Chapter 23
The Fundamental Problem of Dynamics

Julian Barbour

Abstract In a world in which all objects are in relative motion, there arises the problem of equilocality: the identification of points in space that have the same position at different times. Newton recognized this as the fundamental problem of dynamics and to solve it introduced absolute space. Inspired by Mach, Einstein created general relativity in the hope of eliminating this controversial concept, but his indirect approach left the issue unresolved. I will explain how the general method of best matching always leads to dynamical theories with an unambiguous notion of equilocality. Applied to the dynamics of Riemannian 3-geometry, it leads to a radical rederivation of general relativity in which relativity of local scale replaces replaces relativity of simultaneity as a foundational principle. Whereas in the standard spacetime picture there is no unique notion of simultaneity or history, if this alternative derivation leads to the physically correct picture both are fixed in the minutest detail. New approaches to several outstanding problems, including singularities and the origin of time's arrows, are suggested.

23.1 Introduction

In his unpublished *De Gravitatione* [1], Newton addressed what might be called *the fundamental problem of dynamics*: if all motion is relative, how can one identify a point in space that has the same location at different times? This is the problem of *equilocality*. Because he did not present the problem or repeat his arguments in the *Principia*, the issue has attracted little attention. In this paper, I will take direct resolution of the problem as the basis of an alternative derivation of general relativity (for a complementary account, see [2]). The main justification for this are new research avenues that are opened up. It is also interesting to see how Einstein's the-

J. Barbour (✉)
College Farm, South Newington, Banbury, Oxon OX15 4JG, UK
e-mail: julian.barbour@physics.ox.ac.uk

J. Barbour
University of Oxford, Oxford, UK

S. Wuppuluri and G. Ghirardi (eds.), *Space, Time and the Limits of Human Understanding*, The Frontiers Collection, DOI 10.1007/978-3-319-44418-5_23

ory can be derived and radically reformulated following essentially one single idea: the definition of position at different instants of time.

My starting point is Leibniz's notion of space. Although it does not solve the problem, it is a first step. I then describe the creation of dynamical theories by *best matching*, which always leads to a notion of equilocality. Equations that govern the evolution of Riemannian 3-geometries are obtained more or less directly by application of best matching under the condition that only angles, but not lengths, can be compared at spatially separated points. This leads to *shape dynamics* [2]. Remarkably, one recovers not only Einstein's evolution equations in a distinguished foliation but also, in a single package, a *prescription* for how the initial-value problem of general relativity is to be solved and the resulting Cauchy data are to be evolved. To the extent the evolution can be continued, this leads to construction of an Einsteinian spacetime in a foliation by spacelike hypersurfaces of constant mean extrinsic curvature and simultaneously a fibration of the spacetime by timelike curves that pass through equilocal points defined by best matching. It is in this sense that history is fixed in minutest detail.

23.2 The Relational Definition of Position

In his famous correspondence [3] with Clarke, Leibniz rejected Newton's absolute space and time, mainly on the basis the *principle of the identity of indiscernibles*: if two supposedly distinct things or states are in fact indistinguishable, then they are in fact one and the same. This led Leibniz to argue that

> if space was an absolute being, there would something happen for which it would be impossible there should be a sufficient reason... Space is something absolutely uniform; and, without the things placed in it, one point of space does not absolutely differ in any respect whatsoever from another point of space. Now from hence it follows, (supposing space to be something in itself, besides the order of bodies among themselves,) that 'tis impossible there should be a reason, why God, preserving the same situations of bodies among themselves, should have placed them in space after one certain particular manner, and not otherwise; why everything was not placed quite the contrary way, for instance, by changing East into West.

When Clarke objected that "space and time are quantities; which situation and order are not" Leibniz responded

> I will here show, how men come to form to themselves the notion of space. They consider that many things exist at once and they observe in them a certain order of co-existence, according to which the relation of one thing to another is more or less simple. This order, is their *situation* or distance. When it happens that one of those co-existent things changes its relation to a multitude of others, which do not change their relation among themselves; and that another thing, newly come, acquires the same relation to the others, as the former had; we then say, it is come into the place of the former; and this change, we call a motion in that body... And though many, or even all the co-existent things, should change according to certain known rules of direction and swiftness; yet one may always determine the relation of situation, which every co-existent acquires with respect to every other co-existent... And

> supposing, or feigning, that among those co-existents, there is a sufficient number of them, which have undergone no change; then we say, that those which have such a relation to those fixed existents, as others had to them before, have now the *same place* which those others had. And that which comprehends all those places, is called *space*.

Note, first, that Leibniz equates situation with *distance* without saying how that is determined and, second, his definition of space requires "a multitude of others, which do not change their relation among themselves". This means that he had not given a definition of space applicable to the realistic situation in which all bodies of the universe are in motion relative to each other. As I noted, Newton had introduced absolute space precisely to overcome this problem (without explaining the difficulty in the *Principia*). The comment that if all bodies move relative to each other "yet one may always determine the relation of situation, which every co-existent acquires with respect to every other co-existent" is correct *at a given instant* but does not solve the real problem: how can one pair up points whose positions are defined relationally at different times and say they *are at the same place*.

This is the problem of equilocality. Unless it is solved, dynamics (understood as the evolution of relative configurations) has no firm foundation. Consider the principle of least action, which plays a truly essential role in both classical and quantum dynamics. The calculation of the action is impossible if one cannot quantify displacements of particles, which in turn is impossible without a notion of equilocality. The Leibniz–Clarke correspondence gives no guidance on this. In Sect. 23.4, I will show how equilocality can be defined provided certain conditions are met. First it will be helpful to present a notion of space somewhat different from Leibniz's.

23.3 Space as the Order of Coexisting Facts

As we have seen, Leibniz claimed that space is the order of coexisting things and, when pushed, defined order as the *distance* between things. However, Leibniz did not say how distance, which plays a primary role in his notion of space, is to be determined. I have not researched the history of distance determination, which clearly involves measurement and is part of the beautiful discipline of metrology. I will merely note that in the famous lecture given in 1854 in which he introduced his generalization of Euclidean geometry, Riemann said that "measurement consists of placing the quantities that are to be compared on top of each other".

At least since the dawn of agriculture, distance measurement has been been important and remarkably easy thanks to one of what I call 'the gifts of nature', by which I mean the ready availability of measuring rods in the form of straight sticks like bamboo canes or ropes. These have an empirical property of the utmost importance: to a high degree they remain mutually congruent. I can take one short cane as unit, use it to mark notches on as many other long canes as I like, move them around individually over large distances and then bring them back together. The ratios of their lengths, as measured by the notches, will not have changed perceptibly. It was surely thanks to this basic property of rods that Pythagoras' theorem was discovered by the ancient Egyptians at the latest about 4000 years ago.

Now consider this scenario. Suppose $N, N \geq 5$, fixed points in our familiar three-dimensional space. Let the $N(N-1)/2$ distances between them be measured, yielding a corresponding number of positive numbers. These numbers, the measured distances, are *empirical facts*. A priori, there is no reason why they should bear any relation to each other. However, it turns out that, provided $N \geq 5$, they will satisfy certain algebraic relations that will hold to the accuracy with which the measurements have been made and our space is Euclidean. In other words, certain combinations of these distances will, to the corresponding accuracy, be zero. Such a state of affairs is a profound fact and indicates the existence of a controlling law.

The consequences of the law are remarkable. It makes *data compression* possible. Instead of representing the geometrical arrangement of the N points by means of the $N(N-1)/2$ positive numbers, one can express it by means of $3N$ coordinates, conveniently taken to be Cartesian ($\mathbf{r}_a, a = 1, \ldots, N$). The measured distances, the separations $r_{ab} := |\mathbf{r}_a - \mathbf{r}_b|$, are invariant under the Euclidean translations and rotations that can be applied to the $\mathbf{r}_a$. If N is large, the data compression is very significant since the number of distances grows as the square of the number of fixed points (Leibniz's coexisting things), whereas the number of coordinates grows only linearly.

The essential geometry revealed by the possibility of data compression can be taken one step further by the introduction of the root-mean-square length of the system:

$$\ell_{\rm rms} := \sqrt{\sum_{a<b} r_{ab}^2}, \quad r_{ab} := |\mathbf{r}_a - \mathbf{r}_b|. \tag{23.1}$$

If we now divide all the r_{ab} by $\ell_{\rm rms}$, the resulting $\tilde{r}_{ab} = r_{ab}/\ell_{\rm rms}$ still satisfy algebraic relations analogous to the ones I have already described. However, they are now 'liberated' from the arbitrary the unit of length and therefore scale-invariant. The scale-free separations $\tilde{r}_{ab}$ are invariants of the similarity group (Euclidean translations and rotations augmented by dilatations). In modern terms, Leibniz's case against Newton, *as applied to a single configuration*, is that it is the invariants $\tilde{r}_{ab}$ which define reality.

In the light of this discussion, what then is space? Intuitively, many people (including Newton one suspects) think of it as something like a perfectly translucent block of ice. I would argue that this is a mistake. It reifies the data compression of empirical facts found in observable relations into space. The empirical facts and the relations they satisfy are all we need. They ensure the data compression and our intuitive understanding. Here it is worth quoting Piaget [4], who comments that "space is often conceived as an empty box into which bodies are fitted" but says

> space is not a container. It is the totality of the relationships between the bodies we perceive or imagine, or rather, the totality of the relationships we use to endow these bodies with a structure. Space is in fact the logic of the apparent world or at least one of the two essential aspects (the other being time).

This seems to me very close to the scenario I described, the only difference being the quantitative sharpening made possible by the 'gifts' nature gives us in the form of near perfect measuring rods.

In summary, I replace Leibniz's aphorism "space is the order of coexisting things" with space is "the order of coexisting *facts*". There is a 'totality of relationships'. Our intuition gives us a *conceptual space* that enables us to understand the logic of the world and predict its consequences.

23.4 Best Matching

As we have seen, Leibniz failed to give a satisfactory definition of motion in a universe in which all things are in motion relative to each other. However, he did point out that, in any given instant, the distances between all the bodies in the universe will be well defined. I now want to show how equilocality, and with it motion, can defined relationally if the number of bodies in the universe is *finite*. Ironically, the key to this is to use the very thing that Leibniz employed to argue against the reality of absolute space: the possibility, in imagination, to place one and the same relative configuration of the universe in different positions in conceptual space without changing anything observable. The 'moving to different positions and orientations' is achieved mathematically by means of the generators of Euclidean translations and rotations. Scaling (dilation) brings in fascinating issues which I will discuss later.

Best matching does not use the Euclidean generators to move relative configurations in space but *relative to each other*. For simplicity, let us suppose the conceptual space is two dimensional, so we can picture it as a flat table. Let us also consider the simplest possible non-trivial dynamical situation of three distinguishable point particles of masses $m_a, a = 1, 2, 3$, interacting through Newtonian gravity.

We start with a single configuration: a triangle with the particles at its vertices. We can lay the triangle on the table wherever we please and then, like Leibniz, use the generators to move it anywhere else. We choose one position. Now we take another, slightly different triangle. We can lay it on the table in any position we choose. Each position will correspond to certain displacements of the particles. Because of the freedom in the different placings, it seems we cannot say there have been any definite motions. This is the problem of relative motion. But there is one placing of the second triangle relative to the first that is uniquely singled out.

To see that, suppose Cartesian coordinates on the table and let the positions of the particles in the first triangle be $\mathbf{r}_a$ and those of the second in an arbitrary placing be $\bar{\mathbf{r}}_a$. Now consider the quantity

$$\mathrm{d}s_{\mathrm{trial}} = \sqrt{\sum_a m_a |\mathbf{r}_a - \bar{\mathbf{r}}_a|^2}. \tag{23.2}$$

This is a positive definite quantity and, for some position of the second triangle relative to the first, must have a minimum. Let this *best matched* (bm) position be $\mathbf{r}_a + \mathrm{d}\mathbf{r}_a^{\,\mathrm{bm}}$. Then

$$\mathrm{d}s^{\mathrm{bm}} = \sqrt{\sum_a m_a \mathrm{d}\mathbf{r}_a^{\,\mathrm{bm}} \cdot \mathrm{d}\mathbf{r}_a^{\,\mathrm{bm}}} \tag{23.3}$$

defines a metric on the space of relative configurations (relative configuration space: RCS) [5, 6]. As I said, best matching keeps Leibnizian displacement of relative configurations but not to place them differently in space but relative to each other. The freedom that created the problem becomes the solution to it. Best matching applies to any finite number of particles and has important properties:

- $\mathrm{d}s^{\mathrm{bm}}$ (23.3) is independent of the position of the first triangle in the conceptual space. The best-matched pair can be moved around in that space in exactly the way Leibniz imagined moving a single configuration around in absolute space without changing anything observable. The $\mathrm{d}s^{\mathrm{bm}}$ are invariants of the Euclidean group.
- $\mathrm{d}s^{\mathrm{bm}}$ (23.3) is unchanged under swapping of the first for the second triangle. The resulting dynamics is time-reversal symmetric.
- Best matching establishes a unique pairing of any one point on the first triangle with a best-matched point on the second triangle. A notion of equilocality is well defined.
- Best matching brings the centres of mass of the configurations to coincidence and 'squeezes' the relative rotation out of the pair. The instantaneous state of the best-matched system has vanishing momentum $\mathbf{P}$ and angular momentum $\mathbf{L}$.

We can now define best-matched N-body dynamics on the timeless relative configuration space. Suppose two such configurations A and B and any continuous curve joining them in the RCS and for it calculate

$$A_{\mathrm{trial}} = \int_A^B \sqrt{(E - V) \sum_a m_a (\mathbf{r}_a + \mathrm{d}\mathbf{r}_a^{\,\mathrm{bm}}) \cdot (\mathbf{r}_a + \mathrm{d}\mathbf{r}_a^{\,\mathrm{bm}})}, \tag{23.4}$$

where E is a constant and V, a potential, is a function on the RCS. For all such curves, one seeks (as in the standard procedure of the calculus of variations) the one that extremalizes (23.4).

There is now a very interesting way to 'stack' the successive configurations in the conceptual space. Place A anywhere. Then move all the configurations, one after another, into their best-matched position relative to their predecessors (for one of the two possible directions chosen for the advance of time). This is called *horizontal stacking* in [5] and leads to dynamical best-matched evolution in the conceptual space with moreover a uniquely preferred time labelling obtained by *vertical stacking* [5, 7].

When this is all done, it is found, first, that the particles evolve in the stacked conceptual space, which is infinitely many copies of the one needed for a single configuration, exactly as would a system in absolute space and time. Newton's framework is not presupposed but *derived*. Second, the system will have energy E and vanishing angular momentum: $\mathbf{L} = 0$. This latter condition does not follow from Newton's equations and is a *prediction* of the theory. The total momentum $\mathbf{P}$ will also be zero in the stacked frame, but one can always find an inertial frame in which that happens in Newtonian theory. In [5] it was asserted that in such an approach the constant E must also vanish, but the argument for that was flawed.

An argument for $E = 0$ is scale invariance. In Newtonian dynamics both E and $\mathbf{L}$ are conserved. If therefore they vanish at some initial time, they will vanish at all subsequent times. Now to give magnitude to E and $\mathbf{L}$ one needs an external scale, which Leibniz would surely reject. However, vanishing of E and $\mathbf{L}$ remains true whatever the scale (choice of unit). This argument for scale invariance is supported by the principle of sufficient reason: if the energy is to have some value, what reason can one give for it to have one value rather than another? A reason for zero is that it alone is independent of the choice of unit.

Although such an argument is not decisive—it would also require a vanishing cosmological constant—scale invariance comes into consideration in another way. As noted in [8, 9], the N-body problem with $E = \mathbf{L} = 0$ has a very interesting property. In all of its solutions, except for a set of measure zero, there is a point J at which the system's size, as measured by its centre-of-mass moment of inertia, passes through a unique minimum and rises to infinity in both time directions. In [9], this point is called the *Janus point* J by analogy with the Roman god because the two halves of the evolution curve are qualitatively the same either side of J and define arrows of time that point in opposite directions away from J. A further striking property of the point J is that at it one can specify fully scale invariant 'mid-point' data that determine the evolution in either direction away from J [9]. Thus, all the solution-determining information that is encoded in the mid-point data (and conserved by the dynamical evolution) is represented in a form invariant under the action of the similarity group. We recall that this group expresses the essence of Euclidean geometry and leads to the construction of the conceptual space from the 'totality of relationships' that Piaget identified as the true basis of our notion of space.

It may also be mentioned that throughout the 20th century many physicists, including Schrödinger, repeatedly rediscovered a relational mechanics of N mass points based on replacement of the kinetic term $\sum_a m_a \dot{\mathbf{r}}_a \cdot \dot{\mathbf{r}}_a$ in the Newtonian action by

$$W = \sum_{a<b} \frac{\dot{r}_{ab}^2}{r_{ab}}, \quad r_{ab} := |\mathbf{r}_a - \mathbf{r}_b|. \tag{23.5}$$

Such an action, augmented by the Newton potential, leads to a very interesting relational theory, see [10]. However, it suffers from a fatal defect: it predicts anisotropy of effective inertial masses at a level ruled out to many orders of magnitude by the most accurate null experiments, of Hughes–Drever type [11], so far performed in physics. This led Bertotti and myself to abandon our original Leibnizian/Machian proposal based on (23.5) and replace it by the theory of [5] based on best matching, in which there is no mass anisotropy. It is well known that Einstein sought to employ the equivalence principle to implement Mach's call for the replacement of absolute motion by relative motion. It is interesting that isotropy of inertial mass, and with it certain aspects of Lorentz invariance, is now confirmed by generalizations of the Hughes–Drever experiment to many orders of magnitude better than the equivalence principle.

23.5 Equilocality in Dynamical Geometry

It is striking that in creating metric geometry in 1854 Riemann did not take into account his words I cited in Sect. 23.3: "measurement consists of placing the quantities that are to be compared on top of each other". Indeed, the central section of his paper is headed *Metrical relationships that a n-dimensional manifold can have under the assumption that any interval can be measured by any other.* The final words here mean that intervals have a definite length whatever their position in the considered manifold. In other words, intervals at spatially separated points can be said to have the same length even though there is obviously no way in which they can be laid on top of each other to confirm that fact.

The analogy between Riemann's assumption and the implicit assumption of a universal notion of simultaneity at spatially separated points is obvious. The difficulty with simultaneity was first clearly noted by Poincaré in 1898 [12] and resolved in 1905 by him and Einstein. So far as I know, the first person to note the significance of Riemann's assumption was Weyl in 1918 [13]. In 1916 Levi-Civita (soon followed independently by Weyl) had discovered parallel transport. Weyl noted that parallel transport of a vector in a Riemannian space brings it back to its original position with a changed angle but the same length. Weyl called this *rigidity of length* and the last vestige of Euclidean 'distance geometry' (*Ferngeometrie*). To eliminate it, he introduced the notion of parallel transport of length by means of a new 1-form field, for which he coined the term *gauge*. Although his idea was later to play a key role in the discovery of the various gauge theories that underlie the standard model of particle physics, Weyl's initial belief in the identity of his 1-form field and the analogous gauge field in electromagnetism ran into the well known difficulties that Einstein noted.

In fact, Weyl's desire to eliminate 'distance geometry' can be realized, without introduction of any auxiliary field, at the level of three-dimensional Riemannian, i.e., with $+++$ signature, geometry as opposed to the $-+++$ Lorentzian four-geometry with which Weyl worked, no doubt because, as he emphasized in the strongest terms in his book *Space–Time–Matter*, he believed there could be no way back from the four-dimensional world of Einstein and Minkowski.

However, there is a case for taking a step back if one can then take two forward or, as the French say, *reculer pour mieux sauter*. The 'jumping off point' to shape dynamics [2] is that though lengths at spatially separated points cannot be directly compared (any more than clock readings can) *angles are absolute*. Their determination is purely local. Thus, a radian is the angle subtended at the centre of a circle by an arc equal in length to the radius, which can be taken infinitesimally small. Such an angle emerges from an 'order of coexisting facts' and truly belongs to a point.

Bearing this in mind, consider now a Riemannian metric g_{ab} in a three-dimensional manifold. At any point g_{ab} is represented by a symmetric 3×3 matrix. Three of its six coordinates encode coordinate information, two encode information about the angle between curves in the manifold that meet at the considered point, and one is a local scale factor. Following Weyl's argument and by analogy with the objection to simultaneity at spatially separated points, this is the one datum that needs to be questioned: two such scales at spatially separated points cannot be compared. It may already be noted that the two angle degrees of freedom in g_{ab}, which constitute the *conformal* part of the geometry, match the two degrees of freedom per space point associated with the gravitational field in general relativity.

It is well known that Clifford, who had translated Riemann's 1854 paper, mooted the idea that three-dimensional Riemannian geometry could be dynamical (see [14], p. 1202). If we say that only position-independent aspects of geometry are real, as opposed to gauge, then we should look to construct dynamics of conformal 3-metrics (defined as equivalence classes of a Riemannian 3-metrics with respect to conformal transformations). If we assume a spatially closed universe, the dynamical arena will be the space of all conformal 3-geometries on a closed 3-manifold: *conformal superspace*, which is obtained by quotienting Riem (the space of Riemannian 3-geometries) by three-dimensional spatial and conformal transformations. The resulting group is analogous to the similarity group of Euclidean geometry and may be called the *Riemann group*.

The question then arises of whether one can create a dynamics of conformal 3-geometries by best matching with respect to the Riemann group. The answer is yes [15]. The theory turns out to be vacuum general relativity derived in a manner that bears only a remote connection with Einstein's derivation and has some remarkable additions and restrictions that I will list shortly. The basic idea is already clearly suggested by the manner in which we imagined slightly different triangles 'placed on top of each other' and moved relative to each other into their best-matching position. In dynamical geometry, we suppose two 3-metrics $g_{ab}(x)$ and $\bar{g}_{ab}(x)$,

$$\bar{g}_{ab}(x) = g_{ab}(x) + \frac{\partial g_{ab}(x)}{\partial \tau}$$

that differ slightly and imagine them initially placed 'on top of each other' by saying that points in the two metrics with the same coordinate x are equilocal. As quantity to be extremalized by best matching, it is natural, without at this stage worrying about simultaneity at spatially separated points, to take

$$A_{\text{trial}} = \int \mathrm{d}\tau \int \mathrm{d}^3x \sqrt{R\, G^{abcd} \frac{\partial g_{ab}}{\partial \tau} \frac{\partial g_{cd}}{\partial \tau}}, \tag{23.6}$$

where R is the (three-dimensional) scalar curvature and $G^{abcd} = g^{ac}g^{bd} - \lambda g^{ab}g^{cd}$ (λ is an as yet undetermined parameter and τ is a time label). I won't attempt to give a detailed first-principles derivation of the *ansatz* (23.6) except to say that the square root ensures reparametrization invariance and hence the absence of an external time.

What is critical is the taking of the square root before the integration over space. This leads to one quadratic constraint per space point. Its interplay with the constraints that arise from the diffeomorphism and conformal best matching ensures that the geometry has the two expected dynamical degrees of freedom. Another critical point is that the conformal best matching is marginally restricted to transformations that preserve the spatial volume and merely redistribute the local scale factor det g_{ab}. The restriction makes it possible for the universe to expand and necessitates the inclusion of λ in G^{abcd}.

As regards the main things that emerge from this shape-dynamic approach, I simply give the main results with references to their derivations:

- Best matching creates a succession of conformal 3-geometries that, at least in an open neighbourhood, stacks by equilocality into a four-dimensional spacetime that satisfies the Einstein equations [15].
- A point and tangent vector in conformal superspace determine such a succession of conformal 3-geometries [16].
- Best matching also *prescribes* solution of the initial-value problem of general relativity by the method that York [17] found by trial and error in 1972. The restriction to *volume-preserving* conformal transformations explains York's hitherto unexplained scaling law for the trace of the extrinsic curvature [15].
- Best matching imposes a distinguished foliation of the emergent spacetime by surfaces of constant mean extrinsic curvature (CMC surfaces) and ensures its propagation by also requiring a lapse-fixing equation to be satisfied [15].
- The attempt to couple matter fields to the evolving conformal geometry enforces a universal light cone (and with it the value -1 of the DeWitt supermetric in (23.6) [18]. The gauge principle for 1-form fields is also enforced. The taking of the square root at each space point in (23.6) is crucial for these results.

I think it must be agreed that the solution to the equilocality problem, which Newton so clearly formulated in *De Gravitazione* perhaps already 20 years before he wrote the *Principia* (and which Leibniz manifestly failed to solve), is thought provoking. As Clifford's reaction showed, once Riemann had at least partially 'loosened up' geometry, so that it is only locally Euclidean, the idea of making geometry dynamical was very natural. In fact, Riemann effectively created the ADM phase space of dynamical geometry and with it the two infinite-dimensional Lie groups (diffeomorphic and conformal) that act on it. It is especially striking that a theory designed to ensure that at spatially separated points only angles can be compared, ensuring *relativity of local scale*, simultaneously enforces *relativity of simultaneity*. One gets two for the price of one—and the gauge principle for good measure. Note that Lorentz invariance emerges late in the programme and, in contrast to Einstein's route to general relativity, is not a derivational postulate. The status of the equivalence principle is interesting. Both it and isotropy of inertial mass are strongly suggested on empirical grounds, but the extraordinarily high accuracy of Hughes–Drever type experiments make them an even more powerful guide to theory construction—by best matching—than the equivalence principle.

23.6 Caveats and Conclusions

The attentive reader will have noted the caveats “if this alternative derivation leads to the physically correct picture” (in the abstract), “to the extent the evolution can be continued” (in the introduction), and “at least in an open neighbourhood” (first of the final set of bullet points). The fact is that the results of [15], including the crucial unique solvability of the lapse-fixing equation, ensure evolution in conformal superspace and an emergent CMC-foliated spacetime *only in an open neighbourhood*. It is well known that CMC foliations have ‘singularity-avoiding’ tendencies, but there are solutions of general relativity in which the complete spacetime cannot be covered by a CMC foliation. The best known example is the Schwarzschild solution.

However, this is not yet a failure of shape dynamics, which rules out all solutions of general relativity for which space, as in a single Schwarzschild solution, is not closed. It is obvious that the universe contains many collapsed objects. It also appears to have begun very smooth, without any such objects. If shape dynamics is to supplant the spacetime representation of gravity, a major (clearly daunting) research project for it is to establish the extent to which the evolution in conformal superspace, and with it CMC foliation of an emergent spacetime, can be continued. However, it is encouraging that shape dynamics and the solution of what I have called the fundamental problem of dynamics suggest promising new directions of research, some more immediately tractable:

- The various arrows of time may have a dynamical origin and be nothing to do with special conditions at the big bang [8, 9].
- Since only shape degrees of freedom are regarded as physical, while scale is gauge, this suggests reconsideration of the singularity theorems in general relativity. They are generally held to signal the demise of classical spacetime, but that will not be so if the shape evolution remains well behaved.
- Most approaches to quantum gravity assume that space and time become discrete at the Planck length. If best matching and the underlying assumption of continuity that goes with it are foundational, the belief in discreteness may be unfounded.
- In quantization, symmetry with respect to four-dimensional diffeo-morphisms may be inappropriate. Instead, symmetry with respect to three-dimensional diffeomorphisms and conformal transformations is suggested.
- If the approach based on best matching is correct, many solutions allowed in the spacetime representation are ruled out. For example, spatial closure is required and could lead to testable predictions.

References

1. Newton, I. (1962). *De gravitatione*. In A.R. Hall & M.B. Hall (Eds.), *Unpublished scientific papers of Isaac Newton*. Cambridge: Cambridge University Press; discussed in Barbour, J. B. (1989). *Absolute or relative motion?* (Vol. 1, p. 612 ff). Cambridge: Cambridge University Press.

2. Mercati, F. *A shape dynamics tutorial*. arXiv:1409.0105.
3. Alexander, H. G. (Ed.). (1956). *The Leibniz–Clarke correspondence*. Manchester University Press.
4. Piaget, J. (1971) *The child's conception of time*. New York: Ballantine Books, Inc. (translation of French original on 1927).
5. Barbour, J., & Bertotti, B. (1982). Mach's principle and the structure of dynamical theories. *Proceedings of the Royal Society of London A*, *382*, 295.
6. Barbour, J. (1994). The timelessness of quantum gravity: I. *Classical and Quantum Gravity*, *11*, 2853.
7. Barbour, J. The nature of time. arXiv:0903.3489.
8. Barbour, J., Koslowski, T., & Mercati, F. (2014) Identification of a gravitational arrow of time. *Physical Review Letters*, *113*(18), 181101. arXiv:1409.0917.
9. Barbour, J., Koslowski, T., & Mercati, F. Entropy and the typicality of universes. arXiv:1507.06498.
10. Assis, A. K. T. (1995). Weber's law and Mach's principle. In J. Barbour & H. Pfister (Eds.), *Mach's Principle*. *From Newton's bucket to quantum gravity*. Boston: Birkhäuser.
11. Hughes, V. W., et al. (1961). Upper limit for the anisotropy of inertial mass from nuclear magnetic resonance. *Physical Review Letters*, *4*, 342 (1960); Drever, R. W. P. A search for anisotropy of inertial mass using a free precession technique. *Philosophical Magazine*, *6*, 683.
12. Poincarè, H. (1913). La mesure du temps. *Review of Mètaphysics Morale*, *6*, 1 (1898); English translation in Poincaré, H. *The value of science*. New York: Science Press.
13. Weyl, H. (1918). Gravitation und elektrizität. *Sitzungsber. d.Preuss. Akad. d. Wissensch.* (p. 465ff).
14. Misner, C. W., Thorne, K. S., & Wheeler, J. A. (1973). *Gravitation*. San Francisco: WH Freeman and Co.
15. Anderson, E., Barbour, J., Foster, B., Kelleher, B., & Murchadha, N. Ó. (2008). The physical gravitational degrees of freedom. *Classical and Quantum Gravity*, *22*, 1795. arXiv:gr-qc/0407104.
16. Barbour, J., & Murchadha, N. Ó. Conformal superspace: The configuration space of general relativity. arXiv:1009.3559.
17. York, J. W. (1972). The role of conformal 3-geometry in the dynamics of gravitation. *Physical Review Letters*, *28*, 1082.
18. Barbour, J., Foster, B., & Murchadha, N. Ó. (2002). Relativity without relativity. *Classical and Quantum Gravity*, *19*, 3217. arXiv:gr-qc/0012089.