Chapter 18
Geometry and Physical Space

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Geometry as a branch of mathematics studies the properties of points, lines, planes, solids, and their higher dimensional analogues. As applied to physical world, geometry is the study of figures in three-dimensional physical space. Until the mid-nineteenth century, 'geometry' meant Euclidean geometry, the axiomatic theory that forms the basis for the plane geometry will be familiar to most with a high school education, and the postulates on which this theory is based were considered to be indubitable truths about physical points and lines. But with the development of non-Euclidean geometries in the nineteenth century, mathematicians began to distinguish between geometry as a theory of physical space and geometries as theories of mathematical spaces. Doing so raises the question of the status of geometry considered as a theory of physical space.

Euclid presented his geometry in the *Elements* (c. 300 BCE), which gathered together and systematized the geometrical knowledge of the day. The presentation is still the paradigm of an axiomatic theory. The *Elements* starts with 23 definitions, five 'common notions' (essentially logical and arithmetical assumptions), and five postulates (now more commonly known as Euclid's *axioms*), as follows:

1. It is possible to draw a straight line from any point to another point.
2. It is possible to produce a finite straight line continuously in a straight line.
3. It is possible to describe a circle with any center and radius.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines (if extended indefinitely) meet on the side on which the angles which are less than two right angles lie.

(Wolfram Mathworld *Elements*, http://mathworld.wolfram.com/Elements.html (accessed April 2016)

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Of these axioms, the first three can be thought of as idealized claims about what can be done with a ruler and pair of compasses, if we assume that these instruments could be arbitrary large. The fourth, though not about our constructing abilities, still has a very basic and intuitive character that makes it seem appropriate as an axiom rather than something in need of proof. The fifth postulate (which is equivalent to the claim that, given a straight line and a point not on that line, there exists one and only one parallel line through that point), however, was long thought to be different from the rest, looking much more like the *theorems* that can be proved on the basis of these axioms than something so basic as to be axiomatic. So while mathematicians had no doubt that all of these axioms were *true* of points and lines in physical space, there were questions about the appropriateness of the fifth axiom *as an axiom*, and attempts were made to prove it from the other four.

A standard method of mathematical proof is proof by contradiction, or *reductio ad absurdum*: we prove that a proposition P follows from a collection of assumptions by deriving a contradiction (or absurdity) from those assumptions together with the negation of P. So in trying to derive Euclid's parallel postulate, a reasonable approach to take would be to start by assuming the first four axioms and the negation of the parallel postulate, and show that this combination of assumptions leads to a contradiction. This approach was taken by many mathematicians including Gerolamo Saccheri (1667−1733), who derived sufficient bizarre seeming consequences from the assumption that the parallel postulate was false that he published his results under the title *Euclides ab Omni Naevo Vindicatus* ('Euclid Vindicated from All Faults'), declaring that parallel axiom as established [1].

However, Saccheri had not succeeded in proving a formal contradiction from the negation of the parallel postulate, and mathematicians began to suspect that the postulate was indeed independent of the other axioms. In the early 19th century Gauss, Bolyai, and Lobachevsky independently came to the conclusion that the assumption that, given a line and a point not on that line there is more than one parallel line running through it, was logically possible (if not true of physical points and lines). Later Gauss's student Riemann explored the hypothesis that there are no parallel lines, and again came to view this assumption as consistent. The independence of the parallel postulate was finally established in 1868 by Beltrami, whose 'Essay on the Interpretation of non-Euclidean Geometry' presented a model for a two dimensional non-Euclidean geometry on a three dimensional Euclidean surface (a pseudosphere). Before long, models had been given for both Bolyai-Lobachevsky (or hyperbolic) geometry and Riemannian (or elliptic) geometry, and it was clear that it was consistent to assume exactly one parallel (Euclid—surfaces of zero curvature); more than one (Bolyai-Lobachevsky—surfaces of negative curvature) or no parallels (Riemann—surfaces of positive curvature). Looking back on Saccheri's book, it could be recognised that his supposed absurdities derived when assuming the parallel postulate to be false were straightforward theorems of these new non-Euclidean geometries.

The development of non-Euclidean geometries was of ground breaking importance in mathematics in allowing the distinction for the first time between geometry as the theory of mathematically possible spaces, and geometry as the theory of physical space. As Michael Scanlan puts it, in a paper on the proof of the independence of the parallel postulate, "In the past, mathematical practice did not involve a distinction between theory and interpretation. In the eighteenth century, mathematics was seen as the 'abstract' study of certain aspects of nature" [2]. Geometry was a body of truths about physical points and straight lines, and it could reasonably be argued by Immanuel Kant that the truths of geometry are synthetic a priori, substantial (not merely definitional) truths about physical space that are knowable a priori through reflection on our intuition of the nature of our experience of space. The development of non-Euclidean geometries, and their acceptance as part of mathematics, brought to the fore the question of what the proper subject matter of mathematics is. While it could be thought that the points and lines of Euclidean geometry were just slightly idealized abstractions from the physically inscribed points and lines of the diagrams used to convince us of Euclid's proofs, we now had new geometries, with their own terminology of 'points' and 'lines' but with different assumptions about parallels. At most one could be true of (idealized) points and lines in physical space, but all were equally good considered as mathematical theories. The distinction between mathematical spaces and physical space was thus drawn, raising the question of the status of mathematical objects as nonspatiotemporal abstracta.

What interests me here, though, is not so much the status of geometry as the theory of mathematical spaces, but the status of geometry as a theory of physical space. Even once the conceptual possibility of non-Euclidean geometries was recognized, it remained in theory acceptable still to think that Euclidean geometry was knowable a priori to be true as a theory of *physical* points and straight lines. Beltrami's and Klein's models of non-Euclidean geometries showed the consistency of these axiomatic theories by reinterpreting 'point' and particularly 'straight line' to apply to things that were not, by our own lights, *really* straight lines. It remained then possible to argue that, if by point we mean *point in physical space*, and if by straight line we mean *straight line in physical space*, then the mere consistency of alternative geometries should in no way shake our confidence in the truth of Euclid's axioms when understood as a theory of physical points and lines. But in fact, Scanlan tells us, "the mathematicians who originally conceived of non-euclidean geometry, Bolyai, Lobachevsky, and to some extend Gauss, seem all to have conceived of the theory as one which is potentially applicable to physical space" [2], and while our experience locally is Euclidean, the question was raised as to whether on a large scale the Euclidean laws continue to hold. The story is told of Gauss measuring the angles of the triangle formed by three mountain peaks looking for evidence that on a large scale the angles did not add up to two right angles (which is equivalent to the falsity of the parallel postulate), though it is unclear that Gauss's interest here was testing the possibility of curvature in space, as opposed to effects on measurements due to the curvature of the earth's surface. Lobachevsky, however, explicitly conceived of a test of the geometry of space suggesting that one might measure a stellar triangle consisting of the distant star Sirius together with two different positions of the earth at six month intervals, to determine whether the angles were as predicted in a Euclidean or non-Euclidean geometry (see [3], p. 15).

For these mathematicians, then, the question of the correct geometry of physical space was now an empirical matter, to be determined by experiment.

We now know that our best physical theory of space and time is general relativity, according to which spacetime has *variable* curvature (neither Euclidean, nor elliptic, nor hyperbolic, all of which are geometric theories of constant curvatures). One major confirmation of this theory was via the measurement of distant stars. In 1919 the physicist Arthur Eddington travelled to the island of Principe, close to the equator and just off the coast of western Africa, to photograph the solar eclipse of 29 May (see Kennefick [4]). With the sun's light dimmed by the eclipse, it was possible to photograph positions of bright stars from the Hyades cluster beyond the sun, and to achieve measurement results that are generally taken to have confirmed Einstein's prediction of a space curved by the presence of the sun over Newton's assumed flat Euclidean space.

Should we conclude, then, that the question of whether physical space is Euclidean is an empirical one, answerable—and indeed answered in the negative—by experiment? Henri Poincaré argued forcefully against this conclusion, and in favour of the view that the question of the appropriate geometry of physical space is not a priori or *empirical* but rather a matter of convention. Thus at the turn of the twentieth century, Poincaré, well aware of the proposals for empirical tests of geometrical hypotheses, though prior to the development and testing of general relativity, could write:

> If Lobatschewsky's geometry is true, the parallax of a very distant star will be finite. If Riemann's is true, it will be negative. These are the results which seem within the reach of experiment, and it is hoped that astronomical observations may enable us to decide between the two geometries. But what we call a straight line in astronomy is simply the path of a ray of light. If, therefore, we were to discover negative parallaxes, or to prove that all parallaxes are higher than a certain limit, we should have a choice between two conclusions: we could give up Euclidean geometry, or modify the laws of optics, and suppose that light is not rigorously propagated in a straight line. It is needless to add that every one would look upon this solution as the more advantageous. Euclidean geometry, therefore, has nothing to fear from fresh experiments. [5]

According to Poincaré, then, the status of geometry as a theory of space is neither a priori nor empirical, but *conventional*, simply a matter of how we define our terms.

> What, then, are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true, and if the old weights and measures are false; if Cartesian co-ordinates are true and polar coordinates false. One geometry cannot be more true than another; it can only be more convenient. Now, Euclidean geometry is, and will remain, the most convenient [5].

In stating that, in the light of apparent experimental refutation we could choose to alter our hypothesis that light propagates in straight lines rather than altering our geometry, Poincaré's discussion suggests that his conventionalism is simply an application of what has become known as the Quine-Duhem thesis, the claim that, given that no theoretical statement can be tested in isolation, but only against a backdrop of further theoretical assumptions, it is always possible to hold on to any statement in light of recalcitrant experience, simply by adjusting assumptions in our background theory: "Any statement can be held true come what may, if we make drastic enough adjustments elsewhere in the system." (Quine [6], p. 43). But if Poincaré's claim is simply that we could redefine our terms so that 'straight line' doesn't mean 'path taken by a light ray', then this is not a terribly exciting form of conventionalism: there would still be the empirical question of what paths light rays take, and it would still be an empirical matter whether the paths taken by light rays are best described by a Euclidean or a non-Euclidean geometry.

In fact, Poincaré's conventionalism is about more than simply choosing our terms, as is seen in a thought experiment he presents of a world enclosed within a sphere with some rather peculiar properties. In this world, there is a property like temperature, which varies according to the distance from the centre. If R is the radius of the sphere and r the distance of a point in the sphere from the centre, the temperature at this point is proportional to $R^2 - r^2$. Bodies in this world expand and contract at a uniform rate according to changes in temperature, so that a rod that is a metre long by our standards at the centre will get smaller and smaller as it is moved away from the centre, approaching but never reaching zero (the world consists of all points inside the sphere, but not the sphere's boundary). Imagine a plane in this sphere consisting of a great circle of the sphere (i.e. cutting through the centre and with diameter R), and imagine an inhabitant starting at the centre of the plane with a surveyor's wheel with circumference 1 m. As they walk along a radius of the sphere towards the edge, both they and their wheel will contract uniformly. To them, they will feel as though their universe is unbounded of infinite extent as however many metres they travel they will be able to continue. From our perspective this is an error—the universe has finite bounds, but the strange behavior of their measuring instruments mean that the inhabitants are unable to realise this. If surveyors in Poincaré's sphere universe continue to take measurements they will conclude that they are living in a hyperbolic geometry of infinite extent, whereas from our perspective they are living in a Euclidean sphere with physical features that affect their ability to measure.

On one reading of this picture, the possibility of the sphere world presents an epistemic challenge to the claim that the question of the proper geometry of physical space is an empirical matter. On this view, two accounts are available that fit the observed phenomena for the inhabitants within the sphere world. One holds that they and their measuring instruments do not change size as they move around, and that the geometry of their world is hyperbolic. The other holds that they and their measuring instruments change size as they move around, and that the geometry of their world is Euclidean. The inhabitants can't choose between these two hypotheses, so for them the question of the 'true' geometry of physical space cannot be determined empirically (even though as a matter of fact the true geometry is Euclidean). Poincaré's own view, though, is that the lack of knowledge in this case is not because the truth is out there but beyond the inhabitants' grasp, but rather, that there is nothing 'out there' to be known. The epistemic view of the inhabitants' predicament holds that there is a fact of the matter about whether their measuring instruments shrink and grow as they move around or stay the same size, but that this fact is unavailable to the inhabitants. Poincaré, on the other hand, holds that the question of whether the inhabitants shrink or not is itself not an empirical matter, but dependent on a conventional choice for us to make about what we are going to take as counting as 'congruence'. There is no 'God's eye view' which determines what is *really* going on in this example. The two descriptions: the measuring instruments stay the same size and the geometry is hyperbolic, and the measuring instruments change in size and the geometry is Euclidean, are equally good ways of describing the same basic facts. We can choose to define 'congruence' in terms of the behaviour of measuring instruments (so that lengths that measure the same when measured by a meter stick that has been transferred from one to the other are counted as congruent), or we could choose to define it so that measuring whether lengths are congruent depends on knowledge of their distance from the centre. Each choice is a matter of conventional decision, and each leads to a different conclusion about 'the' geometry of the space, so the question of which geometry is correct turns out to be answered by conventional decision rather than empirical investigation.

Poincaré's picture can seem compelling once we consider that we too are in the position of the sphere dwellers. We assume by and large that our measuring instruments remain the same size as we move around in our universe, but an alternative picture according to which we grow and shrink according to location could also be made compatible with our observations. Should we, then, conclude that the question of the correct geometry of physical space can only be made sense of downstream of a conventional decision, and as such, is itself a matter of convention rather than empirical fact? Despite the conventionalist elements of the aforementioned 'Quine-Duhem' thesis, the empiricist response to this conventionalist claim is actually to be found in the work of W.V. Quine. Poincaré's conventionalism depends on holding that there are some elements of our theories that are purely conventional choices about how to set the meanings of terms, that before we can measure and build theories we have to define our terms, and these definitions are a matter of pure convention. Quine argues forcefully against this picture, holding that in the web of beliefs that makes up our best empirically tested theory of the world any element, including those that were originally introduced as conventional definitions to get theorizing going, can be amended in the light of recalcitrant experience. So even though decisions that may seem arbitrary or conventional may need to be made to get theorizing going, those decisions can be revised in the light of recalcitrant experience, making them as empirical as any other elements of our theories. Thus, Quine writes,

> The lore of our fathers is a fabric of sentences. In our hands it develops and changes, through more or less arbitrary and deliberate additions and revisions of our own, more or less directly occasioned by the continuing stimulation of our sense organs. It is a pale grey lore, black with fact and white with convention. But I have found no substantial reasons for concluding that there are any quite black threads in it, or any white ones [7].

In Quine's view, then, the fact that conventional choices about how to use our terms are made on the way to theorizing does not stop our theories—conventions included—from being empirically tested as a package. Indeed, in special and general relativity our previous assumptions about congruence and the behaviour of measuring instruments are challenged; we now adopt a theory according to which our measurements of length are relative to frame of reference (special relativity) and relative to our location with respect to the distribution of mass in the universe (general relativity). In Quine's view, the success of the theoretical package that includes these assumptions is confirmation of the package as a whole. All truths depend in part on the meaning of terms and in part on how the world is, but in this respect, the claim (supported by general relativity) that the spacetime we inhabit has a non-Euclidean geometry of variable curvature is as empirical as anything can be.

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