The Garden of Eden Theorem for Cellular Automata on Group Sets

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Abstract. We prove the Garden of Eden theorem for cellular automata with finite set of states and finite neighbourhood on right amenable left homogeneous spaces with finite stabilisers. It states that the global transition function of such an automaton is surjective if and only if it is pre-injective. Pre-Injectivity means that two global configurations that differ at most on a finite subset and have the same image under the global transition function must be identical.

Keywords: Cellular automata · Group actions · Garden of Eden theorem

The notion of amenability for groups was introduced by John von Neumann in 1929. It generalises the notion of finiteness. A group G is *left* or *right amenable* if there is a finitely additive probability measure on $\mathcal{P}(G)$ that is invariant under left and right multiplication respectively. Groups are left amenable if and only if they are right amenable. A group is *amenable* if it is left or right amenable.

The definitions of left and right amenability generalise to left and right group sets respectively. A left group set $(M, G, \triangleright)$ is *left amenable* if there is a finitely additive probability measure on $\mathcal{P}(M)$ that is invariant under $\triangleright$. There is in general no natural action on the right that is to a left group action what right multiplication is to left group multiplication. Therefore, for a left group set there is no natural notion of right amenability.

A transitive left group action $\triangleright$ of G on M induces, for each element $m_0 \in M$ and each family $\{g_{m_0,m}\}_{m \in M}$ of elements in G such that, for each point $m \in M$, we have $g_{m_0,m} \triangleright m_0 = m$, a right quotient set semi-action $\trianglelefteq$ of G/G_0 on M with defect G_0 given by $m \trianglelefteq gG_0 = g_{m_0,m} g g_{m_0,m}^{-1} \triangleright m$, where G_0 is the stabiliser of m_0 under $\triangleright$. Each of these right semi-actions is to the left group action what right multiplication is to left group multiplication. They occur in the definition of global transition functions of cellular automata over left homogeneous spaces as defined in [5]. A *cell space* is a left group set together with choices of m_0 and $\{g_{m_0,m}\}_{m \in M}$.

A cell space $\mathcal{R}$ is *right amenable* if there is a finitely additive probability measure on $\mathcal{P}(M)$ that is semi-invariant under $\trianglelefteq$. For example cell spaces with finite sets of cells, abelian groups, and finitely right generated cell spaces of

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sub-exponential growth are right amenable, in particular, quotients of finitely generated groups of sub-exponential growth by finite subgroups acted on by left multiplication. A net of non-empty and finite subsets of M is a *right Følner net* if, broadly speaking, these subsets are asymptotically invariant under $\triangleleft$. A finite subset E of G/G_0 and two partitions $\{A_e\}_{e \in E}$ and $\{B_e\}_{e \in E}$ of M constitute a *right paradoxical decomposition* if the map $\_ \triangleleft e$ is injective on A_e and B_e, and the family $\{(A_e \triangleleft e) \cup (B_e \triangleleft e)\}_{e \in E}$ is a partition of M. The Tarski-Følner theorem states that right amenability, the existence of right Følner nets, and the non-existence of right paradoxical decompositions are equivalent. We prove it in [6] for cell spaces with finite stabilisers.

For a right amenable cell space with finite stabilisers we may choose a right Følner net $\mathcal{F} = \{F_i\}_{i \in I}$. The entropy of a subset X of Q^M with respect to $\mathcal{F}$, where Q is a finite set, is, broadly speaking, the asymptotic growth rate of the number of finite patterns with domain F_i that occur in X. For subsets E and E' of G/G_0, an (E, E')-tiling is a subset T of M such that $\{t \triangleleft E\}_{t \in T}$ is pairwise disjoint and $\{t \triangleleft E'\}_{t \in T}$ is a cover of M. If for each point $t \in T$ not all patterns with domain $t \triangleleft E$ occur in a subset of Q^M, then that subset does not have maximal entropy.

The global transition function of a cellular automaton with finite set of states and finite neighbourhood over a right amenable cell space with finite stabilisers, as introduced in [5], is surjective if and only if its image has maximal entropy and it is pre-injective if and only if its image has maximal entropy. This establishes the Garden of Eden theorem, which states that a global transition function as above is surjective if and only if it is pre-injective. This answers a question posed by Sébastien Moriceau at the end of his paper 'Cellular Automata on a G-Set' [4].

The Garden of Eden theorem for cellular automata over $\mathbb{Z}^2$ is a famous theorem by Edward Forrest Moore and John R. Myhill from 1962 and 1963, see the papers 'Machine models of self-reproduction' [2] and 'The converse of Moore's Garden-of-Eden theorem' [3]. This paper is greatly inspired by the monograph 'Cellular Automata and Groups' [1] by Tullio Ceccherini-Silberstein and Michel Coornaert.

In Sect. 1 we introduce E-interiors, E-closures, and E-boundaries of subsets of M. In Sect. 2 we introduce (E, E')-tilings of cell spaces. In Sect. 3 we introduce entropies of subsets of Q^M. And in Sect. 4 we prove the Garden of Eden theorem.

Preliminary Notions. A *left group set* is a triple $(M, G, \triangleright)$, where M is a set, G is a group, and $\triangleright$ is a map from $G \times M$ to M, called *left group action of G on M*, such that $G \to \mathrm{Sym}(M)$, $g \mapsto [g \triangleright \_]$, is a group homomorphism. The action $\triangleright$ is *transitive* if M is non-empty and for each $m \in M$ the map $\_ \triangleright m$ is surjective; and *free* if for each $m \in M$ the map $\_ \triangleright m$ is injective. For each $m \in M$, the set $G \triangleright m$ is the *orbit of m*, the set $G_m = (\_ \triangleright m)^{-1}(m)$ is the *stabiliser of m*, and, for each $m' \in M$, the set $G_{m,m'} = (\_ \triangleright m)^{-1}(m')$ is the *transporter of m to m'*.

A *left homogeneous space* is a left group set $\mathcal{M} = (M, G, \triangleright)$ such that $\triangleright$ is transitive. A *coordinate system for $\mathcal{M}$* is a tuple $\mathcal{K} = (m_0, \{g_{m_0,m}\}_{m \in M})$, where $m_0 \in M$ and for each $m \in M$ we have $g_{m_0,m} \triangleright m_0 = m$. The stabiliser G_{m_0} is denoted by G_0. The tuple $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is a *cell space*. The set $\{gG_0 \mid g \in G\}$ of left cosets of G_0 in G is denoted by G/G_0. The map $\triangleleft\colon M \times G/G_0 \to M$, $(m, gG_0) \mapsto g_{m_0,m} g g_{m_0,m}^{-1} \triangleright m \; (= g_{m_0,m} g \triangleright m_0)$ is a *right semi-action of* G/G_0 *on* M *with defect* G_0, which means that

$$\forall m \in M : m \triangleleft G_0 = m,$$
$$\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall \mathfrak{g}' \in G/G_0 : m \triangleleft g \cdot \mathfrak{g}' = (m \triangleleft gG_0) \triangleleft g_0 \cdot \mathfrak{g}'.$$

It is *transitive*, which means that the set M is non-empty and for each $m \in M$ the map $m \triangleleft \_$ is surjective; and *free*, which means that for each $m \in M$ the map $m \triangleleft \_$ is injective; and *semi-commutes with* $\triangleright$, which means that

$$\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall \mathfrak{g}' \in G/G_0 : (g \triangleright m) \triangleleft \mathfrak{g}' = g \triangleright (m \triangleleft g_0 \cdot \mathfrak{g}').$$

The maps $\iota\colon M \to G/G_0$, $m \mapsto G_{m_0,m}$, and $m_0 \triangleleft \_$ are inverse to each other. Under the identification of M with G/G_0 by either of these maps, we have $\triangleleft\colon (m, \mathfrak{g}) \mapsto g_{m_0,m} \triangleright \mathfrak{g}$.

A left homogeneous space $\mathcal{M}$ is *right amenable* if there is coordinate system $\mathcal{K}$ for $\mathcal{M}$ and there is a finitely additive probability measure μ on M such that

$$\forall \mathfrak{g} \in G/G_0 \forall A \subseteq M : \big((\_ \triangleleft \mathfrak{g})\restriction_A \text{ injective} \implies \mu(A \triangleleft \mathfrak{g}) = \mu(A)\big),$$

in which case the cell space $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is called *right amenable*. When the stabiliser G_0 is finite, that is the case if and only if there is a *right Følner net in* $\mathcal{R}$ *indexed by* $(I, \leq)$, which is a net $\{F_i\}_{i \in I}$ in $\{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\}$ such that

$$\forall \mathfrak{g} \in G/G_0 : \lim_{i \in I} \frac{|F_i \smallsetminus (\_ \triangleleft \mathfrak{g})^{-1}(F_i)|}{|F_i|} = 0.$$

A *semi-cellular automaton* is a quadruple $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$, where $\mathcal{R}$ is a cell space; Q, called *set of states*, is a set; N, called *neighbourhood*, is a subset of G/G_0 such that $G_0 \cdot N \subseteq N$; and δ, called *local transition function*, is a map from Q^N to Q. A *local configuration* is a map $\ell \in Q^N$, a *global configuration* is a map $c \in Q^M$, and a *pattern* is a map $p \in Q^A$, where A is a subset of M. The stabiliser G_0 acts on Q^N on the left by $\bullet\colon G_0 \times Q^N \to Q^N$, $(g_0, \ell) \mapsto [n \mapsto \ell(g_0^{-1} \cdot n)]$, and the group G acts on the set of patterns on the left by

$$\begin{aligned} \blacktriangleright\colon G \times \bigcup_{A \subseteq M} Q^A &\to \bigcup_{A \subseteq M} Q^A, \\ (g, p) &\mapsto \begin{bmatrix} g \triangleright \operatorname{dom}(p) \to Q, \\ \qquad m \mapsto p(g^{-1} \triangleright m). \end{bmatrix} \end{aligned}$$

The *global transition function of* $\mathcal{C}$ is the map $\Delta\colon Q^M \to Q^M$, $c \mapsto [m \mapsto \delta(n \mapsto c(m \triangleleft n))]$.

A *cellular automaton* is a semi-cellular automaton $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$ such that δ is $\bullet$*-invariant*, which means that, for each $g_0 \in G_0$, we have $\delta(g_0 \bullet \_) = \delta(\_)$. Its global transition function is $\blacktriangleright$-equivariant, which means that, for each $g \in G$, we have $\Delta(g \blacktriangleright \_) = g \blacktriangleright \Delta(\_)$.

For each $A \subseteq M$, let $\pi_A\colon Q^M \to Q^A$, $c \mapsto c\restriction_A$.

1 Interiors, Closures, and Boundaries

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a cell space.

In Definition 1 we introduce E-interiors, E-closures, and E-boundaries of subsets of M. In Lemma 3 we define surjective restrictions $\Delta^{-}_{X,A}$ of global transition functions to patterns. And in Theorem 1 we show that right Følner nets are those nets whose components are asymptotically invariant under taking finite boundaries.

Definition 1. *Let A be a subset of M and let E be a subset of G/G_0.*

1. *The set*

$$A^{-E} = \{m \in M \mid m \triangleleft E \subseteq A\} \ \big(= \bigcap_{e \in E} \bigcup_{a \in A} (\_ \triangleleft e)^{-1}(a)\big)$$

is called E-interior of A.

2. *The set*

$$A^{+E} = \{m \in M \mid (m \triangleleft E) \cap A \neq \emptyset\} \ \big(= \bigcup_{e \in E} \bigcup_{a \in A} (\_ \triangleleft e)^{-1}(a)\big)$$

is called E-closure of A.

3. *The set $\partial_E A = A^{+E} \smallsetminus A^{-E}$ is called* E-boundary of A.

Remark 1. Let $\mathcal{R}$ be the cell space $((G, G, \cdot), (e_G, \{g\}_{g \in G}))$, where G is a group and e_G is its neutral element. Then, $G_0 = \{e_G\}$ and $\triangleleft = \cdot$. Hence, the notions of E-interior, E-closure, and E-boundary are the same as the ones defined in [1, Sect. 5.4, Paragraph 2].

Example 1. Let M be the Euclidean unit 2-sphere, that is, the surface of the ball of radius 1 in 3-dimensional Euclidean space, and let G be the rotation group. The group G acts transitively but not freely on M on the left by $\triangleright$ by function application, that is, by rotation about the origin. For each point $m \in M$, its orbit is M and its stabiliser is the group of rotations about the line through the origin and itself.

Furthermore, let m_0 be the north pole $(0, 0, 1)^\intercal$ of M and, for each point $m \in M$, let $g_{m_0,m}$ be a rotation about an axis in the (x, y)-plane that rotates m_0 to m. The stabiliser G_0 of the north pole m_0 under $\triangleright$ is the group of rotations about the z-axis. An element $gG_0 \in G/G_0$ semi-acts on a point m on the right by the induced semi-action $\triangleleft$ by first changing the rotation axis of g such that the new axis stands to the line through the origin and m as the old one stood to the line through the origin and m_0, $g_{m_0,m} g g_{m_0,m}^{-1}$, and secondly rotating m as prescribed by this new rotation.

Moreover, let A be a curved circular disk of radius 3ρ with the north pole m_0 at its centre, let g be the rotation about an axis a in the (x, y)-plane by ρ radians, let E be the set $\{g_0 g G_0 \mid g_0 \in G_0\}$, and, for each point $m \in M$, let E_m be the set $m \triangleleft E$. Because G_0 is the set of rotations about the z-axis and

$m_0 \triangleleft E = g_{m_0,m_0} G_0 g \triangleright m_0 = G_0 \triangleright (g \triangleright m_0)$, the set E_{m_0} is the boundary of a curved circular disk of radius ρ with the north pole m_0 at its centre. And, for each point $m \in M$, because $m \triangleleft E = g_{m_0,m} \triangleright E_{m_0}$, the set E_m is the boundary of a curved circular disk of radius ρ with m at its centre.

The E-interior of A is the curved circular disk of radius 2ρ with the north pole m_0 at its centre. The E-closure of A is the curved circular disk of radius 4ρ with the north pole m_0 at its centre. And the E-boundary of A is the annulus bounded by the boundaries of the E-interior and the E-closure of A.

Essential properties of and relations between interiors, closures, and boundaries are given in the next lemma. The upper bound given in its corollary follows from the last part of Item 7.

Lemma 1. *Let A be a subset of M, let $\{A_i\}_{i \in I}$ be a family of subsets of M, let e be an element of G/G_0, and let E and E' be two subsets of G/G_0.*

1. $A^{-\{G_0\}} = A$, $A^{+\{G_0\}} = A$, *and* $\partial_{\{G_0\}} A = \emptyset$.
2. $A^{-\{G_0,e\}} = A \cap (\_ \triangleleft e)^{-1}(A)$, $A^{+\{G_0,e\}} = A \cup (\_ \triangleleft e)^{-1}(A)$, *and* $\partial_{\{G_0,e\}} A = A \smallsetminus (\_ \triangleleft e)^{-1}(A) \cup (\_ \triangleleft e)^{-1}(A) \smallsetminus A$.
3. $(M \smallsetminus A)^{-E} = M \smallsetminus A^{+E}$ *and* $(M \smallsetminus A)^{+E} = M \smallsetminus A^{-E}$.
4. *Let $E \subseteq E'$. Then, $A^{-E} \supseteq A^{-E'}$, $A^{+E} \subseteq A^{+E'}$, and $\partial_E A \subseteq \partial_{E'} A$.*
5. *Let $G_0 \in E$. Then, $A^{-E} \subseteq A \subseteq A^{+E}$.*
6. *Let $G_0 \in E$ and let A be finite. Then, A^{-E} is finite.*
7. *Let G_0, A, and E be finite. Then, A^{+E} and $\partial_E A$ are finite. More precisely, $|A^{+E}| \leq |G_0| \cdot |A| \cdot |E|$.*
8. *Let $g \in G$ and let $G_0 \cdot E \subseteq E$. Then, $g \triangleright A^{-E} = (g \triangleright A)^{-E}$, $g \triangleright A^{+E} = (g \triangleright A)^{+E}$, and $g \triangleright \partial_E A = \partial_E (g \triangleright A)$.*
9. *Let $m \in M$, let $G_0 \cdot E \subseteq E$, and let $\iota \colon M \to G/G_0$, $m \mapsto G_{m_0,m}$. Then, $m \triangleleft \iota(A^{-E}) = (m \triangleleft \iota(A))^{-E}$, $m \triangleleft \iota(A^{+E}) = (m \triangleleft \iota(A))^{+E}$, and $m \triangleleft \iota(\partial_E A) = \partial_E (m \triangleleft \iota(A))$.*

Corollary 1. *Let G_0 be finite, let A be a finite subset of M, and let $\mathfrak{g}$ be an element of G/G_0. Then, $|(\_ \triangleleft \mathfrak{g})^{-1}(A)| \leq |G_0| \cdot |A|$.*

The restriction $\Delta^-_{X,A}$ of Δ given in Lemma 3 is well-defined according to the next lemma, which itself holds due to the locality of Δ.

Lemma 2. *Let $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$ be a semi-cellular automaton, let Δ be the global transition function of $\mathcal{C}$, let c and c' be two global configurations of $\mathcal{C}$, and let A be a subset of M. If $c\restriction_A = c'\restriction_A$, then $\Delta(c)\restriction_{A^{-N}} = \Delta(c')\restriction_{A^{-N}}$.*

Lemma 3. *Let $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$ be a semi-cellular automaton, let Δ be the global transition function of $\mathcal{C}$, let X be a subset of Q^M, and let A be a subset of M. The map*

$$\begin{aligned} \Delta^-_{X,A} \colon \pi_A(X) &\to \pi_{A^{-N}}(\Delta(X)), \\ p &\mapsto \Delta(c)\restriction_{A^{-N}}, \text{ where } c \in X \text{ such that } c\restriction_A = p, \end{aligned}$$

is surjective. The map $\Delta^-_{Q^M,A}$ is denoted by Δ^-_A.

In the proof of Theorem 1, the upper bound given in Lemma 6 is essential, which itself follows from the upper bound given in Corollary 1 and the inclusion given in Lemma 5, which in turn follows from the equality given in Lemma 4.

Lemma 4. *Let m be an element of M, and let $\mathfrak{g}$ be an element of G/G_0. There is an element $g \in \mathfrak{g}$ such that*

$$\forall \mathfrak{g}' \in G/G_0 : (m \trianglelefteq \mathfrak{g}) \trianglelefteq \mathfrak{g}' = m \trianglelefteq g \cdot \mathfrak{g}',$$

in particular, for said $g \in \mathfrak{g}$, we have $(m \trianglelefteq \mathfrak{g}) \trianglelefteq g^{-1}G_0 = m$.

Lemma 5. *Let A and A' be two subsets of M, and let $\mathfrak{g}$ and $\mathfrak{g}'$ be two elements of G/G_0. Then, for each element $m \in (\_ \trianglelefteq \mathfrak{g})^{-1}(A) \smallsetminus (\_ \trianglelefteq \mathfrak{g}')^{-1}(A')$,*

$$m \trianglelefteq \mathfrak{g} \in \bigcup_{g \in \mathfrak{g}} A \smallsetminus (\_ \trianglelefteq g^{-1} \cdot \mathfrak{g}')^{-1}(A'),$$
$$m \trianglelefteq \mathfrak{g}' \in \bigcup_{g' \in \mathfrak{g}'} (\_ \trianglelefteq (g')^{-1} \cdot \mathfrak{g})^{-1}(A) \smallsetminus A'.$$

Lemma 6. *Let G_0 be finite, let F and F' be two finite subsets of M, and let $\mathfrak{g}$ and $\mathfrak{g}'$ be two elements of G/G_0. Then,*

$$|(\_ \trianglelefteq \mathfrak{g})^{-1}(F) \smallsetminus (\_ \trianglelefteq \mathfrak{g}')^{-1}(F')| \leq \begin{cases} |G_0|^2 \cdot \max\limits_{g \in \mathfrak{g}} |F \smallsetminus (\_ \trianglelefteq g^{-1} \cdot \mathfrak{g}')^{-1}(F')|, \\ |G_0|^2 \cdot \max\limits_{g' \in \mathfrak{g}'} |(\_ \trianglelefteq (g')^{-1} \cdot \mathfrak{g})^{-1}(F) \smallsetminus F'|. \end{cases}$$

Theorem 1. *Let G_0 be finite and let $\{F_i\}_{i \in I}$ be a net in $\{F \subseteq M \mid F \neq \emptyset, F\, finite\}$ indexed by $(I, \leq)$. The net $\{F_i\}_{i \in I}$ is a right Følner net in $\mathcal{R}$ if and only if*

$$\forall E \subseteq G/G_0\, finite : \lim_{i \in I} \frac{|\partial_E F_i|}{|F_i|} = 0.$$

Proof. First, let $\{F_i\}_{i \in I}$ be a right Følner net in $\mathcal{R}$. Furthermore, let $E \subseteq G/G_0$ be finite. Moreover, let $i \in I$. For each $e \in E$ and each $e' \in E$, put $A_{i,e,e'} = (\_ \trianglelefteq e)^{-1}(F_i) \smallsetminus (\_ \trianglelefteq e')^{-1}(F_i)$. For each $\mathfrak{g} \in G/G_0$, put $B_{i,\mathfrak{g}} = F_i \smallsetminus (\_ \trianglelefteq \mathfrak{g})^{-1}(F_i)$. According to Definition 1,

$$\partial_E F_i = \Big(\bigcup_{e \in E} (\_ \trianglelefteq e)^{-1}(F_i)\Big) \smallsetminus \Big(\bigcap_{e' \in E} (\_ \trianglelefteq e')^{-1}(F_i)\Big)$$
$$= \bigcup_{e,e' \in E} (\_ \trianglelefteq e)^{-1}(F_i) \smallsetminus (\_ \trianglelefteq e')^{-1}(F_i) = \bigcup_{e,e' \in E} A_{i,e,e'}.$$

Hence, $|\partial_E F_i| \leq \sum_{e,e' \in E} |A_{i,e,e'}|$.

According to Lemma 6, we have $|A_{i,e,e'}| \leq |G_0|^2 \cdot \max_{g \in e} B_{i,g^{-1} \cdot e'}$. Put $E' = \{g^{-1} \cdot e' \mid e, e' \in E, g \in e\}$. Because E is finite, G_0 is finite, and, for each $e \in E$, we have $|e| = |G_0|$, the set E' is finite. Therefore,

$$\frac{|\partial_E F_i|}{|F_i|} \leq \frac{1}{|F_i|} \sum_{e,e' \in E} |A_{i,e,e'}| \leq \frac{|G_0|^2}{|F_i|} \sum_{e,e' \in E} \max_{g \in e} |B_{i,g^{-1} \cdot e'}|$$
$$\leq \frac{|G_0|^2 \cdot |E|^2}{|F_i|} \max_{e' \in E'} |B_{i,e'}| \leq |G_0|^2 \cdot |E|^2 \cdot \max_{e' \in E'} \frac{|F_i \smallsetminus (\_ \triangleleft e')^{-1}(F_i)|}{|F_i|} \underset{i \in I}{\to} 0.$$

In conclusion, $\lim_{i \in I} \frac{|\partial_E F_i|}{|F_i|} = 0$.

Secondly, for each finite $E \subseteq G/G_0$, let $\lim_{i \in I} \frac{|\partial_E F_i|}{|F_i|} = 0$. Furthermore, let $i \in I$, let $e \in G/G_0$, and put $E = \{G_0, e\}$. According to Item 2 of Lemma 1, we have $F_i \smallsetminus (\_ \triangleleft e)^{-1}(F_i) \subseteq \partial_E F_i$. Therefore,

$$\frac{|F_i \smallsetminus (\_ \triangleleft e)^{-1}(F_i)|}{|F_i|} \leq \frac{|\partial_E F_i|}{|F_i|} \underset{i \in I}{\to} 0.$$

In conclusion, $\{F_i\}_{i \in I}$ is a right Følner net in $\mathcal{R}$. □

2 Tilings

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a cell space.

In Definition 2 we introduce the notion of (E, E')-tilings. In Theorem 2 we show using Zorn's lemma that, for each subset E of G/G_0, there is an (E, E')-tiling. And in Lemma 7 we show that, for each (E, E')-tiling with finite sets E and E', the net $\{|T \cap F_i^{-E}|\}_{i \in I}$ is asymptotic not less than $\{|F_i|\}_{i \in I}$.

Definition 2. *Let T be a subset of M, and let E and E' be two subsets of G/G_0. The set T is called* (E, E')-tiling of $\mathcal{R}$ *if and only if the family $\{t \triangleleft E\}_{t \in T}$ is pairwise disjoint and the family $\{t \triangleleft E'\}_{t \in T}$ is a cover of M.*

Remark 2. Let T be an (E, E')-tiling of $\mathcal{R}$. For each subset F of E and each superset F' of E' with $F' \subseteq G/G_0$, the set T is an (F, F')-tiling of $\mathcal{R}$. In particular, the set T is an $(E, E \cup E')$-tiling of $\mathcal{R}$.

Remark 3. In the situation of Remark 1, the notion of (E, E')-tiling is the same as the one defined in [1, Sect. 5.6, Paragraph 2].

Example 2. In the situation of Example 1, let E' be the set $\{g(g')^{-1}G_0 \mid e, e' \in E, g \in e, g' \in e'\}$ $(= \{g_0 g g_0' g^{-1} G_0 \mid g_0, g_0' \in G_0\})$ and, for each point $m \in M$, let $E'_m = m \triangleleft E'$. Because g^{-1} is the rotation about the axis a by $-\rho$ radians, the set $G_0 g^{-1} \triangleright m_0$ is equal to E_{m_0} and the set $g G_0 g^{-1} \triangleright m_0$ is equal to $E_{g \triangleright m_0}$. Because $m_0 \triangleleft E' = g_{m_0,m_0} G_0 g G_0 g^{-1} \triangleright m_0 = G_0 \triangleright (g G_0 g^{-1} \triangleright m_0) = G_0 \triangleright E_{g \triangleright m_0}$, the set E'_{m_0} is the curved circular disk of radius 2ρ with the north pole m_0 at its centre. And, for each point $m \in M$, because $m \triangleleft E' = g_{m_0,m} \triangleright E'_{m_0}$, the set E'_m is the curved circular disk of radius 2ρ with m at its centre.

If the radius $\rho = \pi/2$, then the circle E_{m_0} is the equator and the curved circular disk E'_{m_0} has radius π and is thus the sphere M, and hence the set $T = \{m_0\}$ is an (E, E')-tiling of $\mathcal{R}$; if the radius $\rho = \pi/4$, then the curved circular disks E'_{m_0} and E'_S, where S is the south pole, have radii $\pi/2$, thus they are hemispheres, and hence the set $T = \{m_0, S\}$ is an (E, E')-tiling of $\mathcal{R}$; if the radius $\rho = \pi/8$, then the curved circular disks E'_{m_0} and E'_S have radii $\pi/4$, and it can be shown with spherical geometry that the set T consisting of the north pole m_0, the south pole S, four equidistant points m_1, m_2, m_3, and m_4 on the equator, and the circumcentres c_1, c_2, $\ldots$, c_8 of the 8 smallest spherical triangles with one vertex from $\{m_0, S\}$ and two vertices from $\{m_1, m_2, m_3, m_4\}$ (see Fig. 1).

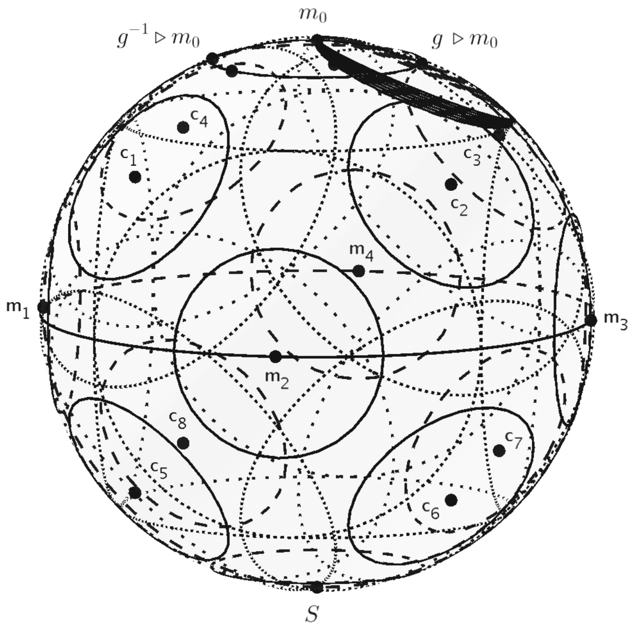

Fig. 1. The points $m_0, S, m_1, m_2, m_3, m_4, c_1, c_2, \ldots, c_8$ constitute an (E, E')-tiling of the sphere; the circles E_m about these points are drawn solid; the boundaries of the curved circular disks E'_m about these points are drawn dotted; the inclined circle about $g \triangleright m_0$ is the rotation $E_{g \triangleright m_0}$ of E_{m_0} by $\pi/8$ about the axis a; and the other inclined circles are rotations $g_0 \triangleright (E_{g \triangleright m_0})$ of $E_{g \triangleright m_0}$ about the z-axis, for a few $g_0 \in G_0$.

Theorem 2. *Let E be a non-empty subset of G/G_0. There is an (E, E')-tiling of $\mathcal{R}$, where $E' = \{g(g')^{-1}G_0 \mid e, e' \in E, g \in e, g' \in e'\}$.*

Proof. Let $\mathcal{S} = \{S \subseteq M \mid \{s \trianglelefteq E\}_{s \in S}$ is pairwise disjoint$\}$. Because $\{m_0\} \in \mathcal{S}$, the set $\mathcal{S}$ is non-empty. Moreover, it is preordered by inclusion.

Let $\mathcal{C}$ be a chain in $(\mathcal{S}, \subseteq)$. Then, $\bigcup_{S \in \mathcal{C}} S$ is an element of $\mathcal{S}$ and an upper bound of $\mathcal{C}$. According to Zorn's lemma, there is a maximal element T in $\mathcal{S}$. By definition of $\mathcal{S}$, the family $\{t \trianglelefteq E\}_{t \in T}$ is pairwise disjoint.

Let $m \in M$. Because T is maximal and $m \trianglelefteq E$ is non-empty, there is a $t \in T$ such that $(t \trianglelefteq E) \cap (m \trianglelefteq E) \neq \emptyset$. Hence, there are e, $e' \in E$ such that $t \trianglelefteq e = m \trianglelefteq e'$.

According to Lemma 4, there is a $g' \in e'$ such that $(m \triangleleft e') \triangleleft (g')^{-1}G_0 = m$, and there is a $g \in e$ such that $(t \triangleleft e) \triangleleft (g')^{-1}G_0 = t \triangleleft g(g')^{-1}G_0$. Therefore, $m = t \triangleleft g(g')^{-1}G_0$. Because $g(g')^{-1}G_0 \in E'$, we have $m \in t \triangleleft E'$. Thus, $\{t \triangleleft E'\}_{t \in T}$ is a cover of M.

In conclusion, T is an (E, E')-tiling of $\mathcal{R}$. □

Lemma 7. *Let G_0 be finite, let $\{F_i\}_{i \in I}$ be a right Følner net in $\mathcal{R}$ indexed by $(I, \leq)$, let E and E' be two finite subsets of G/G_0, and let T be an (E, E')-tiling of $\mathcal{R}$. There is a positive real number $\varepsilon \in \mathbb{R}_{>0}$ and there is an index $i_0 \in I$ such that, for each index $i \in I$ with $i \geq i_0$, we have $|T \cap F_i^{-E}| \geq \varepsilon |F_i|$.*

3 Entropies

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a right amenable cell space, let $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$ be a semi-cellular automaton, and let Δ be the global transition function of $\mathcal{C}$ such that the stabiliser G_0 of m_0 under $\triangleright$, the set Q of states, and the neighbourhood N are finite, and the set Q is non-empty.

In Definition 3 we introduce the entropy of a subset X of Q^M with respect to a net $\{F_i\}_{i \in I}$ of non-empty and finite subsets of M, which is the asymptotic growth rate of the number of finite patterns with domain F_i that occur in X. In Lemma 8 we show that Q^M has entropy $\log |Q|$ and that entropy is non-decreasing. In Theorem 3 we show that applications of global transition functions of cellular automata on subsets of Q^M do not increase their entropy. And in Lemma 9 we show that if for each point t of an (E, E')-tiling not all patterns with domain $t \triangleleft E$ occur in a subset of Q^M, then that subset has less entropy than Q^M.

Definition 3. *Let X be a subset of Q^M and let $\mathcal{F} = \{F_i\}_{i \in I}$ be a net in $\{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\}$. The non-negative real number*

$$\operatorname{ent}_{\mathcal{F}}(X) = \limsup_{i \in I} \frac{\log |\pi_{F_i}(X)|}{|F_i|}$$

is called entropy of X with respect to $\mathcal{F}$.

Remark 4. In the situation of Remark 1, the notion of entropy is the same as the one defined in [1, Definition 5.7.1].

Lemma 8. *Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a net in $\{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\}$. Then,*

1. $\operatorname{ent}_{\mathcal{F}}(Q^M) = \log |Q|$;
2. $\forall X \subseteq Q^M \forall X' \subseteq Q^M : \big(X \subseteq X' \implies \operatorname{ent}_{\mathcal{F}}(X) \leq \operatorname{ent}_{\mathcal{F}}(X')\big)$;
3. $\forall X \subseteq Q^M : \operatorname{ent}_{\mathcal{F}}(X) \leq \log |Q|$.

In the remainder of this section, let $\mathcal{F} = \{F_i\}_{i \in I}$ be a right Følner net in $\mathcal{R}$ indexed by $(I, \leq)$.

Theorem 3. *Let X be a subset of Q^M. Then, $\operatorname{ent}_{\mathcal{F}}(\Delta(X)) \leq \operatorname{ent}_{\mathcal{F}}(X)$.*

Proof. Suppose, without loss of generality, that $G_0 \in N$. Let $i \in I$. According to Lemma 3, the map $\Delta^-_{X,F_i} \colon \pi_{F_i}(X) \to \pi_{F_i^{-N}}(\Delta(X))$ is surjective. Therefore, $|\pi_{F_i^{-N}}(\Delta(X))| \leq |\pi_{F_i}(X)|$. Because $G_0 \in N$, according to Item 5 of Lemma 1, we have $F_i^{-N} \subseteq F_i$. Thus, $\pi_{F_i}(\Delta(X)) \subseteq \pi_{F_i^{-N}}(\Delta(X)) \times Q^{F_i \smallsetminus F_i^{-N}}$. Hence,

$$\begin{aligned}\log|\pi_{F_i}(\Delta(X))| &\leq \log|\pi_{F_i^{-N}}(\Delta(X))| + \log|Q^{F_i \smallsetminus F_i^{-N}}|\\ &\leq \log|\pi_{F_i}(X)| + |F_i \smallsetminus F_i^{-N}| \cdot \log|Q|.\end{aligned}$$

Because $G_0 \in N$, according to Item 5 of Lemma 1, we have $F_i \subseteq F_i^{+N}$. Therefore, $F_i \smallsetminus F_i^{-N} \subseteq F_i^{+N} \smallsetminus F_i^{-N} = \partial_N F_i$. Because G_0, F_i, and N are finite, according to Item 7 of Lemma 1, the boundary $\partial_N F_i$ is finite. Hence,

$$\frac{\log|\pi_{F_i}(\Delta(X))|}{|F_i|} \leq \frac{\log|\pi_{F_i}(X)|}{|F_i|} + \frac{|\partial_N F_i|}{|F_i|}\log|Q|.$$

Therefore, because N is finite, according to Theorem 1,

$$\operatorname{ent}_{\mathcal{F}}(\Delta(X)) \leq \limsup_{i \in I} \frac{\log|\pi_{F_i}(X)|}{|F_i|} + \left(\lim_{i \in I} \frac{|\partial_N F_i|}{|F_i|}\right) \cdot \log|Q| = \operatorname{ent}_{\mathcal{F}}(X). \qquad \square$$

Lemma 9. *Let Q contain at least two elements, let X be a subset of Q^M, let E and E' be two non-empty and finite subsets of G/G_0, and let T be an (E, E')-tiling of $\mathcal{R}$, such that, for each cell $t \in T$, we have $\pi_{t \triangleleft E}(X) \subsetneq Q^{t \triangleleft E}$. Then,* $\operatorname{ent}_{\mathcal{F}}(X) < \log|Q|$.

Corollary 2. *Let Q contain at least two elements, let X be a $\blacktriangleright$-invariant subset of Q^M, and let E be a non-empty and finite subset of G/G_0, such that $\pi_{m_0 \triangleleft E}(X) \subsetneq Q^{m_0 \triangleleft E}$. Then,* $\operatorname{ent}_{\mathcal{F}}(X) < \log|Q|$.

4 Gardens of Eden

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a right amenable cell space and let $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$ be a semi-cellular automaton such that the stabiliser G_0 of m_0 under $\triangleright$, the set Q of states, and the neighbourhood N are finite, and the set Q is non-empty. Furthermore, let Δ be the global transition function of $\mathcal{C}$, and let $\mathcal{F} = \{F_i\}_{i \in I}$ be a right Følner net in $\mathcal{R}$ indexed by $(I, \leq)$.

In Theorem 4 we show that if Δ is not surjective, then the entropy of its image is less than the entropy of Q^M. And the converse of that statement obviously holds. In Theorem 5 we show that if the entropy of the image of Δ is less than the entropy of Q^M, then Δ is not pre-injective. And in Theorem 6 we show the converse of that statement. These four statements establish the Garden of Eden theorem, see Main Theorem 1.

Definition 4. *Let c and c' be two maps from M to Q. The set* $\operatorname{diff}(c, c') = \{m \in M \mid c(m) \neq c'(m)\}$ *is called* difference of c and c'.

Definition 5. *The map Δ is called* pre-injective *if and only if, for each tuple $(c, c') \in Q^M \times Q^M$ such that* $\operatorname{diff}(c, c')$ *is finite and $\Delta(c) = \Delta(c')$, we have $c = c'$.*

In the proof of Theorem 4, the existence of a Garden of Eden pattern, as stated in Lemma 10, is essential, which itself follows from the existence of a Garden of Eden configuration, the compactness of Q^M, and the continuity of Δ.

Definition 6. *1. Let $c\colon M \to Q$ be a global configuration. It is called* Garden of Eden configuration *if and only if it is not contained in $\Delta(Q^M)$.*
2. Let $p\colon A \to Q$ be a pattern. It is called Garden of Eden pattern *if and only if, for each global configuration $c \in Q^M$, we have $\Delta(c){\restriction}_A \neq p$.*

Remark 5. 1. The global transition function Δ is surjective if and only if there is no Garden of Eden configuration.
2. If $p\colon A \to Q$ is a Garden of Eden pattern, then each global configuration $c \in Q^M$ with $c{\restriction}_A = p$ is a Garden of Eden configuration.
3. If there is a Garden of Eden pattern, then Δ is not surjective.

Lemma 10. *Let Δ not be surjective. There is a Garden of Eden pattern with non-empty and finite domain.*

Theorem 4. *Let δ be $\bullet$-invariant, let Q contain at least two elements, and let Δ not be surjective. Then,* $\operatorname{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log |Q|$.

Proof. According to Lemma 10, there is a Garden of Eden pattern $p\colon F \to Q$ with non-empty and finite domain. Let $E = (m_0 \trianglelefteq \_)^{-1}(F)$. Then, $m_0 \trianglelefteq E = F$ and, because $\trianglelefteq$ is free, $|E| = |F| < \infty$. Because p is a Garden of Eden pattern, $p \notin \pi_{m_0 \trianglelefteq E}(\Delta(Q^M))$. Hence, $\pi_{m_0 \trianglelefteq E}(\Delta(Q^M)) \subsetneq Q^{m_0 \trianglelefteq E}$. Moreover, according to [5, Item 1 of Theorem 2], the map Δ is $\blacktriangleright$-equivariant. Hence, for each $g \in G$, we have $g \blacktriangleright \Delta(Q^M) = \Delta(g \blacktriangleright Q^M) = \Delta(Q^M)$. In other words, $\Delta(Q^M)$ is $\blacktriangleright$-invariant. Thus, according to Corollary 2, we have $\operatorname{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log |Q|$. $\square$

In the proof of Theorem 5, the fact that enlarging each element of $\mathcal{F}$ does not increase entropy, as stated in the next lemma, is essential.

Lemma 11. *Let X be a subset of Q^M and let E be a finite subset of G/G_0 such that $G_0 \in E$. Then,* $\operatorname{ent}_{\{F_i^{+E}\}_{i \in I}}(X) \leq \operatorname{ent}_{\mathcal{F}}(X)$.

Theorem 5. *Let* $\operatorname{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log |Q|$. *Then, Δ is not pre-injective.*

Proof. Suppose, without loss of generality, that $G_0 \in N$. Let $X = \Delta(Q^M)$. According to Lemma 11, we have $\operatorname{ent}_{\{F_i^{+N}\}_{i \in I}}(X) \leq \operatorname{ent}_{\mathcal{F}}(X) < \log |Q|$. Hence, there is an $i \in I$ such that

$$\frac{\log |\pi_{F_i^{+N}}(X)|}{|F_i|} < \log |Q|.$$

Thus, $|\pi_{F_i^{+N}}(X)| < |Q|^{|F_i|}$. Furthermore, let $q \in Q$ and let $X' = \{c \in Q^M \mid c{\restriction}_{M \smallsetminus F_i} \equiv q\}$. Then, $|Q|^{|F_i|} = |X'|$. Hence, $|\pi_{F_i^{+N}}(X)| < |X'|$. Moreover, according to Item 3 of Lemma 1, we have $(M \smallsetminus F_i)^{-N} = M \smallsetminus F_i^{+N}$. Hence, for each

$(c, c') \in X' \times X'$, according to Lemma 2, we have $\Delta(c)\restriction_{M \smallsetminus F_i^{+N}} = \Delta(c')\restriction_{M \smallsetminus F_i^{+N}}$. Therefore,

$$|\Delta(X')| = |\pi_{F_i^{+N}}(\Delta(X'))| \leq |\pi_{F_i^{+N}}(\Delta(Q^M))| = |\pi_{F_i^{+N}}(X)| < |X'|.$$

Hence, there are c, $c' \in X'$ such that $c \neq c'$ and $\Delta(c) = \Delta(c')$. Thus, because $\operatorname{diff}(c, c') \subseteq F_i$ is finite, the map Δ is not pre-injective. $\square$

In the proof of Theorem 6, the statement of Lemma 12 is essential, which says that if two distinct patterns have the same image and we replace each occurrence of the first by the second in a configuration, we get a new configuration in which the first pattern does not occur and that has the same image as the original one.

Definition 7. *Identify M with G/G_0 by $\iota\colon m \mapsto G_{m_0,m}$. Let*

$$\blacktriangleleft\colon M \times \bigcup_{A \subseteq M} Q^A \to \bigcup_{A \subseteq M} Q^A, \quad (m, p) \mapsto \begin{bmatrix} m \trianglelefteq \operatorname{dom}(p) \to Q, \\ \qquad m \trianglelefteq a \mapsto p(a). \end{bmatrix}$$

Remark 6. Let A be a subset of M, let p be map from A to Q, and let m be an element of M. Then, $m \blacktriangleleft p = g_{m_0,m} \blacktriangleright p$.

Definition 8. *Identify M with G/G_0 by $\iota\colon m \mapsto G_{m_0,m}$, let A be a subset of M, let p be map from A to Q, let c be map from M to Q, let m be an element of M. The pattern p is said to* occur at m in c *and we write $p \sqsubseteq_m c$ if and only if $m \blacktriangleleft p = c\restriction_{m \trianglelefteq A}$.*

Lemma 12. *Identify M with G/G_0 by $\iota\colon m \mapsto G_{m_0,m}$, let A be a subset of M, let N' be the subset $\{g^{-1} \cdot n' \mid n, n' \in N, g \in n\}$ of G/G_0, and let p and p' be two maps from $A^{+N'}$ to Q such that $p\restriction_{A^{+N'} \smallsetminus A} = p'\restriction_{A^{+N'} \smallsetminus A}$ and $\Delta^-_{A^{+N'}}(p) = \Delta^-_{A^{+N'}}(p')$. Furthermore, let c be a map from M to Q and let S be a subset of M, such that the family $\{s \trianglelefteq A^{+N'}\}_{s \in S}$ is pairwise disjoint and, for each cell $s \in S$, we have $p \sqsubseteq_s c$. Put*

$$c' = c\restriction_{M \smallsetminus (\bigcup_{s \in S} s \trianglelefteq A^{+N'})} \times \coprod_{s \in S} s \blacktriangleleft p'.$$

Then, for each cell $s \in S$, we have $p' \sqsubseteq_s c'$, and $\Delta(c) = \Delta(c')$. In particular, if $p \neq p'$, then, for each cell $s \in S$, we have $p \not\sqsubseteq_s c'$.

Theorem 6. *Let δ be $\bullet$-invariant, let Q contain at least two elements, and let Δ not be pre-injective. Then, $\operatorname{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log|Q|$.*

Proof. Suppose, without loss of generality, that $G_0 \in N$. Identify M with G/G_0 by $\iota\colon m \mapsto G_{m_0,m}$. Because Δ is not pre-injective, there are c, $c' \in Q^M$ such that $\operatorname{diff}(c, c')$ is finite, $\Delta(c) = \Delta(c')$, and $c \neq c'$. Put $A = \operatorname{diff}(c, c')$, put $N' = \{g^{-1} \cdot n' \mid n, n' \in N, g \in n\}$, put $E = A^{+N'}$, and put $p = c\restriction_E$ and $p' = c'\restriction_E$. Because $\Delta(c) = \Delta(c')$, we have $\Delta^-_{A^{+N'}}(p) = \Delta^-_{A^{+N'}}(p')$.

Because N is finite and, for each $n \in N$, we have $|n| = |G_0| < \infty$, the set N' is finite. Moreover, $G_0 \cdot N' \subseteq N'$. According to Item 5 of Lemma 1, because $G_0 \in N'$ and $A \neq \emptyset$, we have $E \supseteq A$ and hence E is non-empty. According to Item 7 of Lemma 1, because G_0, A, and N' are finite, so is E. Because E is non-empty, according to Theorem 2, there is a subset E' of G/G_0 and an (E, E')-tiling T of $\mathcal{R}$. Because G_0 and E are non-empty and finite, so is E'.

Let $Y = \{y \in Q^M \mid \forall t \in T : p \not\sqsubseteq_t y\}$. For each $t \in T$, we have $t \blacktriangleleft p \notin \pi_{t \triangleleft E}(Y)$ and therefore $\pi_{t \triangleleft E}(Y) \subsetneqq Q^{t \triangleleft E}$. According to Lemma 9, we have $\mathrm{ent}_{\mathcal{F}}(Y) < \log |Q|$. Hence, according to Theorem 3, we have $\mathrm{ent}_{\mathcal{F}}(\Delta(Y)) < \log |Q|$.

Let $x \in Q^M$. Put $S = \{t \in T \mid p \sqsubseteq_t x\}$. According to Lemma 12, there is an $x' \in Q^M$ such that $x' \in Y$ and $\Delta(x) = \Delta(x')$. Therefore, $\Delta(Q^M) = \Delta(Y)$. In conclusion, $\mathrm{ent}_{\mathcal{F}}(Q^M) < \log |Q|$. □

Main Theorem 1 (Garden of Eden theorem; Edward Forrest Moore, 1962; John R. Myhill, 1963). *Let $\mathcal{M} = (M, G, \triangleright)$ be a right amenable left homogeneous space with finite stabilisers and let Δ be the global transition function of a cellular automaton over $\mathcal{M}$ with finite set of states and finite neighbourhood. The map Δ is surjective if and only if it is pre-injective.*

Proof. There is a coordinate system $\mathcal{K} = (m_0, \{g_{m_0,m}\}_{m \in M})$ such that the cell space $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is right amenable. Moreover, according to [5, Theorem 1], there is a cellular automaton $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$ such that Q and N are finite and Δ is its global transition function.

In the case that $|Q| \leq 1$, the proof is trivial. Let it be the case that $|Q| > 1$. According to Theorem 4 and Item 1 of Lemma 8, the map Δ is not surjective if and only if $\mathrm{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log |Q|$. And, according to Theorems 5 and 6, we have $\mathrm{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log |Q|$ if and only if Δ is not pre-injective. Hence, Δ is not surjective if and only if it is not pre-injective. In conclusion, Δ is surjective if and only if it is pre-injective. □

Remark 7. In the situation of Remark 1, Main Theorem 1 is [1, Theorem 5.3.1].

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