Continuous Time Gathering of Agents with Limited Visibility and Bearing-only Sensing

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Abstract. A group of mobile agents, identical, anonymous, and oblivious (memoryless), having the capacity to sense only the relative direction (bearing) to neighboring agents within a finite visibility range, are shown to gather to a meeting point in finite time by applying a very simple rule of motion. The agents' rule of motion is: set your velocity vector to be the sum of the two unit vectors in $\mathbb{R}^2$ pointing to your "extremal" neighbours determining the smallest visibility disc sector in which all your visible neighbors reside, provided it spans an angle smaller than π, otherwise, since you are "surrounded" by visible neighbors, simply stay put. Of course, the initial constellation of agents must have a visibility graph that is connected, and provided this we prove that the agents gather to a common meeting point in finite time, while the distances between agents that initially see each other monotonically decreases.

Keywords: Gathering · Bearing-only · Convex polygon

1 Introduction

This paper studies the problem of mobile agent convergence, or robot gathering under severe limitations on the capabilities of the agent-robots. We assume that the agents move in the environment (the plane $\mathbb{R}^2$) according to what they currently "see", or sense in their neighborhood. All agents are identical and indistinguishable (i.e. they are anonymous having no i.d's) and, all of them are performing the same "reactive" rule of motion in response to what they see. Our assumption will be that the agents have a finite visibility range V, a distance beyond which they cannot sense the presence of other agents. The agents within the "visibility disk" of radius V around each agent are defined as its neighbors, and we further assume that the agent can only sense the direction to its neighbors, i.e. it performs a "bearing only" measurement yielding unit vectors pointing toward its neighbor. Therefore, in our setting, each agent senses its neighbors within the visibility disk and sets its motion only according to the distribution of unit vectors pointing to its current neighbors. Figure 1 shows a constellation of agents in the plane ($\mathbb{R}^2$), their "visibility graph" and the visibility disks of some of them, each agent moves based on the set of unit vectors pointing to its neighbors.

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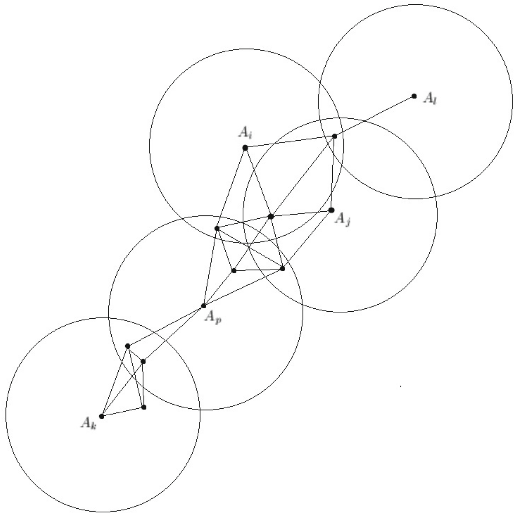

Fig. 1. A constellation of agents in the plane displaying the "visibility disks" of agents A_k, A_l, A_i, A_j, A_p and the visibility graph that they define, having edges connecting pairs of agents that can see each other.

In this paper we shall prove that continuous time limited visibility sensing of directions only and continuous adjustment of agents' velocities according to what they see is enough to ensure the gathering of the agents in finite time to a point of encounter. The literature of robotic gathering is vast and the problem was addressed under various assumptions on the sensing and motion capabilities of the agents. Here we shall only mention papers that deal with gathering assuming continuous time motion and limited visibility sensing, since these are most relevant to our work reported herein. The paper [6] by Olfati-Saber, Fox, and Murray, nicely surveys the work on the topic of gathering (also called consensus) for networked multi-agent systems, where the connections between agents are not necessarily determined by their relative position or distance. This approach to multi-agent systems was indeed the subject of much investigation and some of the results, involving "switching connection topologies" are useful in dealing with constellation-defined visibility-based interaction dynamics too. A lot of work was invested in the analysis of "potential functions" based multi-agent dynamics, where agents are sensing each other through a "distance-based" influence field, a prime example here being the very influential work of Gazi and Passino [7] which analyses beautifully the stability of a clustering process. Interactions involving hard limits on the "visibility distance" in sensing neighbors were analysed in not too many works. Ji and Eggerstedt in [2] analysed such problems using potential functions that are "visibility-distance based barrier functions" and proved connectedness-preservation properties at the expense of making some agents temporarily "identifiable" and "traceable" via a hysteresis process. Ando, Oasa, Suzuki and Yamashita in [8] were the first to deal with hard constraints of limited visibility and analysed the "point convergence" or gathering issue in a discrete-time synchronized setting, assuming agents can see and measure both distances and bearings to neighbors within the visibility range. Subsequently, in a series of papers, Gordon, Wagner, and Bruckstein, in [3–5], analysed gathering with limited visibility and bearing only sensing constraints imposed on the agents. Their work proved gathering or clustering results in discrete-time settings, and also proposed dynamics for the continuous-time settings. In the sequel we shall mention the continuous time motion model they analysed and compare it to our dynamic rule of motion. In our work, as well as most of the papers mentioned above one assumed that the agents can directly control their velocity with no acceleration constraints. We note that the literature of multi-agent systems is replete with papers assuming more complex and realistic dynamics for the agents, like unicycle motions, second order systems and double integration models relating the location to the controls, and seek sensor based local control-laws that ensure gathering or the achievement of some desired configuration. However we feel that it is still worthwhile exploring systems with agents directly controlling their velocity based on very primitive sensing, in order to test the limits on what can be achieved by agents with such simple, reactive behaviours.

2 The Gathering Problem

We consider N agents located in the plane ($\mathbb{R}^2$) whose positions are given by $\{P_k = (x_k, y_k)^T\}_{k=1,2,\ldots,N}$, in some absolute coordinate frame which is unknown to the agents. We define the vectors

$$u_{ij} = \begin{cases} \frac{P_j - P_i}{\|P_j - P_i\|} & 0 < \|P_j - P_i\| \leq V \\ 0 & \|P_j - P_i\| = 0 \text{ or } \|P_j - P_i\| > V \end{cases}$$

hence u_{ij} are, if not zero, the unit vectors from P_i to all P_j's which are neighbors of P_i in the sense of being at a distance less than V from P_i, i.e. P_j's obeying:

$$\|P_j - P_i\| \triangleq [(P_j - P_i)^T (P_j - P_i)]^{1/2} \leq V$$

Note that we have $u_{ij} = -u_{ji}, \forall(i, j)$. For each agent P_i, let us define the special vectors, u_i^+ and u_i^- (from among the vectors u_{ij} defined above). Consider the nonzero vectors from the set $\{u_{ij}\}_{j=1,2,\ldots,N}$. Anchor a moving unit vector $\bar{\eta}(\theta)$ at P_i pointing at some arbitrary neighbor, i.e. at $u_{ik} \neq 0$, $\bar{\eta}(0) = u_{ik}$ and rotate it clockwise, sweeping a full circle about P_i. As $\bar{\eta}(\theta)$ goes from $\eta(0)$ to $\eta(2\pi)$ it will encounter all the possible u_{ij}'s and these encounters define a sequence of angles $\alpha_1, \alpha_2, \ldots, \alpha_r$ that add to $2\pi = \alpha_1 + \ldots + \alpha_r$ (α_k = angle from k-th to (k+1)-th encounter with a u_{ij}, α_r = angle from last encounter to first one again, see Fig. 2). If none of the angles $\{\alpha_1, \ldots, \alpha_r\}$ is bigger than π, set $u_i^+ = u_i^- = 0$. Otherwise define $u_i^+ = u_{i(m)}$ and $u_i^- = u_{i(n)}$, the unit vectors encountered when entering and exiting the angle $\alpha_b > \pi$ bigger than π.

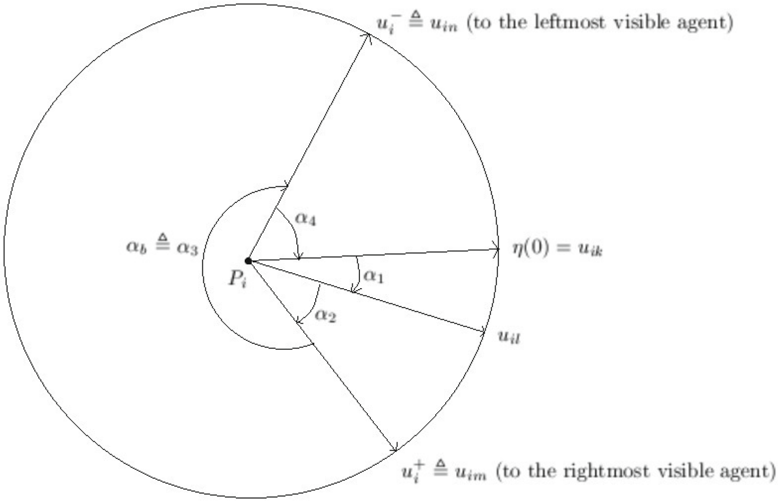

Fig. 2. Leftmost and rightmost visible agents of agent located at P_i.

One might call u_i^- the pointer to the "leftmost visible agent" from P_i and u_i^+ the pointer to the "rightmost visible agent" among the neighbors of P_i. If P_i has nonzero right and leftmost visible agents it means that all its visible neighbors belong to a disk sector defined by an angle less than π, and P_i will be movable. Otherwise we call it "surrounded" by neighbors and, in this case, it will stay in place while it remains surrounded. The dynamics of the multi-agent system will be defined as follows.

$$\frac{dP_i}{dt} = v_0(u_i^+ + u_i^-) \text{ for } i = 1, \ldots, N \tag{1}$$

Note that the speed of each agent is in the span of $[0, 2v_0]$. With this we have defined a local, distributed, reactive law of motion based on the information gathered by each agent. Notice that the agents do not communicate directly, are all identical, and have limited sensing capabilities, yet we shall show that, under the defined reactive law of motion, in response to what they can "see" (which is the bearings to their neighboring agents), the agents will all come together while decreasing the distance between all pairs of visible agents. A simulated example of such a system is given in Fig. 3.

Assume that we are given an initial configuration of N agents placed in the plane in such a way that their visibility graph is connected. This just means that there is a path (or a chain) of mutually visible neighbors from each agent to any other agent. Our first result is that while agents move according to the above described rule of motion, the visibility graph will only be supplemented with new edges and old "visibility connections" will never be lost.

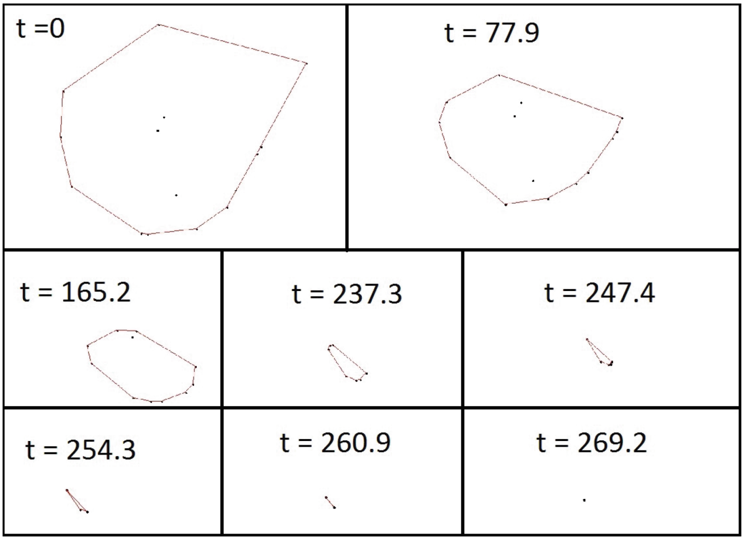

Fig. 3. Simulated evolution at different snapshots of a system composed of 15 agents obeying the laws of (1) with a random initial constellation. The convex hull of the set of agents is also represented.

2.1 Connectivity is Never Lost

We shall show that

Theorem 1. *A multi-agents system under the dynamics*

$$\{\dot{P}_i = v_0(u_i^+ + u_i^-)\}_{i=1,...,N})$$

ensures that pairs of neighboring agents at $t = 0$ *(i.e. agents at a distance less than* V*) will remain neighbors forever.*

Proof. To prove this result we shall consider the dynamics of distances between pairs of agents. We have that the distance Δ_{ij} between P_i and P_j is

$$\Delta_{ij} = \|P_j - P_i\| = [(P_j - P_i)^T(P_j - P_i)]^{1/2}$$

hence

$$\begin{aligned}
\tfrac{d}{dt}\Delta_{ij}^{(t)} &= \tfrac{1}{\|P_j - P_i\|}(P_j - P_i)^T(\dot{P}_j - \dot{P}_i) \\
&= u_{ij}^T(\dot{P}_j - \dot{P}_i) \\
&= -u_{ij}^T\dot{P}_i - u_{ji}^T\dot{P}_j \\
&= -v_0 u_{ij}^T(u_i^+ + u_i^-) - v_0 u_{ji}^T(u_j^+ + u_j^-)
\end{aligned}$$

where we used the dynamics (1). However for every agent P_i we have either $u_i^+ + u_j^- \triangleq 0$ if agent is surrounded, or $u_i^+ + u_i^-$ is in the direction of the center of the disk sector in which all neighbors (including P_j) reside. Therefore the inner product $u_{ij}^T(u_i^+ + u_i^-) =< u_{ij}, (u_i^+ + u_i^-) >$ will necessary be positive, hence

$$\frac{d}{dt}\Delta_{ij}^{(t)} = -(v_0 * positive + v_0 * positive) \leq 0$$

This shows that distances between neighbors can only decrease (or remain the same). Hence agents never lose neighbors under the assumed dynamics.

2.2 Finite-Time Gathering

We have seen that the dynamics of the system (1) ensures that agents that are neighbors at $t = 0$ will forever remain neighbors. We shall next prove that, as time passes, agents acquire new neighbors and in fact will all converge to a common point of encounter. We prove the following:

Theorem 2. *A multi-agent system with dynamics given by (1) gathers all agents to a point in $\mathbb{R}^2$, in finite time.*

Proof. We shall rely on a Lyapunov function $L(P_1, \ldots, P_N)$, a positive function defined on the geometry of agent constellations which becomes zero if and only if all agents' locations are identical. We shall show that, due to the dynamics of the system, the function $L(P_1, \ldots, P_N)$ decreases to zero at a rate bounded away from zero, ensuring finite time convergence. The function L will be defined as the perimeter of the convex hull of all agents' locations, $CH\{P_i(t)\}_{i=1,\ldots,N}$. Indeed, consider the set of agents that are, at a given time t, the vertices of the convex hull of the set $\{P_i(t)\}_{i=1,\ldots,N}$. Let us call these agents $\{\tilde{P}_k(t)\}$ for $k = 1, \ldots, K \leq N$. For every agent $\tilde{P}_k$ on the convex hull (i.e. for every agent that is a corner of the convex polygon defining the convex hull), we have that all other agents, are in a region (wedge) determined by the half lines from $\tilde{P}_k$ in the directions $\tilde{P}_k\tilde{P}_{k-1}$ and $\tilde{P}_k\tilde{P}_{k+1}$, a wedge with an opening angle say θ_k. Since clearly $\theta_k \leq \pi$ for all k we must have that agent $\tilde{P}_k$ has all its visible neighbors in a wedge of its visibility disk with an angle $\alpha_k \leq \theta_k \leq \pi$ hence its u_k^+ and u_k^- vectors will not be zero, causing the motion of $\tilde{P}_k$ towards the interior of the convex hull. This will ensure the shrinking of the convex hull, while it exists, and the rate of this shrinking will be determined by the evolution of the constellation of agents' locations. Let us formally prove that indeed, the convex hull will shrink to a point in finite time. Consider the perimeter $L(t)$ of $CH\{P_i(t)\}_{i=1,\ldots,N}$

$$L(t) = \sum_{k=1}^{K(t)} \Delta_{k,k+1} = \sum_{k=1}^{K(t)} [(\tilde{P}_{k+1})(t) - \tilde{P}_k(t))^T(\tilde{P}_{k+1}(t) - \tilde{P}_k(t))]^{1/2}$$

where the indices are considered modulo $K(t)$.

We have, assuming that K remains the same for a while, that

$$\frac{d}{dt}L(t) = \sum_{k=1}^{K} \frac{d}{dt}\Delta_k = -\sum_{k=1}^{K} \left(v_0 \tilde{u}_{k,k+1}^T(u_k^+ + u_k^-) + v_0 \tilde{u}_{k,k+1}^T(u_{k+1}^+ + u_{k+1}^-)\right)$$

but note that $\tilde{u}_{k,k+1}$ does not necessarily lie between u_k^+ and u_k^- anymore, since, in fact, $\tilde{P}_k$ and $\tilde{P}_{k+1}$ might not even be neighbors.

Now let us consider $\frac{d}{dt}L(t)$ and rewrite it as follows

$$\frac{d}{dt}L(t) = -v_0 \sum_{k=1}^{K} \tilde{u}_{k,k+1}^T(u_k^+ + u_k^-) - v_0 \sum_{k=1}^{K} \tilde{u}_{k+1,k}^T(u_{k+1}^+ + u_{k+1}^-)$$

Rewriting the second term above, by moving the indices k by -1 we get

$$\frac{d}{dt}L(t) = -v_0 \sum_{k=1}^{K} \tilde{u}_{k,k+1}^T(u_k^+ + u_k^-) - v_0 \sum_{k=1}^{K} \tilde{u}_{k,k-1}^T(u_k^+ + u_k^-)$$

This yields

$$\frac{d}{dt}L(t) = -v_0 \sum_{k=1}^{K} < u_k^+, \tilde{u}_{k,k+1} + \tilde{u}_{k,k-1} > -v_0 \sum_{k=1}^{K} < u_k^-, \tilde{u}_{k,k+1} + \tilde{u}_{k,k-1} >$$

Note that we have here inner products between unit vectors, yielding the cosines of the angles between them. Therefore, defining $\theta_k =$ the angle between $\tilde{u}_{k,k-1}$ and $\tilde{u}_{k,k+1}$ (i.e. the interior angle of the convex hull at the vertex k, see Fig. 4), and the angles:

$$\begin{aligned}
\alpha_k^+ &\triangleq \gamma(u_k^+, \tilde{u}_{k,k+1}) \\
\beta_k^+ &\triangleq \gamma(\tilde{u}_{k,k-1}, u_k^+) \\
\alpha_k^- &\triangleq \gamma(\tilde{u}_{k,k-1}, u_k^-) \\
\beta_k^- &\triangleq \gamma(u_k^-, \tilde{u}_{k,k+1})
\end{aligned}$$

we have $\alpha_k^+ + \beta_k^+ = \alpha_k^- + \beta_k^- = \theta_k$ and all these angles are between 0 and π.

Using these angles we can rewrite

$$\frac{d}{dt}L(t) = -\sum_{k=1}^{K} v_0(\cos\alpha_k^+ + \cos\beta_k^+) - \sum_{k=1}^{K} v_0(\cos\alpha_k^- + \cos\beta_k^-)$$

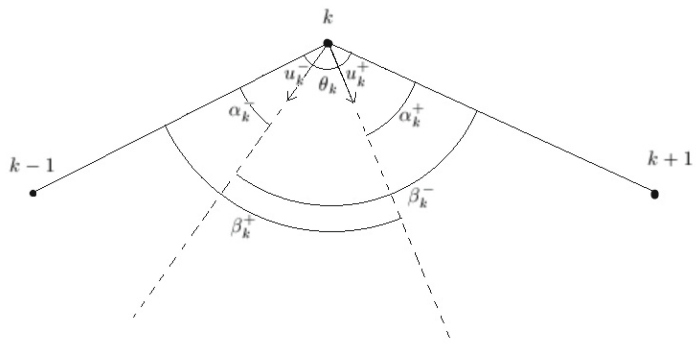

Fig. 4. Angles at a vertex of the convex hull.

Now, using the inequality (proved in Appendix 1)

$$\begin{array}{c} \cos\alpha + \cos\beta \geq 1 + \cos(\alpha+\beta) \\ 0 \leq \alpha, \beta, \alpha+\beta \leq \pi \end{array} \tag{2}$$

we obtain that

$$-\frac{d}{dt}L(t) \geq 2v_0 \sum_{i=1}^{K}(1+\cos\theta_i) \tag{3}$$

For any convex polygon we have the following result (see the detailed proof in Appendix 1):

Lemma 1. *For any convex polygon with K vertices and interior angles $\theta_1, \ldots, \theta_K$, with $(\theta_1 + \ldots + \theta_K) = (K-2)\pi$ we have that*

$$\sum_{k=1}^{K}\cos(\theta_i) \geq \begin{cases} 1+(K-1)\cos\left(\frac{(K-2)\pi}{K-1}\right) & 2 \leq K \leq 6 \\ K\cos\left(\frac{(K-2)\pi}{K}\right) & K \geq 7 \end{cases} \tag{4}$$

Therefore, we obtain from (3) and (4) that

$$-\frac{d}{dt}L(t) \geq \mu(K) \tag{5}$$

where

$$\begin{aligned} \mu(K) = 2v_0 &\left(K + \begin{Bmatrix} 1+(K-1)\cos\left(\frac{(K-2)\pi}{K-1}\right) & 2 \leq K \leq 6 \\ K\cos\left(\frac{(K-2)\pi}{K}\right) & K \geq 7 \end{Bmatrix}\right) \\ = \quad & 2v_0 K\left(1 - \max\left\{\cos\left(\frac{2\pi}{K}\right), \frac{K-1}{K}\cos\left(\frac{\pi}{K-1}\right) - \frac{1}{K}\right\}\right) \end{aligned}$$

Note here that, since $(1 - max\{\ldots\}) > 0$ we have that the rate of decrease in the perimeter of the configuration is srictly positive while the convex hull of the agent location is not a single point.

The argument outlined so far assumed that the number of agents determining the convex hull of their constellation is a constant K. Suppose however that in the course of evolution some agents collide and/or some agents become "exposed" as vertices of the convex hull, and hence K may jump to some different integer value. At a collision between two agents we assume that they merge and thereafter continue to move as a single agent. Since irrespective of the value of K the perimeter decreases at a rate which is strictly positive and bounded away from zero we have effectively proved that in finite time the perimeter of the convex hull will necessarily reach 0. This concludes the proof of Theorem 2.

Figure 5 shows the bound as a function of K assuming $v_0 = 1$. Note that we always have $K \leq N$, and $\mu(K)$ is a decreasing function of K, hence we have an upper bound on the time of convergence for any configuration of N agents given by $\frac{L(0)}{\mu(N)}$.

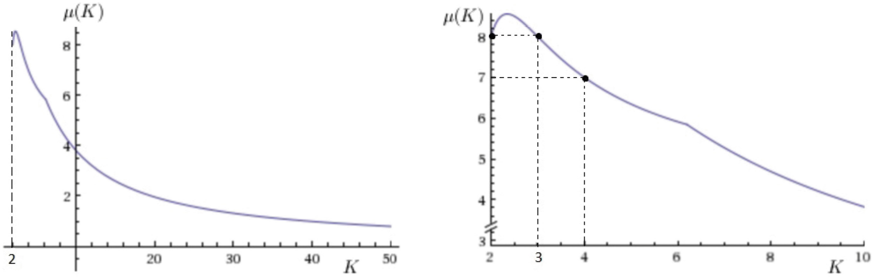

Fig. 5. Graph of the bound $\mu(K)$ of (5). The graph on the right is a zoom on small values of K. Note the change of curve between $K = 6$ and $K = 7$, due to the "interesting" discontinuity in the geometric result exhibited in Eq. (4).

The inequalities of (2) and of (4) become equalities for particular configurations of the agents (for example a regular polygon in which each pair of adjacent neighbors are visible to each other, if $K \geq 7$). In this case, the bound in (3) yields the exact rate of convergence of the convex-hull perimeter as long as K remains the same.

3 Generalizations

All the above analysis can be generalized for dynamics of the form

$$\frac{dP_i}{dt} = f(P^{(i)})(u_i^+ + u_i^-) \text{ for } i = 1, \dots, N \qquad (6)$$

$f(P^{(i)}) \geq 0$ is some positive function of the configuration of the neighbours seen by agent i. This generalization also guarantees that the rule of motion is locally defined and reactive, and defined in the same way for all agents. The dynamics (1) corresponds to a particular case of (6), with $f(P^{(i)}) = v_0 = \textit{constant}$ for all agents.

It is easy to slightly change the proofs above in order to show that Theorem 1 (ensuring that connectivity is not lost) is still valid as long as $f(P^{(i)}) \geq 0$ for all i, and that Theorem 2 (ensuring finite time gathering) is also valid as long as $f(P^{(i)}) \geq \epsilon > 0$ for all i.

Note that in the work of Gordon et al. [3], a constant speed for the agents was considered, and this corresponds to setting $f(P^{(i)}) = \frac{1}{||u_i^+ + u_i^-||}$ for a mobile agent i, rather than v_0. Given that in this case $f(P^{(i)}) \geq \frac{1}{2}$, the conditions for Theorems 1 and 2 are verified, and hence the dynamics with constant speed also ensures convergence to a single point without pairs of initially visible agents losing connectivity. We therefore also have a proof for the convergence of the algorithm that was proposed in the above-mentioned paper.

4 Concluding Remarks

We have shown that a very simple local control on the velocity of agents in the plane, based on limited visibility and bearing only sensing of neighbors ensures their finite time gathering. We provided a very simple geometric proof that finite time gathering is achieved, and provided precise bounds on the rate of decrease of the perimeter of the agent configuration's convex hull. These bounds are based on a geometric lower bound on the sum of cosines of the interior angles of an arbitrary convex planar polygon, that is interesting on its own right (a curious breakpoint occurring in the bound at 7 vertices). Our result may be regarded as a convergence proof for a highly nonlinear autonomous dynamic system, naturally handling dynamic changes in its dimension (the events when two agents meet and merge). The reader is refered to [1] for a more complete analysis of these results including various simulations illustrating them.

Appendix 1: Proof of Lemma 1

We shall first prove the following facts:

Fact a. Let $0 \leq a \leq b \leq \pi$ and $0 \leq a + b \leq \pi$. Then we have

$$\begin{aligned}\sqrt{2(1+\cos(a+b))} = 2\cos\left(\tfrac{a+b}{2}\right) \geq \cos(a) + \cos(b) \geq \\ 2\cos^2\left(\tfrac{a+b}{2}\right) = 1 + \cos(a+b)\end{aligned}$$

Proof. The function cosine is decreasing in $[0, \pi]$, and given that $\frac{a+b}{2} \geq \frac{b-a}{2}$:

$$1 \geq \cos\left(\frac{b-a}{2}\right) \geq \cos\left(\frac{a+b}{2}\right)$$

multiplying by $2\cos\left(\frac{a+b}{2}\right) \geq 0$:

$$\begin{aligned}2\cos\left(\tfrac{a+b}{2}\right) &\geq 2\cos\left(\tfrac{a+b}{2}\right)\cos\left(\tfrac{b-a}{2}\right) \geq 2\cos^2\left(\tfrac{a+b}{2}\right) \\ 2\cos\left(\tfrac{a+b}{2}\right) &\geq \cos(a) + \cos(b) \qquad\quad \geq 1 + \cos(a+b)\end{aligned}$$

A direct consequence is the following fact.

Fact b. Let $0 \leq a, b \leq \pi$. Then

$$\cos(a) + \cos(b) \geq \begin{cases} 1 + \cos(a+b) & : a + b \leq \pi \\ 2\cos\left(\frac{a+b}{2}\right) & : a + b \geq \pi \end{cases}$$

Proof. The first line is already part of Fact a. The second line can be proven by using the left inequality of Fact a with $\pi - a$ and $\pi - b$, noticing that $0 \leq \pi - a \leq \pi$, $0 \leq \pi - b \leq \pi$, and $\pi - a + \pi - b \leq \pi$ for $a + b \geq \pi$.

Now we can prove Lemma 1. Suppose any given initial configuration of the polygon with interior angles $0 \leq x_1, \ldots, x_n \leq \pi$. We then have $x_1 + \ldots + x_n = (n-2)\pi$.

Now repeat the following step: Go through all the pairs of non-zero values (x_i, x_j). As long as there is still a pair verifying $x_i + x_j \le \pi$, transform it from (x_i, x_j) to $(0, x_i + x_j)$. When there are no such pairs, then among all the non-zero values, take the minimum value and the maximum value, say x_i and x_j (they must verify $x_i + x_j \ge \pi$ due to the previously applied process), and transform the pair from (x_i, x_j) to $\left(\frac{x_i+x_j}{2}, \frac{x_i+x_j}{2}\right)$.

Repeat the above process until convergence. We prove that the process converges and that we can get as close as desired to a configuration where all non-zero values are equal. Note that after each step, the sum of the values equals $(n-2)\pi$, and that the values of all x_i's remain between 0 and π.

The number of values that the above process set to zero must be less or equal to 2 in order to have the sum of the n positive values equal to $(n-2)\pi$. Therefore it is guaranteed that after a finite number of iterations, there will be no pairs of nonzero values whose sum is less than π (otherwise this would allow us to add a zero value without changing the sum).

Once in this situation, all we do is replacing pairs of "farthest" non-zero values (x_i, x_j) with the pair $\left(\frac{x_i+x_j}{2}, \frac{x_i+x_j}{2}\right)$. Let us show that all the nonzero values converge to the same value, specifically to their average.

Let k be the number of remaining non-zero values after the iteration t_0 which sets the "last value" to zero. Denote these values at the i-th iteration by $(x_1^{(i)}, \ldots, x_k^{(i)})$. Define:

$$m = \frac{x_1^{(i)} + \ldots + x_k^{(i)}}{k} = \frac{(n-2)\pi}{k}$$

$$E_i = (x_1^{(i)} - m)^2 + \ldots + (x_k^{(i)} - m)^2$$

Without loss of generality, suppose that at the i-th iteration the extreme values were x_1 and x_2 and so we transformed $(x_1^{(i)}, x_2^{(i)})$ into $\left(x_1^{(i+1)} = \frac{x_1^{(i)}+x_2^{(i)}}{2}, x_2^{(i+1)} = \frac{x_1^{(i)}+x_2^{(i)}}{2}\right)$. So we have:

$$\begin{aligned} E_{i+1} - E_i &= 2\left(\frac{x_1^{(i)}+x_2^{(i)}}{2} - m\right)^2 - (x_1^{(i)} - m)^2 - (x_2^{(i)} - m)^2) \\ &= -\tfrac{1}{2}(x_1^{(i)} - x_2^{(i)})^2 \end{aligned}$$

But $x_1^{(i)}$ and $x_2^{(i)}$ being the extreme values, we have for any $1 \le l \le k$:

$$(x_1^{(i)} - x_2^{(i)})^2 \ge (x_l^{(i)} - m)^2$$

and by summing over l we get that:

$$k(x_1^{(i)} - x_2^{(i)})^2 \ge E_i$$

Hence

$$\begin{array}{rcl} E_{i+1} - E_i &=& -\frac{1}{2}(x_1^{(i)} - x_2^{(i)})^2 \leq -\frac{E_i}{2k} \\ E_{i+1} &\leq& \left(1 - \frac{1}{2k}\right) E_i \\ 0 \leq \quad E_i &\leq& \left(1 - \frac{1}{2k}\right)^{i-t_0} E_{t_0} \end{array}$$

proving that E_i converges to zero, i.e. all the non-zero values converge to m.

At each step of the above described process, according to Fact b, the sum of cosines can only decrease. Therefore from any given configuration we can get as close as possible to a configuration in which all non-zero values are equal, without increasing the sum of the cosines. Hence, the minimum value must be reached in a configuration in which all non-zero values are equal.

Since there can be at most only two zero values, the minimum value of the sum of the cosines is the minimum of the following:

- $2 + (n-2)\cos\left(\frac{(n-2)\pi}{n-2}\right) = -(n-4)$ (case with 2 zeros)
- $1 + (n-1)\cos\left(\frac{(n-2)\pi}{n-1}\right)$ (case with 1 zero)
- $n\cos\left(\frac{(n-2)\pi}{n}\right)$ (case with no zero)

An analytical comparison of these values depending on n leads to the result stated in Lemma 1.

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