Probability Properties of Interest Rate Models

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Abstract. The processes of short-term interest rates generate changes in most market indices, as well as form the basis of determining the value of marketable assets and commercial contracts. They play a special role in calculating the term structure of the yield. Therefore, the development of mathematical models of these processes is extremely interesting for financial analysts and researchers of market issues. There are many versions of change of short-term risk-free interest rates in the framework of the theory of diffusion processes. However, there is still no such model, which would be the basis for building a term structure of yields close to that existing in a real financial market. It is interesting to analyze the existing models in order to clarify features of models in a probabilistic sense in more detail than has been done by their creators and users. Such an analysis will be made here for the family of models used by the authors in three well-known papers [1–3], where they were applied for the fitting of the real time series of yield.

Keywords: Yield · Short-term risk-free interest rates · Term structure

1 Introduction

All the models considered belong to the class of diffusion models, that generate processes $X(t)$, described by the equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t),\ t > t_0,\ X(t_0) = X_0, \tag{1}$$

where a specific determination of drift $\mu(x)$ and volatility $\sigma(x)$ defines one or another particular model. Some models, such as models: Vasicek, Cox-Ingersoll-Ross, geometric Brownian motion, Ahn-Gao, are well documented in the literature, but nevertheless their properties are listed here for convenience of comparison with other, less well-known or not investigated models. The analysis is the first part of the work devoted to the explanation of the most suitable short-term rate models to determine the term structure of a zero-coupon yield that is reproducing the actually observed yield, as far as possible, the best way.

2 The Vasicek Model [4]

For $\mu(x) = k(\theta - x)$, $\sigma^2(x) = \sigma^2$ the Eq. (1) generates the Ornstein-Uhlenbeck process that is known in finance literature as the Vasicek model. Probability

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density for this process is normal with the expectation $E[X] = \theta$ and the variance $Var\,[X] = \dfrac{\sigma^2}{2k}$:

$$f(x) = \sqrt{\frac{k}{\pi}}\frac{1}{\sigma}e^{-k\frac{(x-\theta)^2}{\sigma^2}}. \tag{2}$$

3 The CIR Model [5]

When the functions of drift and volatility are $\mu(x) = k(\theta - x)$ and $\sigma^2(x) = \sigma^2 x$ from (1) for the short-term interest rate $r(t)$ a nonnegative process is obtained. In financial literature such a process is named the Cox-Ingersoll-Ross model (the CIR model).

$$dr(t) = k(\theta - r(t))dt + \sqrt{2kD\frac{r(t)}{\theta}}dW(t),$$

where θ and D are the stationary expectation and variance respectively.

The CIR process has a gamma distribution with the scale parameter $c = \dfrac{2k}{\sigma^2}$ and the form parameter $q = \dfrac{2k\theta}{\sigma^2}$. So

$$f(x) = \frac{c^q x^{q-1}}{\Gamma(q)}e^{-cx},\ q > 0,\ x > 0. \tag{3}$$

The moments of this distribution are calculated by the formula

$$E[X^m] = \frac{\Gamma(m+q)}{c^m\Gamma(q)},$$

and important numerical characteristics are the expectation $E[X]$, the variance $Var\,[X]$, the skewness S and the kurtosis K:

$$\begin{aligned} E[X] &= \frac{q}{c} = \theta, \\ Var\,[X] \equiv D &= \frac{q}{c^2} = \frac{\sigma^2\theta}{2k}, \\ S &\equiv \frac{E\left[(X - E[X])^3\right]}{Var\,[X]^{\frac{3}{2}}} = 2\sqrt{q}, \\ K &\equiv \frac{E\left[(X - E[X])^4\right]}{Var\,[X]^2} = 3 + \frac{6}{q}. \end{aligned}$$

4 The Duffie-Kan Model [6]

In the Duffie-Kan model the rate $r(t)$ is generated by Eq. (1) with functions $\mu(x) = k(\theta - x)$ and $\sigma(x) = \sqrt{\gamma x + \delta} \equiv \sqrt{2kD\dfrac{x - r_0}{\theta - r_0}}$:

$$dr(t) = (\alpha r(t) + \beta)dt + \sqrt{\gamma r(t) + \delta}dW(t),\ \gamma r(0) + \delta > 0,$$

where $k = -\alpha > 0$, $\theta = -\dfrac{\beta}{\alpha} > 0$, $D = \dfrac{\beta\gamma - \alpha\delta}{2\alpha^2} > 0$, $r_0 = -\dfrac{\delta}{\gamma} < \theta$.

The process $r(t)$ has the stationary probability density $f(x)$ which is a shifted gamma density with the shift parameter r_0, the scale parameter c and the form parameter q, i.e.

$$f(x) = \frac{c^q (x - r_0)^{q-1}}{\Gamma(q)} e^{-c(x-r_0)}, \; r_0 < x < \infty, \tag{4}$$

where $q = \dfrac{(\theta - r_0)^2}{D}$, $c = \dfrac{(\theta - r_0)}{D} > 0$, r_0 is the limit bottom value of interest rate $r(t)$.

The important numerical characteristics of the stationary density

$$\begin{aligned} E[X] &= \frac{q}{c} = \theta, \\ Var[X] \equiv D &= \frac{q}{c^2}, \\ S &= 2\sqrt{q}, \\ K &= 3 + \frac{6}{q}. \end{aligned}$$

5 The Ahn-Gao Model [2]

In the Ahn-Gao model it is assumed that drift and volatility are nonlinear functions $\mu(x) = k(\theta - x)x$ and $\sigma^2(x) = \sigma^2 x^3$. Such a process has the stationary probability density $f(x)$ of form

$$f(x) = \frac{c^q}{\Gamma(q) x^{1+q}} e^{-\frac{c}{x}}, \; x > 0, \tag{5}$$

where the scale parameter $c = \dfrac{2k\theta}{\sigma^2}$ and the form parameter $q = 2 + \dfrac{2k}{\sigma^2}$. The process of the Ahn-Gao model can be obtained from the CIR process by transformation $X_{AG} = \dfrac{1}{X_{CIR}}$. The important numerical characteristics of the stationary density of process are determined by formulae

$$\begin{aligned} E[X] &= \frac{c}{q-1} = \frac{2k\theta}{2k + \sigma^2}, \\ Var[X] &= \frac{c^2}{(q-1)^2(q-2)} = \frac{2k\sigma^2\theta^2}{(2k + \sigma^2)^2}, \\ S &= 4\frac{\sqrt{q-2}}{q-3}, \\ K &= 3\frac{(q-2)(q+5)}{(q-3)(q-4)}. \end{aligned}$$

6 The BDT Model [7]

The Black-Derman-Toy (BDT) model

$$dr(t) = [\alpha_1 r(t) - \alpha_2 r(t)\ln r(t)]dt + \beta r(t)dW(t),\ \alpha_2 > 0,$$

by transformation $Y(t) = \ln r(t)$ reduces to linear form

$$dY(t) = \left(\alpha_1 - \frac{\beta^2}{2} - \alpha_2 Y(t)\right)dt + \beta dW(t).$$

This equation allows a stationary solution and process $Y(t)$ that is found in explicit form

$$Y(t) = \frac{1}{\alpha_2}\left(\alpha_1 - \frac{\beta^2}{2}\right) + \xi(t),\ \xi(t) = \beta \int_{-\infty}^{t} e^{-\alpha_2 s} dW(s),$$

where $\xi(t)$ is a stochastic Gaussian process with zero expectation, variance $Var\,[\xi(t)] = \dfrac{\beta^2}{2\alpha_2}$ and covariance $Cov\,[t_1, t_2] = \dfrac{\beta^2}{2\alpha_2} e^{-\alpha_2|t_2-t_1|}$. Thus the BDT model generates a log-normal process and allows a stationary regime. The leading stationary moments of the interest rate are calculated by formulae

$$\begin{aligned}
E[r] &= e^{\frac{1}{\alpha_2}\left(\alpha_1 - \frac{\beta^2}{4}\right)},\\
Var\,[r] &= (\lambda - 1)e^{\frac{2}{\alpha_2}\left(\alpha_1 - \frac{\beta^2}{4}\right)},\ \lambda = e^{\frac{\beta^2}{2\alpha_2}},\\
S &= (\lambda + 2)\sqrt{\lambda - 1},\\
K &= \lambda^4 + 2\lambda^3 + 3\lambda^2 - 3.
\end{aligned}$$

7 The Ait-Sahalia Model [8]

Ait-Sahalia has tested the based models of short interest rates (including those described here) by fitting them to the actually time series of rates. It was found that an acceptable level of goodness-of-fit of all these rates was rejected because of the drift and volatility properties. As a result he proposed the following functions of drift and diffusion

$$\mu(r) = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_{-1}\frac{1}{r},\ \sigma^2(r) = \beta_0 + \beta_1 r + \beta_2 r^2.$$

In this model, the non-linear functions of drift and diffusion allow a wide variety of forms. To $\sigma^2(r) > 0$ for any r, it is necessary that the diffusion function parameters ensure the fulfilment of inequalities

$$\beta_0 > 0,\ \beta_2 > 0,\ \gamma^2 \equiv 4\beta_0\beta_2 - \beta_1^2 \geqslant 0.$$

Relevant to this function a probability density is given by the expression

$$f(x) = Nx^B(\beta_0 + \beta_1 x + \beta_2 x^2)^{C-1} e^{Ax + G\mathrm{arctg}\,(E+Fx)},\ x > 0,$$

where N is the normalization constant,

$$A = \frac{2\alpha_2}{\beta_2} < 0,\ B = \frac{2\alpha_{-1}}{\beta_0} > 0,\ C = \frac{\alpha_1}{\beta_2} - \frac{\alpha_2\beta_1}{\beta_2^2} - \frac{\alpha_{-1}}{\beta_0},$$

$$G = \frac{2\left(2\alpha_0 + \frac{\alpha_2\beta_1^2}{\beta_2^2} - \frac{\alpha_1\beta_1}{\beta_2} - \frac{2\alpha_2\beta_0}{\beta_2} - \frac{\alpha_{-1}\beta_1}{\beta_0}\right)}{\gamma},\ E = \frac{\beta_1}{\gamma},\ F = \frac{\beta_2}{\gamma}.$$

Since the density $f(x)$ at $x \to 0$ has order $O(x^B)$, $B > 0$, and at $x \to \infty$ its order is $O(x^{B+C}e^{Ax})$, $A < 0$, then for every finite m the moments $E[X^m]$ exist, but their analytical expressions cannot be obtained, and they can be calculated only numerically.

8 The CKLS Model [1]

In the Chan-Karolyi-Longstaff-Sanders (CKLS) model it is assumed that $\mu(x) = k(\theta - x)$, $\sigma^2(x) = \sigma^2 x^3$. It turns out that a random process corresponding to this model has a stationary density

$$f(x) = \frac{n}{x^3} e^{-c\left(\left(\frac{\theta}{x}\right)^2 - 2\frac{\theta}{x}\right)},\ x > 0, \tag{6}$$

where $c = \dfrac{k}{\theta\sigma^2}$, n is normalization constant. Note that such random process has only the first stationary moment $E[X] = \theta$.

9 The Unrestricted Model I [2]

In "unrestricted model I"

$$dr = (\alpha_1 + \alpha_2 r + \alpha_3 r^2)dt + \sqrt{\alpha_4 + \alpha_5 r + \alpha_6 r^3}dW \tag{7}$$

all the preceding models are embedded, that is, at a certain setting parameters $\{\alpha\}$ can get any of the previous models. The table in this case has the form

Restrictions of parameters	Model	Equation of processes
$\alpha_3 = \alpha_5 = \alpha_6 = 0$	Vasicek	$dr = k(\theta - r)dt + \sigma dW$
$\alpha_3 = \alpha_4 = \alpha_6 = 0$	CIR	$dr = k(\theta - r)dt + \sigma\sqrt{r}dW$
$\alpha_3 = \alpha_6 = 0$	Duffie-Kan	$dr = k(\theta - r)dt + \sqrt{\alpha + \beta r}dW$
$\alpha_1 = \alpha_4 = \alpha_5 = 0$	Ahn-Gao	$dr = k(\theta - r)rdt + \sigma r^{1.5}dW$
$\alpha_3 = \alpha_4 = \alpha_5 = 0$	CKLS	$dr = k(\theta - r)dt + \sigma r^{1.5}dW$

Stationary probability density "unrestricted I" process has the form

$$f(x) = \frac{c(w)}{\sigma^2(x)} e^{\int\limits_w^x \frac{2\mu(u)}{\sigma^2(u)} du} = \frac{c(w)}{\alpha_4 + \alpha_5 x + \alpha_6 x^3} e^{\int\limits_w^x \frac{2\left(\alpha_1 + \alpha_2 u + \alpha_3 u^2\right)}{\alpha_4 + \alpha_5 u + \alpha_6 u^3} du},$$

where $c(w)$ is the normalization constant, w is a fixed number from the set of possible values of a random process, the specific value of which does not play some role.

Getting the explicit form of expression for $f(x)$ is possible, but it will be quite cumbersome in a general case, and we restrict ourselves to the case when the values of the parameters $\{\alpha\}$ provide the performance properties of the probability density $f(x)$. First, we note that the volatility of the real process needs to be a real function, so $\sigma^2(r) = \alpha_4 + \alpha_5 r + \alpha_6 r^3 \geqslant 0$ for all values of r. At the same time analytic properties of the probability density depend on the type of the roots of equation $\alpha_4 + \alpha_5 r + \alpha_6 r^3 = 0$, $\alpha_6 > 0$. The sign of the discriminant $\Delta = \left(\frac{\alpha_5}{3\alpha_6}\right)^3 + \left(\frac{\alpha_4}{2\alpha_6}\right)^2$ specifies the number of real and complex roots of the equation. When $\Delta > 0$, there is one real and two complex conjugate roots. When $\Delta < 0$, there are three different real roots. When $\Delta = 0$, real roots are multiples.

Let $\Delta > 0$ and the real root is $r = r_0$, then we can write

$$\alpha_4 + \alpha_5 r + \alpha_6 r^3 = \alpha_6 (r - r_0)\left(r^2 + pr + q\right),$$

where r_0, p and q are a relatively sophisticated analytical expression and because of that are not listed here. However, if $\alpha_4 = 0$, then $r_0 = 0$, $p = 0$, $q = \frac{\alpha_5}{\alpha_6}$. In this case, the probability density is given by

$$f(x) = \frac{c(w)}{\alpha_6 x \left(x^2 + \frac{\alpha_5}{\alpha_6}\right)} e^{\int\limits_w^x \frac{2\left(\alpha_1 + \alpha_2 u + \alpha_3 u^2\right)}{\alpha_6 u \left(u^2 + \frac{\alpha_5}{\alpha_6}\right)} du}$$

$$= n x^{\frac{2\alpha_1}{\alpha_5} - 1} \left(\alpha_6 x^2 + \alpha_5\right)^{\frac{\alpha_3}{\alpha_6} - \frac{\alpha_1}{\alpha_5} - 1} e^{\frac{2\alpha_2}{\sqrt{\alpha_5 \alpha_6}} \operatorname{arctg}\left[x\sqrt{\frac{\alpha_6}{\alpha_5}}\right]}, \quad (8)$$

where n is the normalization constant. For the existence of the probability density its parameters must satisfy the inequalities: $\frac{\alpha_1}{\alpha_5} > 1$, $\frac{\alpha_3}{\alpha_6} < 1$. In order to at the same time there exist stationary moments it is necessary for the expectation $\frac{\alpha_3}{\alpha_6} < 0.5$, for variance $\frac{\alpha_3}{\alpha_6} < 0$, for the third moment $\frac{\alpha_3}{\alpha_6} < -0.5$ and for the fourth moment $\frac{\alpha_3}{\alpha_6} < -1$.

If $\Delta < 0$, denote the roots of the equation $r_0 > r_1 > r_2$ so

$$\alpha_4 + \alpha_5 r + \alpha_6 r^3 = \alpha_6 (r - r_0)(r - r_1)(r - r_2).$$

Then the probability density is expressed in the form

$$f(x) = n(x - r_0)^{\frac{2\left(\alpha_1+\alpha_2 r_0+\alpha_3 r_0^2\right)}{\alpha_6(r_0-r_1)(r_0-r_2)}-1} \times (x - r_1)^{-\frac{2\left(\alpha_1+\alpha_2 r_1+\alpha_3 r_1^2\right)}{\alpha_6(r_0-r_1)(r_1-r_2)}-1} (x - r_2)^{\frac{2\left(\alpha_1+\alpha_2 r_2+\alpha_3 r_2^2\right)}{\alpha_6(r_0-r_2)(r_1-r_2)}-1}. \quad (9)$$

In this case the inequalities must be performed

$$2(\alpha_1 + \alpha_2 r_0 + \alpha_3 r_0^2) > \alpha_6(r_0 - r_1)(r_0 - r_2),\ \frac{\alpha_3}{\alpha_6} < 1.$$

For the existence of the m-th moment other than that necessary to perform the conditions $\frac{m}{2} + \frac{\alpha_3}{\alpha_6} < 1$. Unfortunately, the analytical expression of the normalization constant n and moments $E[r^m]$ is very cumbersome and they includes hypergeometric functions. Under these assumptions the process with such a density has a bottom line equal to the largest root, i.e. $r(t) \geqslant r_0$.

10 The Unrestricted Model II [1]

In the "unrestricted model II" process of short rate follows the equation

$$dr = k(\theta - r)dt + \sigma r^\gamma dW,\ \gamma > 0. \quad (10)$$

Therefore $\mu(x) = k(\theta - x)$, $\sigma^2(x) = \sigma^2 x^{2\gamma}$ and the stationary density $f(x)$ has form

$$f(x) = \frac{n}{x^{2\gamma}} e^{\frac{1}{x^{2\gamma}}\left(\frac{qx}{1-2\gamma} - \frac{cx^2}{2-2\gamma}\right)},\ x > 0, \quad (11)$$

where $q = \frac{2k\theta}{\sigma^2}$, $c = \frac{2k}{\sigma^2}$, n is the normalization constant. The values of parameter γ, allowing the convergence of the integral of $f(x)$ on the interval $(0, \infty)$, determined by the inequality $\gamma > 0.5$. At the same time, there are two critical points: $\gamma = 0.5$ (in this case, the model is transformed into a short-term rate model CIR) and $\gamma = 1$, when the probability density is reduced to a form that corresponds to process of the Brennan-Schwartz model [9]

$$f(x) = \frac{q^{1+c}}{x^{2+c}\Gamma(1 + c)} e^{-\frac{q}{x}},\ x > 0. \quad (12)$$

When $\gamma = 1.5$, the "unrestricted model II" is known as the CKLS model. The Vasicek model is also a model embedded in the "unrestricted model II" at $\gamma = 0$. For existence of moments of order m, it is necessary the fulfilment of inequality $2\gamma > m + 1$. Unfortunately, the expression for the probability density in the general case does not allow the calculation of moments in analytical form, although for referred particular cases they are simply calculated. For the model CIR

$$E[X^m] = \frac{\Gamma(m + q)}{c^m \Gamma(q)},$$

for Brennan-Schwartz model

$$E[X^m] = q^m \frac{\Gamma(1+c-m)}{\Gamma(1+c)},$$

the moments of order m exist if the inequality $m < 1 + c$ is fulfilled. So that

Model	γ	$E[X]$	$Var\,[X]$	Skewness	Kurtosis
Vasicek	0	θ	$\frac{\sigma^2}{2k}$	0	3
CIR	0.5	$\frac{q}{c} = \theta$	$\frac{q}{c^2} = \frac{\sigma^2\theta}{2k}$	$2\sqrt{q}$	$3 + \frac{6}{q}$
Brennan-Schwartz	1.0	$\frac{q}{c} = \theta$	$\frac{\theta^2}{c-1}$	$\frac{4\sqrt{c-1}}{c-2}$	$\frac{3(c-1)(c+6)}{(c-2)(c-3)}$
CKLS	1.5	$\frac{q}{c} = \theta$	not exist	not exist	not exist

Even before the appearance of the "unrestricted model II" models were used, which then turned out to be special cases of this model. This is the model of the CIR (1980) [10], which is obtained from the Eq. (10), if we assume that $\gamma = 1.5$ and $k = 0$. Another particular version is the CEV model, i.e. the model of constant elasticity of variance that was proposed J. Cox and S. Ross (1976) [11], as in Eq. (10) made $\theta = 0$. The properties of the processes generated by these models can be understood by considering the limiting transition $k \to 0$ in the first model or $\theta \to 0$ in the second. When k and θ are still finite the stationary regimes in the models exist and the probability density of processes for these models is expressed in the form (11). However, in the limiting case $k = 0$ or $\theta = 0$ stationary regimes of processes no longer exist, and the probability density cannot be expressed in the form (11), and can be obtained as solutions of partial differential equations

$$\frac{\partial f(x,t|y,s)}{\partial t} - \frac{1}{2}\frac{\partial^2[\sigma^2 x^3 f(x,t|y,s)]}{\partial x^2} = 0$$

for the CIR model (1980) and

$$\frac{\partial f(x,t|y,s)}{\partial t} + \beta\frac{\partial[x f(x,t|y,s)]}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2[x^{2\gamma} f(x,t|y,s)]}{\partial x^2} = 0$$

for the CEV model at the boundary condition for both equations

$$\lim_{t \to s} f(x,t|y,s) = \delta(x-y).$$

Unfortunately, these equations cannot be solved analytically, but we can say that for $k = 0$ or $\theta = 0$ the process generated by the Eq. (10) becomes unsteady for the CIR model (1980) with the constant expectation and increasing with time variance, and for the CEV model changing with time both the expectation and the variance.

The other non-stationary models are as following.

11 The Merton Model [12]

$$dr(t) = \alpha dt + \sigma dW(t)$$

generates a nonstationary Gaussian process

$$r(t) = r(0) + \alpha t + \sigma W(t)$$

with a linearly varying expectation and linearly increasing variance

$$E\left[r|r(0)\right] = r(0) + \alpha t,\ \ Var\left[r\right] = \sigma^2 t.$$

12 The Dothan Model [13]

The equation of the Dothan model

$$dr = \sigma r dW$$

is solved in explicit form:

$$r(t) = r(0)e^{-0.5\sigma^2 t + \sigma W(t)},$$

which implies that a random process generated by the model has a log-normal distribution and is non-stationary. The expectation is steady, but the variance increases exponentially with time

$$E[r|r(0)] = r(0),\ \ Var\left[r|r(0)\right] = r(0)^2\left(e^{\sigma^2 t} - 1\right).$$

13 The GBM Model [14]

The GBM model is a model of process geometric Brownian motion

$$dr = \beta r dt + \sigma r dW$$

was introduced into the modern financial analysis by P. Samuelson (1965). It generates a non-stationary process of geometric Brownian motion

$$r(t) = r(0)e^{(\beta - 0.5\sigma^2)t + \sigma W(t)}.$$

In this case, the probability density of the interest rate is log-normal. Unlike BDT model, which also generates a log-normal process, moments of $r(t)$ in the GBM model is not constant but increases exponentially with time, in particular,

$$\begin{aligned} E[r|r(0)] &= r(0)e^{\beta t}, \\ Var\left[r|r(0)\right] &= r(0)^2(\lambda - 1)e^{2\beta t},\ \lambda = e^{\sigma^2 t}, \\ S &= (\lambda + 2)\sqrt{\lambda - 1}, \\ K &= \lambda^4 + 2\lambda^3 + 3\lambda^2 - 3. \end{aligned}$$

Expressions for skewness and kurtosis formally coincide with the expressions of these characteristics of the BDT model, but parameter λ here is not constant and increases exponentially with time.

14 Conclusion

As mentioned above, the process of short-term rates is the basis for building a term structure of the yield of zero-coupon bonds. This explains the interest in the analysis of the processes of short-term rates. In the literature there are many articles that made empirical attempts to find a model of short-term rates, for which a term structure closest to the actual observed structure is obtained [1–3]. On the other hand there is also empirical evidence in the literature that the famous model of short-term rates do not provide an acceptable level of goodness-of-fit [8]. Therefore there is a need for analytical studies to determine the degree of risk in the use of a particular model of short-term rates of the yield. As a necessary basis for this information is needed about the probability properties of the short-term rate processes, expressed analytically. This is the subject of this paper that shall be considered as the first stage of this work.

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