Forward Kinematic Analysis of the 3-RPRS Parallel Manipulator

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Abstract This paper presents a comprehensive analytical solution to the forward kinematic problem of a newly introduced spatial parallel manipulator, namely, the 3-RPRS. The manipulator has three legs with two actuators in each, which connect a moving triangular platform to a fixed base. Loop-closure equations are formed to find the unknown passive rotary joint angle in each leg. These equations are subsequently reduced to a single univariate polynomial equation of degree 16. The coefficients of this equation are obtained as closed-form functions of the architecture parameters of the manipulator and the input joint angles, and therefore the analysis covers all possible architectures and configurations. Furthermore, it is found that the polynomial has only the even powers, therefore leading to 8 pairs of solutions, each pair being mirrored at the base platform. The theoretical developments are illustrated via a numerical example. The results obtained are validated by checking the residues of the original loop-closure equations, thereby establishing the correctness of the formulation as well as the results.

Keywords 3-RPRS manipulator · Loop-closure equations · Forward kinematic univariate · Closed-form solution

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1 Introduction

Parallel manipulators have attracted a lot of research attentions in the past few decades for their superior performance over their serial counterparts in terms of the load-carrying capacity, rigidity and accuracy. While some manipulators are very well established in terms of their applications in the industries, such as the Stewart platform manipulator (introduced in 1954) [6], and the Delta robot (introduced in 1980s) [4], many more continue to emerge. For instance, the Agile Eye [2] is a three degree of freedom (DoF) spherical parallel manipulator, mainly developed for camera orientation. The mechanical architecture of the robot allows it to achieve high velocities and acceleration. This manipulator has been immensely popular since its introduction in 1993. A partially decoupled four DoF parallel manipulator with the acronym PAMINSA has been presented in [3]. The specialty of this robot is the decoupling of the displacements along the horizontal and vertical axes, enabling static balancing, and reducing the loads on the actuators.

A new six DoF spatial parallel manipulator was introduced in [7]. The manipulator has three legs mounted on a circualr guide at the base, which allows it to exhibit large (i.e., kinematically unbounded) yaw motions, which is not very common in platform-type spatial parallel manipulators. In [7] only the inverse kinematic analysis of the manipulator was presented. In the present paper, the forward kinematic problem of the manipulator is studied in the *closed-form*, to the extent permissible mathematically. The *loop-closure* constraints are derived in the joint space. Through a sequence of elimination of variables, the set of equations is reduced to a single equation in one of the passive joint variables. This equation, called the *forward kinematics univariate* (FKU) (in accordance with [5]), when transformed to its algebraic form, turns out to be of degree 16. However, using symbolic simplifications[1] performed in the computer algebra system `Mathematica`, it is revealed that the FKU is actually of degree 8 in the square of the remaining variable. Thus, it is established that there are essentially 8 possible poses of the manipulator, the remaining being mirror reflections of the same, at the base platform. The theoretical calculations are illustrated through a numerical example, and the validity of the results obtained is demonstrated by computing the norm of the residues of the constraint equations.

The rest of the paper is organised as follows: Sect. 2 covers the mathematical formulation, starting from the geometrical description of the manipulator, then forming the loop-closure equations and finally presenting a detailed solution procedure for the forward kinematic problem. Section 3 presents a numerical case study. The correctness of the results obtained has been verified by checking the residues of the loop-closure equations. The conclusions are presented in Sect. 4.

[1]Interested readers can find the details of the used simplification scheme in [1, 5].

2 Mathematical Formulation

The geometry of the manipulator is described in this section, followed by the formulation of its forward kinematics.

2.1 Geometry

The geometry of the 3-RPRS is shown in Fig. 1. It consists of a fixed base and a moving top platform, connected by three *legs* of identical architecture, namely, RPRS. The first two joints in each leg are *active* (i.e., actuated), together imparting general six DoF spatial motion to the top platform. The first joint imparts a rotation about the axis Z_0. Physically the rotary motion is achieved by mounting the links on a circular guide. The next joint is prismatic, with its axis orthogonal to the said circular guide. A rigid strut of length l is connected to the prismatically actuated link at the base, through a passive (i.e., unactuated) revolute joint situated at the point $\boldsymbol{b}_i$, $i = 1, 2, 3$, which are given by:

$$\boldsymbol{b}_i = [d_i \cos\theta_i, d_i \sin\theta_i, 0]^\top. \tag{1}$$

In Eq. (1), d_i denotes the extent of the actuation of the prismatic point, and θ_i denotes the orientation of the ith prismatic joint axis, given as the in-plane CCW rotation about the Z_0 axis, measured from the X_0 axis. The vector along the ith strut is expressed in the local frame of the leg as (see the frame $X_{l3}Y_{l3}Z_{l3}$ in Fig. 1):

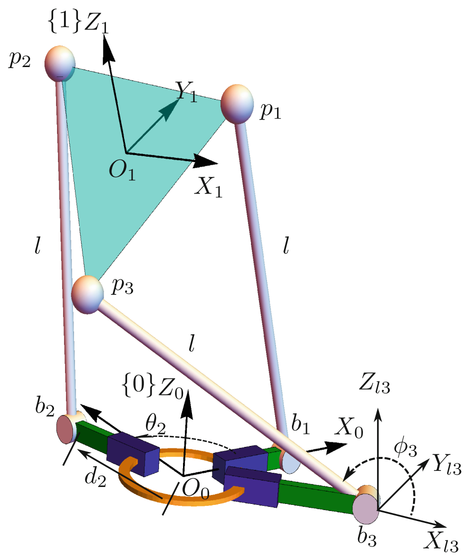


Fig. 1 Schematic representation of the manipulator

$$\boldsymbol{l}_i = [l \cos \phi_i, 0, l \sin \phi_i]^\top. \tag{2}$$

The upper extremity of the ith strut is connected to the end-effector platform by a passive spherical joint at the point $\boldsymbol{p}_i$, where the location of $\boldsymbol{p}_i$ in the base-frame $\{0\}$ is given by:

$$\boldsymbol{p}_i = \boldsymbol{b}_i + \boldsymbol{R}_i \left(\theta_i\right) \boldsymbol{l}_i. \tag{3}$$

In Eq. (3), $\boldsymbol{R}_i \left(\theta_i\right)$ is the *rotation matrix* corresponding to a counter-clockwise rotation about the positive Z_0 axis, by an angle θ_i. The end-effector is an equilateral triangle, with each side having length s.

The complete set of joint variables describing the configuration of the manipulator is given by $\boldsymbol{q} = [\theta_1, \theta_2, \theta_3, d_1, d_2, d_3, \phi_1, \phi_2, \phi_3]^\top$, of which $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3, d_1, d_2, d_3]^\top$ are designated as the *active* variables, and $\boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3]^\top$ are the *passive* rotary joint variables. In terms of these variables, the forward kinematic problem reduces to the task of finding $\boldsymbol{\phi}$, for given geometric parameters l, s, which have already been defined above, and the known inputs, $\boldsymbol{\theta}$. The first step towards that is to find the three *independent* equations involving $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$, which are derived next.

2.2 Loop-Closure Equations

The manipulator is required to maintain *loop-closure* at *every* instance, leading to the constraints that apply on the motion of the top platform, and the manipulator in general. In this case, the loop-closure constraints can be stated as follows: the tip points of each leg, $\boldsymbol{p}_i$, move in space in such a manner, that the distance between any two of them is held constant, and equal to s (as the top platform is a an equilateral triangle of side s). This statement is translated into the desired three equations in $\boldsymbol{\phi}$ in the following:

$$\left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right)^\top \cdot \left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right) - s^2 = 0 \Rightarrow f_1\left(\phi_1, \phi_2\right) = 0, \tag{4}$$

$$\left(\boldsymbol{p}_2 - \boldsymbol{p}_3\right)^\top \cdot \left(\boldsymbol{p}_2 - \boldsymbol{p}_3\right) - s^2 = 0 \Rightarrow f_2\left(\phi_2, \phi_3\right) = 0, \tag{5}$$

$$\left(\boldsymbol{p}_3 - \boldsymbol{p}_1\right)^\top \cdot \left(\boldsymbol{p}_3 - \boldsymbol{p}_1\right) - s^2 = 0 \Rightarrow f_3\left(\phi_3, \phi_1\right) = 0. \tag{6}$$

Equations (4)–(6) represent the loop-closure equations, which are solved next to obtain the values of the unknown passive joint rotation, ϕ_i.

2.3 Solution

Equations (4)–(6) are linear in the cosine and sine of (ϕ_1, ϕ_2), (ϕ_2, ϕ_3) and (ϕ_3, ϕ_1), respectively. Solving for $\cos\phi_1$ and $\sin\phi_1$ simultaneously from Eqs. (4) and (6) and substituting in the identity $\sin^2\phi_1 + \cos^2\phi_1 = 1$, the unknown ϕ_1 is eliminated from the system of equations, while leading to a new equation in the remaining variables:

$$g(\phi_2, \phi_3) = 0. \tag{7}$$

Also, the common solution of $f_1 = 0, f_3 = 0$ in ϕ_1 is obtained *uniquely* as

$$\phi_1 = \text{atan2}(\cos\phi_1, \sin\phi_1), \tag{8}$$

where $\text{atan2}(\cos(\cdot), \sin(\cdot))$ denotes the *two-argument* arctangent function. At this stage, there remain two equations involving the sine and cosine of ϕ_2, ϕ_3: $f_2 = 0$, and $g = 0$. As g is found to be quadratic in these functions, it is easier to work with the algebraic forms of these equations henceforth. These are converted to polynomials in t_2 using the standard tangent half-angle substitution: $t_2 = \tan(\phi_2/2)$. After the transformation, $g = 0$ becomes a quartic equation in t_2, (say, $h_1(t_2, \phi_3) = 0$) while $f_2 = 0$ turns into a quadratic,[2] say, $h_2(t_2, \phi_3) = 0$. These steps are shown schematically in (9).

$$\left.\begin{array}{l} f_1\left(\phi_1, \phi_2\right) = 0 \\ f_3\left(\phi_1, \phi_3\right) = 0 \end{array}\right) \xrightarrow{\times\phi_1} \begin{array}{l} g(\phi_2, \phi_3) = 0 \xrightarrow{\phi_2\to t_2} h_1(t_2, \phi_3) = 0; \\ f_2(\phi_2, \phi_3) = 0 \xrightarrow{\phi_2\to t_2} h_2(t_2, \phi_3) = 0. \end{array} \tag{9}$$

The symbol '$\xrightarrow{\times x}$' implies the elimination of the variable x from two simultaneous equations in x, and '$\xrightarrow{\phi_2\to t_2}$' implies the conversion of a trigonometric expression in ϕ_2 to its algebraic equivalent, in terms of $t_2 = \tan(\phi/2)$. The quadratic nature of h_2 in t_2 suggests an easy means for obtaining t_2 in the closed-form, as well as eliminating t_2 from the equations $h_i = 0$, as shown below. Firstly, $h_1(t_2, \phi_3)$ is divided by $h_2(t_2, \phi_3)$, treating both as polynomials in t_2:

$$h_1(t_2, \phi_3) = \alpha(t_2, \phi_3)h_2(t_2, \phi_3) + \beta(t_2, \phi_3), \tag{10}$$

where α, β are the quotient and the remainder polynomials, respectively. The underlying assumption made here is that the leading coefficient of the divisor $h_2(t_2, \phi_3)$

[2] It may be noted here is that the last remaining variable, ϕ_3, is retained in its trigonometric form while ϕ_2 alone is converted to t_2. This helps in further symbolic computations required in the derivation of the FKU (see [5] for the details).

is non-zero.[3] Evidently, β can be at the most *linear* in t_2, and $h_1 = 0, h_2 = 0$ means $\beta = 0$. Therefore, one finds an expression of t_2 *uniquely* in terms of all the known entities and ϕ_3, and this value of t_2 is *guaranteed* to satisfy the equations $h_1 = 0$, $h_2 = 0$ *simultaneously*.

$$\beta(t_2, \phi_3) = 0 \Rightarrow c_1(\phi_3)t_2 + c_2(\phi_3) = 0;$$
$$t_2 = -c_2(\phi_3)/c_1(\phi_3), \text{ assuming } c_1(\phi_3) \neq 0. \quad (11)$$

From t_2, the passive variable ϕ_2 is found by inverting the half-tangent formula:

$$\phi_2 = 2\arctan(t_2). \quad (12)$$

Equation (12) yields a unique solution of ϕ_2 for each value of ϕ_3. The solutions for ϕ_3 are computed next, to complete the solution of the forward kinematics problem.

The common solution t_2 from Eq. (11) is substituted in the equation $h_2(t_2, \phi_3) = 0$, to reduce it to an equation with ϕ_3 as the sole remaining unknown. This equation is converted to its algebraic form in terms of $t_3 = \tan(\phi_3/2)$, and the FKU equation $\eta\left(t_3\right) = 0$ is obtained as a polynomial in the unknown t_3. The FKU is of degree 16 in t_3. However, upon rigorous symbolic simplification, it is found that the FKU has the following form:

$$a_0t_3^{16} + a_1t_3^{14} + a_2t_3^{12} + a_3t_3^{10} + a_4t_3^8 + a_5t_3^6 + a_6t_3^4 + a_7t_3^2 + a_8 = 0. \quad (13)$$

The coefficients, a_i, are functions of the architecture parameters and known inputs, and are therefore known *exactly*, in the closed form. These can therefore be evaluated to an arbitrary level of desired accuracy, and the FKU may be solved to find all the real values of t_3. This is illustrated in the next section. The numerical form of the FKU after substituting the values of the architecture parameters and inputs is given in Eq. (14) as a monic polynomial:

$$t_3^{16} - 21.4575t_3^{14} - 86.4879t_3^{12} + 2667.8104t_3^{10} + 608.5886t_3^8$$
$$- 28338.0681t_3^6 + 44125.4058t_3^4 - 21515.7426t_3^2 + 3376.6233 = 0. \quad (14)$$

3 Numerical Results

To illustrate the mathematical developments presented above, the following numerical values are chosen for the architecture parameters: $l = 2.2$, and s = 2. The numerical values of the active variables[4] are chosen as $d_1 = 1.2, d_2 = 1.5, d_3 = 1.7, \theta_1 = 0$,

[3]The manipulator is singular when h_2 becomes linear in t_2, a case which is not discussed in this paper.

[4]All angles are in radians, and lengths in meters, unless mentioned otherwise explicitly.

$\theta_2 = 1.832, \theta_3 = 5.497$. The given input vector along with $l = 2.2$ and $s = 2$ results in 16 distinct solutions for t_3 of which 8 are real and the other 8 are complex. The real values of the passive joint angles for the given set of input variables is enumerated in Table 1. In order to validate the obtained solutions numerically, all the numerical inputs are results are substituted back to the original loop-closure equations, Eqs. (4)–(6), and the residue of the vector $\boldsymbol{f} = [f_1, f_2, f_2]^\top$ is computed:

$$e = \|\boldsymbol{f}\|. \tag{15}$$

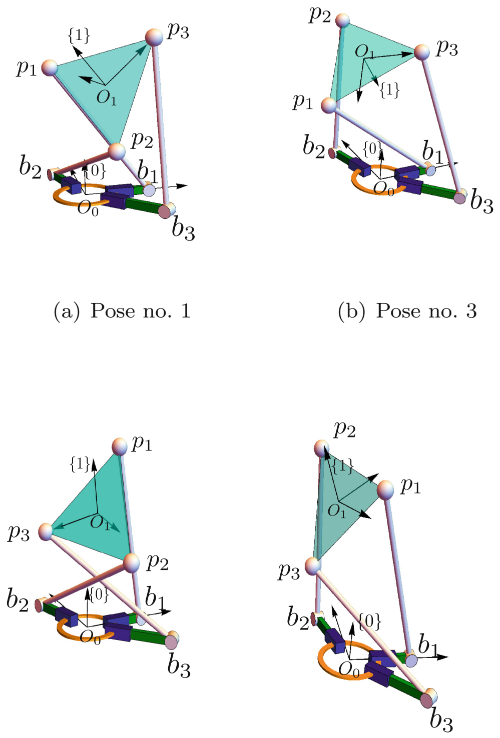
The error e is tabulated in the last column of Table 1 for each of the real branches of the solution. It can be seen from Table 1, that there are actually 4 distinct solutions (i.e., no. 1, 3, 5, 7), while the other 4 are mirror images of them (i.e., no. 2, 4, 6, 8, respectively) (Fig. 2).

4 Conclusions

This paper has presented an analytical study of the forward kinematics of the 3-RPRS manipulator. The manipulator has three legs, connecting a moving platform in the form of an equilateral triangle to a fixed base. Each of the legs have a rotary and a prismatic actuator in series, together imparting the full six spatial motions onto the moving platform. The legs contain a single passive variable, the determination of which is the objective of the forward kinematic analysis. Three loop-closure equations are formed to find the three unknowns, and these equations are subsequently reduced to a single polynomial of degree 16. Further, this polynomial is shown to have only even-powered terms, therefore revealing that the manipulator can have at most 8 poses for a given set of inputs, and the corresponding mirror images about the fixed base. The coefficients of the polynomial are obtained in closed-form, for a generic architecture of the manipulator, thus solving the forward kinematics problem comprehensively. The results were numerically validated by substituting, in the loop-closure constraint equations and evaluating the residue of each equation. The resulting residues were found to be very close to zero, thereby providing necessary validation of the correctness of the solutions.

Table 1 Forward kinematic solutions for input variables $d_1 = 1.2, d_2 = 1.5, d_3 = 1.7, \theta_1 = 0, \theta_2 = 1.832, \theta_3 = 5.497$

No	$t_{3,real}$	ϕ_1	ϕ_2	ϕ_3	x	y	z	α	β	γ	$e \times 10^{-13}$
1	4.2833	1.8570	1.4448	2.6828	−0.2775	0.8376	2.3942	0.8512	−0.9805	0.7172	3.0
2	−4.2833	−1.8570	−1.4448	−2.6828	−0.2775	0.8376	−2.3942	0.8512	0.9805	0.7172	3.0
3	3.6324	1.7548	2.6770	2.6043	0.1121	−0.1738	1.9429	−2.3911	0.8534	1.3053	0.5
4	−3.6324	−1.7548	−2.6770	−2.6043	0.1121	−0.1738	−1.9429	2.3911	0.8534	1.3053	0.5
5	1.4431	2.5006	1.6025	1.9296	−0.3704	0.3000	2.5337	−2.7312	0.5902	−2.1014	3.3
6	−1.4431	−2.5006	−1.6025	−1.9296	−0.3704	0.3000	−2.5337	2.7312	0.5902	−2.1014	3.3
7	1.0184	2.2008	2.7546	1.6462	0.2684	−0.7589	2.1825	1.1851	−0.6321	−2.1613	0.7
8	−1.0184	−2.2008	−2.7546	−1.6462	0.2684	−0.7589	−2.1825	−1.1851	0.6321	−2.1613	0.7



(a) Pose no. 1 (b) Pose no. 3

(c) Pose no. 5 (d) Pose no. 7

Fig. 2 Configurations depicting the real solutions for the given inputs (numbered as per Table 1). **a** Pose no. 1, **b** Pose no. 3, **c** Pose no. 5, **d** Pose no. 7

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