Various Strategies of Elastic Forces Evaluation in the Absolute Nodal Coordinate Formulation

R. Bulín and M. Hajžman

Abstract This paper deals with the description of the absolute nodal coordinate formulation (ANCF) which is suitable for the flexible bodies modelling considering large deformation. As it is shown for beam ANCF elements, this formulation leads to the nonlinear expression of elastic forces, which could be the main disadvantage of the ANCF. The evaluation of these forces can be done by numerical integration in each computational step or by analytical derivation with the help of a software for symbolic operations. The computational performance of various elastic forces evaluation strategies is investigated using a benchmark problem of falling flexible pendulum.

Keywords Absolute nodal coordinate formulation · Elastic forces · Benchmark · Flexible beam

1 Introduction

An absolute nodal coordinate formulation (ANCF) [3] is a suitable approach to the modelling of beams, cables, wires and fibers in the framework of robot, manipulator and mechanism design. The ANCF is based on the usage of global displacements and slopes as nodal coordinates. It belongs to the modern approaches of flexible multibody dynamics and allows to model flexible bodies performing a large motion including deformation.

Dynamical models employing ANCF are characterized by constant mass matrices and highly nonlinear stiffness matrices. Therefore the issue of the formulation of elastic forces is very important and it is the motivation for the work presented in this paper. There are several types of ANCF beam elements [2]. This

R. Bulín (✉) · M. Hajžman
University of West Bohemia, Pilsen, Czech Republic
e-mail: rbulin@kme.zcu.cz

M. Hajžman
e-mail: mhajzman@kme.zcu.cz

J. Beran et al. (eds.), *Advances in Mechanism Design II*,
Mechanisms and Machine Science 44, DOI 10.1007/978-3-319-44087-3_23

paper is focused on a lower order ANCF element, which can be used for modelling of thin cables, fibers and wires. This element respects bending and axial stiffness and can be also extended by adding torsional stiffness [4].

2 Absolute Nodal Coordinate Formulation

A spatial lower order ANCF element of length l with two nodes uses components of position vector $\mathbf{r}$ of nodes and its derivation with respect to local parameter $x \in \langle 0, l \rangle$ (slopes) as the nodal coordinates. This can be expressed as

$$\mathbf{e} = \left[\mathbf{r}^{(i)T}, \mathbf{r}_{,x}^{(i)T}, \mathbf{r}^{(j)T}, \mathbf{r}_{,x}^{(j)T}\right]^T, \tag{1}$$

where $\mathbf{e}$ is the vector of nodal coordinates, $\mathbf{r}^{(i)}$ is the position vector of node i and $\mathbf{r}_{,x}^{(i)} = \frac{\partial \mathbf{r}^{(i)}}{\partial x} = \boldsymbol{\tau}^{(i)}$ represents the slope at node i (note, that each element has node i and j). This implies that each element has 12 degrees of freedom, 6 at each node. Global position $\mathbf{r} = \left[r_x, r_y, r_z\right]^T$ of an arbitrary beam point determined by parameter x can be written as

$$\mathbf{r}(x) = \mathbf{S}(x)\mathbf{e}, \quad \mathbf{S}(x) = [s_1\mathbf{I}, s_2\mathbf{I}, s_3\mathbf{I}, s_4\mathbf{I}], \tag{2}$$

where $\mathbf{S}$ is the global shape function matrix of size 3×12, $\mathbf{I}$ is the identity matrix of size 3×3 and the shape functions can be derived in the form

$$\begin{aligned} &s_1 = 1 - 3\xi^2 + 2\xi^3, \quad s_2 = l\left(\xi - 2\xi^2 + \xi^3\right), \\ &s_3 = 3\xi^2 - 2\xi^3, \qquad s_4 = l\left(-\xi^2 + \xi^3\right), \qquad \xi = x/l. \end{aligned} \tag{3}$$

It must be noted, that a cubic polynomials in x are employed to describe all three components of the displacement and the element can be considered as isoparametric.

Standard procedures (e.g. the Lagrange equations or the principle of virtual work) can be used in order to derive a mathematical model of the spatial ANCF element. Kinetic energy of the element with material density ρ is

$$E_k = \frac{1}{2}\int_0^l \rho A \dot{\mathbf{r}}^T \dot{\mathbf{r}} \mathrm{d}x = \frac{1}{2}\dot{\mathbf{e}}^T \int_0^l \rho A \mathbf{S}^T \mathbf{S} \mathrm{d}x \, \dot{\mathbf{e}} = \frac{1}{2}\dot{\mathbf{e}}^T \mathbf{M}_e \dot{\mathbf{e}}, \tag{4}$$

where $\mathbf{M}_e$ is the element mass matrix.

Strain energy E_p of the element is used for the derivation of elastic forces in the ANCF beam model and the form of an adopted elasticity model determines the complexity of the whole model. The most common approach employs the separation of the strain energy of longitudinal deformation E_{pl} and the strain energy of transverse (bending) deformation E_{pt} as

$$E_p = E_{pl} + E_{pt} = \frac{1}{2}\int_0^l EA\varepsilon_x^2 \mathrm{d}x + \frac{1}{2}\int_0^l EI\kappa^2 \mathrm{d}x, \tag{5}$$

where E is the Young modulus, A is the area of the cross-section and I is the second moment of the area about a transverse axis. In this particular case, it is assumed that the second moments of the area for both transverse axes z and y are equal, so it is applied $I = I_{zz} = I_{yy}$. The axial strain ε_x and the curvature κ can be in general case expressed as [2]

$$\varepsilon_x = \frac{1}{2}\left(\mathbf{r}_{,x}^T \mathbf{r}_{,x} - 1\right) = \frac{1}{2}\left(f^2 - 1\right), \quad \kappa = \left|\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2}\right| = \frac{\left|\mathbf{r}_{,x} \times \mathbf{r}_{,xx}\right|}{\left|\mathbf{r}_{,x}\right|^3}, \tag{6}$$

where $f = \frac{\mathrm{d}s}{\mathrm{d}x}$ represents the deformation gradient for longitudinal strain and ds is the infinitesimal arc length. Note, that the axial strain is defined by Green strain tensor. The strain energy leads to nonlinear elastic forces that must be evaluated in each integration step.

3 Evaluation of Elastic Forces

Based on the general expression of axial strain (6), the vector of longitudinal elastic forces of the element e has the form

$$\begin{aligned}\mathbf{Q}_l^e = \frac{\partial E_{pl}}{\partial \mathbf{e}} &= EA\int_0^l \left(\frac{\partial \varepsilon_x}{\partial \mathbf{e}}\right)^T \varepsilon_x \mathrm{d}x \\ &= EA\int_0^l \left(\mathbf{S}_{,x}^T \mathbf{S}_{,x}\mathbf{e}\right)\left[\frac{1}{2}\left(\mathbf{e}^T \mathbf{S}_{,x}^T \mathbf{S}_{,x}\mathbf{e} - 1\right)\right]\mathrm{d}x = \mathbf{K}_l(\mathbf{e})\mathbf{e}.\end{aligned} \tag{7}$$

The integral in Eq. (7) can be derived analytically or by using some software for symbolic operations (MATLAB R2012a was used in this work). It can be shown that the resultant nonlinear longitudinal stiffness matrix is a full matrix and its elements are quadratic functions of the nodal coordinates. The evaluation of such a vector of elastic forces in each time step can be computationally demanding. That is why it is suitable to approximate the integral in (7) by using the Gaussian quadrature. As it is noted in [1], the matrix $\mathbf{K}_l(\mathbf{e})$ is not unique and another derivation of this matrix in case of planar ANCF beam is shown there. It is based on the separation of vector of nodal coordinates $\mathbf{e}$ to sum of two vectors—arbitrary rigid-body displacement and flexible deformation. This approach leads to simpler form of matrix $\mathbf{K}_l(\mathbf{e})$ in case of planar elements and is referred to as L2 model.

The vector of transverse elastic forces of element e has the form

$$\mathbf{Q}_t^e = \frac{\partial E_{pt}}{\partial \mathbf{e}} = EI \int_0^l \left(\frac{\partial \kappa}{\partial \mathbf{e}}\right)^T \kappa \, \mathrm{d}x. \tag{8}$$

Since the general expression of curvature κ in Eq. (6) is of a complex form, the integral in Eq. (8) is difficult to solve even with the help of the software for symbolic operations. But, as it is mentioned in [2], the derivation of curvature $\frac{\partial \kappa}{\partial \mathbf{e}}$ can be obtained in closed form. The vector of transverse elastic forces can be then evaluated using Gaussian quadrature with the use of several precomputed terms. As it is described in [1], significant simplification can be achieved when the longitudinal deformation within the element is assumed constant while developing the vector of transverse elastic forces. Then, the deformation gradient $f = \frac{\mathrm{d}s}{\mathrm{d}x}$ has a constant value $\bar{f}$ and the curvature can be simplified as

$$\kappa = \left|\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2}\right| = \left|\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}x^2} \cdot \frac{\mathrm{d}^2 x}{\mathrm{d}s^2}\right| = \frac{1}{\bar{f}^2}\mathbf{r}_{,xx} = \frac{1}{\bar{f}^2}\mathbf{S}_{,xx}\mathbf{e}. \tag{6}$$

This approach is referred to as T2 model and it is recommended to use sufficient number of elements to meet the mentioned assumption.

4 Benchmark Problem of a Flexible Pendulum

An in-house software for the numerical simulation of chosen mechanical systems with flexible beams modelled by spatial lower order ANCF beam elements was created in MATLAB. A falling flexible pendulum as a standard benchmark example for the testing of the created code was implemented and several simulation results are described in this section. The scheme of the pendulum is in Fig. 1 and its parameters are l = 2 m, ρ = 4000 kg m^3, a = 0.01 m, E = 10^8 Pa, g = 9.81 m s^{-2}. The simulation time is 2 s. The equations of motion were solved using *ode23t* function in MATLAB with implicit error settings. The numerical solution was performed using HP Compaq Elite 8300 with Intel Core i5-3570 CPU and 16 GB RAM.

Three strategies of the elastic forces evaluation were tested. First model denoted as GL-T uses Gaussian quadrature to evaluate both longitudinal and transverse elastic forces (L is the number of Gaussian points used for determination of $\mathbf{Q}_l^e$ and T is the number of points for $\mathbf{Q}_t^e$). Second model denoted as ST2 uses symbolically determined longitudinal elastic forces and T2 model [1] for transverse forces. The

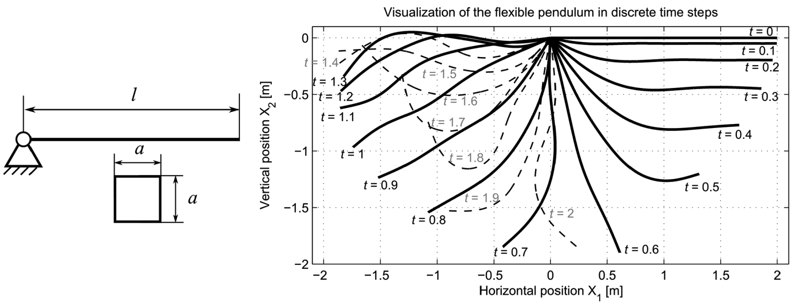

Fig. 1 The scheme of the flexible pendulum with its cross-section and the visualization of the pendulum in discrete time steps

last model is denoted as L2T2 and the strategy of evaluation of the elastic forces is evident from the name of the model and corresponds with [1].

The visualization of the flexible pendulum obtained from solution of the G10-10 model is shown in Fig. 1. The computational times for each model and maximum difference over time from G10-10 model are summarized in Table 1. It is obvious, that results from G2-2 model are rather different, because it uses only two Gaussian points to approximate cubic terms in general. Other differences between models are rather small. It seems reasonable to use more Gaussian points for longitudinal force approximation (5 points), because their evaluation is relatively fast. The vertical displacement of the pendulum tip for selected models and the difference of vertical displacement of the tip is shown in Fig. 2.

Table 1 The summary of computational demands for various approaches to elastic forces

Model	Number of elements	Computational time (s)	Computational time for longitudinal forces (s)	Computational time for transverse forces (s)	Maximum difference from G10-10 model (m)
G2-2	10	107.5	14.6	60.5	3.36×10^{-1}
G3-3	10	185.7	26.3	116.1	4.16×10^{-3}
G4-4	10	233.4	33.8	154.1	1.84×10^{-4}
G5-5	10	285.2	42.2	192.9	3.29×10^{-5}
G10-10	10	492.5	76.5	367.2	Reference model
ST2	10	355.0	261.1	48.1	1.02×10^{-4}
L2T2	10	109.8	14.8	50.3	1.02×10^{-2}
L2T2	20	481.4	67.5	233.4	7.72×10^{-3}

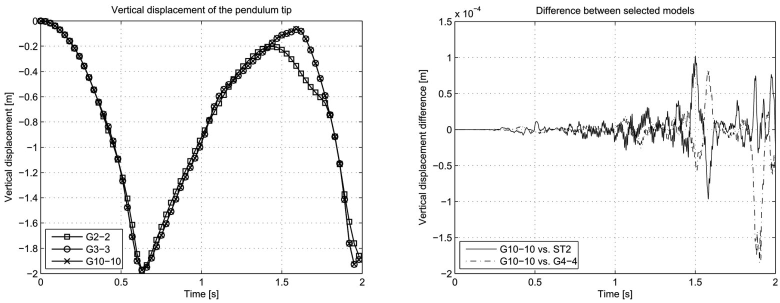

Fig. 2 The vertical displacement of the pendulum tip for selected models and the difference of vertical displacement of the tip for selected models

5 Conclusions

In this paper, the ANCF beam element suitable for problems of flexible multibody dynamics was described and various strategies of evaluation of the elastic forces were shown. According to resultant computational times of various models it is recommended to use five Gaussian points to approximate longitudinal elastic forces and at least three points for transverse forces. Another option for transverse force evaluation is to use T2 model [1]. The symbolical evaluation of longitudinal forces leads to full stiffness matrix, whose evaluation is relatively slow.

Acknowledgments The first author was supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports and the second author was supported by the project 15-20134S of the Czech Science Foundation.

References

1. Berzeri, M., Shabana, A.A.: Development of simple models for the elastic force in the absolute nodal co-ordinate formulation. J. Sound Vib. **235**, 539–565 (2000)
2. Gerstmayr, J., Shabana, A.A.: Analysis of thin beams and cables using absolute nodal co-ordinate formulation. Nonlinear Dyn. **45**, 109–130 (2006)
3. Shabana, A.A.: Dynamics of Multibody Systems, 3rd edn. Cambridge UP (2005)
4. Yoo, W.-S., Dmitrochenko, O., Park, S.-J., Lim, O.-K.: A new thin spatial beam element using the absolute nodal coordinates: application to a rotating strip. Mech. Based Des. Struct. Mach. **33**, 339–422 (2005)