Chapter 1
Space as a Source and as an Object of Knowledge: The Transformation of the Concept of Space in the Post-Kantian Philosophy of Geometry

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1.1 Introduction

This paper deals with the transformation of the concept of space in the post-Kantian philosophy of geometry from the second half of the nineteenth century to the early twentieth century. Kant famously characterized space and time as forms of intuitions, which lie at the foundations of the apodictic knowledge of mathematics. The success of his philosophical account of space was due not least to the fact that Euclidean geometry was widely considered to be a model of apodictic certainty at that time. However, such later scientific developments as non-Euclidean geometries and the general theory of relativity called into question the certainty of Euclidean geometry and posed the problem of reconsidering space not so much as a source of knowledge, but as an open question for empirical research.

The first section offers a discussion of the main objections against Kant's view of space as a source of knowledge. The opposed view of space as an object of knowledge emerged in geometrical empiricism, a tradition that can be traced back to such mathematicians as Carl Friedrich Gauss, Nikolai Lobachevsky, Bernhard Riemann, Richard Dedekind, and Felix Klein, and that found one of its clearest expressions in the epistemological writings of the physiologist and physicist Hermann von Helmholtz. The second section provides a general introduction to the new phase of this debate inaugurated by Einstein's general theory of relativity of 1915. On the one hand, Einstein relied on geometrical empiricism for the view that geometry has an empirical meaning. On the other hand, he distanced himself from the received view of space by claiming that the general covariance of his field equations removed from space and time the last remnant of physical objectivity. In the third section,

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I discuss two different strategies for a philosophical account of the transformation of the concept of space from a Kantian perspective: (1) Hermann Weyl vindicated the foundational role of the Kantian concept of space by observing that any coordinate assignment, even in Einstein's relativistic space-time theory, presupposes the ideal perspective of the transcendental subject for its setting; (2) Ernst Cassirer emphasized the heuristic aspect of the concept of space as a hypothetical system of mathematical relations. Although both strategies offer a possible philosophical account of space informed by the sciences, I argue that Cassirer's focus on the structure of spatial notions, rather than their subjective origin, had the advantage of reflecting a variety of uses of spatial concepts in human culture and art, which he considered to be no less essential than scientific concepts to a philosophical account of the concept of space.

1.2 Three Roads to Geometrical Empiricism in the Nineteenth Century

1.2.1 The Philosophical

The first motivation for geometrical empiricism was to overcome some of the philosophical difficulties of Kant's account of space in the *Critique of Pure Reason*. Kant characterized space and time by distinguishing these concepts from sensations, on the one side, and general concepts, on the other. The first distinction is in terms of form and matter of appearance: "Since that within which the sensations can alone be ordered and placed in a certain form cannot itself be in turn sensation, the matter of all appearance is only given to us a posteriori, but its form must all lie ready for it in the mind a priori, and can therefore be considered separately from all sensation" (Kant [1], A20/B34). More specifically, space and time are forms of intuition, according to Kant, insofar as the order of appearance is directly present to the mind in the localization of objects in space and time. Therefore, he claimed that space is a necessary representation, which lies at the foundation of all outer intuitions (A24/B38). Kant went on to argue that space differs from general concepts in the way in which it is related to its parts: whereas general concepts contain a finite collection of possible instantiations under them, any limitation of space, including infinite division, lies in a single concept of space as one of its parts. Therefore, he characterized space as an intuition that lies at the origin of knowledge concerning external reality. The principles of (Euclidean) geometry offered the first example of conceptual knowledge derived from intuition with apodictic certainty (A25/B39).

One classical objection to this view is that it presupposed the syllogistic logic of Kant's time, according to which logical reasoning is restricted to finite domains and construction in intuition is necessary to justify existential assumptions concerning infinite domains (e.g., the parallel to a given line from a point outside it and incommensurable magnitudes).[1] After the emergence of mathematical logic in the nineteenth century, it became possible to account for the same distinction in terms of two different means of logical proof or rules of quantification. This approach to Kant's intuition, which is known as "logical," goes back to Cassirer [3] and was given a modern formulation by Hintikka [4]. Alternative approaches include the phenomenological approach advocated by Parsons [5], among others, based on the fact that immediacy is no less essential than singularity to the Kantian conception of space. But how to accommodate the former feature of intuition with the discovery of non-Euclidean geometry in the nineteenth century? Not only does such a discovery contradict the view that Euclidean geometry is evidently true, but the question arises whether geometric knowledge can be true at all.

In order to overcome this difficulty, Friedman [6] argued for mediating between these two approaches in line with Helmholtz's geometrical empiricism. On the other hand, Helmholtz ruled out the view of geometrical axioms as evident truths by explaining how basic geometric notions are derived by observation and experience of rigid motions. On the other hand, he inferred a naturalized form of intuition by considering the possible changes of perspective of a perceiving subject. As Helmholtz [7, p. 162] put it, "Kant's doctrine of the a priori given forms of intuition is a very fortunate and clear expression of the state of affairs; but these forms must be devoid of content and free to an extent sufficient for absorbing any content whatsoever that can enter the relevant form of perception." Helmholtz believed that the form/content distinction deserved a new formulation after the emergence of experimental psychology of vision, on the one hand, and non-Euclidean geometry, on the other.

1.2.2 The Natural

Helmholtz's naturalization of the Kantian theory of perception goes back to his 1855 lecture "Über das Sehen des Menschen." On that occasion—Helmholtz was delivering the Kant Memorial Lecture as a Professor of Physiology at the University of Königsberg—he maintained that Kant's view that the perception of physical objects presupposes some subjective forms of intuition received an empirical confirmation in Johannes Müller's theory of specific nerve energies. Müller showed that sensuous qualities depend not so much on the perceived object as on the constitution of our nerves in the case of perceptions usually associated with light. His theory accounted for the fact that the same visual sensation can have different causes (e.g., an electric current or a blow to the eye). Vice versa, light does not necessarily cause visual sensations, but, for example, ultraviolet rays cause only chemical reactions.

[1] Several examples are discussed in Friedman [2, Ch. 1].

Helmholtz was Müller's student in Berlin and relied on the same experimental methodology. However, he developed an original approach to visual perception in his *Handbuch der physiologischen Optik* (1867). Helmholtz advocated a theory of local signs, according to which sensations are signs for their stimuli. According to him, visual perception depends on our capacity to interpret those signs by drawing unconscious inferences from nerve stimuli to existing objects. Such a capacity deserves an empirical explanation in terms of psychological—rather than purely physiological—processes. Therefore, Helmholtz called his approach "empirist" and contrasted it with nativist views.[2] In this connection, he distanced himself from Kant's conception of space and time as pure intuitions, whose form can be defined independently of any empirical content. Not only did Kant overlook the empirical conditions for the formation of these concepts, but his theory of pure sensibility led his followers to assume that there are innate laws grounded in the form of spatial intuition, that is, the axioms of (Euclidean) geometry (Helmholtz [9], p. 456).

As it emerges most clearly in Helmholtz's later paper "Die Tatsachen in der Wahrnehmung" (1878), Helmholtz's criticism did not rule out the possibility of generalizing the Kantian notion of form of spatial intuition to all possible combinations of contents. After the passage quoted above, Helmholtz went on to say that "the axioms of geometry limit the form of intuition of space in such a way that it can no longer absorb every thinkable content, if geometry is at all supposed to be applicable to the actual world. If we drop them, the doctrine of the transcendentality of the form of intuition is without any taint. Here Kant was not critical enough in his critique; but this is admittedly a matter of theses coming from mathematics, and this bit of critical work had to be dealt with by the mathematicians" (Helmholtz [7], p. 162).[3] Furthermore, Helmholtz believed that even his empiricist epistemology relied ultimately on the assumption of a lawful course of nature, which sometimes he presented as a condition of the possibility of experience in Kant's sense.[4]

I turn back to the latter issue in connection with Cassirer's theory of the symbolic forms of experience. The following section offers a brief overview of different positions on geometrical empiricism in nineteenth-century mathematics.

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[2]See [8, Ch. 5]. Helmholtz addressed two different questions. The first concerned the two-dimensionality of vision. At the time Helmholtz was writing, the dominant view endorsed by Johannes Müller, among others, was that a two-dimensional, spatial representation is primitively given in vision. In this view, only the perceptions of depth and of distance (i.e., the kind of perceptions that presuppose three-dimensionality) have to be learned. By contrast, Helmholtz sought to derive all spatial representations from the association of nonspatial sensations. The second question concerned the singularity of vision. Helmholtz called nativist Müller, Ewald Hering, and all those who derived the singularity of vision from the supposition of an anatomical connection between the two retinas.

[3]For a discussion of Helmholtz's claims about the "transcendental" status of space, see Biagioli [10].

[4]See, e.g., Helmholtz [7, p. 142]. Notice, however, that there were important turning points in Helmholtz's relation to Kant in this regard. Cf. Hatfield [8] and Hyder [11].

1.2.3 The Mathematical

The third road to geometrical empiricism goes back to the first attempts to rethink the question concerning the form of physical space after the discovery of non-Euclidean geometry.[5] Different hypotheses about the existence and the number of parallel lines to a given line had been explored in the eighteenth-century by such mathematicians as Girolamo Saccheri, Johann Heinrich Lambert, and Adrien-Marie Legendre. However, it was only in the 1820s that János Bolyai and Nikolai Lobachevsky, independently of each other, developed a system of geometry based on the denial of Euclid's parallel postulate. Lobachevsky was also one of the first to address the question whether non-Euclidean theorems (e.g., the proposition that the sum of the angles in a triangle is less than 180°, which followed from the denial of the parallel postulate) can be tested empirically by using astronomical measurements.

Although such an experiment proved to be impractical, much of the debate that followed from Riemann to Einstein concerned the possibility to explore the link between geometry and experience. As Gauss (in [13], p. 87) put it in a letter to Bessel dated April 9, 1830, "the theory of space has an entirely different place in knowledge from that occupied by pure mathematics. There is lacking throughout our knowledge of it the complete persuasion of necessity (also of absolute truth) which is common to the latter; we must add in humility that if number is exclusively the product of our mind, space has a reality outside our mind and we cannot completely prescribe its laws." On the one hand, Gauss's remark is reminiscent of Newton's view of geometry as the part of mechanics that deals with measurement in the *Philosophiæ Naturalis Principia Mathematica* [14]. On the other hand, by 1830, Gauss had enough knowledge of non-Euclidean geometry to see that there might be different possible hypotheses when it comes to the laws of space.[6]

A further development of this tradition is found in Gauss's student's, Bernhard Riemann, habilitation lecture of [17] "Über die Hypothesen, welche der Geometrie zu Grunde liegen" and in a series of papers published by Helmholtz between 1868 and 1878 and later collected by Paul Hertz and Moritz Schlick in the centenary edition of Helmholtz's *Epistemological Writings* (1977).[7] Both Riemann and Helmholtz started with a general notion of space as a manifold, comparable with such empirical manifolds as color and tone systems, in order to then pose the question of the necessary and sufficient conditions for introducing metrical relations. In this regard, Riemann showed the possibility of formulating infinitely many

[5]For an introductory account of the discovery of non-Euclidean geometry and its prehistory, see Engel and Stäckel [12].

[6]Sartorius von Waltershausen [15, p. 81] reported that even Gauss made an attempt to test the Euclidean hypothesis during his geodetic work. However, this interpretation is controversial and it was only after Bolyai's and Lobachevsky's works that the question arose whether the geometry of space could be non-Euclidean (see [16]).

[7]For a comprehensive account of nineteenth-century philosophy of geometry, see Torretti [18].

geometrical hypotheses based on Gauss's geometry of surfaces. Riemann foreshadowed Einstein's general theory of relativity by considering even the hypothesis of variably curved spaces and by articulating the conjecture of a metric, whose coefficients would depend on the local distribution of matter and forces.

The greater generality of Riemann's inquiry notwithstanding (Helmholtz restricted his consideration to manifolds of constant curvature), Helmholtz was one of the first to draw attention to the possibility of a physical interpretation of non-Euclidean geometry. He identified the fundamental precondition for the possibility of measurement as the requirement that the points of a system in motion remain fixedly linked or the free mobility of rigid bodies. He considered this to be a fact ascertained by observation and experiment, beginning with our ability to localize objects in space by performing congruent displacements. He then showed that the metrical geometry that is implicit in spatial perception and in measurement includes both Euclidean and non-Euclidean cases as possible mathematical specifications. He agreed with the conclusion of Riemann's inquiry, insofar as he believed his argument to prove the empirical—rather than a priori—origin of geometrical axioms.

The standard formulation of geometrical empiricism, in this sense, is found in Helmholtz's 1870 public lecture "Über den Ursprung und die Bedeutung der geometrischen Axiome," which initiated a philosophical and mathematical debate on the status of geometrical axioms.[8] A further development of Helmholtz's view of axioms as empirical generalizations is found in Pasch [20], which contains one of the first axiomatic treatments of geometry. The notion of axiom was progressively weakened with the emergence of the axiomatic method. Both mathematicians and philosophers called axioms "postulates" to emphasize the conceptual nature of the axioms and the possibility of formulating different hypotheses when it comes to physical space (Klein, some of the positions advocated by the Peano School, and Cassirer). In the twentieth century, it became common usage to refer to axioms as definitions in disguise (Poincaré) or implicit definitions (Hilbert, Schlick) of geometrical concepts as objects, whose existence depends solely on the coherence of the system of relations established by the axioms.

To sum up, with the advancement of the axiomatic method, geometrical empiricism lost its significance for a clarification of the questions concerning the origin and meaning of geometrical axioms. It became ever more clear that an appropriate understanding of how axioms work in modern mathematics presupposes a sharp separation between geometries as axiomatic systems and interpreted geometry or the theory of space and space-time. However, the tradition discussed above remained an important reference in the debate concerning physical and philosophical interpretations of the mathematical structures under consideration.

In order to highlight this point, the following section discusses the related problem of establishing geometrical and physical invariants.

[8]On the discussion of Helmholtz's view in neo-Kantianism, see Biagioli [19].

1.3 Invariants and Symmetries

The first characterization of geometrical objects in terms of invariants of transformation groups goes back to Felix Klein's "Vergleichende Betrachtungen über neuere geometrische Forschungen" [21], which is best known as "Erlanger Programm," after Klein's appointment as Professor at the University of Erlanger in the same year. An essential contribution to the implementation of such a research program is found in Sophus Lie's *Theorie der Transformationsgruppen*, which appeared in three volumes between 1888 and 1893.[9] The third volume contains a critique of Helmholtz's mathematical reasoning, along with the theorem that became the standard derivation of a metric of constant curvature from a set of necessary and sufficient assumptions about infinitesimal free mobility.

The fundamental ideas of the group-theoretical approach emerged from the observation that some geometrical properties are invariant under specific types of geometrical transformations. Such congruent displacements as translations, for example, leave invariant parallelism, lengths and measure of angles. In modern terminology, these are called the invariants of the Euclidean group. However, the same invariants might not be preserved by other transformations. Collineations in projective (and non-Euclidean) geometry, for example, leave invariant such properties as, of three or more points: to lie on the same line; of curves: to be a conic. But such transformations are known to alter such invariants of the Euclidean group as parallelism and absolute measurements.

Klein arrived at the idea of a group-theoretical classification of geometry by applying the basic notions of the algebraic theory of groups introduced by Evariste Galois and Camille Jordan to projective geometry—which flourished in the nineteenth century after the works of Jean-Victor Poncelet, Jakob Steiner, Christian von Staudt, and Arthur Cayley, among others.[10] Transformations form a group if: (i) the product of any two elements of the group also belongs to the group; (ii) there is in the group a neutral element (i.e., an element that leaves the other elements unchanged when combined with them); (iii) for every transformation in the group, there is in the group the inverse transformation.

Such mathematicians as Klein, Lie, and Poincaré showed that the same notions offered a general point of view for the comparison of geometrical researches. The significance of their classificatory ideas lies not least in the fact that the same ideas offered a point of comparison between different physical theories. Klein [25], for example, used the same approach to account for the shift from classical mechanics to special relativity. Whereas Galilean transformations preserve the invariant quantities of Newtonian mechanics (i.e., acceleration, force, mass, and therefore time, length, and simultaneity), the Lorentz transformations preserve the velocity of light, but not length and time (simultaneity).

[9] On Klein's relationship to Lie and the reception of the Erlanger Programm, see Hawkins [22] and Rowe [23].

[10] On the development of group theory from Galois to Klein, see Wussing [24].

The shift from special to general relativity seems to mark a break with this tradition, insofar as the space-time of general relativity is variably curved, the local value of curvature depending on the distribution of mass and energy, as first suggested by Riemann's conjecture. In the context of general relativity, such a correlation between space, time, and matter followed from Einstein's theory of gravitation. It is worth noting that, nevertheless, Einstein attached much importance to Helmholtz's interpretation of non-Euclidean geometry via the free mobility of rigid bodies for the development of his own ideas. In Einstein [26] he claimed that, without the view of geometry set forth by Helmholtz, he would have been unable to formulate the theory of relativity. But how are we to understand this comparison? Not only did Einstein's general relativity presuppose a different mathematical approach for the determination of the measure of curvature, but it implied a different concept of space. By considering hypotheses other than Euclidean geometry in the characterization of physical space, Helmholtz conceived of space as an object of research, whose properties can be determined empirically, to the required degree of approximation. In general relativity it became impossible to ask about the geometrical structure of space or space-time per se, insofar as geometrical properties depend on the distribution of matter. In classical mechanics and special relativity, the idea of space and time as objects endowed with a particular structure depends on the fact that inertial systems provide a privileged frame of reference. By contrast, Einstein's principle of relativity—in its various formulations—presupposes that the laws of physics apply to systems in any kind of motion. One of the conditions for this is that the form of natural laws remains unchanged under arbitrary changes of space-time values or general covariance. As Einstein [27, p. 117] put it, general covariance "takes away from space and time the last remnant of physical objectivity." Space-time coincidences, on the other hand, become fully objective in the sense of independence from the observer's perspective.

Einstein's claim has been much discussed at that time and in more recent literature.[11] The concluding section of this paper deals with Weyl's and Cassirer's interpretations as two different ways to account for the transformation of the concept of space from geometrical empiricism to general relativity and to provide new answers to the question concerning subjective and objective aspects of knowledge.

1.4 Subjectivity and Objectivity

1.4.1 Hermann Weyl

Hermann Weyl was both a great mathematician, whose contributions ranged from analysis, algebra, topology, differential geometry, and fundamental physics, and one of the few philosopher-scientists of the twentieth century to defend a Kantian

[11] For a discussion of different positions, see Norton [28] and Ryckman [29].

conception of space in line with Husserl's phenomenological approach to the a priori.[12] In his main work on the general theory of relativity, *Raum-Zeit-Materie*, which appeared in four editions between 1918 and 1921, Weyl [30, p. 98]) accounted for the shift to the new theory by saying that the space of general relativity "is nothing more than a three-dimensional manifold devoid of all form; it acquires a definite form only through the advent of the material content filling it and determining its metric relations." With reference to his metrical infinitesimal geometry, nevertheless, Weyl argued that the empirical determination of metrical relations did not rule out the possibility of a priori knowledge, insofar as the borderline between a priori and a posteriori was set somewhere else. In Weyl's account: whereas the metric at a point P, along with the metrical relation of P to its neighboring points, is everywhere the same and retains the status of an a priori condition of experience, the mutual orientation of the metrics in different points is a posteriori.

In *Raum-Zeit-Materie*, Weyl advocated a phenomenological conception of the a priori in order to deal with problem of coordination. Referring to Einstein's remark about general covariance, Weyl [30, p. 8] wrote: "[T]he objectivity of things conferred by the exclusion of the ego and its data derived directly from intuition, is not entirely satisfactory; the co-ordinate system which can only be specified by an individual act (and then only approximately) remains as an inevitable residuum of this elimination of the percipient." Whereas Einstein focused on the objectivity achieved in general relativity when it comes to determine space-time coincidences, Weyl's emphasis is on the fact that a residuum of subjectivity still plays the role of a necessary presupposition for the setting of the coordinate system. In other words, Weyl vindicated the Kantian conception of space as a source of knowledge by identifying the minimal phenomenological structure that is required for space-time coordination.

1.4.2 Ernst Cassirer

Ernst Cassirer started his career as one of the leading figures of the Marburg School of neo-Kantianism and became known as the founder of the philosophy of symbolic forms.[13] This contained one of the most promising attempts in twentieth-century philosophy to develop a unified perspective on human culture, including natural and social sciences, the humanities, and the arts as different expressions of the subject/object relation. Cassirer identified a range of different symbolic forms in which such a relation can be articulated. The broader scope of this view notwithstanding, one of Cassirer's starting point is found in his 1921 book *Zur*

[12]See Ryckman [29, Chaps. 5 and 6].

[13]On the development of Cassirer's thought from neo-Kantianism to the philosophy of symbolic forms, see Ferrari [31].

Einstein'schen Relativitätstheorie. Commenting on the significance of general covariance for the formulation of Einstein's principle of relativity, Cassirer observed that general relativity confirmed the tendency of every physical theory to abstract from what is immediately given in perception in order to gain the appropriate conceptual expression and understanding of the facts of experience. As Duhem [32, p. 322] put it, in order to provide the basis for the development of physical theory, empirical facts have to be transformed and put into a "symbolic form."

Regarding the spatial order of experience as a symbolic form, even before general relativity Cassirer emphasized that the mathematical concept of space provides a combination of different hypotheses. He referred, in particular, to the classic cases of manifolds of constant curvature investigated by Helmholtz, Klein, and Lie. Despite the difference between these cases and the variably curved space-time of general relativity, Cassirer argued for a further generalization of the previous system of hypotheses in order to attain to complete objectivity in facing the problem of coordination. His interpretation focused on the fact that, by generalizing the principle of relativity, Einstein needed not single out any privileged systems of reference.

Without entering into the details of Cassirer's interpretation, for my present purpose I limit myself to point out that the example of Einstein's space-time theory played an important role in Cassirer's conception of his own philosophical project. He wrote in his concluding chapter on the problem of redefining physical reality in symbolic terms rather than as absolute reality: "It is the task of systematic philosophy, which extends far beyond the theory of knowledge, to free the idea of the world from this one-sidedness. It has to grasp the whole system of symbolic forms, the application of which produces for us the concept of an ordered reality, and by virtue of which subject and object, ego and world are separated and opposed to each other in definite form, and it must refer each individual in this totality to its fixed place" (Cassirer [33], p. 447).

The relevant point for my brief comparison with Weyl is that Cassirer articulated a different strategy for vindicating the core ideas of the Kantian theory of space. Instead of identifying the a priori of space as a subjective, but necessary source of knowledge, Cassirer investigated the different ways in which the concept of a spatial order is used to articulate the subject/object relation in anthropological, scientific, and artistic contexts. A number of examples are found in Cassirer's main works on the philosophy of symbolic forms. A very clear characterization of this approach for a wider audience is found in his 1931 paper [34] "Mythischer, ästhetischer und theoretischer Raum," which was delivered during the fourth *Congress für Ästhetik und allgemeine Kunstwissenschaft* in Hamburg. In this paper, he argued that the variety of spatial concepts notwithstanding, space as a symbolic form is characterized by its purely relational nature, which is a necessary precondition for the determination of things and meanings. Cassirer referred, for example, to the mythical significance of spatial orientation in ancient myths and in religious representations of the afterlife. The example of spatial and temporal concepts in the arts shows even more clearly the potential of the symbolic forms in inventing a new constellation of meanings.

In Cassirer's view, the clarification of the symbolic function of space and time in modern physics opened the door to looking at these concepts from different perspectives, while deepening Kant's insight into the nature of space and time as orders of appearance rather than perceived objects. However, Cassirer's proposal is not to return to Kant's original view of space as a source of knowledge, but to investigate how thought can anticipate experience in virtue of the creative and hypothetical character of our systems of concepts. My suggestion is that the philosophy of symbolic forms offers an original synthesis between the two opposed ways to consider space as a source and an object of knowledge. As Cassirer [33], p. 426) put it, empiricism and idealism meet in certain presuppositions with regard to the doctrine of empirical space and of empirical time: "Both here grant to experience the decisive role, and both teach that every exact measurement presupposes universal empirical laws."

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