}{\lambda}\right)^{1-D} \left(G\left(\log\frac{1}{t/\lambda}\right) - 2(t/\lambda)^D\right) \\ &= t^{1-D}\left(\lambda^D G(\log\lambda + \log t^{-1}) - 2t^D\right).\end{aligned}$$

This completes the proof of the lemma. □

Relative tube zeta functions have a scaling property which is analogous to that obtained in Proposition 2.2.22 for the tube zeta functions of bounded sets. We leave the proof to the interested reader. It suffices to use Lemma 4.6.10(a).

Proposition 4.6.11 (Scaling property of relative tube zeta functions). *Let (A,Ω) be a relative fractal drum and let $\delta > 0$. Let us denote by $\tilde\zeta_{A,\Omega}(s;\delta)$ the associated relative fractal zeta function defined by Equation* (4.5.1)*. Then, for any $\lambda > 0$, we have $D(\tilde\zeta_{\lambda A,\lambda\Omega}) = D(\tilde\zeta_{A,\Omega};\delta) = \overline{\dim}_B(A,\Omega)$ and*

$$\tilde\zeta_{\lambda A,\lambda\Omega}(s;\lambda\delta) = \lambda^s \tilde\zeta_{A,\Omega}(s;\delta), \tag{4.6.11}$$

for all $s \in \mathbb{C}$ *such that* $\operatorname{Re} s > \overline{\dim}_B(A,\Omega)$*. Furthermore, if* $\omega \in \mathbb{C}$ *is a simple pole of* $\tilde{\zeta}_{A,\Omega}(s;\delta)$*, meromophically extended to a connected open neighborhood of the critical line* $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$ *(as usual, we keep the same notation for the extended function), then*

$$\operatorname{res}(\tilde{\zeta}_{\lambda A,\lambda\Omega},\omega) = \lambda^{\omega} \operatorname{res} \tilde{\zeta}_{A,\Omega}(\,\cdot\,;\delta),\omega). \tag{4.6.12}$$

One proof of Proposition 4.6.11 would rely on the functional equation (4.5.2) combined with Theorem 4.1.40, the scaling property of the distance zeta function.

In the proof of Theorem 4.6.9, we shall use the following simple fact. If a function $G(\tau) = H(\tau,\tau,\dots)$ is transcendentally quasiperiodic with respect to a sequence of quasiperiods $(T_i)_{i\geq 1}$, it is clear that for any fixed sequence of real numbers $\mathbf{d} := (d_i)_{i\geq 1}$, the corresponding function

$$G_{\mathbf{d}}(\tau) = H(d_1+\tau, d_2+\tau,\dots)$$

is quasiperiodic with respect to the same sequence of quasiperiods.

Proof of Theorem 4.6.9. The proof is divided into three steps.

Step 1: First of all, note that the generalized Cantor sets $C^{(m_i,a_i)}$ are well defined, since $m_i a_i = m_i^{1-1/D} < 1$ for each $i \geq 1$; see Definition 3.1.1. Furthermore,

$$|\Omega| = \sum_{i=1}^{\infty} |\Omega_i| \leq C_1 \sum_{i=1}^{\infty} m_i^{1-1/D} c_i^{1/D} \leq C_1 \sum_{i=1}^{\infty} c_i^{1/D} \leq C_1 \sum_{i=1}^{\infty} c_i < \infty,$$

where we have assumed without loss of generality that $c_i \leq 1$ for all $i \geq 1$. Using Lemma 4.6.10, we have

$$\begin{aligned}
|A_t \cap \Omega| &= \sum_{i=1}^{\infty} |(A_i)_t \cap \Omega_i| = t^{1-D} \sum_{i=1}^{\infty} |\Omega_i|^D \left(G_i\Big(\log|\Omega_i| + \log\frac{1}{t}\Big) - 2t^D \right) \\
&= t^{1-D}\Big(G\Big(\log\frac{1}{t}\Big) - 2|\Omega| t^D \Big),
\end{aligned}$$

where

$$G(\tau) := \sum_{i=1}^{\infty} |\Omega_i|^D G_i(\log|\Omega_i| + \tau)$$

and the functions $G_i = G_i(\tau)$ are T_i-periodic, with $T_i := \log(1/a_i)$, for all $i \geq 1$. This shows that $G(\tau) = H(\tau,\tau,\dots)$, where

$$H((\tau_i)_{i\geq 1}) := \sum_{i=1}^{\infty} |\Omega_i|^D G_i(\log|\Omega_i| + \tau_i).$$

Note that the last series is well defined, and that so is the series defining $G(\tau)$. Indeed, letting $\mathscr{M}_i = \mathscr{M}^{*D}(C^{(m_i,a_i)})$ and using Proposition 3.1.2, we see that

$$0 < G_i(\tau) \leq \mathscr{M}_i = \left(\frac{2(m_i-1)}{1-m_i a_i}\right)^{1-D} \frac{m_i}{m_i-1}(1-a_i) \leq C m_i^{1-D}, \tag{4.6.13}$$

where C is a positive constant independent of i, since $m_i \to \infty$ and $m_i a_i \to 0$ as $i \to \infty$. Therefore,

$$\sum_{i=1}^{\infty} |\Omega_i|^D G_i(\tau_i) \le \sum_{i=1}^{\infty} (C_1^D m_i^{D-1} c_i)\,(C m_i^{1-D}) = C C_1^D \sum_{i=1}^{\infty} c_i < \infty.$$

In particular,

$$\mathscr{M}^{*D}(A,\Omega) \le C C_1^D \sum_{i=1}^{\infty} c_i < \infty.$$

On the other hand, since $(A_1,\Omega_1) \supset (A,\Omega)$, we can use Lemma 4.6.10(a) (with $r := D$) and Proposition 3.1.2 to obtain that

$$\mathscr{M}_*^D(A,\Omega) \ge \mathscr{M}_*^D(A_1,\Omega_1) = |\Omega_1|^D \mathscr{M}_*^D(C^{(m_1,a_1)}) = |\Omega_1|^D \frac{1}{D}\left(\frac{2D}{1-D}\right)^{1-D} > 0.$$

Step 2: Let n be any fixed positive integer. Since the set of real numbers

$$\{\log m_1, \dots, \log m_n\}$$

is rationally independent, we conclude from Baker's theorem (see Theorem 3.1.14 above or [Ba, Theorem 2.1]) that the set of real numbers $\{1, \log m_1, \dots, \log m_n\}$ is algebraically independent. Dividing all of these numbers by D, and using $D = (\log m_i)/T_i$, where $T_i = \log(1/a_i)$ for all i (see Proposition 3.1.2), we deduce that

$$\left\{\frac{1}{D}, \frac{\log m_1}{D}, \dots, \frac{\log m_n}{D}\right\} = \left\{\frac{1}{D}, T_1, \dots, T_n\right\}$$

is algebraically independent as well. Since n is arbitrary, this proves that the relative fractal drum (A,Ω) is transcendentally ∞-quasiperiodic, in the sense of Definition 4.6.7.

Step 3: To prove the last claim, note that the critical line $\{\operatorname{Re} s = D\}$ contains the union of the set of poles $\mathscr{P}_i := \mathscr{P}(\tilde{\zeta}_{A_i,\Omega_i}, \mathbb{C}) = D + \mathbf{p}_i \mathbb{i}\mathbb{Z}$ of the tube zeta functions $\tilde{\zeta}_{A_i,\Omega_i}$, $i \ge 1$. Since the integers m_i are all distinct, we have that $m_i \to \infty$ as $i \to \infty$, and therefore, $\mathbf{p}_i = 2\pi/T_i = 2\pi D/\log m_i \to 0$. This proves that the union $\cup_{i\ge1}\mathscr{P}_i$, as a set of nonisolated singularities of $\tilde{\zeta}_{A,\Omega} = \sum_{i\ge1} \tilde{\zeta}_{A_i,\Omega_i}$ (see Lemma 4.5.9), is dense in the critical line $\{\operatorname{Re} s = D\}$. Since we have a dense set of nonisolated singularities of $\zeta_{A,\Omega}$ along the critical line, then in fact, each point on the line is a nonisolated singularity. Indeed, reasoning by contradiction, if any point (say, s_0) on the critical line is a removable singularity, then there is a punctured connected open neighborhood of s_0 in which the fractal zeta function $\zeta_{A,\Omega}$ is holomorphic, and hence, the same is true along the corresponding punctured open interval (along the critical line) containing the singularity s_0, which is impossible. (For more details, see the proof of Lemma 4.6.12 just below.) It follows, in particular, that (4.6.8) holds, as desired.

We note that the above argument can be summarized as follows: The set of nonisolated singularities along $L := \{\mathrm{Re}\, s = D\}$ is closed in the critical line L. Since the latter set is already known to be dense in L, it follows that it must be all of L. This argument is the content of Lemma 4.6.12 just below. □

At the end of the proof of Step 3 of Theorem 4.6.9, we have used the following lemma.[47]

Lemma 4.6.12. *In Step* 3 *of the proof of Theorem 4.6.9 above, the set of nonisolated singularities of* $\zeta_{A,\Omega}$ *along the critical line* $L := \{\mathrm{Re}\, s = D\}$ *is both closed and dense in, and therefore coincides with, L.*

Proof. We already know from the first part of Step 3 of the proof of Theorem 4.6.9 that the set of nonisolated singularities of $\zeta_{A,\Omega}$ along L is dense in L. Therefore, all we need to show is that it is also closed in L. Equivalently, we must show that the set of removable singularities of $\zeta_{A,\Omega}$ along L is open in L.

For this purpose, assume that there exists $s_0 \in L$ which is a removable singularity of $\zeta_{A,\Omega}$. By definition, this means that there exists an open disk $U := B_\rho(s_0)$ in $\mathbb{C}$ centered at s_0 and such that $\zeta_{A,\Omega}$ is holomorphic in the punctured disk $U \setminus \{s_0\}$. (Upon resolution of the singularity at s_0, we could take all of U instead and hence, all of I just below.) Therefore, if $I := U \cap L$ is the corresponding open interval along the critical line $L := \{\mathrm{Re}\, s = D\}$, then $\zeta_{A,\Omega}$ cannot have any nonremovable singularity in the punctured interval $I \setminus \{s_0\}$, and therefore consists entirely of removable singularities.

This establishes the fact that the set of removable singularities along L is open in L, and thereby concludes the proof of the lemma. In particular, we have shown that every point of the line L is a nonisolated nonremovable singularity of $\zeta_{A,\Omega}$; i.e, L is a natural barrier for $\zeta_{A,\Omega}$. More specifically, L is a (meromorphic) natural boundary for $\zeta_{A,\Omega}$, in the sense of part (*ii*) of Definition 1.3.8 in Subsection 1.3.2. □

It is noteworthy that the sequence $\mathscr{M}^{*D}(C^{(m_i,a_i)},(0,1))$ appearing in Theorem 4.6.9 is divergent. More precisely, it is easy to deduce from the equality in (4.6.13) that

$$\mathscr{M}^{*D}(C^{(m_i,a_i)},(0,1)) \sim (2m_i)^{1-D} \quad \text{as} \quad i \to \infty.$$

The conditions of Theorem 4.6.9 are satisfied if, for example, $m_i := p_i$ for every $i \geq 1$ (that is, $(m_i)_{i\geq 1}$ is the sequence of prime numbers $(p_i)_{i\geq 1}$, written in increasing order), and if $C_1 := 1$ and $c_i := 2^{-i}$ for all $i \geq 1$. More general choices of the sequence $(m_i)_{\geq 1}$ can be found in Theorem 4.6.13 below; see also Remark 4.6.15.

Theorem 4.6.9 shows that a result about the meromorphic extensions of distance relative zeta functions, obtained in Theorem 4.5.2 for a class of Minkowski nonmeasurable RFDs satisfying a periodicity condition, cannot be extended to transcendentally quasiperiodic RFDs with infinitely many quasiperiods. For quasiperiodic sets and RFDs with finitely many quasiperiods, such extensions are also possible. See, for example, Theorem 2.3.43 and its obvious extension to the context of RFDs.

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[47] It will be apparent to the reader that, in the statement of Lemma 4.6.12, the closedness statement is of a general nature.

4.6.2 Hyperfractals and Transcendentally ∞-Quasiperiodic Fractal Strings and Sets

The following result provides some sufficient conditions on the sequence $(m_i)_{i\geq 1}$, for the rational independence to hold in condition (4.6.6). It complements Theorem 3.1.15.

Theorem 4.6.13. *Let $m_i \geq 2$ be given integers, $i \geq 1$, and let $S_i := \operatorname{supp}(m_i)$ be their corresponding supports (see Definition 4.6.4). Assume that*

$$i \mapsto \max S_i \quad \text{is increasing.} \tag{4.6.14}$$

Let $D \in (0,1)$, and define the relative fractal drum $(A,\Omega) = \cup_{i=1}^{\infty}(A_i,\Omega_i)$, where $A_i := 2^{-i}C^{(m_i,a_i)} + \inf\Omega_i$, $a_i := m_i^{-1/D}$, and the family of open intervals $(\Omega_i)_{i\geq 1}$ is disjoint, with $|\Omega_i| := 2^{-i}$ for all $i \geq 1$. Then the relative fractal drum (A,Ω) is transcendentally quasiperiodic and with infinite order of quasiperiodicity. Furthermore,

$$D(\zeta_{A,\Omega}) = D_{\mathrm{mer}}(\zeta_{A,\Omega}) = D_{\mathrm{hol}}(\zeta_{A,\Omega}), \tag{4.6.15}$$

and moreover, all of the points on the critical line $\{\operatorname{Re} s = D\}$ are nonisolated singularities of $\zeta_{A,\Omega}$.

In order to prove this result, we shall use the following auxiliary lemma.

Lemma 4.6.14. *Let $(m_i)_{i\geq 1}$ be a sequence of integers, $m_i \geq 2$, such that the sequence of the associated exponent vectors $(\mathbf{e}(m_i))_{i\geq 1}$ is rationally linearly independent. Then the sequence $(\log m_i)_{i\geq 1}$ is rationally linearly independent as well.*

Proof. Let n be a fixed positive integer. Since the vectors $\mathbf{e}(m_1),\dots,\mathbf{e}(m_n)$ are rationally linearly independent, then using Steps 1 and 2 of the proof of Theorem 3.1.15, we conclude that the numbers $\log m_1,\dots,\log m_n$ are rationally linearly independent as well. □

Proof of Theorem 4.6.13. Condition (4.6.14) ensures that any pair of numbers m_i and m_j, $i \neq j$, has different corresponding sets of prime factors. From this we can easily conclude that the exponent vectors $\mathbf{e}(m_i)$, $i \geq 1$, are rationally linearly independent. Indeed, assume that

$$k_1\mathbf{e}(m_1) + \cdots + k_n\mathbf{e}(m_n) = 0, \tag{4.6.16}$$

for some $n \in \mathbb{N}$, where the coefficients k_i are integers. Let $s_i := \max S_i$. By looking at the s_n-th component of (4.6.16), we immediately obtain that $k_n = 0$. We then apply the same reasoning to the s_{n-1}-th component in order to obtain $k_{n-1} = 0$, and so on.

Using Lemma 4.6.14, we conclude that the sequence of integers $(\log m_i)_{i\geq 1}$ is rationally linearly independent. The claim then follows from Theorem 4.6.9. □

Remark 4.6.15. It is easy to see that condition (4.6.14) in Theorem 4.6.13 can be relaxed. More specifically, it suffices to assume that $i \mapsto \max S_i$ be injective. Indeed, if the map is injective, then each member of the set $\{\max S_i : i \in \mathbb{N}\}$ has multiplicity 1, and after a suitable permutation, we can obtain (4.6.14).[48]

In Theorems 4.6.9 and 4.6.13, we have constructed a transcendentally quasiperiodic relative fractal drum (A,Ω) with infinite order of quasiperiodicity. In particular, (A,Ω) has infinitely many algebraically incommensurable quasiperiods $T_i = \frac{1}{D}\log m_i$, $i \geq 1$.

The following corollary (Corollary 4.6.17) shows that there exist bounded fractal strings $\mathscr{L} = (\ell_j)_{j\geq 1}$ with infinitely many algebraically incommensurable quasiperiods (i.e., with infinitely many incommensurable quasifrequencies). We see from the proof of this result that $\mathscr{L}$ can be effectively constructed.

Definition 4.6.16. As we know, any bounded fractal string $\mathscr{L} = (\ell_j)_{j\geq 1}$ can be naturally identified with a relative fractal drum $(A_{\mathscr{L}},\Omega_{\mathscr{L}})$ in $\mathbb{R}$, where

$$A_{\mathscr{L}} := \Big\{a_k := \sum_{j\geq k} \ell_j : k \geq 1\Big\}, \quad \Omega_{\mathscr{L}} := \bigcup_{k=1}^{\infty} (a_{k+1}, a_k),$$

with $|\Omega_{\mathscr{L}}| = \sum_{j=1}^{\infty} \ell_j < \infty$. Let $n \in \mathbb{N} \cup \{\infty\}$ be fixed. We say that *a bounded fractal string* $\mathscr{L} = (\ell_j)_{j\geq 1}$ *is n-quasiperiodic* if the corresponding relative fractal drum $(A_{\mathscr{L}},\Omega_{\mathscr{L}})$ is n-quasiperiodic. The *order of quasiperiodicity of a bounded fractal string* $\mathscr{L}$ is defined as the order of quasiperiodicity of the corresponding relative fractal drum $(A_{\mathscr{L}},\Omega_{\mathscr{L}})$; see Definitions 4.6.7 and 4.6.8.

In addition, we say that *a bounded fractal string* $\mathscr{L} = (\ell_j)_{j\geq 1}$ *is transcendentally* (resp., *algebraically*) *∞-quasiperiodic* if the corresponding relative fractal drum $(A_{\mathscr{L}},\Omega_{\mathscr{L}})$ is transcendentally (resp., algebraically) ∞-quasiperiodic.

The family $\mathscr{L}_{\rm qp}$ of all ∞-quasiperiodic fractal strings is the disjoint union of the family $\mathscr{L}_{\rm aqp}$ of algebraically ∞-quasiperiodic fractal strings and the family $\mathscr{L}_{\rm tqp}$ of transcendentally ∞-quasiperiodic fractal strings:

$$\mathscr{L}_{\rm qp}(\infty) = \mathscr{L}_{\rm aqp}(\infty) \cup \mathscr{L}_{\rm tqp}(\infty).$$

If we let

$$\overline{\mathscr{L}}_{\rm qp} := \mathscr{L}_{\rm qp} \cup \mathscr{L}_{\rm qp}(\infty), \quad \overline{\mathscr{L}}_{\rm tqp} := \mathscr{L}_{\rm tqp} \cup \mathscr{L}_{\rm tqp}(\infty), \quad \overline{\mathscr{L}}_{\rm aqp} := \mathscr{L}_{\rm aqp} \cup \mathscr{L}_{\rm aqp}(\infty),$$

where $\mathscr{L}_{\rm qp}$, $\mathscr{L}_{\rm tqp}$ and $\mathscr{L}_{\rm aqp}$ are defined by (3.1.30) on page 202, then

$$\overline{\mathscr{L}}_{\rm qp} = \overline{\mathscr{L}}_{\rm tqp} \cup \overline{\mathscr{L}}_{\rm aqp}.$$

We expect that the family $\overline{\mathscr{L}}_{\rm aqp}$ is nonempty.

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[48] We wish to thank Tomislav Šikić for this remark.

Corollary 4.6.17. (a) *There exists an effectively constructible bounded fractal string* $\mathscr{L}=(\ell_j)_{j\geq 1}$ *in* $\mathbb{R}$ *which is transcendentally* ∞*-quasiperiodic* (*see Definition 4.6.7*)*, such that*

$$D(\zeta_{\mathscr{L}})=D_{\rm hol}(\zeta_{\mathscr{L}})=D_{\rm mer}(\zeta_{\mathscr{L}}) \tag{4.6.17}$$

and all of the points on the critical line $\{\mathrm{Re}\, s=D\}$ *are nonisolated singularities of the geometric zeta function* $\zeta_{\mathscr{L}}$*; in other words, the fractal string* $\mathscr{L}$ *is also maximally hyperfractal* (*in the sense of Definition 4.6.23*(*iii*) *below and the comment following it*)*.*

(b) *In particular, there exists an effectively constructible bounded subset* A_0 *of* $\mathbb{R}$*, which is transcendentally* ∞*-quasiperiodic, such that*

$$D(\zeta_{A_0})=D_{\rm hol}(\zeta_{\mathscr{L}})=D_{\rm mer}(\zeta_{A_0}) \tag{4.6.18}$$

and all of the points on the critical line $\{\mathrm{Re}\, s=D\}$ *are nonisolated singularities of the distance zeta function* ζ_{A_0} (*as well as of the tube zeta function* $\tilde{\zeta}_{A_0}$)*; in other words, the bounded set* A_0 *is also maximally hyperfractal* (*in the sense of Definition 4.6.23*(*iii*) *below and the comment following it*)*.*

Proof. (a) It suffices to note that each relative subdrum (A_i,Ω_i) of (A,Ω), defined in Theorem 4.6.13, can be viewed as a fractal string $\mathscr{L}_i$ (i.e., Cantor's string) associated with a generalized Cantor set $A_i=C^{(m_i,a_i)}$. Therefore, the relative fractal drum $(A,\Omega)=\cup_{i\geq 1}(A_i,\Omega_i)$ can be viewed as a bounded fractal string $\mathscr{L}=\sqcup_{i\geq 1}\mathscr{L}_i$.

(b) To prove this, it suffices to associate a new RFD (A_0,Ω_0) to the fractal string $\mathscr{L}$ from (a). Its construction can be found in Definition 4.6.16. ☐

Remark 4.6.18. Note that the set A_0 in Corollary 4.6.17(b) does not coincide with the set A from the relative fractal drum (A,Ω), associated with the fractal string $\mathscr{L}$. Indeed, A is a union of a countable family of Cantor sets (therefore, an uncountable set), whereas A_0 is a decreasing sequence of positive real numbers converging to zero. Here, A_0 is generated by the union of a sequence of generalized Cantor strings $\mathscr{L}_i$, $i\geq 1$, and each $\mathscr{L}_i$ is generated by a generalized relative Cantor drum.

Remark 4.6.19. For $N\geq 2$, one can readily extend Corollary 4.6.17 to obtain an explicitly constructible maximally hyperfractal and transcendentally ∞-quasiperiodic fractal spray in $\mathbb{R}^N$, and correspondingly, a bounded subset A of $\mathbb{R}^N$ having those same exact properties. Indeed, it suffices to proceed exactly as in the passage from Example 5.1 to Example 5.1' in [Lap1]. Namely, for example, if $A_0\subset\mathbb{R}$ is the bounded set obtained in part (b) of Corollary 4.6.17, simply let $A:=A_0\times[0,1]^{N-1}$, now viewed as a bounded subset of $\mathbb{R}^N$. (See also Subsection 4.6.4.)

Remark 4.6.20. There is a classic example of a function which is holomorphic on the open unit disk in $\mathbb{C}$ and is such that each of its points on the boundary is a nonisolated singularity. See Problem 6.2.18 on page 558.

Example 4.6.21. Concerning Lemma 4.6.14, there are many other ways to ensure the rational independence of $\log m_i$, $i \geq 1$. For example, if $m_1, \ldots, m_n$ are the positive integers defined by

$$\begin{aligned} m_1 &= p_{j_1}^k p_{j_2} \cdots p_{j_n} \\ m_2 &= p_{j_1} p_{j_2}^k \cdots p_{j_n} \\ &\vdots \\ m_n &= p_{j_1} p_{j_2} \cdots p_{j_n}^k, \end{aligned}$$

where $k \geq 2$ is a fixed integer, then these integers have identical supports, and their exponent vectors are given by

$$\begin{aligned} \mathbf{e}(m_1) &= (k, 1, 1, \ldots, 1), \\ \mathbf{e}(m_2) &= (1, k, 1, \ldots, 1), \\ &\vdots \\ \mathbf{e}(m_n) &= (1, 1, \ldots, 1, k), \end{aligned}$$

where we have truncated the exponent vectors outside of their supports. It is easy to see that these vectors are rationally linearly independent. Indeed, we have $a := \frac{1}{k+n-1}(\mathbf{e}(m_1) + \cdots + \mathbf{e}(m_n)) = (1, 1, \ldots, 1)$, and therefore, the vectors

$$\frac{1}{k-1}(\mathbf{e}(m_1) - a), \ldots, \frac{1}{k-1}(\mathbf{e}(m_n) - a)$$

form the standard basis of $\mathbb{Q}^n$.

Example 4.6.22. Let $(P_j)_{j\geq 1}$ be a partition of the set of all prime numbers, such that each set P_j is finite. Applying the construction from Example 4.6.21 on each P_j, with $k = k_j \geq 2$, we obtain an infinite sequence of integers $(m_i)_{i\geq 1}$ such that the associated sequence $(\mathbf{e}(m_i))_{i\geq 1}$ of their exponent vectors is rationally linearly independent.

Alternatively, we can also use the constructions from Remark 4.6.21 and from (4.6.14) intermittently, applied on the elements of the sequence of sets $(P_j)_{j\geq 1}$.

In Subsection 4.6.4, we will extend the construction carried out in the present subsection to obtain maximally hyperfractal sets in $\mathbb{R}^N$ of arbitrarily prescribed dimension $D \in (N-1, N)$, for any $N \geq 1$.

4.6.3 Fractality, Hyperfractality and Complex Dimensions

The following definition is closely related to the the notion of fractality (given in [Lap-vFr3], Sections 12.1.1 and 12.1.2, including Figures 12.1–12.3), as will be explained in Remark 4.6.24 below. At this point, the reader may wish to review the definition of a (*meromorphic*) *partial natural boundary* and that of a (*meromorphic*) *natural boundary* (and correspondingly, of a *partial domain of meromorphy* and of a *domain of meromorphy*) given, respectively, in part (i) and in part (ii) of Definition 1.3.8 of Subsection 1.3.2 on page 39 (and as strengthened in Remark 1.3.9).

Definition 4.6.23. (*Hyperfractality*). Let A be a bounded subset of $\mathbb{R}^N$ and let $D := \overline{\dim}_B A$. Then:

(i) The set A is a *hyperfractal* (or is *hyperfractal*) if there is a screen $\boldsymbol{S}$ (see page 95 above or Definition 5.1.1 on page 411 below) which is a (meromorphic) partial natural boundary for the associated tube (or equivalently, if $D < N$, distance) zeta function of A. This means, in particular, that the fractal zeta function cannot be meromorphically continued to any connected open neighborhood of S (or, equivalently, of the associated window $\boldsymbol{W}$); see Definition 1.3.8(i) for the precise definition of a (meromorphic) partial natural boundary. (See also both parts of Remark 1.3.9.) Equivalently, the interior $\mathring{\boldsymbol{W}}$ of the window is a partial domain of meromorphy for the fractal zeta function of A.

(ii) The set A is a *strong hyperfractal* (or is *strongly hyperfractal*) if the critical line $\{\operatorname{Re} s = D\}$ is a (meromorphic) partial natural boundary of the associated fractal zeta function; that is, if we can choose $S = \{\operatorname{Re} s = D\}$ in (i).[49] Equivalently, the open right half-plane $\{\operatorname{Re} s > D\}$ is a partial domain of meromorphy for $\tilde{\zeta}_A$ (or equivalently, if $D < N$, for ζ_A), also in the sense of Definition 1.3.8(i).

(iii) Finally, the set A is *maximally hyperfractal* if it is strongly hyperfractal and every point of the critical line $\{\operatorname{Re} s = D\}$ is a nonisolated singularity of the fractal zeta function of A. In that case, the critical line $\{\operatorname{Re} s = D\}$ is a meromorphic natural boundary of the fractal zeta function; see Definition 1.3.8(ii) for the precise definition of a partial natural boundary. In short, the fractal zeta function of A cannot be extended meromorphically (and, a fortiori, holomorphically) to any punctured (and connected) open neighborhood of s, given any point s of the critical line. Equivalently, the open right half-plane $\{\operatorname{Re} s > D\}$ is a domain of meromorphy for $\tilde{\zeta}_A$ (or equivalently, if $D < N$, for ζ_A), also in the sense of Definition 1.3.8(ii)

An analogous definition can be provided (in the obvious manner) where instead of A, we have a fractal string $\mathscr{L} = (\ell_j)_{j\geq 1}$ in $\mathbb{R}$ or, more generally, a relative fractal drum (A, Ω) in $\mathbb{R}^N$.

Remark 4.6.24. (*Complex dimensions and the definition of fractality*). In [Lap-vFr1–3], a geometric object is said to be "fractal" if the associated zeta function has at least one nonreal complex pole (with positive real part); i.e., the object has at least one nonreal complex dimension.[50] (See [Lap-vFr3, Sections 12.1

[49] Recall from Theorem 2.1.11(a) that since $D = \overline{\dim}_B A$, the fractal zeta function ζ_A is holomorphic (and hence, meromorphic) in the window $\boldsymbol{W} = \{\operatorname{Re} s > D\}$, in that case.

[50] Then, clearly, it has at least two nonreal complex conjugate complex dimensions.

and 12.2] for a detailed discussion.) In [Lap-vFr2, Lap-vFr3], in order, in particular, to take into account some possible situations pertaining to random fractals (see [HamLap], partly described in [Lap-vFr3, Section 13.4]), the definition of fractality (within the context of the theory of complex dimensions) was extended so as to allow for the case described in part (i) of Definition 4.6.23 just above, namely, the existence of a partial natural boundary along a screen. See [Lap-vFr3, Subsection 13.4.3].

We note that in [Lap-vFr3] (and the other aforementioned references), the term "hyperfractal" was not used to refer to case (i) (or to any other situation). More important, except for fractal strings and in very special higher-dimensional situations (such as suitable fractal sprays), one did not have to our disposal (as we now do, thanks to the general theory developed in this book and in [LapRaŽu1–8]) a general definition of "fractal zeta function" associated with an arbitrary bounded subset of $\mathbb{R}^N$, for every $N \geq 1$. Therefore, we can now define the "fractality" of any bounded subset of $\mathbb{R}^N$ (including Julia sets and the Mandelbrot set) and, more generally, of any relative fractal drum, by the presence of a nonreal complex dimension or else by the "hyperfractality" (in the sense of part (i) of Definition 4.6.23) of the geometric object under consideration. Here, "complex dimension" is understood as a (visible) pole of the associated fractal zeta function (the distance or tube zeta function of a bounded subset or a relative fractal drum of $\mathbb{R}^N$, or else, as was the case in most of [Lap-vFr3], the geometric zeta function of a fractal string).

Much as in [Lap-vFr1–3] and [Lap3–8], this terminology (concerning fractality, hyperfractality, and complex dimensions), can be extended to 'virtual geometries', as well as to (absolute or) relative fractal drums, noncommutative geometries, dynamical systems, and arithmetic geometries, via suitably associated 'fractal zeta functions', be they absolute or relative distance or tube zeta functions, spectral zeta functions, dynamical zeta functions, or arithmetic zeta functions (or their logarithmic derivatives thereof).

As we have seen in Theorem 4.6.9 and Corollary 4.6.17, there exist bounded sets A_0, fractal strings $\mathscr{L}$ and RFDs (A,Ω), that are maximally hyperfractal. In other words, all the points on the critical line $\{\operatorname{Re} s = D\}$ are nonisolated singularities of the corresponding zeta functions. (See Problem 6.2.20.) Furthermore, the construction provided in Subsection 4.6.4 below will show that for any integer $n \geq 1$ there exists a maximally hyperfractal bounded subset of $\mathbb{R}^N$, of arbitrary prescribed dimension $D \in (N-1,N)$; see Corollary 4.6.28. In addition, we recall that in Example 3.3.7, we have constructed a fractal string $\mathscr{L}_\infty$ whose associated fractal zeta function has a countable set of essential singularities on the critical line; see Equation (3.3.32). Such a fractal string is therefore strongly hyperfractal, in the sense of part (ii) of Definition 4.6.23 (and as strengthened in part (b) of Remark 1.3.9). It is worth pointing out that this construction was generalized to a whole class of strongly hyperfractal RFDs which are not maximally hyperfractal; see Example 4.2.10 of Subsection 4.2.2 above.

Corollary 4.6.17 provides a partial answer to a part of [Lap-vFr3, Problem 13.146, p. 473] (building on open problems proposed toward the end of [HamLap]). Note that in Corollary 4.6.17(b) we have constructed a (deterministic) hyperfractal A_0 on the real line, which is just a bounded countable set on the real line (more precisely, a bounded decreasing sequence converging to zero; see Remark 4.6.18). In this sense, A_0 may be viewed as being fairly simple. Recall, however, that it has been (effectively) constructed by means of a countable family of generalized Cantor sets, and in this sense, this sequence (as well as the corresponding hyperfractal string) is extremely complex. Also, we stress that in this construction, we do not use any random fractal sets. Random fractal strings, along with the associated random zeta functions and complex dimensions, are the object of the work of Ben Hambly and the first author in [HamLap], which is surveyed in [Lap-vFr3, Section 13.4] where the aforementioned open problem can be found.

In short, the latter problem asks whether almost surely, and within a suitably defined class of random fractals, the associated (pointwise) random fractal zeta functions have a (meromorphic) partial natural boundary. Our present work now enables us to give a proper meaning to the notion of 'fractal zeta functions' in higher dimensions, and hence to adapt it to random fractals (by naturally extending the notions introduced for random fractal strings in [HamLap]). Moreover, in the deterministic setting, the examples constructed here indicate that in some cases, one can obtain a much stronger conclusion; namely, the partial natural boundary can consist solely of nonisolated singularities. In turn, in the random setting, one may complete the above open problem (from [HamLap] and [Lap-vFr3]) by asking whether, almost surely, the random fractals within a suitable class are *maximally hyperfractal*, and hence, admit the critical line as a (meromorphic) natural boundary (for the associated fractal zeta function).

Given $d \in \mathbb{R}$ such that $d \le D$, Definition 4.6.23 (or its obvious conterpart for a fractal string $\mathscr{L}$ or a relative fractal drum (A,Ω)) can be extended as follows. In the analog of case (i), A is said to be *hyperfractal* (respectively, *strictly hyperfractal*) *in dimension d* if the screen $\boldsymbol{S}$ can be chosen so that $\sup S = d$ (respectively, $\max S$ exists and $\max S = d$).[51] In the analog of case (ii), A is said to be *strongly hyperfractal in dimension d* if the vertical line $\{\operatorname{Re} s = d\}$ is a (meromorphic) partial natural boundary of the associated zeta function (that is, if we can choose $S = \{\operatorname{Re} s = d\}$ in the counterpart of (i)). Finally, A is said to be *maximally hyperfractal in dimension d* if it is strongly hyperfractal in dimension d and every point of the vertical line $\{\operatorname{Re} s = d\}$[52] is a nonisolated singularity of the zeta function. Therefore, $\{\operatorname{Re} s = D\}$ is a (meromorphic) natural boundary (for the associated fractal zeta function).

It would be interesting to consider the following open problem, which complements in a different direction the problem about random fractals stated above in the discussion following Remark 4.6.24. Namely, one may ask whether given a (deterministic or random) relative fractal drum which is hyperfractal or even, maximally

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[51] As in [Lap-vFr2], we adopt the following notation: $\sup S := \sup_{t\in\mathbb{R}} S(t)$, and similarly for $\max S$ (when it exists). See the definition of a screen on page 95.

[52] Except possibly for some points in a small neighborhood of d in that line.

hyperfractal (with respect to the standard power law gauge function, $h \equiv 1$), one can sometime find another (non power law) gauge function h (in the sense of Definition 6.1.4 of Section 6.1 below) for which the associated zeta function no longer has a partial natural boundary. We note that in order to address this problem, one should be ready to work with analytic functions on suitable Riemann surfaces rather than just on $\mathbb{C}$ or on the Riemann sphere $\widetilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. See [EsLapRRo] and [ElLapMacRo] where a related open problem is raised in connection with the 'multifractal zeta functions' of [LapRo1, LapLéRo, LéMen, ElLapMacRo]. We plan to develop this point of view in a later work, especially in connection with the results of Chapter 5 below, on fractal tube formulas and Minkowski measurability criteria.

We will pursue the discussion of fractality in Subsection 5.5.4, in connection with the devil's staircase (the graph of the Cantor function) and a suitable version thereof studied in Example 5.5.14. See, especially, Remark 5.5.15 and the comments surrounding it. We will also revisit this issue (the notion of fractality and the related notions of critical fractality, subcritical fractality, and more general fractality in dimension $d \in \mathbb{R}$, all introduced in Subsection 5.5.4) in various places, including in Subsection 5.5.6, when discussing Example 5.5.22 (the $1/2$-square fractal), Example 5.5.23 (the $1/3$-square fractal), and Example 5.5.25 (the geometric progression fractal string).

4.6.4 Maximal Hyperfractals in Euclidean Spaces

The aim of this subsection is to show that, given a maximal hyperfractal set A in $\mathbb{R}^N$, the sets of the form $A \times [0,1]^m$ will also be maximally hyperfractal for any positive integer m. The main result is stated in Theorem 4.6.27 below. It will enable us, in particular, to obtain an N-dimensional analog of part (b) of Corollary 4.6.17 above; see Corollary 4.6.28 below.

Lemma 4.6.25. *Assume that $f = f(s)$ is a Dirichlet-type integral (DTI) such that $D_{\rm hol}(f) \in \mathbb{R}$ and the corresponding critical line of holomorphic continuation $\{\operatorname{Re} s = D_{\rm hol}(f)\}$ consists entirely of nonisolated singularities. Assume that a $(\mathbb{C}$-valued$)$ function $g = g(s)$ is holomorphic on the open right half-plane $\{\operatorname{Re} s > \alpha\}$, where $\alpha \in \mathbb{R} \cup \{-\infty\}$ and $\alpha < D_{\rm hol}(f)$. Then $D_{\rm hol}(f+g) = D_{\rm hol}(f)$ and hence, the critical line $\{\operatorname{Re} s = D_{\rm hol}(f+g)\}$ of holomorphic continuation corresponding to the function $f+g$ also consists entirely of nonisolated singularities.*

Proof. Since f is holomorphic on $\{\operatorname{Re} s > D_{\rm hol}(f)\}$, and by definition, the holomorphicity lower bound $D_{\rm hol}(f)$ is optimal (i.e., it is the infimum of all $\beta \in \mathbb{R}$ such that f is holomorphic on $\{\operatorname{Re} s > \beta\}$), it then follows that $D_{\rm hol}(f) = D_{\rm hol}(f+g)$; indeed, by hypothesis, g is holomorphic on the open right half-plane $\{\operatorname{Re} s > \alpha\}$ containing $\{\operatorname{Re} s > D_{\rm hol}(f)\}$.

In order to prove the second claim, we argue by contradiction and assume that some complex number s_0 with $\operatorname{Re} s_0 = D_{\rm hol}(f+g)$ is a removable singularity of

$f+g$. Then, since g is holomorphic at s_0 (because $\alpha < D_{\mathrm{hol}}(g)$), it would follow that s_0 is a removable singularity of the function $f = (f+g) - g$ as well. However, this would contradict the assumption according to which the holomorphy critical line $\{\mathrm{Re}\, s = D_{\mathrm{hol}}(f)\}$ consists of nonisolated singularities. □

Remark 4.6.26. Actually, a slightly more general result holds. Indeed, it suffices to assume that the function $g = g(s)$ appearing in Lemma 4.6.25 is holomorphic on an open subset of $\mathbb{C}$ containing the *closed* right half-plane $\{\mathrm{Re}\, s \geq D_{\mathrm{hol}}(f)\}$.

Theorem 4.6.27. *Assume that A is a maximally hyperfractal subset of $\mathbb{R}^N$ and let d be a positive integer. Then the set $A \times [0,1]^d$ is also maximally hyperfractal.*

Proof. By part (a) of Theorem 2.2.32, we can write

$$\zeta_{A\times[0,1]^d}(s) = \zeta_A(s-d) + g(s) \tag{4.6.19}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > \overline{\dim}_B A + d$, where

$$g(s) := \sum_{k=1}^{d} \binom{d}{k} \zeta_A(s-d+k)$$

is holomorphic on $\{\mathrm{Re}\, s > \overline{\dim}_B A + d - 1\}$. By hypothesis, the critical line of holomorphic continuation of the function $f(s) := \zeta_A(s-d)$ is the vertical line $\{\mathrm{Re}\, s = \overline{\dim}_B A + d\}$ and consists entirely of nonisolated singularities. On the other hand, the function $g(s)$ is holomorphic on $\{\mathrm{Re}\, s > \alpha := \overline{\dim}_B A + d - 1\}$, since this is the case of the functions $\zeta_A(s-d+k)$ for $k = 1, 2, \ldots, d$. (Here, we have also used the easily verified fact that $\overline{\dim}_B A$ does not depend on N, the embedding dimension; see also Proposition 4.7.6 below for a more general context.) Since $\alpha < \overline{\dim}_B A + d$, the claim now follows from Lemma 4.6.25. □

The identity (4.6.19) implies that

$$\zeta_{A\times[0,1]^d}(s) \sim \zeta_A(s-d), \tag{4.6.20}$$

which we call the *shift property* of the distance zeta function with respect to the Cartesian product of A with the d-dimensional cube $[0,1]^d$. Furthermore, the set $A \times [0,1]^d$ is called the *fractal grill* generated by A.

Corollary 4.6.28. *Let N be any positive integer. Then, for any $D \in (N-1, N)$, there is an explicitly constructible maximally hyperfractal subset A of $\mathbb{R}^N$ such that $\dim_B A = D$.*

Proof. Let $A_{\mathscr{L}}$ be a maximally hyperfractal set in $\mathbb{R}$ of the sort constructed in part (b) of Corollary 4.6.17 above. It then suffices to let $A := A_{\mathscr{L}} \times [0,1]^{N-1}$ and to apply Theorem 4.6.27 to the set $A_{\mathscr{L}} \subset \mathbb{R}$ instead of A and with $d = N-1$. □

Actually, by considering $A_{\mathscr{L}} \times [0,1]^d$, the Cartesian product of $A_{\mathscr{L}}$ by $[0,1]^d$, with $1 \leq d \leq N-1$, the same proof as the one just above shows that in the statement of Corollary 4.6.28, we may assume that $\overline{\dim}_B A \in (d, N)$, for any $d = 1, \ldots, N-1$.

4.7 Complex Dimensions and Embeddings Into Higher-Dimensional Spaces

In this section, we obtain useful results concerning relative fractal drums and bounded subsets of $\mathbb{R}^N$ embedded into higher-dimensional spaces. In particular, we show that the complex dimensions (and their multiplicities) of a bounded set (or, more generally, of a relative fractal drum) are independent of the dimension of the ambient space. (See Theorem 4.7.3 and Theorem 4.7.10, respectively.) In addition, we apply some of these results in order to calculate the complex dimensions of the Cantor dust.

4.7.1 Embeddings Into Higher Dimensions in the Case of Bounded Sets

We begin this subsection by stating a result which (along with the subsequent result, Theorem 4.7.2) will be key to the developments in this section. We first work with bounded sets, in Subsection 4.7.1, and then with general RFDs, in Subsection 4.7.2.

Proposition 4.7.1. *Let $A \subseteq \mathbb{R}^N$ be a bounded set and let $\overline{D} := \overline{\dim}_B A$. Then, for the tube zeta functions of A and $A \times \{0\} \subseteq \mathbb{R}^{N+1}$, the following equality holds:*

$$\tilde{\zeta}_{A\times\{0\}}(s;\delta) = 2\int_0^{\pi/2} \frac{\tilde{\zeta}_A(s;\delta\sin\tau)}{\sin^{s-N-1}\tau}\,\mathrm{d}\tau, \tag{4.7.1}$$

for all $s \in \mathbb{C}$ such that $\mathrm{Re}\, s > \overline{D}$.

Proof. First of all, it is well known and easy to check directly from the definitions (see Equations (1.3.1) and (1.3.4)) that $\overline{\dim}_B(A \times \{0\}) = \overline{\dim}_B A$, from which we conclude that the tube zeta functions of A and $A \times \{0\}$ are both holomorphic in the right half-plane $\{\mathrm{Re}\, s > \overline{D}\}$. Furthermore, we use the fact (see [Res, Proposition 6]) that for every $t > 0$, we have

$$|(A\times\{0\})_t|_{N+1} = 2\int_0^t |A_{\sqrt{t^2-u^2}}|_N\,\mathrm{d}u, \tag{4.7.2}$$

where as before, $|\cdot|_N$ denotes the N-dimensional Lebesgue measure. After having made the change of variable $u := t\cos v$, this yields

$$|(A\times\{0\})_t|_{N+1} = 2t\int_0^{\pi/2} |A_{t\sin v}|_N \sin v\,\mathrm{d}v. \tag{4.7.3}$$

Finally, for the tube zeta function of $A\times\{0\}$, we can write successively:

$$
\begin{aligned}
\tilde{\zeta}_{A\times\{0\}}(s;\delta) &= \int_0^\delta t^{s-N-2}|(A\times\{0\})_t|_{N+1}\,\mathrm{d}t\\
&= 2\int_0^\delta t^{s-N-1}\,\mathrm{d}t\int_0^{\pi/2}|A_{t\sin v}|_N\sin v\,\mathrm{d}v\\
&= 2\int_0^{\pi/2}\sin v\,\mathrm{d}v\int_0^\delta t^{s-N-1}|A_{t\sin v}|_N\,\mathrm{d}t\\
&= 2\int_0^{\pi/2}\sin^{N+1-s}v\,\mathrm{d}v\int_0^{\delta\sin v}\tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau\\
&= 2\int_0^{\pi/2}\frac{\tilde{\zeta}_A(s;\delta\sin v)}{\sin^{s-N-1}v}\,\mathrm{d}v,
\end{aligned}
$$

where we have used the Fubini–Tonelli theorem in order to justify the interchange of integrals (in the third equality), as well as made another change of variable (in the fourth equality), namely, $\tau := t\sin v$. This completes the proof of the proposition. □

Theorem 4.7.2. *Let $A\subseteq\mathbb{R}^N$ be a bounded set and let $\overline{D}:=\overline{\dim}_B A$. Then, we have the following equality between $\tilde{\zeta}_A$, the tube zeta function of A, and $\tilde{\zeta}_{A_M}$, the tube zeta function of $A_M := A\times\{0\}\cdots\times\{0\}\subseteq\mathbb{R}^{N+M}$, with $M\in\mathbb{N}$ arbitrary:*

$$
\tilde{\zeta}_{A_M}(s;\delta) = \frac{(\sqrt{\pi})^M\,\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+M-s}{2}+1\right)}\tilde{\zeta}_A(s;\delta)+E(s;\delta), \tag{4.7.4}
$$

initially valid for all $s\in\mathbb{C}$ such that $\mathrm{Re}\,s>\overline{D}$. *Here, the* error function $E(s) := E(s;\delta)$ (*initially defined in the case when $M=1$ by the integral on the right-hand side of Equation* (4.7.7) *below*) *admits a meromorphic extension to all of $\mathbb{C}$. The possible poles* (*in* $\mathbb{C}$) *of $E(s;\delta)$ are located at $s_k := N+2+2k$ for every $k\in\mathbb{N}_0$, and all of them are simple.* (*It follows that $\tilde{\zeta}_A$ is well defined at each s_k.*) *Moreover, we have that for each $k\in\mathbb{N}_0$,*[53]

$$
\mathrm{res}(E(\,\cdot\,;\delta),s_k) = \frac{(-1)^{k+1}\,(\sqrt{\pi})^M}{k!\,\Gamma\left(\frac{M}{2}-k\right)}\tilde{\zeta}_A(s_k;\delta). \tag{4.7.5}
$$

More specifically, if M is even, then all of the poles s_k of $E(s;\delta)$ for $k\geq M/2$ are canceled; i.e., the corresponding residues in (4.7.5) *are equal to zero. On the other hand, if M is odd, there are no such cancellations and all of the residues in* (4.7.5) *are nonzero; so that all the s_k's are simple poles of $E(s;\delta)$ in that case.*

[53] We refer to Theorem 4.7.3 for more precise information about the domain of validity of the approximate functional equation (4.7.4), and to Corollary 4.7.4 for information about the relationship between the (visible) poles of $\tilde{\zeta}_A$ and $\tilde{\zeta}_{A_M}$.

Proof. We will prove the theorem in the case when $M = 1$. The general case when $M \in \mathbb{N}$ then follows immediately by induction. From Proposition 4.7.1 we have that for $\mathrm{Re}\, s > \overline{\dim}_B A$, formula (4.7.1) holds. In turn, this latter identity can be written as

$$\begin{aligned}\tilde{\zeta}_{A\times\{0\}}(s;\delta) &= 2\tilde{\zeta}_A(s;\delta)\int_0^{\pi/2}\frac{\mathrm{d}\tau}{\sin^{s-N-1}\tau}\\ &\quad -2\int_0^{\pi/2}\frac{\mathrm{d}v}{\sin^{s-N-1}v}\int_{\delta\sin v}^{\delta}\tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau \\ &= \tilde{\zeta}_A(s;\delta)\cdot \mathrm{B}\left(\frac{N-s}{2}+1,\frac{1}{2}\right)+E(s;\delta),\end{aligned} \tag{4.7.6}$$

where $\mathrm{B}(u,v)$ denotes the *Euler beta function* and

$$E(s;\delta) := -2\int_0^{\pi/2}\frac{\mathrm{d}v}{\sin^{s-N-1}v}\int_{\delta\sin v}^{\delta}\tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau. \tag{4.7.7}$$

By using the functional equation which links the beta function with the gamma function (namely, $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ for all $x,y>0$ and hence, upon meromorphic continuation, for all $x,y\in\mathbb{C}$), we obtain that (4.7.4) holds (with $M=1$) for all $s\in\mathbb{C}$ such that $\mathrm{Re}\, s > \overline{\dim}_B A$.

By looking at the expression for $E(s;\delta)$ in (4.7.7), we see that the integrand is holomorphic for every $v\in(0,\pi/2)$ since the integral $\int_{\delta\sin v}^{\delta}\tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau$ is equal to $\tilde{\zeta}_A(s;\delta)-\tilde{\zeta}_A(s;\delta\sin v)$, which is an entire function. Furthermore, if we assume that $\mathrm{Re}\, s < N+1$, then since $\tau\mapsto\tau^{\mathrm{Re}\, s-N-1}$ is decreasing, we have the following estimate:

$$\begin{aligned}|E(s;\delta)| &\le 2\int_0^{\pi/2}\sin^{N+1-\mathrm{Re}\, s}v\,\mathrm{d}v\int_{\delta\sin v}^{\delta}\tau^{\mathrm{Re}\, s-N-1}|A_\tau|_N\,\mathrm{d}\tau\\ &\le 2|A_\delta|_N\int_0^{\pi/2}\sin^{N+1-\mathrm{Re}\, s}v\,\mathrm{d}v\int_{\delta\sin v}^{\delta}\tau^{\mathrm{Re}\, s-N-1}\,\mathrm{d}\tau\\ &\le 2\delta^{\mathrm{Re}\, s-N-1}|A_\delta|_N\int_0^{\pi/2}\sin^{N+1-\mathrm{Re}\, s}v\,\sin^{\mathrm{Re}\, s-N-1}v\int_{\delta\sin v}^{\delta}\mathrm{d}\tau\\ &= 2\delta^{\mathrm{Re}\, s-N}|A_\delta|_N\int_0^{\pi/2}(1-\sin v)\,\mathrm{d}v\\ &= 2\delta^{\mathrm{Re}\, s-N}|A_\delta|_N\left(\frac{\pi}{2}-1\right).\end{aligned} \tag{4.7.8}$$

Hence,

$$|E(s;\delta)| \le 2\delta^{\mathrm{Re}\, s-N}|A_\delta|\left(\frac{\pi}{2}-1\right). \tag{4.7.9}$$

We conclude from this inequality that for $s_0\in\{\mathrm{Re}\, s<N+1\}$, the condition $(3')$ of Remark 2.1.48 is satisfied, which implies, in light of Theorem 2.1.47, that $E(s;\delta)$ is holomorphic on the open half-plane $\{\mathrm{Re}\, s<N+1\}$.

On the other hand, we know that both of the tube zeta functions $\tilde{\zeta}_A$ and $\tilde{\zeta}_{A_M}$ are holomorphic on $\{\mathrm{Re}\, s > \overline{\dim}_B A\}\supseteq\{\mathrm{Re}\, s>N\}$. The fact that $E(s;\delta)$ is meromorphic on $\mathbb{C}$, as well as the statement about its poles, now follows from Equation (4.7.4) (with $M = 1$) and the fact that the gamma function is nowhere vanishing in $\mathbb{C}$. (In fact, $1/\Gamma(s)$ is an entire function with zeros at the nonpositive integers.) More specifically, the locations of the poles of $E(s;\delta)$ must coincide with the locations of the poles $s_k = N+2+2k$, for $k \in \mathbb{N}_0$, of $\Gamma((N-s)/2+1)$ since the left-hand side of (4.7.4) is holomorphic on $\{\mathrm{Re}\, s > \overline{\dim}_B A\}$ and because $\tilde{\zeta}_A(s_k) > 0$ (since it is defined as the integral of a positive function). Note that since $N \geq \overline{D}$, we have $s_k > \overline{D}$, and hence, $\tilde{\zeta}_A$ is well defined at s_k, for each $k \in \mathbb{N}_0$.

Finally, by multiplying (4.7.4) by $(s-s_k)$, taking the limit as $s \to s_k$ and then using the fact that the residue of the gamma function at $-k$ is equal to $(-1)^k/k!$, we deduce that (4.7.5) holds, as desired.

Furthermore, if M is odd, there are no cancellations between the poles of the numerator and of the denominator in (4.7.4) since an integer cannot be both even and odd; i.e., the residues are nonzero for each $k \in \mathbb{N}_0$. On the other hand, if M is even, then it is clear that all of the residues at s_k for $k \geq M/2$ are equal to zero; i.e., the corresponding poles at s_k cancel out with the poles of the denominator in (4.7.4). □

Theorem 4.7.2 has as an important consequence, namely, the fact that the notion of complex dimensions does not depend on the dimension of the ambient space.

Theorem 4.7.3. *Let $A \subseteq \mathbb{R}^N$ be a bounded set and A_M be its embedding into $\mathbb{R}^{N+M}$, with $M \in \mathbb{N}$ arbitrary. Then, the tube zeta function $\tilde{\zeta}_A$ of A has a meromorphic extension to a given connected open neighborhood U of the critical line $\{\mathrm{Re}\, s = \overline{\dim}_B A\}$ if and only if the analogous statement is true for the tube zeta function $\tilde{\zeta}_{A_M}$ of A_M. Furthermore, in that case, the approximate functional equation (4.7.4) remains valid for all $s \in U$. In addition, the multisets[54] of the poles of $\tilde{\zeta}_A$ and $\tilde{\zeta}_{A_M}$ located in U coincide; i.e., $\mathscr{P}(\tilde{\zeta}_A, U) = \mathscr{P}(\tilde{\zeta}_{A_M}, U)$.[55] Consequently, neither the values nor the multiplicities of the complex dimensions of A depend on the dimension of the ambient space.*

Proof. This is a direct consequence of Theorem 4.7.2 and the principle of analytic continuation. More precisely, identity (4.7.4) is valid for all $s \in \mathbb{C}$ such that $\mathrm{Re}\, s > \overline{\dim}_B A$ and the function $E(s;\delta)$ is meromorphic on all of $\mathbb{C}$. Furthermore, according to Theorem 4.7.2, the poles of $E(s;\delta)$ belong to $\{\mathrm{Re}\, s \geq N+2\}$, which implies that the function $s \mapsto E(s;\delta)$ is holomorphic on $\{\mathrm{Re}\, s < N+2\}$. Identity (4.7.4) then remains valid if any of the two zeta functions involved (namely, $\tilde{\zeta}_A$ or $\tilde{\zeta}_{A_M}$) has a meromorphic continuation to some connected open neighborhood of the critical line $\{\mathrm{Re}\, s = \overline{\dim}_B A\}$. This completes the proof of the theorem. □

Corollary 4.7.4. *Let $A \subseteq \mathbb{R}^N$ be a bounded set (with $\overline{D} := \overline{\dim}_B A$) such that its tube zeta function $\tilde{\zeta}_A$ has a meromorphic continuation to a connected open neighborhood U of the critical line $\{\mathrm{Re}\, s = \overline{\dim}_B A\}$. Furthermore, suppose that $s = \overline{D}$ is a simple*

[54] In these multisets, each pole is counted according to its multiplicity.

[55] Recall that the bounded sets A and A_M have the same upper Minkowski dimension, $\overline{\dim}_B A = \overline{\dim}_B A_M$, and hence, the same critical line $\{\mathrm{Re}\, s = \overline{\dim}_B A\}$.

pole of $\tilde{\zeta}_A$. *Let* $A_M \subseteq \mathbb{R}^{N+M}$ *be the canonical embedding of A into* $\mathbb{R}^{N+M}$, *with* $M \in \mathbb{N}$ *arbitrary, as in Theorem 4.7.2. Then*

$$\operatorname{res}(\tilde{\zeta}_{A_M}, \overline{D}) = \frac{(\sqrt{\pi})^M \, \Gamma\left(\frac{N-\overline{D}}{2}+1\right)}{\Gamma\left(\frac{N+M-\overline{D}}{2}+1\right)} \operatorname{res}(\tilde{\zeta}_A, \overline{D}). \tag{4.7.10}$$

We point out here that the above corollary is compatible with the dimensional invariance of the normalized Minkowski content, obtained in [Kne] (see also [Res]). More specifically, if in the above corollary, we assume, in addition, that $\overline{D}$ is the only pole of the tube zeta function of A on the critical line $\{\operatorname{Re} s = \overline{D}\}$ (i.e., $\overline{D}$ is the only complex dimension of A with real part $\overline{D}$), then, according to Theorem 5.4.2 of Chapter 5 below (the "sufficient condition for Minkowski measurability"), A and $A \times \{0\}$ are Minkowski measurable with Minkowski dimension $D := \overline{D}$ and have respective Minkowski contents satisfying the following identity:

$$\frac{\mathscr{M}^D(A)}{\pi^{\frac{D-N}{2}}\,\Gamma\left(\frac{N-D}{2}+1\right)} = \frac{\mathscr{M}^D(A \times \{0\})}{\pi^{\frac{D-N-1}{2}}\,\Gamma\left(\frac{N+1-D}{2}+1\right)}. \tag{4.7.11}$$

4.7.2 Embeddings Into Higher Dimensions in the Case of Relative Fractal Drums

The observations made in the previous subsection in the context of bounded subsets of $\mathbb{R}^N$ can also be extended to the more general context of relative fractal drums (RFDs) in $\mathbb{R}^N$. More specifically, let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$ and let

$$(A \times \{0\}, \Omega \times (-1,1))$$

be its natural embedding into $\mathbb{R}^{N+1}$. We want to connect the relative tube zeta functions of these two RFDs; the following lemma will be needed for this purpose.

Lemma 4.7.5. *Let* (A, Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *and fix* $\delta \in (0,1)$. *Then we have*

$$\big|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))\big|_{N+1} = 2\int_0^\delta |A_{\sqrt{\delta^2-u^2}} \cap \Omega|_N \, \mathrm{d}u. \tag{4.7.12}$$

Proof. We proceed much as in the proof of [Res, Proposition 6]. Namely, if we let $(x,y) \in \mathbb{R}^N \times \mathbb{R} \equiv \mathbb{R}^{N+1}$ and define

$$V := \{(x,y) : d_{N+1}((x,y), A \times \{0\}) \le \delta\} \cap \{(x,y) : x \in \Omega, |y| \le 1\}, \tag{4.7.13}$$

where $(x,y) \in \mathbb{R}^N \times \mathbb{R} \simeq \mathbb{R}^{N+1}$ and for any $k \in \mathbb{N}$, d_k denotes the Euclidean distance in $\mathbb{R}^k$. It is clear that the following equality holds:

$$d_{N+1}((x,y), A \times \{0\}) = \sqrt{d_N(x,A)^2 + y^2}.$$

This implies that for a fixed $y \in [-\delta, \delta] \subset \mathbb{R}$, we have

$$\begin{aligned} V_y :&= \left\{ x \in \mathbb{R}^N : d_{N+1}((x,y), A \times \{0\}) \le \delta \right\} \\ &= \left\{ x \in \mathbb{R}^N : d_N(x,A) \le \sqrt{\delta^2 - y^2} \right\}. \end{aligned} \tag{4.7.14}$$

(Note that if $|y| > \delta$, then V_y is empty.) Finally, Fubini's theorem implies that

$$\begin{aligned} \big|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))\big|_{N+1} &= \int_V \mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\delta}^{\delta} \mathrm{d}y \int_{V_y \cap \{x \in \mathbb{R}^N : x \in \Omega\}} \mathrm{d}x \\ &= 2\int_0^{\delta} |A_{\sqrt{\delta^2 - y^2}} \cap \Omega|_N \,\mathrm{d}y, \end{aligned}$$

which completes the proof of the lemma. □

The above lemma will eventually yield (in Theorem 4.7.10 below) an RFD analog of Proposition 4.7.1 from Subsection 4.7.1 above. First, however, we will show that the upper and lower relative box dimensions of an RFD are independent of the ambient space dimension.

Proposition 4.7.6. *Let (A, Ω) be an RFD in $\mathbb{R}^N$ and let*

$$(A,\Omega)_M := (A_M, \Omega \times (-1,1)^M) \tag{4.7.15}$$

be its embedding into $\mathbb{R}^{N+M}$, for some $M \in \mathbb{N}$. Then we have that

$$\overline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega)_M \tag{4.7.16}$$

and

$$\underline{\dim}_B(A,\Omega) = \underline{\dim}_B(A,\Omega)_M. \tag{4.7.17}$$

Proof. We only prove the proposition in the case when $M = 1$, from which the general result then easily follows by induction. It is clear that for $0 < \delta < 1$, we have

$$\begin{aligned} (A \times \{0\})_\delta \cap (\Omega \times (-1,1)) &\subseteq (A \times \{0\})_\delta \cap (\Omega \times (-\delta,\delta)) \\ &\subseteq (A_\delta \cap \Omega) \times (-\delta,\delta); \end{aligned}$$

so that

$$|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1} \le 2\delta |A_\delta \cap \Omega|_N. \tag{4.7.18}$$

This observation, in turn, implies that for every $r \in \mathbb{R}$, we have

$$\frac{|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1}}{\delta^{N+1-r}} \leq \frac{2|A_\delta \cap \Omega|_N}{\delta^{N-r}}. \tag{4.7.19}$$

Furthermore, by successively taking the upper and lower limits as $\delta \to 0^+$ in Equation (4.7.19) just above, we obtain the following inequalities involving the r-dimensional upper and lower relative Minkowski contents, respectively:

$$\mathscr{M}^{*r}(A,\Omega)_1 \leq 2\mathscr{M}^{*r}(A,\Omega) \quad \text{and} \quad \mathscr{M}_*^r(A,\Omega)_1 \leq 2\mathscr{M}_*^r(A,\Omega). \tag{4.7.20}$$

In light of the definition of the relative upper and lower box (or Minkowski) dimensions (see Equation (4.1.4) and Equation (4.1.6), along with the text surrounding them), we deduce that

$$\overline{\dim}_B(A,\Omega)_1 \leq \overline{\dim}_B(A,\Omega) \quad \text{and} \quad \underline{\dim}_B(A,\Omega)_1 \leq \underline{\dim}_B(A,\Omega). \tag{4.7.21}$$

On the other hand, for geometric reasons, we have that

$$(A_{\delta/2} \cap \Omega) \times \left(-\frac{\delta\sqrt{3}}{2}, \frac{\delta\sqrt{3}}{2}\right) \subseteq (A \times \{0\})_\delta \cap (\Omega \times (-1,1));$$

so that

$$\delta\sqrt{3}|A_{\delta/2} \cap \Omega|_N \leq |(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1}. \tag{4.7.22}$$

Much as before, this inequality implies that for every $r \in \mathbb{R}$, we have

$$\frac{\sqrt{3}|A_{\delta/2} \cap \Omega|_N}{2^{N-r}(\delta/2)^{N-r}} \leq \frac{|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1}}{\delta^{N+1-r}} \tag{4.7.23}$$

and by successively taking the upper and lower limits as $\delta \to 0^+$, we obtain that

$$\frac{\sqrt{3}\mathscr{M}^{*r}(A,\Omega)}{2^{N-r}} \leq \mathscr{M}^{*r}(A,\Omega)_1 \quad \text{and} \quad \frac{\sqrt{3}\mathscr{M}_*^r(A,\Omega)}{2^{N-r}} \leq \mathscr{M}_*^r(A,\Omega)_1. \tag{4.7.24}$$

Finally, this completes the proof because (again in light of Equation (4.1.4) and Equation (4.1.6), along with the text surrounding them), (4.7.24) implies the reverse inequalities for the upper and lower relative box dimensions in (4.7.21). □

Remark 4.7.7. Observe that it follows from Proposition 4.7.6 (combined with part (b) of Theorem 4.1.7) that the RFDs (A,Ω) and $(A,\Omega)_M$ have the same upper Minkowski dimension, $\overline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega)_M$, and hence, the same critical line $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$. This fact will be used implicitly in the statement of Proposition 4.7.8 as well as in the statements of Theorems 4.7.9 and 4.7.10 just below.

We can now state the desired results for embedded RFDs and their relative fractal zeta functions. In light of Lemma 4.7.5 and Proposition 4.7.6, the proofs follow the same steps as in the corresponding results established in Subsection 4.7.1 about bounded subsets of $\mathbb{R}^N$ (namely, Proposition 4.7.1 and Theorem 4.7.2, respectively), and for this reason, we will omit them.

Proposition 4.7.8. *Fix $\delta \in (0,1)$ and let (A,Ω) be an RFD in $\mathbb{R}^N$, with $\overline{D} := \overline{\dim}_B(A,\Omega)$. Then, for the relative tube zeta functions of (A,Ω) and $(A,\Omega)_1 := (A\times\{0\},\Omega\times(-1,1))$, the following equality holds*:

$$\tilde{\zeta}_{A\times\{0\},\Omega\times(-1,1)}(s;\delta) = 2\int_0^{\pi/2} \frac{\tilde{\zeta}_{A,\Omega}(s;\delta\sin\tau)}{\sin^{s-N-1}\tau}\,\mathrm{d}\tau, \tag{4.7.25}$$

for all $s\in\mathbb{C}$ such that $\mathrm{Re}\,s > \overline{D}$.

Theorem 4.7.9. *Fix $\delta \in (0,1)$ and let (A,Ω) be an RFD in $\mathbb{R}^N$, with $\overline{D} := \overline{\dim}_B(A,\Omega)$. Then, we have the following equality between $\tilde{\zeta}_{A,\Omega}$, the tube zeta function of (A,Ω), and $\tilde{\zeta}_{A_M,\Omega\times(-1,1)^M}$, the tube zeta function of the relative fractal drum $(A,\Omega)_M := (A_M,\Omega\times(-1,1)^M)$ in $\mathbb{R}^{N+M}$, where $M\in\mathbb{N}$ is arbitrary*:

$$\tilde{\zeta}_{A_M,\Omega\times(-1,1)^M}(s;\delta) = \frac{\left(\sqrt{\pi}\right)^M\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+M-s}{2}+1\right)}\tilde{\zeta}_{A,\Omega}(s;\delta) + E(s;\delta), \tag{4.7.26}$$

initially valid for all $s\in\mathbb{C}$ such that $\mathrm{Re}\,s > \overline{D}$.[56] *Here, the* error function $E(s) := E(s;\delta)$ *is meromorphic on all of $\mathbb{C}$. Furthermore, the possible poles (in $\mathbb{C}$) of $E(s;\delta)$ are located at $s_k := N+2+2k$ for every $k\in\mathbb{N}_0$, and all of them are simple. (It follows that $\tilde{\zeta}_A$ is well defined at each s_k.) Moreover, we have that for each $k\in\mathbb{N}_0$,*

$$\operatorname{res}(E(\,\cdot\,;\delta),s_k) = \frac{(-1)^{k+1}\left(\sqrt{\pi}\right)^M}{k!\,\Gamma\left(\frac{M}{2}-k\right)}\tilde{\zeta}_{A,\Omega}(s_k;\delta). \tag{4.7.27}$$

More specifically, if M is even, then all of the poles s_k of $E(s;\delta)$ for $k\geq M/2$ are canceled; i.e., the corresponding residues in (4.7.27) are equal to zero. On the other hand, if M is odd, there are no such cancellations and all of the residues in (4.7.27) are nonzero; so that all of the s_k's are simple poles of $E(s;\delta)$ in that case.

We deduce at once from Theorem 4.7.9 the following key result about the invariance of the complex dimensions of a relative fractal drum with respect the dimension of the ambient space. This result extends Theorem 4.7.3 to general RFDs.

Theorem 4.7.10. *Let (A,Ω) be an RFD in $\mathbb{R}^N$ and let the RFD $(A,\Omega)_M := (A_M,\Omega\times(-1,1)^M)$ be its embedding into $\mathbb{R}^{N+M}$, for some arbitrary $M\in\mathbb{N}$. Then, the tube zeta function $\tilde{\zeta}_{A,\Omega}$ of (A,Ω) has a meromorphic extension to a given connected open neighborhood U of the critical line* $\{\mathrm{Re}\,s = \overline{\dim}_B(A,\Omega)\}$ *if and only if*

56 See Theorem 4.7.10 for more precise information about the domain of validity of the approximate functional equation (4.7.26).

the analogous statement is true for the tube zeta function $\tilde{\zeta}_{(A,\Omega)_M} := \tilde{\zeta}_{A_M,\Omega\times(-1,1)^M}$ *of* $(A,\Omega)_M$. (*See Remark 4.7.7 just above.*) *Furthermore, in that case, the approximate functional equation* (4.7.26) *remains valid for all* $s \in U$. *In addition, the multisets of the poles of* $\tilde{\zeta}_{A,\Omega}$ *and* $\tilde{\zeta}_{(A,\Omega)_M}$ *belonging to U coincide; i.e.,*

$$\mathscr{P}(\tilde{\zeta}_{A,\Omega},U) = \mathscr{P}(\tilde{\zeta}_{(A,\Omega)_M},U). \tag{4.7.28}$$

Consequently, neither the values nor the multiplicities of the complex dimensions of the RFD (A,Ω) *depend on the dimension of the ambient space.*

Remark 4.7.11. In the above discussion about embedding RFDs into higher-dimensional spaces, we can also make similar observations if we embed (A,Ω) as a 'one-sided' RFD, for example of the form $(A\times\{0\},\Omega\times(0,1))$, a fact which can be more useful when decomposing a relative fractal drum into a union of relative fractal subdrums in order to compute its distance (or tube) zeta function.[57] This observation follows immediately from the above results for 'two-sided' embeddings of RFDs since, by symmetry, we have

$$\tilde{\zeta}_{A\times\{0\},\Omega\times(-1,1)}(s) = 2\,\tilde{\zeta}_{A\times\{0\},\Omega\times(0,1)}(s). \tag{4.7.29}$$

We note that when using the above formulas, one only has to be careful to take into account the factor 2. Furthermore, we can also embed (A,Ω) as

$$(A\times\{0\},\Omega\times(-\alpha,\alpha)) \quad \text{or} \quad (A\times\{0\},\Omega\times(0,\alpha)), \tag{4.7.30}$$

for some $\alpha > 0$, but in that case, the corresponding formulas will only be valid for all $\delta \in (0,\alpha)$.

We could now use the functional equation (2.2.23) connecting the tube and distance zeta functions, in order to translate the above results in terms of $\zeta_{A,\Omega} := \zeta_{A,\Omega}(\,\cdot\,;\delta)$, the (relative) distance zeta function of the RFD (A,Ω). However, we will instead use another approach because it gives some additional information about the resulting error function. More specifically, consider the *Mellin zeta function of a relative fractal drum*, to be introduced and studied in Section 5.4 below (see Definition 5.4.6). Here, we state some of its properties (see Theorems 5.4.7, 5.4.9 and 5.4.10) which will be needed in the following discussion.

The Mellin zeta function of an RFD (A,Ω) with $\overline{\dim}_B(A,\Omega) < N$ is initially defined by

$$\zeta^{\mathfrak{M}}_{A,\Omega}(s) = \int_0^{+\infty} t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t, \tag{4.7.31}$$

for all $s \in \mathbb{C}$ located in a suitable vertical strip. In fact, in light of Theorem 5.4.7, the above Lebesgue integral is absolutely convergent (and hence, convergent) for all $s \in \mathbb{C}$ such that $\operatorname{Re} s \in (\overline{\dim}_B(A,\Omega),N)$. Moreover, the relative distance and Mellin zeta functions of (A,Ω) are connected by the functional equation

[57] See Subsection 4.2.3 for examples of such decompositions in the case of the relative Sierpiński gasket and carpet, as well as of their higher-dimensional analogs.

$$\zeta_{A,\Omega}(s) = (N-s)\zeta^{\mathfrak{M}}_{A,\Omega}(s), \tag{4.7.32}$$

on every open connected set $U \subseteq \mathbb{C}$ to which any of the two zeta functions has a meromorphic continuation. Observe that in (4.7.32), the parameter δ is absent. Indeed, this means implicitly that the functional equation (4.7.32) is valid only for the parameters $\delta > 0$ for which $\Omega \subseteq A_\delta$ is satisfied; that is, when the equality $\zeta_{A,\Omega}(s;\delta) = \int_\Omega d(x,A)^{s-N}\,\mathrm{d}x$ is satisfied.

We will now embed the relative fractal drum (A,Ω) of $\mathbb{R}^N$ into $\mathbb{R}^{N+1}$ as

$$(A\times\{0\},\Omega\times\mathbb{R}).$$

Strictly speaking, this is not a relative fractal drum in $\mathbb{R}^{N+1}$ since there does not exist $\delta > 0$ such that $\Omega\times\mathbb{R} \subseteq (A\times\{0\})_\delta$. On the other hand, observe that Lemma 4.7.5 is now valid for every $\delta > 0$; that is,

$$\big|(A\times\{0\})_\delta \cap (\Omega\times\mathbb{R})\big|_{N+1} = 2\int_0^\delta |A_{\sqrt{\delta^2-u^2}}\cap\Omega|_N\,\mathrm{d}u. \tag{4.7.33}$$

Proposition 4.7.12. *Let (A,Ω) be an RFD in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega) < N$. Then the function $F = F(s)$, defined by the integral*

$$F(s) := \int_0^{+\infty} t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1}\,\mathrm{d}t, \tag{4.7.34}$$

is holomorphic inside the vertical strip $\{\overline{\dim}_B(A,\Omega) < \mathrm{Re}\,s < N\}$.

Proof. We split the integral into two integrals: $F(s) = \int_0^1 + \int_1^{+\infty}$. According to Proposition 4.7.6, the first integral,

$$\begin{aligned}&\int_0^1 t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1}\,\mathrm{d}t\\ &=\int_0^1 t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times(-1,1))\big|_{N+1}\,\mathrm{d}t,\end{aligned}$$

defines a holomorphic function on the right half-plane $\{\mathrm{Re}\,s > \overline{\dim}_B(A,\Omega)\}$.

In order to deal with the second integral, we observe that

$$\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1} \le 2t|\Omega|_N,$$

and consequently, deduce that

$$\left|\int_1^{+\infty} t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1}\,\mathrm{d}t\right| \le 2|\Omega|_N\int_1^{+\infty} t^{\mathrm{Re}\,s-N-1}\,\mathrm{d}t = \frac{2|\Omega|_N}{N-\mathrm{Re}\,s},$$

for all $s\in\mathbb{C}$ such that $\mathrm{Re}\,s < N$. In light of Theorem 2.1.47 and Remark 2.1.48, the latter inequality implies that the integral over $(1,+\infty)$ defines a holomorphic function on the left half-plane $\{\mathrm{Re}\, s < N\}$. Therefore, it follows that $F(s)$ is holomorphic in the vertical strip $\{\overline{\dim}_B(A,\Omega) < \mathrm{Re}\, s < N\}$ and the proof of the proposition is complete. ☐

In light of the above proposition, we continue to use the convenient notation $\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}}$ for the integral appearing on the right-hand side of (4.7.34) although, as was noted earlier, $(A\times\{0\},\Omega\times\mathbb{R})$ is not technically a relative fractal drum in $\mathbb{R}^{N+1}$; see Remark 4.7.11 above. The following result is the counterpart of Theorem 4.7.2 in the present, more general context.

Theorem 4.7.13. *Let* (A,Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *such that* $\overline{D} := \overline{\dim}_B(A,\Omega) < N$*. Then, for every* $a > 0$*, the following approximate functional equation holds*:

$$\zeta_{A\times\{0\},\Omega\times(-a,a)}(s) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}\right)}{\Gamma\left(\frac{N+1-s}{2}\right)}\zeta_{A,\Omega}(s) + E(s;a), \tag{4.7.35}$$

initially valid for all $s\in\mathbb{C}$ *such that* $\mathrm{Re}\, s > \overline{D}$*. Here, the* error function $E(s) := E(s;a)$ *is initially given (for all* $s\in\mathbb{C}$ *such that* $\mathrm{Re}\, s < N$*) by*

$$E(s;a) := (s-N-1)\int_a^{+\infty} t^{s-N-2}|(A\times\{0\})_t\cap\Omega\times(\mathbb{R}\setminus(-a,a))|_{N+1}\,\mathrm{d}t, \tag{4.7.36}$$

and admits a meromorphic extension to all of $\mathbb{C}$*, with a set of simple poles equal to* $\{N+2k : k\in\mathbb{N}_0\}$*.*

Moreover, Equation (4.7.35) *remains valid on any connected open neighborhood of the critical line* $\{\mathrm{Re}\, s = \overline{D}\}$ *to which* $\zeta_{A,\Omega}$ *(or, equivalently,* $\zeta_{A\times\{0\},\Omega\times(-a,a)}$*) can be meromorphically continued.*

Proof. In a completely analogous way as in the proof of Theorem 4.7.2, we obtain that

$$\tilde{\zeta}_{A\times\{0\},\Omega\times\mathbb{R}}(s;\delta) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)}\tilde{\zeta}_{A,\Omega}(s;\delta) + \tilde{E}(s;\delta), \tag{4.7.37}$$

now valid for all $\delta > 0$ (see Equation (4.7.33) above and the discussion preceding it). Furthermore, the error function $\tilde{E}(s) := \tilde{E}(s;\delta)$ is holomorphic on $\{\mathrm{Re}\, s < N+1\}$ and

$$|\tilde{E}(s,\delta)| \le 2\delta^{\mathrm{Re}\, s-N}|A_\delta\cap\Omega|_N\left(\frac{\pi}{2}-1\right) \tag{4.7.38}$$

for all $s\in\mathbb{C}$ such that $\mathrm{Re}\, s < N+1$. See the proof of Theorem 4.7.2 and Equation (4.7.8) in order to derive the above estimate. The estimate (4.7.38) now implies that the sequence of holomorphic functions $\tilde{E}(\,\cdot\,;n)$ tends to 0 as $n\to\infty$, uniformly on every compact subset of $\{\mathrm{Re}\, s < N\}$, since $|A_n\cap\Omega| = |\Omega|$ for all n sufficiently large. Furthermore, we also have that $\tilde{\zeta}_{A,\Omega}(\,\cdot\,;n)\to\zeta^{\mathfrak{M}}_{A,\Omega}$ and

$$\tilde{\zeta}_{A\times\{0\},\Omega\times\mathbb{R}}(s;n)\to\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}} \quad \text{as} \quad n\to\infty, \tag{4.7.39}$$

uniformly on every compact subset of $\{\overline{D} < \mathrm{Re}\, s < N\}$. This implies that by taking the limit in (4.7.37) as $\delta \to +\infty$, we obtain the following functional equality between holomorphic functions:

$$\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}}(s) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)}\,\zeta^{\mathfrak{M}}_{A,\Omega}(s), \tag{4.7.40}$$

valid in the vertical strip $\{\overline{D} < \mathrm{Re}\, s < N\}$. We can obtain this equality even more directly by applying Lebesgue's dominated convergence theorem to a counterpart of (4.7.25).

Moreover, according to (4.7.32) and (4.7.40), we have the functional equation

$$\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}}(s) = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)}\,\zeta_{A,\Omega}(s), \tag{4.7.41}$$

from which we deduce that the right-hand side admits a meromorphic extension to the right half-plane $\{\mathrm{Re}\, s > \overline{D}\}$, with simple poles located at the simple poles of $\Gamma((N-s)/2)$; that is, at $s_k := N+2k$ for all $k \in \mathbb{N}_0$. (Observe that in the above ratio of gamma functions, there are no cancellations between the poles of the numerator and of the denominator; indeed, an integer cannot be both even and odd.) From this we conclude that by the principle of analytic continuation, the same property also holds for the left-hand side of (4.7.41) and, furthermore, the left-hand side has a meromorphic extension to any domain $U \subseteq \mathbb{C}$ to which the right-hand side can be meromorphically extended.

In order to complete the proof of the theorem, we now observe that for any $a > 0$, since

$$\begin{aligned}\big|(A\times\{0\})_t \cap (\Omega\times\mathbb{R})\big| &= \big|(A\times\{0\})_t \cap (\Omega\times(-a,a))\big| \\ &\quad + \big|(A\times\{0\})_t \cap (\Omega\times(\mathbb{R}\setminus(-a,a)))\big|,\end{aligned}$$

the left-hand side of (4.7.41) can be split into two parts, as follows:

$$\begin{aligned}\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}}(s) &= \zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times(-a,a)}(s) \\ &\quad + \int_a^{+\infty} t^{s-N-2}\big|(A\times\{0\})_t \cap (\Omega\times(\mathbb{R}\setminus(-a,a)))\big|\,\mathrm{d}t \\ &= \frac{\zeta_{A\times\{0\},\Omega\times(-a,a)}(s)}{N+1-s} - \frac{E(s;a)}{N+1-s}.\end{aligned}$$

We then combine this observation with (4.7.41) to obtain (4.7.35). From the theory developed in this chapter (see Theorem 4.1.7), we know that $\zeta_{A\times\{0\},\Omega\times(-a,a)}(s)$ is holomorphic on the open right half-plane $\{\mathrm{Re}\, s > \overline{D}\}$. Furthermore, much as in the proof of Proposition 4.7.12, we can show that $E(s) := E(s;a)$ defines a holomorphic function on the open left half-plane $\{\mathrm{Re}\, s < N\}$. This fact, along with the functional equation (4.7.35), now ensures that $E(s;a)$ admits a meromorphic continuation to all of $\mathbb{C}$, with a set of simple poles equal to $\{N+2k : k \in \mathbb{N}_0\}$. (Note that $\zeta_{A,\Omega}(s) > 0$

for all $s \in [N, +\infty)$, which implies that there are no zero-pole cancellations on the right-hand side of (4.7.35).) This completes the proof of Theorem 4.7.13. ☐

We note that in Example 4.7.15 below, we actually want to embed (A, Ω) into $\mathbb{R}^{N+1}$, as $(A \times \{0\}, \Omega \times (0,a))$ for some $a > 0$. By looking at the proof of the above theorem and using a suitable symmetry argument, we can obtain the following result, which deals with this type of embedding.

Theorem 4.7.14. *Let* (A, Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *such that* $\overline{D} := \overline{\dim}_B(A, \Omega) < N$. *Then, the following approximate functional equation holds*:

$$\zeta_{A\times\{0\},\Omega\times(0,a)}(s) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}\right)}{2\Gamma\left(\frac{N+1-s}{2}\right)}\zeta_{A,\Omega}(s) + E(s;a), \tag{4.7.42}$$

initially valid for all $s \in \mathbb{C}$ *such that* $\operatorname{Re} s > \overline{D}$. *Here, the* error function $E(s) := E(s;a)$ *is initially given* (*for all* $s \in \mathbb{C}$ *such that* $\operatorname{Re} s < N$) *by*

$$E(s;a) := (s-N-1)\int_a^{+\infty} t^{s-N-2}|(A\times\{0\})_t \cap \Omega \times (\mathbb{R}\setminus(0,a))|_{N+1}\,\mathrm{d}t, \tag{4.7.43}$$

and admits a meromorphic continuation to all of $\mathbb{C}$, *with a set of simple poles equal to* $\{N + 2k : k \in \mathbb{N}_0\}$.

Moreover, Equation (4.7.42) *remains valid on any connected open neighborhood of the critical line* $\{\operatorname{Re} s = \overline{D}\}$ *to which* $\zeta_{A,\Omega}$ (*or, equivalently,* $\zeta_{A\times\{0\},\Omega\times(0,a)}$) *can be meromorphically continued.*

Example 4.7.15. (*Complex dimensions of the Cantor dust RFD*). In this example, we will consider the relative fractal drum consisting of the Cantor dust contained in $[0,1]^2$ and compute its distance zeta function. More precisely, let $A := C^{(1/3)} \times C^{(1/3)}$ be the Cantor dust (i.e., the Cartesian product of the ternary Cantor set $C := C^{1/3}$ by itself; see Figure 1.2 of Subsection 1.1) and let $\Omega := (0,1)^2$. We will not obtain an explicit formula in a closed form but we will instead use Theorem 4.7.14 in order to deduce that the distance zeta function of the Cantor dust has a meromorphic continuation to all of $\mathbb{C}$.

More interestingly, we will also show that the set of complex dimensions of the Cantor dust is a subset of the union of a periodic set contained in the critical line $\{\operatorname{Re} s = \log_3 4\}$ and the set of complex dimensions of the Cantor set (which is a periodic set contained in the critical line $\{\operatorname{Re} s = \log_3 2\}$). *This fact is significant because it shows that in this case, the distance* (*or tube*) *zeta function also detects the 'lower-dimensional' fractal nature of the Cantor dust.*

Note that, as is well known, the Minkowski dimension of the RFD (or Cantor string) $(C, (0,1))$ is given by $\dim_B(C, (0,1)) = \log_3 2$ (see [Lap-vFr1, Subsection 1.2.2] or Equation (2.2.17) in Example 2.2.6 above). Furthermore, it will follow from the discussion below that, as might be expected since $(A, \Omega) = (C, (0,1)) \times (C, (0,1))$,

$$\dim_B(A, \Omega) = 2\dim_B(C, (0,1)) = \log_3 4. \tag{4.7.44}$$

Consequently, it follows that the critical line of the RFD in $\mathbb{R}$ ('Cantor string') $(C,(0,1))$ is the vertical line $\{\mathrm{Re}\, s = \log_3 2\}$, while the critical line of the RFD in $\mathbb{R}^2$ ('Cantor dust') (A,Ω) is the vertical line $\{\mathrm{Re}\, s = \log_3 4\}$, as was stated in the previous paragraph.

The construction of the RFD (A,Ω) can be carried out by beginning with the unit square and removing the open middle-third 'cross', and then iterating this procedure ad infinitum. (See Figure 1.2 on page 10.) This procedure implies that we can subdivide the Cantor dust into a countable union of RFDs which are scaled down versions of two base (or generating) RFDs, denoted by (A_1,Ω_1) and (A_2,Ω_2). The first one of these base RFDs, (A_1,Ω_1), is defined by $\Omega_1 := (0,1/3)^2$ and by A_1 being the union of the four vertices of the closure of Ω_1 (namely, of the square $[0,1/3]^2$). Furthermore, the second base RFD, (A_2,Ω_2), is defined by $\Omega_2 := (0,1/3)\times(0,1/6)$ and by A_2 being the ternary Cantor set contained in $[0,1/3]\times\{0\}$.

At the n-th step of the iteration, we have exactly 4^{n-1} RFDs of the type $(a_nA_1,a_n\Omega_1)$ and $8\cdot 4^{n-1}$ RFDs of the type $(a_nA_2,a_n\Omega_2)$, where $a_n := 3^{-n}$ for each $n\in\mathbb{N}$. This observation, together with the scaling property of the relative distance zeta function (see Theorem 4.1.40), yields successively (for all $s\in\mathbb{C}$ with $\mathrm{Re}\, s$ sufficiently large):

$$\begin{aligned}\zeta_{A,\Omega}(s) &= \sum_{n=1}^{\infty} 4^{n-1}\zeta_{a_nA_1,a_n\Omega_1}(s) + 8\sum_{n=1}^{\infty} 4^{n-1}\zeta_{a_nA_2,a_n\Omega_2}(s)\\ &= \left(\zeta_{A_1,\Omega_1}(s)+8\zeta_{A_2,\Omega_2}(s)\right)\sum_{n=1}^{\infty} 4^{n-1}\cdot 3^{-ns}\\ &= \frac{1}{3^s-4}\left(\zeta_{A_1,\Omega_1}(s)+8\zeta_{A_2,\Omega_2}(s)\right).\end{aligned}\tag{4.7.45}$$

Moreover, for the relative distance zeta function of (A_1,Ω_1), we have

$$\begin{aligned}\zeta_{A_1,\Omega_1}(s) &= 8\int_0^{1/6}\mathrm{d}x\int_0^x\left(\sqrt{x^2+y^2}\right)^{s-2}\mathrm{d}y\\ &= 8\int_0^{\pi/4}\mathrm{d}\theta\int_0^{1/6\cos\theta} r^{s-1}\,\mathrm{d}r\\ &= \frac{8}{6^s s}\int_0^{\pi/4}\cos^{-s}\theta\,\mathrm{d}\varphi = \frac{8I(s)}{6^s s},\end{aligned}\tag{4.7.46}$$

where $I(s) := \int_0^{\pi/4}\cos^{-s}\theta\,\mathrm{d}\theta$ is easily seen to be an entire function (by means of Theorem 2.1.45 with $\varphi(\theta) := \cos^{-1}\theta$ for $\theta\in(0,\pi/4)$).[58] Consequently, $\zeta_{A,\Omega}$ admits a meromorphic continuation to all of $\mathbb{C}$ and we have

$$\zeta_{A,\Omega}(s) = \frac{8}{3^s-4}\left(\frac{I(s)}{6^s s}+\zeta_{A_2,\Omega_2}(s)\right),\tag{4.7.47}$$

[58] In fact, $I(s) = 2^{-1}\mathrm{B}_{1/2}(1/2,(1-s)/2)$, where $\mathrm{B}_x(a,b) := \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ is the *incomplete beta function*.

for all $s \in \mathbb{C}$. Furthermore, let $\zeta_{C,(0,1)}$ be the relative distance zeta function of the Cantor middle-third set constructed inside $[0,1]$; see Example 5.5.3 in Chapter 5 below. From Theorem 4.7.14 and the scaling property of the relative distance zeta function (Theorem 4.1.40), we now deduce that

$$\begin{aligned}\zeta_{A_2,\Omega_2}(s) &= \frac{\sqrt{\pi}\Gamma\left(\frac{1-s}{2}\right)}{2\Gamma\left(\frac{2-s}{2}\right)}\zeta_{3^{-1}C,3^{-1}(0,1)}(s) + E(s;6^{-1}) \\ &= \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\frac{\sqrt{\pi}}{6^s s(3^s-2)} + E(s;6^{-1}),\end{aligned} \tag{4.7.48}$$

where $E(s;6^{-1})$ is meromorphic on all of $\mathbb{C}$ with a set of simple poles equal to $\{2k+1 : k \in \mathbb{N}_0\}$; so that for all $s \in \mathbb{C}$, we have

$$\zeta_{A,\Omega}(s) = \frac{8}{s(3^s-4)}\left(\frac{I(s)}{6^s} + \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\frac{\sqrt{\pi}}{6^s s(3^s-2)} + E(s;6^{-1})\right). \tag{4.7.49}$$

Formula (4.7.49) implies that $\mathscr{P}(\zeta_{A,\Omega})$, the set of all complex dimensions (in $\mathbb{C}$) of the 'relative' Cantor dust, is a subset of

$$\left(\log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \{0\} \tag{4.7.50}$$

and consists of simple poles of $\zeta_{A,\Omega}$. Of course, we know that $\log_3 4 \in \mathscr{P}(\zeta_{A,\Omega})$, but we can only conjecture that the other poles on the critical line $\{\mathrm{Re}\, s = \log_3 4\}$ are in $\mathscr{P}(\zeta_{A,\Omega})$ since it may happen that there are zero-pole cancellations in (4.7.49). On the other hand, since it is known that the Cantor dust is not Minkowski measurable (see [FaZe]), we can deduce from the sufficient condition for Minkowski measurability obtained in Theorem 5.4.2 of Chapter 5 below that there must exist at least two other (necessarily nonreal) poles $s_{\pm k_0} = \log_3 4 \pm \frac{2k_0\pi\mathbb{i}}{\log 3}$ of $\zeta_{A,\Omega}$, for some $k_0 \in \mathbb{N}$.[59] From (4.7.49) we cannot even claim that $0 \in \mathscr{P}(\zeta_{A,\Omega})$ for sure, but we can see that all of the principal complex dimensions of the Cantor set are elements of $\mathscr{P}(\zeta_{A,\Omega})$; i.e., $\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z} \subseteq \mathscr{P}(\zeta_{A,\Omega})$. We conjecture that we also have $\log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z} \subseteq \mathscr{P}(\zeta_{A,\Omega})$; that is, we conjecture that $\mathscr{P}_c(\zeta_{A,\Omega}) = \log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}$.

The above example can be easily generalized to Cartesian products of any finite number of generalized Cantor sets, in which case we conjecture that the set of complex dimensions of the product is contained in the union of sets of complex dimensions of each of the factors, modulo any zero-pole cancellations which may occur. In light of this and other similar examples, it would be interesting to obtain some results about zero-free regions for fractal zeta functions. We leave this problem as a possible subject for future investigations.

[59] Indeed, according to Theorem 5.4.2, $D := \log_3 4$ cannot be the only complex dimension of (A,Ω) on the critical line $\{\mathrm{Re}\, s = D\}$ since otherwise, the Cantor dust would be Minkowski measurable, which is a contradiction.