# A Tverberg Type Theorem for Matroids

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*In memory of Jirka Matousek*

**Abstract** Let $b(M)$ denote the maximal number of disjoint bases in a matroid $M$. It is shown that if $M$ is a matroid of rank $d+1$, then for any continuous map $f$ from the matroidal complex $M$ into $\mathbb{R}^d$ there exist $t \geq \sqrt{b(M)}/4$ disjoint independent sets $\sigma_1, \ldots, \sigma_t \in M$ such that $\bigcap_{i=1}^{t} f(\sigma_i) \neq \emptyset$.

## 1 Introduction

Tverberg's theorem [15] asserts that if $V \subset \mathbb{R}^d$ satisfies $|V| \geq (k-1)(d+1)+1$, then there exists a partition $V = V_1 \cup \cdots \cup V_k$ such that $\bigcap_{i=1}^{k} \operatorname{conv}(V_i) \neq \emptyset$. Tverberg's theorem and some of its extensions may be viewed in the following general context. For a simplicial complex $X$ and $d \geq 1$, let the *affine Tverberg number* $T(X,d)$ be the maximal $t$ such that for any affine map $f : X \to \mathbb{R}^d$, there exist disjoint simplices $\sigma_1, \ldots, \sigma_t \in X$ such that $\bigcap_{i=1}^{t} f(\sigma_i) \neq \emptyset$. The *topological Tverberg number* $TT(X,d)$ is defined similarly where now $f : X \to \mathbb{R}^d$ can be an arbitrary continuous map.

Let $\Delta_n$ denote the $n$-simplex and let $\Delta_n^{(d)}$ be its $d$-skeleton. Using the above terminology, Tverberg's theorem is equivalent to $T(\Delta_{(k-1)(d+1)}, d) = k$ which

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is clearly the same as $T(\Delta^{(d)}_{(k-1)(d+1)}, d) = k$. Similarly, the topological Tverberg theorem of Bárány, Shlosman and Szűcs [2] states that if $p$ is prime then $TT(\Delta_{(p-1)(d+1)}, d) = p$. Schöneborn and Ziegler [14] proved that this implies the stronger statement $TT(\Delta^{(d)}_{(p-1)(d+1)}, d) = p$. This result was extended by Özaydin [13] for the case when $p$ is a prime power. The question whether the topological Tverberg theorem holds for every p that is not a prime power had been open for long. Very recently, and quite surprisingly, Frick [7] has constructed a counterexample for every non-prime power $p$. His construction is built on work by Mabillard and Wagner [10]. See also [4] and [1] for further counterexamples.

There is a colourful version of Tverberg theorem. To state it let $n = r(d+1) - 1$ and assume that the vertex set $V$ of $\Delta_n$ is partitioned into $d + 1$ classes (called colours) and that each colour class contains exactly $r$ vertices. We define $Y_{r,d}$ as the subcomplex of $\Delta_n$ (or $\Delta^{(d)}_n$) consisting of those $\sigma \subset V$ that contain at most one vertex from each colour class. The colourful Tverberg theorem of Živaljević and Vrećica [16] asserts that $TT(Y_{2p-1,d}, d) \geq p$ for prime $p$ which implies that $TT(Y_{4k-1,d}, d) \geq k$ for arbitrary $k$. A neat and more recent theorem of Blagojević, Matschke, and Ziegler [5] says that $TT(Y_{r,d}, d) = r$ if $r + 1$ is a prime, which is clearly best possible. Further information on Tverberg's theorem can be found in Matoušek's excellent book [11].

Let $M$ be a matroid (possibly with loops) with rank function $\rho$ on the set $V$. We identify $M$ with the simplicial complex on $V$ whose simplices are the independent sets of $M$. It is well known (see e.g. Theorem 7.8.1 in [3]) that $M$ is $(\rho(V) - 2)$-connected. Note that both $\Delta^{(d)}_n$ and $Y_{r,d}$ are matroids of rank $d + 1$. In this note we are interested in bounding $TT(M, d)$ for a general matroidal complex $M$. Let $b(M)$ denote the maximal number of pairwise disjoint bases in $M$. Our main result is the following

**Theorem 1** *Let $M$ be a matroid of rank $d + 1$. Then*

$$TT(M, d) \geq \sqrt{b(M)}/4 \ .$$

In Sect. 2 we give a lower bound on the topological connectivity of the deleted join of matroids. In Sect. 3 we use this bound and the approach of [2, 16] to prove Theorem 1.

## 2 Connectivity of Deleted Joins of Matroids

We recall some definitions. For a simplicial complex $Y$ on a set $V$ and an element $v \in V$ such that $\{v\} \in Y$, denote the *star* and *link* of $v$ in $Y$ by

$$\mathrm{st}(Y, v) = \{\sigma \subset V : \{v\} \cup \sigma \in Y\}$$

$$\mathrm{lk}(Y, v) = \{\sigma \in \mathrm{st}(Y, v) : v \notin \sigma\}.$$

For a subset $V' \subset V$ let $Y[V'] = \{\sigma \subset V' : \sigma \in Y\}$ be the induced complex on $V'$. We regard $\mathrm{st}(Y, v)$, $\mathrm{lk}(Y, v)$ and $Y[V']$ as complexes on the original set $V$ (keeping in mind that not all elements of $V$ have to be vertices of these complexes). Let $f_i(Y)$ denote the number of $i$-simplices in $Y$. Let $X_1, \ldots, X_k$ be simplicial complexes on the same set $V$ and let $V_1, \ldots, V_k$ be $k$ disjoint copies of $V$ with bijections $\pi_i : V \to V_i$. The *join* $X_1 * \cdots * X_k$ is the simplicial complex on $\bigcup_{i=1}^k V_i$ with simplices $\bigcup_{i=1}^k \pi_i(\sigma_i)$ where $\sigma_i \in X_i$. The *deleted join* $(X_1 * \cdots * X_k)_\Delta$ is the subcomplex of the join consisting of all simplices $\bigcup_{i=1}^k \pi_i(\sigma_i)$ such that $\sigma_i \cap \sigma_j = \emptyset$ for $1 \leq i \neq j \leq k$. When all $X_i$ are equal to $X$, we denote their deleted join by $X_\Delta^{*k}$. Note that $\mathbb{Z}_k$ acts freely on $X_\Delta^{*k}$ by cyclic shifts.

**Claim 2** *Let $M_1, \ldots, M_k$ be matroids on the same set $V$, with rank functions $\rho_1, \ldots, \rho_k$. Suppose $A_1, \ldots, A_k$ are disjoint subsets of $V$ such that $A_i$ is a union of at most $m$ independent sets in $M_i$. Then $Y = (M_1 * \cdots * M_k)_\Delta$ is $(\lceil \frac{1}{m+1} \sum_{i=1}^k |A_i| \rceil - 2)$-connected.*

*Proof* Let $c = \lceil \frac{1}{m+1} \sum_{i=1}^k |A_i| \rceil - 2$. If $k = 1$ then $\rho_1(V) \geq \left\lceil \frac{|A_1|}{m} \right\rceil$ and hence $Y = M_1$ is $(\left\lceil \frac{|A_1|}{m} \right\rceil - 2)$-connected. For $k \geq 2$ we establish the Claim by induction on $f_0(Y) = \sum_{i=1}^k f_0(M_i)$. If $f_0(Y) = 0$ then all $A_i$'s are empty and the Claim holds. We henceforth assume that $f_0(Y) > 0$ and consider two cases:

(a) If $M_i = M_i[A_i]$ for all $1 \leq i \leq k$ then $Y = M_1 * \cdots * M_k$ is a matroid of rank

$$\sum_{i=1}^k \rho_i(V) \geq \sum_{i=1}^k \left\lceil \frac{|A_i|}{m} \right\rceil \geq \left\lceil \frac{\sum_{i=1}^k |A_i|}{m} \right\rceil .$$

Hence $Y$ is $(\left\lceil \frac{\sum_{i=1}^k |A_i|}{m} \right\rceil - 2)$-connected.

(b) Otherwise there exists an $1 \leq i_0 \leq k$ such that $M_{i_0} \neq M_{i_0}[A_{i_0}]$. Choose an element $v \in V - A_{i_0}$ such that $\{v\} \in M_{i_0}$. Without loss of generality we may assume that $i_0 = 1$ and that $v \notin \bigcup_{i=1}^{k-1} A_i$. Let $S = \bigcup_{i=1}^k V_i$ and let $Y_1 = Y[S - \{\pi_1(v)\}]$, $Y_2 = \mathrm{st}(Y, \pi_1(v))$. Then

$$Y_1 = (M_1[V - \{v\}] * M_2 * \cdots * M_k)_\Delta .$$

Noting that $f_0(Y_1) = f_0(Y) - 1$ and applying the induction hypothesis to the matroids $M_1[V - \{v\}], M_2, \ldots, M_k$ and the sets $A_1, \ldots, A_k$, it follows that $Y_1$ is $c$-connected. We next consider the connectivity of $Y_1 \cap Y_2$. Write $A_1 = \bigcup_{j=1}^t C_j$ where $t \leq m$, $C_j \in M_1$ for all $1 \leq j \leq t$, and the $C_j$'s are pairwise disjoint. Since $\{v\} \in M_1$, it follows that there exist $\{C_j'\}_{j=1}^t$ such that $C_j' \subset C_j$, $|C_j'| \geq |C_j| - 1$, and $C_j' \in \mathrm{lk}(M_1, v)$ for all $1 \leq j \leq t$. Let

$$M_i' = \begin{cases} \mathrm{lk}(M_1, v) & i = 1, \\ M_i[V - \{v\}] & 2 \leq i \leq k, \end{cases}$$

and

$$A'_i = \begin{cases} \bigcup_{j=1}^{t} C'_j & i = 1, \\ A_i & 2 \le i \le k-1, \\ A_k - \{v\} & i = k. \end{cases}$$

Observe that

$$Y_1 \cap Y_2 = \mathrm{lk}(Y, \pi_1(v)) = (M'_1 * \cdots * M'_k)_\Delta$$

and that $A'_i$ is a union of at most $m$ independent sets in $M'_i$ for all $1 \le i \le k$. Noting that $f_0(Y_1 \cap Y_2) \le f_0(Y) - 1$ and applying the induction hypothesis to the matroids $M'_1, \ldots, M'_k$ and the sets $A'_1, \ldots, A'_k$, it follows that $Y_1 \cap Y_2$ is $c'$-connected where

$$\begin{aligned} c' &= \left\lceil \frac{1}{m+1} \sum_{i=1}^{k} |A'_i| \right\rceil - 2 \\ &= \left\lceil \frac{1}{m+1} \left( \sum_{j=1}^{t} |C'_j| + \sum_{i=2}^{k-1} |A_i| + |A_k - \{v\}| \right) \right\rceil - 2 \\ &\ge \left\lceil \frac{1}{m+1} \left( |A_1| - m + \sum_{i=2}^{k-1} |A_i| + |A_k| - 1 \right) \right\rceil - 2 = c - 1. \end{aligned}$$

As $Y_1$ is $c$-connected, $Y_2$ is contractible and $Y_1 \cap Y_2$ is $(c-1)$-connected, it follows that $Y = Y_1 \cup Y_2$ is $c$-connected. □

Let $M$ be a matroid on $V$ with $b(M) = b$ disjoint bases $B_1, \ldots, B_b$. Let $I_1 \cup \cdots \cup I_k$ be a partition of $[b]$ into almost equal parts $\lfloor \frac{b}{k} \rfloor \le |I_i| \le \lceil \frac{b}{k} \rceil$. Applying Claim 2 with $M_1 = \cdots = M_k = M$ and $A_i = \cup_{j \in I_i} B_j$, we obtain:

**Corollary 3** *The connectivity of $M_\Delta^{*k}$ is at least*

$$\frac{b\rho(V)}{\lceil \frac{b}{k} \rceil + 1} - 2 \ .$$

We suggest the following:

**Conjecture 4** *For any $k \ge 1$ there exists an $f(k)$ such that if $b(M) \ge f(k)$ then $M_\Delta^{*k}$ is $(k\rho(V) - 2)$-connected.*

*Remark* Let $M$ be the rank 1 matroid on $m$ points $M = \Delta_{m-1}^{(0)}$. The chessboard complex $C(k, m)$ is the $k$-fold deleted join $M_\Delta^{*k}$. Chessboard complexes play a key role in the works of Živaljević and Vrećica [16] and Blagojević, Matschke, and Ziegler [5] on the colourful Tverberg theorem. Let $k \ge 2$. Garst [9] and Živaljević and Vrećica [16] proved that $C(k, 2k-1)$ is $(k-2)$-connected. On the other hand, Friedman and Hanlon [8] showed that $\tilde{H}_{k-2}(C(k, 2k-2); \mathbb{Q}) \neq 0$, so $C(k, 2k-2)$ is not $(k-2)$-connected. This implies that the function $f(k)$ in Conjecture 4 must satisfy $f(k) \geq 2k - 1$.

## 3 A Tverberg Type Theorem for Matroids

We recall some well-known topological facts (see [2]). For $m \geq 1, k \geq 2$ we identify the sphere $S^{m(k-1)-1}$ with the space

$$\left\{ (y_1, \ldots, y_k) \in (\mathbb{R}^m)^k : \sum_{i=1}^{k} |y_i|^2 = 1 \ , \ \sum_{i=1}^{k} y_i = 0 \in \mathbb{R}^m \right\} \ .$$

The cyclic shift on this space defines a $\mathbb{Z}_k$ action on $S^{m(k-1)-1}$. The action is free for prime $k$.

The *k-fold deleted product* of a space $X$ is the $\mathbb{Z}_k$-space given by

$$X_D^k = X^k - \{(x, \ldots, x) \in X^k : x \in X\} \ .$$

For $m \geq 1$ define a $\mathbb{Z}_k$-map

$$\phi_{m,k} : (\mathbb{R}^m)_D^k \to S^{m(k-1)-1}$$

by

$$\phi_{m,k}(x_1, \ldots, x_k) = \frac{(x_1 - \frac{1}{k}\sum_{i=1}^{k} x_i, \ldots, x_k - \frac{1}{k}\sum_{i=1}^{k} x_i)}{(\sum_{j=1}^{k} |x_j - \frac{1}{k}\sum_{i=1}^{k} x_i|^2)^{1/2}} \ .$$

We'll also need the following result of Dold [6] (see also Theorem 6.2.6 in [12]):

**Theorem 5 (Dold)** *Let $p$ be a prime and suppose $X$ and $Y$ are free $\mathbb{Z}_p$-spaces such that* $\dim Y = k$ *and $X$ is $k$-connected. Then there does not exist a $\mathbb{Z}_p$-map from $X$ to $Y$.*

*Proof of Theorem 1* Let $M$ be a matroid on the vertex set $V$, and let $f : M \to \mathbb{R}^d$ be a continuous map. Let $b = b(M)$ and choose a prime $\sqrt{b}/4 \leq p \leq \sqrt{b}/2$. We'll show that there exist disjoint simplices (i.e. independent sets) $\sigma_1, \ldots, \sigma_p \in M$ such that $\bigcap_{i=1}^{p} f(\sigma_i) \neq \emptyset$. Suppose for contradiction that $\bigcap_{i=1}^{p} f(\sigma_i) = \emptyset$ for all such choices of $\sigma_i$'s. Then $f$ induces a continuous $\mathbb{Z}_p$-map

$$f_* : M_{\Delta}^{*p} \to (\mathbb{R}^{d+1})_D^p$$

as follows. If $x_1, \ldots, x_p$ have pairwise disjoint supports in $M$ and $(t_1, \ldots, t_p) \in \mathbb{R}^p_+$ satisfies $\sum_{i=1}^p t_i = 1$ then

$$f_*(t_1\pi_1(x_1) + \cdots + t_p\pi_p(x_p)) = (t_1, t_1 f(x_1), \ldots, t_p, t_p f(x_p)) \in (\mathbb{R}^{d+1})^p_D \ .$$

Hence $\phi_{d+1,p} f_*$ is a $\mathbb{Z}_p$-map between the free $\mathbb{Z}_p$-spaces $M_\Delta^{*p}$ and $S^{(d+1)(p-1)-1}$. This however contradicts Dold's Theorem since by Corollary 3 the connectivity of $M_\Delta^{*p}$ is at least

$$\frac{b(d+1)}{\lceil \frac{b}{p} \rceil + 1} - 2 \geq (d+1)(p-1) - 1$$

by the choice of $p$.

□

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