Free Vibration Frequency Spectrum of Four-Planetary Gearing Box

L. Půst, L. Pešek and A. Radolfová

Abstract Dynamic analysis of multi-mesh planetary gearings is very important for reduction of noise and vibration. Splitting of force flow into several planet wings is the main advantage of planetary gearings. As it can be devaluated by unequal load sharing on individual planet stages, the floating sun gear and flexible pins of planet gears are applied. This paper shows that dynamic model of such a gearing box is very complicated with many multiple eigenfrequencies. The gained frequency spectrum with multiple eigenvalues is derived and analyzed.

Keywords Planetary gearing · Negative stiffness · Frequency spectrum · Multiple eigenvalues

1 Introduction

Modal and spectral dynamic properties of planetary gearboxes are more complicated than parallel-axis gear transmission systems and therefore they need deeper dynamic analysis. The main advantage of planetary gearing is in splitting of force flow into several planet stages and so minimizing of weight. In order to prevent unequal load sharing on planet stages, floating sun gear and flexible pins of planet gears are applied. Dynamic model of such a gearing box is very complicated mainly since it has several multiple eigenfrequencies in its spectrum.

In this paper, the solution of spectral properties of the plane type of gearings with four planetary subsystems and with fixed planet carrier is presented. As the all

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wheels have helical gearings, fluctuation of teeth contact stiffness over a mesh cycle can be neglected. Steady contacts in gearings are asserted by means of preloading due to the constant moment loads on the sun and on outer rig gears.

Application of floating sun gear causes that in addition to the deformations in mesh contact in the direction of tangent to the base circle there is also a radial motion component perpendicular to this tangential deformation. Restoring forces at displacement in this radial direction are usually not taken into account at mathematical modelling. The new radial stiffness of gear contact has been therefore introduced both for external and internal tooth systems.

2 Type of Investigated Gearbox

The plane type of gearings with four planetary subsystems and with fixed planet carrier has been solved—Fig. 1. All the wheels have helical gearings. As these gearings have a very small variation of contact stiffness, teeth contact stiffness is supposed to be constant. The main aim of the complex study is analysis of influence of planetary pins compliance (stiffness k_c) and of free (or weekly supported, stiffness k_c) axis of sun wheel on gearbox dynamic properties. The second aim is to prepare the theoretical base for evaluation of measurements data gained at the planned experiments on new gearing box prototype.

3 Radial Stiffness of Two Gearing Wheels

Mutual radial motion of two gearing wheels changes pressure angle α as shown in Fig. 2, where $r_3^* = r_3 \cos(\alpha)$, $r_2^* = r_2 \cos(\alpha)$ are radiuses of base, (r_3, r_2 of pitch) circles. The radial shift Δy determines change $\Delta\alpha$ of pressure angle α. The radial component F_r of the contact force F increases:

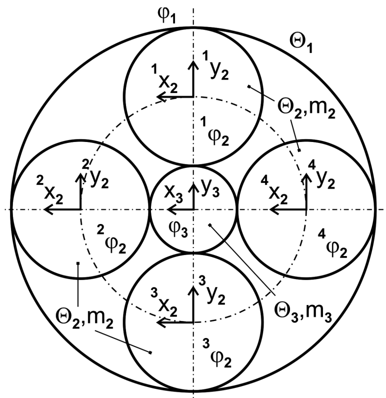
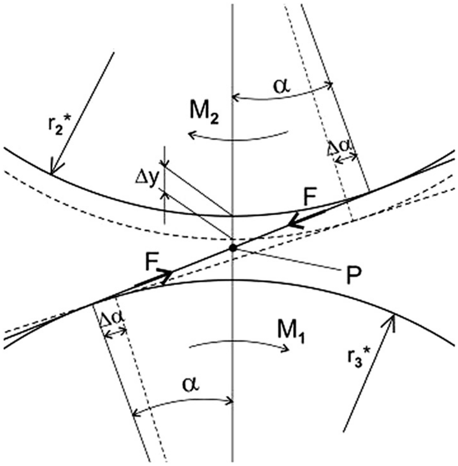


Fig. 1 Four-planetary gearing

Fig. 2 External mesh



$$\Delta F_r = F(\sin(\alpha - |\Delta\alpha|) - \sin(\alpha)) \cong -F\cos(\alpha)\sin(|\Delta\alpha|) \tag{1}$$

Also the change of radial shift Δy is connected with $\Delta\alpha$. The approaching of base circles Δy_c and of axes Δy_a of both wheels are the same $\Delta y_c = \Delta y_a = \Delta y$:

$$\Delta y = \frac{r_3^* + r_2^*}{\cos(\alpha)} - \frac{r_3^* + r_2^*}{\cos(\alpha - |\Delta\alpha|)} \cong \frac{(r_3^* + r_2^*)\sin(\alpha)\sin(|\Delta\alpha|)}{\cos^2(\alpha)}. \tag{1a}$$

The ratio of ΔF_r and Δy, gives negative radial stiffness k_r:

$$k_r = \frac{\Delta F_r}{\Delta y} = \frac{\Delta F_r}{\Delta y_c} = \frac{\Delta F_r}{\Delta y_a} = \frac{-F\cos^3(\alpha)}{(r_3^* + r_2^*)\sin(\alpha)} = \frac{-F\cos^2(\alpha)}{(r_3 + r_2)\sin(\alpha)}. \tag{2}$$

Similar relations are valid also for the radial contact stiffness between ring and planetary wheel that is an internal gearing contact, where the pressure angle α increases $\Delta\alpha > 0$ at wheels penetrating $\Delta y = \Delta y_c$. In such a case the radial contact stiffness k_r is positive. However, the approaching of base circles Δy_c at internal gearing contact is connected with increase of wheel axes distance $\Delta y_a = -\Delta y_c$ and therefore the radial contact stiffness $k_r = \frac{\Delta F_r}{\Delta y_a}$ of this internal gearing has again negative sign similar to the external mesh.

In the mesh, there are perpendicular friction forces besides pressure forces. These forces act along the whole length of pressure line. The friction forces in the addendum part of the pressure line have opposite direction than in the dedendum part, they are roughly in balance and therefore the friction forces result is small and can be neglected in the following solution.

3.1 Radial Free Vibration

The first step in solution of dynamics of the four planetary gearing system (Fig. 1) is analysis of one separate planet wheel. We shall use the same value for external and internal tooth contacts $k_r = -40{,}000$ N/m. Equations of motion of free radial vibration of one planet wing, at assumption that the ring wheel axis is stiff, are

$$m_3\ddot{y}_3 - k_r(y_3 - y_2) + k_3 y_3 = 0, \quad m_2\ddot{y}_2 + k_r y_3 + k_c y_2 = 0. \tag{3}$$

It is a 2DOF system, which can serve as a mathematical model of the upper planet subsystem labeled by a left upper index " ^{1} " in Fig. 1. Mathematical models of other three planet subsystems labeled by " $^{2,\,3,\,4}$ " have similar structure but the variables must be exchanged according to the orientation of planet wings [1].

3.2 Free Tangential Vibration

Tangential motion and rotation of wheels are influenced by stiffness of tooth meshes $k_1,\ k_2$ and stiffness of flexible wheel's pins $k_3,\ k_c$. As planet and sun wheel both rotate and translate, the masses of planet and of sun wheel as well their moments of inertia must be considered. Differential equations of one planet wing motion are described in [2]. Here the direct derivation of mathematical model of the entire four planetary gearing box is applied.

4 Free Vibration of Four-Planetary Gearing Box

Mathematical description of free vibration of gearing box can be constructed by means of Lagrange equations written for n generalized coordinates of motion q_{j}

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \quad j = 1, \ldots, n \tag{4}$$

where T is kinetic energy of investigated system, V is potential energy.

Let us investigate dynamic properties of a gearing box shown in Fig. 1 without any external driving and braking aggregate. The coordinate vector q is:

$$q = [\varphi_1\ {}^1y_2\ {}^1x_2\ {}^1\varphi_2\ {}^2y_2\ {}^2x_2\ {}^2\varphi_2\ {}^3y_2\ {}^3x_2\ {}^3\varphi_2\ {}^4y_2\ {}^4x_2\ {}^4\varphi_2\ y_3\ x_3\ \varphi_3]^T. \tag{5}$$

Kinetic energy of the four-planets gearing system with 16 DOF is

$$T = \frac{1}{2}\begin{bmatrix} \Theta_1\dot{\varphi}_1^2 + m_2({}^1\dot{y}_2^2 + {}^1\dot{x}_2^2 + {}^2\dot{y}_2^2 + {}^2\dot{x}_2^2 + {}^3\dot{y}_2^2 + {}^3\dot{x}_2^2 + {}^4\dot{y}_2^2 + {}^4\dot{x}_2^2) \\ + \Theta_2({}^1\dot{\varphi}_2^2 + {}^2\dot{\varphi}_2^2 + {}^3\dot{\varphi}_2^2 + {}^4\dot{\varphi}_2^2) + m_3(\dot{y}_3^2 + \dot{x}_3^2) + \Theta_3\dot{\varphi}_3^2 \end{bmatrix}. \tag{6}$$

Potential energy V of the same 16 DOF gearing system is a function of all angular and transversal coordinates given in the coordinate vector q and contains also tooth mesh stiffness parameters $k_1,\ k_2$, stiffness k_c of flexible planet pins and stiffness k_3 of sun gear support. The axis of ring wheel is supposed to be sufficiently stiff with no transversal displacements $(y_1 = 0,\ x_1 = 0)$. Potential energy consists of radial and potential parts $V = V_{rad} + V_{\tan}$. The complete potential energy is

$$V = \frac{1}{2}\begin{bmatrix} k_r\left[(y_3 - {}^1y_2)^2 + (x_3 - {}^2x_2)^2 + (y_3 - {}^3y_2)^2 + (x_3 - {}^4x_2)^2\right] \\ + k_r\left[{}^1y_2^2 + {}^2x_2^2 + {}^3y_2^2 + {}^4x_2^2\right] + k_c({}^1y_2^2 + {}^1x_2^2 + {}^2y_2^2 + {}^2x_2^2 + {}^3y_2^2 + {}^3x_2^2 + {}^4y_2^2 + {}^4x_2^2) \\ + k_1[(r_1\varphi_1 - r_2{}^1\varphi_2 - {}^1x_2)^2 + (r_1\varphi_1 - r_2{}^2\varphi_2 + {}^2y_2)^2 + (r_1\varphi_1 - r_2{}^3\varphi_2 - {}^3x_2)^2 \\ + (r_1\varphi_1 - r_2{}^4\varphi_2 + {}^4y_2)^2] + k_2[(r_3\varphi_3 - x_3 - r_2{}^1\varphi_2 + {}^1x_2)^2 \\ + (r_3\varphi_3 + y_3 - r_2{}^2\varphi_2 + {}^2y_2)^2 + (r_3\varphi_3 + x_3 - r_2{}^3\varphi_2 - {}^3x_2)^2 \\ + (r_3\varphi_3 + y_3 - r_2{}^4\varphi_2 + {}^4y_2)^2] + k_3(y_3^2 + x_3^2) \end{bmatrix} \tag{7}$$

Introducing expressions (6) and (7) into Lagrange equations. (4) we get 16 differential equations of motion

$$\begin{aligned}
&\Theta_1\ddot{\varphi}_1 + k_1r_1[4r_1\varphi_1 - r_2({}^1\varphi_2 + {}^2\varphi_2 + {}^3\varphi_2 + {}^4\varphi_2) - {}^1x_2 + {}^2y_2 + {}^3x_2 - {}^4y_2] = 0,\\
&m_2{}^1\ddot{y}_2 + (k_c + 2k_r){}^1y_2 - k_ry_3 = 0,\\
&m_2{}^1\ddot{x}_2 - k_1r_1\varphi_1 + (k_1 - k_2)r_2{}^1\varphi_2 + (k_1 + k_2 + k_c){}^1x_2 + k_2r_3\varphi_3 - k_2x_3 = 0,\\
&\Theta_2{}^1\ddot{\varphi}_2 - k_1r_2r_1\varphi_1 + (k_1 + k_2)r_2{}^1\varphi_2 - (k_2 - k_1)r_2{}^1x_2 + k_2r_2x_3 - k_2r_2r_3\varphi_3 = 0,\\
&m_2{}^2\ddot{y}_2 + k_1r_1\varphi_1 - (k_1 - k_2)r_2{}^2\varphi_2 + (k_1 + k_2 + k_c){}^2y_2 - k_2r_3\varphi_3 - k_2y_3 = 0,\\
&m_2{}^2\ddot{x}_2 + (k_c + 2k_r){}^2x_2 - k_rx_3 = 0,\\
&\Theta_2{}^2\ddot{\varphi}_2 - k_1r_2r_1\varphi_1 + (k_1 + k_2)r_2{}^2\varphi_2 + (k_2 - k_1)r_2{}^2y_2 - k_2r_2y_3 - k_2r_2r_3\varphi_3 = 0,\\
&m_2{}^3\ddot{y}_2 + (k_c + 2k_r){}^3y_2 - k_ry_3 = 0,\\
&m_2{}^3\ddot{x}_2 + k_1r_1\varphi_1 - (k_1 - k_2)r_2{}^3\varphi_2 + (k_1 + k_2 + k_c){}^3x_2 - k_2r_3\varphi_3 - k_2x_3 = 0,\\
&\Theta_2{}^3\ddot{\varphi}_2 - k_1r_2r_1\varphi_1 + (k_1 + k_2)r_2{}^3\varphi_2 + (k_2 - k_1)r_2{}^3x_2 - k_2r_2x_3 - k_2r_2r_3\varphi_3 = 0,\\
&m_2{}^4\ddot{y}_2 - k_1r_1\varphi_1 + (k_1 - k_2)r_2{}^4\varphi_2 + (k_1 + k_2 + k_c){}^4y_2 + k_2r_3\varphi_3 - k_2y_3 = 0,\\
&m_2{}^4\ddot{x}_2 + (k_c + 2k_r){}^4x_2 - k_rx_3 = 0,\\
&\Theta_2{}^4\ddot{\varphi}_2 - k_1r_2r_1\varphi_1 + (k_1 + k_2)r_2{}^4\varphi_2 + (k_1 - k_2)r_2{}^4y_2 + k_2r_3y_3 - k_2r_2r_3\varphi_3 = 0,\\
&m_3\ddot{y}_3 + k_2[-r_2({}^2\varphi_2 - {}^4\varphi_2) + 2y_3 - {}^2y_2 - {}^4y_2] + k_3y_3 + k_r(2y_3 - {}^1y_2 - {}^3y_2) = 0\\
&m_3\ddot{x}_3 + k_2[r_2({}^1\varphi_2 - {}^3\varphi_2) + 2x_3 - {}^1x_2 - {}^3x_2] + k_3x_3 + k_r(2x_3 - {}^2x_2 - {}^4x_2) = 0\\
&\Theta_3\ddot{\varphi}_3 + k_2r_3(4r_3\varphi_3 - r_2{}^1\varphi_2 - r_2{}^2\varphi_2 - r_2{}^3\varphi_2 - r_2{}^4\varphi_2 + {}^1x_2 - {}^2y_2 - {}^3x_2 + {}^4y_2) = 0,
\end{aligned} \tag{8}$$

These equations of motion can be rewritten into a matrix form

$$M\ddot{q} + Kq = 0 \tag{9}$$

with the coordinate vector q given by (5) and with the diagonal inertia matrix M

$$M = diag[\backslash\Theta_1, m_2, m_2, \Theta_2, m_2, m_2, \Theta_2, m_2, m_2, \Theta_2, m_2, m_2, \Theta_2, m_3, m_3, \Theta_3\backslash] \tag{10}$$

and with the full stiffness matrix K which is of order 16.

The roots of characteristic determinant

$$\left|-\Omega^2 M + K\right| = 0 \tag{11}$$

give eigen-frequencies of investigated planetary gearbox.

If the parameters of the example of planetary gearbox are:

Tangential mesh stiffness $k_1 = k_2 = 4e + 9$ N/m, radial stiffness $k_r = -4e + 4$ N/m, planet pin stiffness $k_c = 5e + 9$ N/m, radiuses $r_1 = 0.3$ m, $r_2 = 0.12$ m, $r_3 = 0.06$ m, masses $m_1 = 250$ kg, $m_2 = 42$ kg, $m_3 = 25$ kg, moments of inertia $\Theta_1 = 200$ kgm^2, $\Theta_2 = 0.5$ kgm^2, $\Theta_3 = 0.05$ kgm^2, then by means of program "eig" in system Matlab we get eigenfrequencies of investigated planetary gearing—see Table 1.

The first eigen-frequency has zero value and corresponds to the revolution of all gearing wheels. The remaining fifteen non-zero eigen-frequencies correspond to the vibrations superposed on this rotation. There are three twofold frequencies 856, 2673, 3780 Hz and one fourfold frequency 1779 Hz.

The used program "eig" in Matlab system ascertains corresponding modes of vibrations in a normalized form. In the case when all the eigenvalues are distinct, one mode shape orthogonal to the rest of eigenmodes belongs to each one of them. But there are some multi-fold eigen-frequencies in the planet gearings frequency spectrum, which need special mode shape procedure [3, 4]. There is no difficulty for computer programs to extract multiple eigenvalues, but it makes certain complication in ascertaining of eigenvectors. If the system has a repeated eingen value, we get a corresponding number of different, independent eigenvectors. Any linear combination of these vectors is also an eigenvector. Therefore the eigenvector matrix U is not unique. Different procedures are proposed in literature, the simplest one seems to be the perturbation method [4, p. 382] based on splitting the multiple

Table 1 Eigen frequencies

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
Hz	0.0	856	856	1225	1779	1779	1779	1779
	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
Hz	2415	2673	2673	2869	2947	3780	3780	5908

eigenvalue into several separated eigenfrequencies located close to each other and having separate mode shapes. If in the above mentioned mathematical model of planetary gearing is completed with moderately increasing e.g. stiffness k_c of flexible planet pins, then the frequency spectrum differs a little from the original in Table 1, all eigenvalues are distinct and eigenmodes can be easily determined.

5 Conclusion

It is shown that the solution of vibrations of planetary gearing box with the weakly supported sun wheel needs to include radial gear mesh stiffness into mathematical model and that this stiffness is negative. After deriving 16 differential equations of gearing motion, the free frequency spectrum is ascertained. Several multiple eigenfrequencies were discovered and the method for ascertaining of adjoined eigenmode shapes is indicated.

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