Chapter 10
The Stabilizing Virtues of Monetary Policy on Endogenous Bubble Fluctuations

Lise Clain-Chamosset-Yvrard and Thomas Seegmuller

Abstract We explore the stabilizing role of monetary policy on the existence of endogenous fluctuations when the economy experiences a rational bubble. Considering an overlapping generations model, expectation-driven fluctuations are explained by a portfolio choice between three assets (capital, bonds and money), credit market imperfections and a collateral effect. They occur under a positive bubble on bonds. The key mechanism relies on the existence of gaps between the returns on assets due to financial distortions. Then, we study the stabilizing role of the monetary policy. Such a policy managed by a (standard) Taylor rule has no clear stabilizing virtues.

Keywords Indeterminacy · Rational bubble · Cash-in-advance constraint · Monetary policy · Taylor rule

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10.1 Introduction

In recent years, asset prices have experienced large fluctuations, and the financial sphere of the economy had strong effects on the real one, as illustrating during the last financial crisis. Some empirical contributions shed light on the excessive asset price volatility, and reveal that asset prices fluctuate more than their fundamental value (see Shiller 1981, 1989, and LeRoy and Porter 1981). One explanation for this excessive volatility is the existence and the fluctuations of asset bubbles.

A large body of theoretical literature explores the role of credit market imperfections in the existence and dynamics of rational bubbles (Farhi and Tirole 2012; Martin and Ventura 2012; Wang and Wen 2012). Despite the fact that most of these contributions deal with credit constraints at the level of entrepreneurs, some empirical studies highlight the existence of credit constraints faced by consumers underlying the role of collateral on their behavior (Campbell and Mankiw 1989; Jappelli 1990; Iacoviello 2004). Such types of credit market imperfections affect the portfolio choices between different existing assets, but also explain the gaps between their returns. We think that such credit market imperfections may be a main transmission channel between the financial and the real spheres. In this paper, we argue that credit constraints faced by consumer play a crucial role to explain expectation-driven fluctuations of speculative bubbles, as illustrated during the recent subprime crisis. This idea already appears in Bosi and Seegmuller (2010) and Clain-Chamosset-Yvrard and Seegmuller (2015).[1] They assume that the share of consumption financed by credit is positively correlated to a collateral. Because of this type of credit market distortions, the portfolio choices are no longer constant through time. A change in agents' expectations generates a new trade-off between asset holdings promoting equilibrium indeterminacy, and thus the occurrence of expectation-driven fluctuations. We enrich these contributions by considering both capital and the stabilizing role of a monetary policy conducted through a Taylor rule.

Indeed, economic fluctuations based on consumer credit constraints open the door to new policy tools for stabilizing issues. A stabilizing policy must dampen or eliminate the mechanism source of indeterminacy. Since our explanation of expectation-driven fluctuations relies on a trade-off between different assets, namely capital, bonds and money, relevant stabilizing policies are those reducing the gaps between their returns. Monetary policy appears to be a natural policy tool, since it affects the opportunity cost of money holdings through the level of the nominal interest rate. In addition, contrary to most of the literature, we analyze the stabilizing role of the monetary policy when bubble fluctuations occur in an economy with both production and a positive bubble (Grandmont 1985, 1986; Bernanke and Woodford 1997; Benhabib et al. 2001; Sorger 2005; and Rochon and Polemarchakis 2006).

[1] Only few other contributions have analyzed the existence of bubble fluctuations with an interplay between the real and the financial spheres of the economy (Michel and Wigniolle 2003, 2005; Bosi and Seegmuller 2010; Wigniolle 2014).

We consider a simple overlapping generations (OLG) model with capital accumulation to highlight the role of consumers' credit market imperfections and collateral in an economy characterized by a rational bubble.[2] Households save through bonds, money and capital. Bonds are sold by the monetary authority to supply money. Because of a binding cash-in-advance (CIA) constraint, money is held by households to finance a share of their consumption in the second period of their life. Despite the fact that capital is used for the production, it also serves as a collateral: Holding more capital increases the amount of collateral, and thus allows each household to reduce the share of consumption financed through money. It is important to note that the three assets have different returns. Bonds have larger return than capital because this latter is used as a collateral to relax the consumers' credit constraint, and also a larger return than money because we focus on equilibria with binding constraints. As a direct implication, the Fisher relationship is not satisfied.[3] The violation of this relationship will induce some portfolio choices that promote indeterminacy, and therefore endogenous fluctuations.

We prove the existence of a steady state characterized by a positive rational bubble on bonds (Tirole 1985). In contrast to several existing papers (Farmer 1986; Benhabib and Laroque 1988; Rochon and Polemarchakis 2006), expectation-driven fluctuations occur in the neighborhood of such a steady state with a positive rational asset bubble under gross substitutability and reasonable values of input substitution, without requiring arbitrarily large increasing returns to scale (Cazzavillan and Pintus 2005). This result is obtained because of the role of collateral.

Since expectation-driven fluctuations are mainly driven by the portfolio choices between capital, money and bonds and the violation of the Fisher relationship, a policy may have a stabilizing virtue if it is able to reduce the gaps between the different returns on assets. Therefore, the monetary authority could play an active role in stabilizing the economy by manipulating the nominal interest rate. Following Bernanke (2010) who argues that a rule which responds to expected inflation is relevant to describe the US monetary policy, we consider that the nominal interest rate is determined according to a Taylor rule on expected inflation. We show however that the stabilizing results are mitigated. A weakly active policy can even promote endogenous fluctuations for some relevant parameter configurations. One explanation is that such a rule does not significantly modify the nominal interest rate, and therefore does not alter so much the portfolio choices. More generally, this result provides an adding argument emphasizing that standard policy tools are not so relevant in some circumstances. In our case, these circumstances are the existence of a bubble and consumers' credit constraints affected by a collateral.

This paper is organized as follows. In the next section, we present the model. The intertemporal equilibrium is defined in Sect. 10.3. Section 10.4 is devoted to the

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[2]Our work is close to the framework developed by Rochon and Polemarchakis (2006). However, our analysis differs in two main points: First, we take into account the role of collateral on the consumption behavior; Second, we analyze a monetary policy that could fit better the practices of central banks, instead of an interest rate pegging.

[3]Recall that the Fisher relationship means that the gross real interest rate is equal to the gross nominal interest rate deflated by the gross inflation rate.

steady state analysis. In Sect. 10.5, we analyze the occurrence of expectation-driven fluctuations when there is a positive bubble and the stabilizing role of monetary policy. Concluding remarks are provided in Sect. 10.6, and all the proofs are gathered in a final Appendix.

10.2 The Model

We consider an OLG model with production in discrete time ($t = 0, 1, ..., +\infty$). This economy consists of identical two period-lived households, firms, a monetary authority and a government.

10.2.1 Households

There is no population growth, and at each date t, a generation of unit size is born and lives for two periods.

An household derives utility from consumption of final good when young (c_t) and old (d_{t+1}). Her preferences are represented by an additively separable life-cycle utility function:

$$u(c_t) + \beta v(d_{t+1}) = \frac{c_t^{1-\varepsilon_u}}{1-\varepsilon_u} + \beta \frac{d_{t+1}^{1-\varepsilon_v}}{1-\varepsilon_v}, \quad \beta > 0 \tag{10.1}$$

where $\varepsilon_u > 0$ and $\varepsilon_v > 0$ denote respectively the degrees of concavity of $u(c_t)$ and $v(d_{t+1})$. We further note that $\varepsilon_v < 1$ implies gross substitutability meaning that savings are an increasing function of the global return on portfolio.[4]

In her first period of life, the household is young and supplies one unit of labor inelastically remunerated at the wage w_t. With this wage, she can consume an amount c_t of final good at price p_t, and save through a diversified portfolio of nominal

[4] As we will see below, the consumer problem has the following structure:

$$\begin{aligned} max \ \frac{c_t^{1-\varepsilon_u}}{1-\varepsilon_u} &+ \beta \frac{d_{t+1}^{1-\varepsilon_v}}{1-\varepsilon_v} \\ st. \quad c_t + s_t &= w_t \\ d_{t+1} &= \tilde{R}_{t+1} s_t + \Delta_{t+1}, \end{aligned}$$

where s_t represents global savings of a household, $\tilde{R}_{t+1}$ the global return on her portfolio, w_t her labor income and Δ_{t+1} a monetary transfer. From this problem, we obtain:

$$\frac{\mathrm{d}s_t}{\mathrm{d}\tilde{R}_{t+1}} \frac{\tilde{R}_{t+1}}{s_t} = \frac{1 - \varepsilon_v \tilde{R}_{t+1} s_t / (\tilde{R}_{t+1} s_t + \Delta_{t+1})}{\varepsilon_u s_t/(w_t - s_t) + \varepsilon_v \tilde{R}_{t+1} s_t / (\tilde{R}_{t+1} s_t + \Delta_{t+1})},$$

which is positive for $\varepsilon_v < 1$.

balances M_{t+1} needed for a transaction motive, productive capital per capita k_{t+1} (with rental factor R_{t+1})[5] and nominal bonds B_{t+1} (with nominal interest rate i_{t+1}). In our framework, bonds denote nominal debts issued by the monetary authority in order to inject money in the economy. In contrast to asset papers with no fundamental value considered as freely disposed of, these bonds can have a negative nominal value ($B_{t+1} < 0$).

In her second period of life, the household is old. She uses her remunerated savings and her monetary transfer Δ_{t+1} received from the monetary authority to purchase an amount d_{t+1} of final good at price p_{t+1}. The first and second-period budget constraints are written as follows:

$$p_t c_t + M_{t+1} + B_{t+1} + p_t k_{t+1} \leq p_t w_t \tag{10.2}$$

$$\begin{aligned} p_{t+1} d_{t+1} \leq\ & M_{t+1} + (1 + i_{t+1})\, B_{t+1} \\ & + p_{t+1} R_{t+1} k_{t+1} + \Delta_{t+1} \end{aligned} \tag{10.3}$$

The household has to pay cash a part of the second period consumption d_{t+1}: Her money demand is rationalized by a cash-in-advance (CIA) constraint. We use a constraint of the type introduced by Hahn and Solow (1995), i.e. $\gamma p_{t+1} d_{t+1} \leq M_{t+1}$, but we extend it to capture the role of collateral:

$$\gamma(k_{t+1}) p_{t+1} d_{t+1} \leq M_{t+1} \tag{10.4}$$

A binding cash-in-advance constraint means that a share $\gamma(k_{t+1}) \in (0, 1)$ of her second-period consumption has to be paid cash, i.e. with nominal balances M_{t+1}. As underlined in Rochon and Polemarchakis (2006) and Clain-Chamosset-Yvrard and Seegmuller (2015), the household can consume the remaining share $1 - \gamma(k_{t+1})$ of consumption on credit when old. Indeed, because she holds $B_{t+1} + p_{t+1} k_{t+1}$ in her portfolio when young, the household knows that she will have her remunerated savings from bonds and capital $(1 + i_{t+1}) B_{t+1}/p_{t+1} + R_{t+1} k_{t+1}$ at the next period, in addition to the transfer from the monetary authority Δ_{t+1}/p_{t+1}. As a result, she can consume on credit by borrowing from a bank or a financial institution an amount equal to $(1 + i_{t+1}) B_{t+1}/p_{t+1} + R_{t+1} k_{t+1} + \Delta_{t+1}/p_{t+1}$, that she will pay back at the end of her second period of life. In the following, we refer to $1 - \gamma(k_{t+1})$ as the credit share.

Furthermore, we assume that the credit share is increasing with the amount of physical capital held by a household. Through this assumption, we assert that capital acts as a collateral for the household. Since a collateral is by definition an asset that a household offers a bank or a financial institution to secure a loan, we argue that the value of physical capital k_{t+1} can be pledged as a collateral, rather than the capital income $R_{t+1} k_{t+1}$. If the household fails to repay the loan, the financial institution can seize its physical capital to recover its losses, and thus become the owner of this capital. Since the financial institution takes less risk, it would be easier for the

[5] We assume a full capital depreciation within a period.

household to obtain credit from the bank or the financial institution by holding more capital in her portfolio, and thus to reduce her need of cash in her second period of life.

This is also in accordance with some empirical studies which, focusing on U.S data, underline the negative correlation between money holdings and wealth (see Wolff 1998). In our framework, k_{t+1} can be seen as a proxy of household's wealth. In any case, it is a simple way to introduce credit market imperfections and to capture the role of collateral on consumption behavior of the household as highlighted by empirical studies (among others, Campbell and Mankiw 1989; Iacoviello 2004).[6]

The following assumption summarizes the properties of the function $\gamma(k)$:

Assumption 1 $\gamma(k) \in (0, 1)$ is a continuous function defined on $[0, +\infty)$, C^2 on $(0, +\infty)$, decreasing ($\gamma'(k) \leq 0$).

For further references, we define the following elasticities:

$$\eta_1(k) \equiv \frac{[1-\gamma(k)]' k}{1-\gamma(k)} = -\frac{\gamma'(k)k}{1-\gamma(k)} \geq 0, \tag{10.5}$$

$$\eta_2(k) \equiv -\frac{[1-\gamma(k)]'' k}{[1-\gamma(k)]'} = -\frac{\gamma''(k)k}{\gamma'(k)} \tag{10.6}$$

Example The following function satisfies these properties:

$$\gamma(k) = 1 - \frac{a + bk^{\epsilon}}{1 + ck^{\epsilon}}, \tag{10.7}$$

with $a \in (0, 1)$, $c > 1$, $b \in (ac, c)$ and $\epsilon > 0$. Using this example, $\eta_1(k)$ and $\eta_2(k)$ are given by:

$$\eta_1(k) = \frac{b - ca}{a + bk^{\epsilon}} \frac{\epsilon k^{\epsilon}}{1 + ck^{\epsilon}} \geq 0 \quad \textit{and} \quad \eta_2(k) = 1 + \epsilon \left(\frac{2ck^{\epsilon}}{1 + ck^{\epsilon}} - 1 \right)$$

When collateral does not matter ($\eta_1(k_{t+1}) = 0$), and γ tends to 0, money is no longer needed and the credit market distortion disappears, whereas when $\gamma > 0$, there is a need of cash. When collateral matters ($\eta_1(k_{t+1}) > 0$), the households are aware of the credit share function: They are able to relax the CIA constraint by increasing capital holdings.

[6]This manner of introducing a collateral effect differs from models with borrowing/collateral constraint *à la* Kiyotaki and Moore (1997). First, borrowing is typically used to finance investment project in these models with collateral constraint, whereas in our paper borrowing finances consumption. Second, our CIA constraint implies a limit on the borrowing's share of total expenditures instead of the borrowing capacity itself. Indeed, using the second-period budget constraint and introducing $A_{t+1} = (1 + i_{t+1})B_{t+1}/p_{t+1} + R_{t+1}k_{t+1} + \Delta_{t+1}/p_{t+1}$ as the amount of borrowing, we can rewrite our CIA constraint as follows $A_{t+1}/(p_{t+1}d_{t+1}) \leq 1 - \gamma(k_{t+1})$. Finally, our limit is nonlinear and increasing with collateral, whereas the borrowing limit of a standard collateral constraint is exogenous or linear with collateral.

Using $\pi_{t+1} \equiv p_{t+1}/p_t$ and introducing the real variables $m_{t+1} \equiv M_{t+1}/p_{t+1}$, $b_{t+1} \equiv B_{t+1}/p_{t+1}$ and $\delta_{t+1} \equiv \Delta_{t+1}/p_{t+1}$, the constraints (10.2)–(10.4) can be rewritten as follows:

$$c_t + \pi_{t+1}m_{t+1} + \pi_{t+1}b_{t+1} + k_{t+1} \leq w_t \tag{10.8}$$

$$d_{t+1} \leq m_{t+1} + \left(1 + i_{t+1}\right) b_{t+1} + R_{t+1}k_{t+1} + \delta_{t+1} \tag{10.9}$$

$$\gamma\left(k_{t+1}\right) d_{t+1} \leq m_{t+1} \tag{10.10}$$

An household derives her optimal consumption choice (c_t, d_{t+1}) and her optimal portfolio choice $(k_{t+1}, m_{t+1}, b_{t+1})$ by maximizing her utility function (10.1) under her budget and cash-in-advance constraints (10.8)–(10.10).

Assumption 2 Let $\tilde{\varepsilon}_u \equiv c\dfrac{1+i}{\pi}\dfrac{i\eta_1(1-\gamma)}{\eta_2 d(1+i\gamma)^2}$.[7] For all $t \geq 0$, we assume $i_t > 0$, $\eta_2(k_t) > 0$ and $\varepsilon_u > \tilde{\varepsilon}_u$.

Since the conditions in Assumption 2 rely on endogenous variables, as i_t, π_t, $\gamma(k_t)$, Assumption 2 can seem quite strong. Nevertheless, as we are interested in the occurrence of fluctuations in the vicinity of a steady state, Assumption 2 will be supposed satisfied at the steady state. By continuity, it will also hold in the neighborhood of this steady state. Note that we provide a numerical example satisfying $n_2(k_t) > 0$ at the normalized steady state that we consider for the dynamic analysis. In addition, since ε_u is a free preference parameter, Assumption 2 can always be satisfied.

We can derive the following Lemma[8]:

Lemma 1 *Under Assumptions 1 and 2, constraints (10.8)–(10.10) are binding and the second-order conditions are satisfied.*

Lemma 1 requires that the function of the credit share $1 - \gamma(k_{t+1})$ is concave: Capital holdings increase, at a decreasing rate, the fraction of second-period consumption purchased on credit. Moreover, the CIA constraint is binding if the nominal interest rate i_{t+1} is strictly positive ($i_{t+1} > 0$).

Under Assumptions 1 and 2, the optimal households' behavior is summarized by the following equations:

$$\frac{u'(c_t)}{\beta v'(d_{t+1})} = \frac{1+i_{t+1}}{\pi_{t+1}}\frac{1}{1+i_{t+1}\gamma(k_{t+1})} \tag{10.11}$$

$$R_{t+1} = \frac{1+i_{t+1}}{\pi_{t+1}} - i_{t+1}\eta_1(k_{t+1})\frac{1-\gamma(k_{t+1})}{k_{t+1}}d_{t+1} \tag{10.12}$$

[7] For simplicity, the arguments of the functions and the time subscripts are omitted.

[8] The proof of Lemma 1 is given in a technical appendix available on https://sites.google.com/site/liseclainchamosset.

When collateral does not matter ($\eta_1(k) = 0$), Eqs. (10.11) and (10.12) rewrite:

$$\frac{u'(c_t)}{\beta v'(d_{t+1})} = \frac{(1 + i_{t+1})/\pi_{t+1}}{1 + i_{t+1}\gamma} \tag{10.13a}$$

and

$$R_{t+1} = \frac{1 + i_{t+1}}{\pi_{t+1}} \tag{10.13b}$$

We note that as γ tends to 0, we obtain the intertemporal trade-off found in Diamond (1965) and Tirole (1985), in which there are no credit market distortions in the economy (see Eq. (10.13a)). As $\gamma > 0$, a distortion exists: Old households now have to pay cash γ in order to consume an additional unit of final good, and money entails an opportunity cost. Nevertheless, when collateral does not matter, capital and bonds are perfect substitutes (see Eq. (10.13b)).

When collateral matters ($\eta_1(k) > 0$), capital and bonds are no longer perfect substitutes. Households can now decrease their need of cash by holding more capital in their portfolio. As a consequence, the return on capital is lower than the return on bonds.

The endogeneity of the credit share ensures the portfolio choices to be no longer constant through time. The trade-off between assets is endogenous and depends on the amount of collateral held by the households. A change in expected inflation generates a portfolio effect, i.e. a new trade-off between asset holdings. This portfolio effect is the key mechanism through which expectation-driven fluctuations may occur. Since the portfolio choices are the explanation for fluctuations, we will also focus on a stabilizing policy designed to dampen the portfolio effect by modifying the different returns on assets.

10.2.2 Firms

A representative competitive firm produces the final good using capital and labor under a constant returns to scale technology $f(K/L)L$. Using $k = K/L$, the intensive production function $f(k)$ satisfies:

Assumption 3 $f(k)$ is a continuous function defined on $[0, +\infty)$ and C^2 on $(0, +\infty)$, strictly increasing ($f'(k) > 0$) and strictly concave ($f''(k) < 0$). Defining $\alpha(k) \equiv f'(k)k/f(k) \in (0, 1)$ as the capital share in total income and $\sigma(k) \equiv \left[\frac{f'(k)k}{f(k)} - 1\right]\frac{f'(k)}{kf''(k)} > 0$ as the elasticity of capital-labor substitution, we further assume $f'(1) < 1$, $\lim_{k \to 0^+} f'(k) > 1$ and $\sigma(k) > 1 - \alpha(k)$.

Note that at the end of Sect. 10.4.2, we provide a numerical example that satisfies Assumption 3. The competitive firm takes the prices as given and maximizes the profits $f(K_t/L_t)L_t - w_t L_t - R_t K_t$:

$$R_t = f'(k_t) \equiv R(k_t) \tag{10.14}$$

$$w_t = f(k_t) - k_t f'(k_t) \equiv w(k_t) \tag{10.15}$$

Hence, the interest rate and wage elasticities are respectively equal to $\epsilon_R(k) \equiv R'(k)k/R(k) = -(1-\alpha(k))/\sigma(k)$ and $\epsilon_w(k) \equiv w'(k)k/w(k) = \alpha(k)/\sigma(k)$. The inequality $\sigma(k) > 1-\alpha(k)$ involves capital income $R_t k_t$ being increasing with k_t, which is not a restrictive assumption.

10.2.3 Monetary Authority

For implementing monetary policy, the monetary authority (central bank) uses open market operations defined as the purchase or sale of bonds in exchange for nominal balances.[9] At time t, the central bank creates nominal balances M_{t+1}, which offer liquidity at the next period $t+1$.[10] The money growth factor $\mu_t = M_{t+1}/M_t$ can be written as follows:

$$\mu_t = \pi_{t+1}\frac{m_{t+1}}{m_t} \tag{10.16}$$

In order to supply M_{t+1} in the economy at $t+1$, the central bank buys bonds from old households, and pays for them in cash through open market operations. The profits made by central bank Δ_t at time t are given by:

$$\Delta_t = B_{t+1} + M_{t+1} - (1+i_t)B_t - M_t \tag{10.17}$$

These profits are distributed as dividends to the old households at time t. The budget constraint of the monetary authority at time t is written as follows:

$$B_{t+1} + M_{t+1} = (1+i_t)B_t + M_t + \Delta_t = (1+i_t)(B_t + M_t) \tag{10.18}$$

or in real terms:

$$\pi_{t+1}(b_{t+1} + m_{t+1}) = (1+i_t)(b_t + m_t) \tag{10.19}$$

Introducing the variable $\theta_t \equiv (1+i_t)(b_t + m_t)$, Eq. (10.19) can be rewritten as follows:

$$\pi_{t+1}\theta_{t+1} = (1+i_{t+1})\theta_t \tag{10.20}$$

[9]To study the existence of expectation-driven fluctuations in an OLG model without collateral, Rochon and Polemarchakis (2006) use similar open market operations to issue money in the economy.

[10]Placing a part of their savings in the form of nominal balances in their first period of life, young households have the opportunity to obtain liquidity in their second period of life.

Note that if θ_t is positive at the equilibrium, then a part of bonds, which are purely unbacked public assets (intrinsically useless), has a positive value. Let $b_t = \bar{b}_t + \tilde{b}_t$, where $\bar{b}_t$ denotes the real counterpart of money, and $\tilde{b}_t$ the real value of unbacked public assets. As $\bar{b}_t + m_t = 0$, $\theta_t = (1 + i_t)(b_t + m_t) > 0$ is equivalent to $\tilde{b}_t > 0$. We can argue that there is a bubble on bonds when $\tilde{b}_t > 0$. Therefore, $\theta_t > 0$ pertains to a situation in which a positive bubble on bonds exists.[11] When $\theta_t = 0$, all bonds are the counterpart of money. In this case, all money in the economy corresponds to inside money: No bubbles on bonds exist. When $\theta_t < 0$, there is an excess of households' debt.

In addition, the monetary authority chooses the nominal interest rate i_{t+1} as the monetary instrument, and implements the following interest rate rule:

$$1 + i_{t+1} = \left(1 + i^*\right) \left(\frac{\pi_{t+1}}{\pi^*}\right)^{\phi}, \tag{10.21}$$

where $\phi \geq 0$ is a measure of monetary policy responses to expected inflation. Furthermore, i^* and π^* are respectively the stationary values of the nominal interest rate and the inflation of an existing stationary equilibrium chosen as the targets by the monetary authority.

When $\phi = 0$, the central bank decides to fix the level of the nominal interest rate at its stationary level i^*. When $\phi > 0$, Eq. (10.21) depicts a Taylor interest rate rule, which responds to expected inflation. Note that according to Bernanke (2010), a rule which responds to expected inflation is more relevant to describe the US monetary policy than a rule responding to observed inflation. For $\phi \in (0, 1)$, the rule weakly reacts to expected inflation. An increase (decrease) in the inflation raises (depresses) the nominal interest rate less than proportionally, involving a decrease (increase) in the real interest rate. For $\phi > 1$, the rule strongly reacts to expected inflation. An increase (decrease) in the inflation raises (depresses) the nominal interest rate more than proportionally, involving an increase (decrease) in the real interest rate. Following Benhabib et al. (2001), we define a rule with $\phi \in (0, 1)$ as a passive one, and a rule with $\phi > 1$ as an active one.

10.3 Intertemporal Equilibrium

At the intertemporal equilibrium, the budget and cash-in-advance constraints of households are given by:

$$c_t + \frac{\pi_{t+1}}{1 + i_{t+1}} \theta_{t+1} + k_{t+1} = w(k_t) \tag{10.22}$$

$$d_{t+1} = \theta_{t+1} + f'(k_{t+1}) k_{t+1} \tag{10.23}$$

[11] Alternatively, $\theta_t > 0$ corresponds to a situation in which the outside money is positive. A positive outside money indicates that there is fiat money in circulation in the economy. In the literature on rational bubble, the bubble is often considered as being fiat money.

$$\gamma(k_{t+1})d_{t+1} = m_{t+1} \tag{10.24}$$

The budget constraints of the monetary authority is as follows:

$$\pi_{t+1} = (1 + i_{t+1})\,\frac{\theta_t}{\theta_{t+1}} \tag{10.25}$$

Substituting Eq. (10.25) into the first-period budget constraint Eq. (10.22), we get:

$$c_t + \theta_t + k_{t+1} = w(k_t) \tag{10.26}$$

Using Eqs. (10.16), (10.24) and (10.25), we deduce the money growth factor:

$$\mu_t = (1 + i_{t+1})\,\frac{\theta_t}{\theta_{t+1}}\,\frac{\gamma(k_{t+1})}{\gamma(k_t)}\,\frac{\theta_{t+1} + f'(k_{t+1})k_{t+1}}{\theta_t + f'(k_t)k_t} \tag{10.27}$$

Substituting Eqs. (10.22) and (10.26) into Eq. (10.11), and Eqs. (10.23) and (10.25) into Eq. (10.12), the consumers' intertemporal trade-off and the no-arbitrage condition are respectively given by:

$$\begin{cases} \theta_t \dfrac{u'(f(k_t) - f'(k_t)k_t - \theta_t - k_{t+1})}{\beta v'(\theta_{t+1} + f'(k_{t+1})k_{t+1})} = \dfrac{\theta_{t+1}}{1 + i_{t+1}\gamma(k_{t+1})} \\[2ex] \dfrac{\theta_{t+1}}{\theta_t} = \dfrac{1 + i_{t+1}}{\pi_{t+1}} = f'(k_{t+1})H_{t+1}(k_{t+1}, \theta_t), \end{cases} \tag{10.28}$$

$$with \quad H_{t+1}(k_{t+1}, \theta_t) \equiv \frac{1 + i_{t+1}\eta_1(k_{t+1})\left[1 - \gamma(k_{t+1})\right]}{1 - \theta_t i_{t+1}\eta_1(k_{t+1})\left[1 - \gamma(k_{t+1})\right]/k_{t+1}} \tag{10.29}$$

When collateral does not matter ($\eta_1(k) = 0$), we obtain $H_{t+1}(k_{t+1}, \theta_t) = 1$. Therefore, the Fisher equation ($(1 + i_{t+1})/\pi_{t+1} = f'(k_{t+1})$) holds at the intertemporal equilibrium. This means that the return on real asset (capital) is equal to the return on nominal asset (bonds) deflated by the inflation factor. The role of collateral ($\eta_1(k) > 0$) implies the violation of the Fisher equation ($H_{t+1}(k_{t+1}, \theta_t) > 1$). As capital serves as a collateral, its return becomes lower than the real return on bonds ($f'(k_{t+1}) < (1 + i_{t+1})/\pi_{t+1}$)). This violation of the Fisher equation will induce some portfolio choices that promote indeterminacy, a source of expectation-driven fluctuations.

Interestingly, the level of nominal interest rate can offset the collateral effect (see Eq. (10.29)). Hence, considering a monetary policy conducted through an usual interest rate rule, like a Taylor one, could a priori be relevant to stabilize macroeconomic fluctuations.

From the budget constraint of the monetary authority Eq. (10.25) and the monetary rule Eq. (10.21), we deduce that:

$$\frac{\pi_{t+1}}{\pi^*} = \left(\frac{\theta_t}{\theta_{t+1}}\right)^{\frac{1}{1-\phi}} \tag{10.30}$$

Substituting Eq. (10.30) into Eq. (10.21), we obtain the nominal interest rate at the equilibrium:

$$i_{t+1} = \left(1 + i^*\right)\left(\frac{\theta_t}{\theta_{t+1}}\right)^{a_\phi} - 1, \quad where \quad a_\phi \equiv \frac{\phi}{1-\phi} \in (-\infty, -1) \cup [0, +\infty) \tag{10.31}$$

Note that when $a_\phi \in (-\infty, -1)$, the monetary rule is active. When $a_\phi \in (0, +\infty)$, the rule is passive.

Definition 1 Under Assumptions 1–3, an intertemporal equilibrium with perfect foresight is a sequence $(k_t, \theta_t) \in \mathbb{R}_+ \times \mathbb{R}$, $t = 0, 1, ..., +\infty$, such that the dynamic system (10.28) is satisfied, where $H_{t+1}(k_{t+1}, \theta_t)$ is defined by (10.29), i_{t+1} by Eq. (10.31), and $k_0 > 0$ is given.

Taking into account Eqs. (10.29) and (10.31), we note that k_t is the only predetermined variable of the two-dimensional dynamic system (10.28). The intertemporal sequence of k_t and θ_t enables us to determine all the other variables, namely c_t, d_t, m_t and b_t.

10.4 Steady State Analysis

From the system (10.28), we deduce that two kinds of steady state exist: $\theta = 0$ and $\theta \neq 0$. Since we are interested in fluctuations with a positive bubble, we will focus on steady states with $\theta > 0$. A steady state with a positive bubble is a solution $(k, \theta) \in \mathbb{R}^2_{++}$ that satisfies the following system:

$$\begin{cases} \dfrac{u'(f(k) - f'(k)k - k - \theta)}{\beta v'(\theta + f'(k)k)} = \dfrac{1}{1 + i^*\gamma(k)} \\[2ex] f'(k)H(k, \theta) = 1 \end{cases} \tag{10.32}$$

$$with \quad H(k, \theta) = \frac{1 + i^*\eta_1(k)\left[1 - \gamma(k)\right]}{1 - \theta i^*\eta_1(k)\left[1 - \gamma(k)\right]/k} \tag{10.33}$$

Under a constant credit share ($\eta_1(k) = 0$), we see from the system (10.32) that the steady state is unique, and the monetary policy does not affect the production side. Indeed, the second equation of the system (10.32) reduces to $f'(k) = 1$. Under Assumption 3, this gives a unique stationary solution in k and superneutrality of money holds. When collateral matters ($\eta_1(k) > 0$), the superneutrality of money is canceled. Because of collateral, the monetary sphere affects the real one.

10.4.1 Existence

From Eq. (10.32), a steady state with $\theta > 0$ is a solution $k \in \mathbb{R}_{++}$ satisfying:

$$\begin{cases} \dfrac{u'\left(c(k)\right)}{\beta v'\left(d(k)\right)} = \dfrac{1}{1 + i^* \gamma(k)} \\ \theta = \dfrac{1 - f'(k)\left\{1 + i^* \eta_1(k)[1 - \gamma(k)]\right\}}{i^* \eta_1(k)[1 - \gamma(k)]/k} \end{cases} \tag{10.34}$$

$$with \quad c(k) = f(k) - k - \frac{k[1 - f'(k)]}{i^* \eta_1(k)[1 - \gamma(k)]} \text{ and } d(k) = \frac{k[1 - f'(k)]}{i^* \eta_1(k)[1 - \gamma(k)]}$$

From these equations, we deduce that $d(k) > 0$ implies $f'(k) < 1$, and from Eqs. (10.25) and (10.27), $1 + i^* = \pi = \mu > 1$.

Assumption 4

$$\frac{1 - f'(k^*)}{f'(k^*)} > i^* \eta_1(k^*) \left[1 - \gamma(k^*)\right] \tag{10.35}$$

Under Assumption 4, any bubble can be positive (see Eq. (10.34)). Note also that this assumption is satisfied by the example provided at the end of Sect. 10.4.2.

Proposition 1 *Let $\overline{k}$ be defined by $c\left(\overline{k}\right) = 0$ and $\underline{k} \in (0, \overline{k})$ by $f'(\underline{k}) = 1$. Under Assumptions 1–4, there exists a steady state characterized by $k^* \in \left(\underline{k}, \overline{k}\right)$ and $0 < \theta^* < f(k^*) - k^* - f'(k^*)k^*$.*

Proof See Appendix "Proof of Proposition 1".

Proposition 1 indicates that a steady state with a positive bubble exists. When collateral does not matter ($\eta_1(k) = 0$), we can see from Eq. (10.33) that the steady state is at the golden rule ($R(k) = 1$). As the well-known result of Tirole (1985), a positive rational asset bubble crowds out capital. When collateral matters ($\eta_1(k) > 0$), this economy experiences an over-accumulation of capital at the steady state ($R(k) < 1$). The existence of collateral incites households to hold more capital in their portfolio in order to relax the cash-in-advance constraint, and therefore, the capital return decreases.

Regarding the monetary policy, we recall that the central bank chooses the value of an existing steady state for its target. Since the steady state k^* always exists, we assume that the central bank selects this steady state as a target, i.e. $\pi^* = 1 + i^*$.[12]

[12] Indeed, our analysis does not exclude multiplicity of steady states. See Clain-Chamosset-Yvrard and Seegmuller (2013) for more details.

10.4.2 Normalized Steady State

In order to facilitate the analysis of local dynamics (Sect. 10.5), we establish the existence of a normalized steady state $k^* = 1$ (NSS). We follow the procedure introduced by Cazzavillan et al. (1998), and use the scaling parameter β to give conditions for the existence of such a steady state.

Assumption 5 Let $\nu(\eta_1) = i^* \eta_1(1) \left[1 - \gamma(1)\right]$, we assume:

$$f(1) > 1 + \frac{1 - f'(1)}{\nu(\eta_1)}$$

Assumption 5 ensures that the first period consumption at the normalized steady state is positive (i.e. $c(1) > 0$), and it is satisfied when the productivity is sufficiently large. Note that we show that a numerical example satisfies Assumption 5 at the end of Sect. 10.4.2.

Proposition 2 *Under Assumptions 1–5, there exists a unique value $\beta^* > 0$ given by*

$$\beta^* = \frac{u'\left[f(1) - 1 - (1 - f'(1))/\nu(\eta_1)\right]}{v'\left[(1 - f'(1))/\nu(\eta_1)\right]}\left[1 + i^* \gamma\,(1)\right]$$

such that $k^* = 1 \in \left(\underline{k}, \overline{k}\right)$ is a steady state of the dynamic system (10.28). Moreover, there is a positive bubble $(\theta^* > 0)$ if $1 - f'(1)\left[1 + \nu(\eta_1)\right] > 0$.

Thereafter, we set $\beta = \beta^*$ so that $k^* = 1$. We further note $c^* \equiv c(1)$, $\gamma \equiv \gamma(1)$, $\eta_1 \equiv \eta_1\,(1)$, $\eta_2 \equiv \eta_2\,(1)$, $\psi \equiv f'(1)$, $\alpha = \alpha(1)$ and $\sigma = \sigma(1)$.

To convince the reader that all our assumptions leading to our results are compatible, consider the following example:

Example For a non-empty set of parameter values $(a, b, c, \epsilon, A, \alpha, \sigma)$, the function $\gamma(k)$ given by Eq. (10.7) in Sect. 10.2.1 and the production function $f(k) = A\left(\alpha k^{(\sigma-1)/\sigma} + 1 - \alpha\right)^{\sigma/(\sigma-1)}$ evaluated at the normalized steady state $k^* = 1$ fit all the requirements imposed by Assumptions 2–5.

Indeed, there exist critical values $\underline{\sigma}$, $\bar{\alpha}$, $\underline{A}$ and $\bar{A}$ such that for $a < 1$, $c > 1$, $b \in (ac, c)$, $\epsilon > 0$, $\sigma > \underline{\sigma}$, $\alpha \in (1 - \sigma, \bar{\alpha})$ and $A \in (\underline{A}, \bar{A})$, the function $\gamma(k)$ and the production function $f(k)$ evaluated at $k^* = 1$ satisfy Assumptions 2–5.

Proof See Appendix "Proof of Example in Section 10.4.2".

10.5 Expectation-Driven Fluctuations and the Stabilizing Role of Monetary Policy

We now study the emergence of expectation-driven fluctuations with a speculative bubble and an interplay between the financial and the real spheres. We show that when no Taylor rule is implemented ($\phi = 0$), local indeterminacy can occur in the neighborhood of the normalized steady state with a positive bubble under not restrictive conditions, namely gross substitutability and a not too weak input substitution, because of the credit market distortion. The violation of the Fisher relationship and the resulting portfolio choice between bonds, capital and money are the key ingredients to explain these fluctuations. When the Taylor rule is implemented ($\phi > 0$), we analyze the stabilizing role of the monetary policy. We will see that it is quite mitigated.

To study the existence of local indeterminacy, we introduce the following additional assumption:

Assumption 6 Let $\bar{\eta}_1 > 0$ and $\bar{\eta}_2 > 0$.[13] We assume $\eta_1 < \bar{\eta}_1$ and $\eta_2 > \bar{\eta}_2$.

Example Note that $\eta_1 = (b - ca)\epsilon/[(a + b)(1 + c)]$ and $\eta_2 = 1 + \epsilon[2c/(1 + c) - 1]$ at the normalized steady state $k^* = 1$. Recall that $a < 1$, $c > 1$, $b \in (ac, c)$ and $\epsilon > 0$.

For b close to ca, η_1 is close to zero, and thus $\eta_1 < \bar{\eta_1}$ is satisfied. As shown in Appendix "Proofs of Proposition 3, Corollaries 1 and 2" (see Eq. (10.54)), $\epsilon > \bar{\epsilon}$ is equivalent to $\eta_2 > \bar{\eta_2}$. Thus, the function $\gamma(k)$ evaluated at $k^* = 1$ satisfies Assumption 6.

To derive our different results, we start by linearizing the dynamic system (10.28) around the normalized steady state $k^* = 1$, and obtain the following lemma[14]:

Lemma 2 *Under Assumptions 1–6, the characteristic polynomial, evaluated at the steady state* $k^* = 1$*, writes* $P(X) \equiv X^2 - TX + D = 0$*, where* T *and* D *are respectively the trace and the determinant of the associated Jacobian matrix. We have:*

$$1 - T + D = \varepsilon_{dk} \frac{1 - \psi\,[1 + \nu(\eta_1)]}{\xi_1} \frac{\varepsilon_v - \varepsilon_v^s}{\varepsilon_v - \bar{\varepsilon}_v} \tag{10.36}$$

$$1 + T + D = \frac{\xi_3}{\xi_1} \frac{\varepsilon_v - \varepsilon_v^f}{\varepsilon_v - \bar{\varepsilon}_v} \tag{10.37}$$

$$1 - D = \frac{\varepsilon_v - \varepsilon_v^h}{\varepsilon_v - \bar{\varepsilon}_v} \tag{10.38}$$

[13] The expressions of $\bar{\eta}_1$ and $\bar{\eta}_2$ are given in Appendix "Proofs of Proposition 3, Corollaries 1 and 2".

[14] The proof of Lemma 2 is given in a technical appendix available on https://sites.google.com/site/liseclainchamosset.

where $\varepsilon_v \neq \bar{\varepsilon}_v$, and the expressions of $\xi_1 \equiv \xi_1(a_\phi)$, $\xi_3 \equiv \xi_3(a_\phi)$, $\bar{\varepsilon}_v \equiv \bar{\varepsilon}_v(a_\phi)$, $\varepsilon_v^f \equiv \varepsilon_v(a_\phi)$, $\varepsilon_v^h \equiv \varepsilon_v^h(a_\phi)$, ε_v^s and ε_{dk} are given in Appendix "Proofs of Proposition 3, Corollaries 1 and 2".

We recall that when $1 - T + D = 0$, one eigenvalue is equal to 1. When $1 + T + D = 0$, one eigenvalue is equal to -1. When $1 - T + D > 0$, $1 + T + D > 0$ and $D = 1$, the characteristic roots are complex conjugates with modulus equal to 1. All eigenvalues are inside the unit circle, when the following conditions are satisfied (i) $1 - T + D > 0$, (ii) $1 + T + D > 0$ and (iii) $D < 1$. In other words, when conditions (i)-(iii) are satisfied, the steady state is a sink, i.e. locally indeterminate. The steady state is a saddle point when $1 - T + D < 0$ (resp. > 0) and $1 + T + D > 0$ (resp. < 0). It is a source otherwise.

A (local) bifurcation arises when at least one eigenvalue crosses the unit circle. Therefore, a bifurcation occurs when either (iv) $1 - T + D = 0$, or (v) $1 + T + D = 0$, or (vi) $1 - T + D > 0$, $1 + T + D > 0$ and $D = 1$. According to a continuous change of ε_v, a pitchfork bifurcation emerges when ε_v goes through ε_v^s, defined by $1 - T + D = 0$.[15] A flip bifurcation occurs when ε_v goes through ε_v^f, defined by $1 + T + D = 0$. Finally, a Hopf bifurcation arises, as ε_v goes through ε_v^h, defined by $D = 1$, but we still keep $1 - T + D > 0$ and $1 + T + D > 0$.

The next proposition summarizes the local dynamic properties of the model:

Proposition 3 *Let $\varepsilon_v \neq \bar{\varepsilon}_v$, $\psi < 1/(1 + i^*\gamma)$ and Assumptions 1–6 hold.*

1. When $\varepsilon_u \in \left(\tilde{\varepsilon}_u, \varepsilon_u^s\right)$, the following holds:

- *if $a_\phi \in (-\infty, \hat{a}_\phi)$, then the steady state is locally determinate for $\varepsilon_v < max\{\varepsilon_v^h, \varepsilon_v^s\}$, undergoes a pitchfork (resp. Hopf) bifurcation for $\varepsilon_v = \varepsilon_v^s$ (resp. $\varepsilon_v = \varepsilon_v^h$), is locally indeterminate for $\varepsilon_v > max\{\varepsilon_v^h, \varepsilon_v^s\}$.*
- *if $a_\phi \in (\hat{a}_\phi, \bar{a}_\phi)$, then the steady state is locally indeterminate for $\varepsilon_v < \varepsilon_v^f$, undergoes a flip bifurcation for $\varepsilon_v = \varepsilon_v^f$, is locally determinate for $\varepsilon_v^f < \varepsilon_v < max\{\varepsilon_v^h, \varepsilon_v^s\}$, undergoes a pitchfork (resp. Hopf) bifurcation for $\varepsilon_v = \varepsilon_v^s$ (resp. $\varepsilon_v = \varepsilon_v^h$), is locally indeterminate for $\varepsilon_v > max\{\varepsilon_v^h, \varepsilon_v^s\}$.*
- *if $a_\phi \in (\bar{a}_\phi, -1[$, then the steady state is locally indeterminate for $\varepsilon_v < \varepsilon_v^s$, undergoes a pitchfork bifurcation for $\varepsilon_v = \varepsilon_v^s$, is locally determinate for $\varepsilon_v^s < \varepsilon_v < max\{\varepsilon_v^f, \varepsilon_v^h\}$, undergoes a flip (resp. Hopf) bifurcation for $\varepsilon_v = \varepsilon_v^f$ (resp. $\varepsilon_v = \varepsilon_v^h$), is locally indeterminate for $\varepsilon_v > max\{\varepsilon_v^f, \varepsilon_v^h\}$.*
- *if $a_\phi \in [0, \tilde{a}_\phi)$, then the steady state is locally indeterminate for $\varepsilon_v < \varepsilon_v^s$, undergoes a pitchfork bifurcation for $\varepsilon_v = \varepsilon_v^s$, is locally determinate for $\varepsilon_v \in (\varepsilon_v^s, \varepsilon_v^f)$, undergoes a flip bifurcation for $\varepsilon_v = \varepsilon_v^f$, is locally indeterminate for $\varepsilon_v > \varepsilon_v^f$.*
- *if $a_\phi \in (\tilde{a}_\phi, +\infty)$, then the steady state is locally indeterminate for $\varepsilon_v < \varepsilon_v^s$, undergoes a pitchfork bifurcation for $\varepsilon_v = \varepsilon_v^s$, is locally determinate for $\varepsilon_v > \varepsilon_v^s$.*

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[15] Indeed, we have an odd number of steady states. We prove the existence of at least three steady states in the online technical appendix at https://sites.google.com/site/liseclainchamosset. See also Clain-Chamosset-Yvrard and Seegmuller (2013).

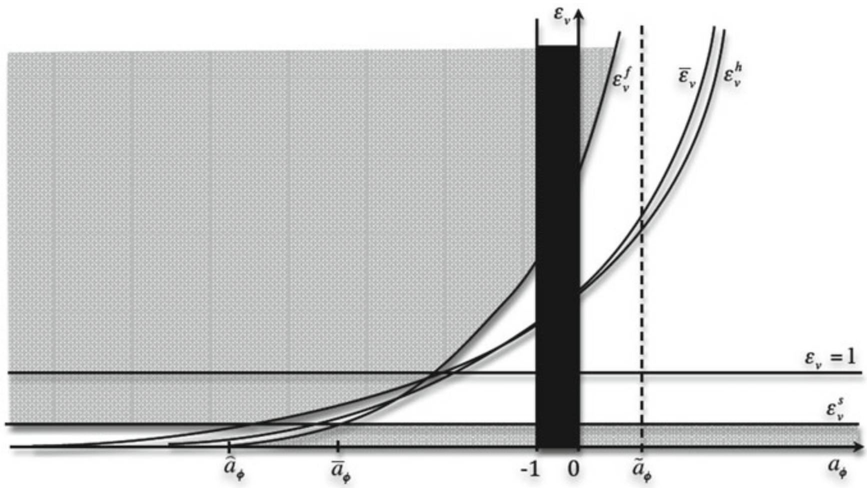

Fig. 10.1 The stabilizing role of monetary policy

2. *When* $\varepsilon_u > \varepsilon_u^s$*, the steady state is locally determinate for* $\varepsilon_v < max\{\varepsilon_v^f, \varepsilon_v^h\}$*, undergoes a flip (resp. Hopf) bifurcation for* $\varepsilon_v = \varepsilon_v^f$ *(resp.* $\varepsilon_v = \varepsilon_v^h$*), and is locally indeterminate for* $\varepsilon_v > max\{\varepsilon_v^f, \varepsilon_v^h\}$*.*

Proof See Appendix "Proofs of Proposition 3, Corollaries 1 and 2".

Figure 10.1 provides a qualitative illustration of the local dynamic properties of the model when $\varepsilon_v^s > 0$. Under Assumptions 1–6, $\bar{\varepsilon}_v$, ε_v^f, ε_v^h are increasing functions of a_ϕ, while ε_v^s does not depend on a_ϕ. Furthermore, $\bar{\varepsilon}_v$, ε_v^f, ε_v^h and ε_v^s intersect as indicated in Fig. 10.1 under some parameter conditions. Note that Fig. 10.1 is precisely constructed using Appendix "Proofs of Proposition 3, Corollaries 1 and 2". From Lemma 2 and plotting the different critical and bifurcation values, $\bar{\varepsilon}_v$, ε_v^s, ε_v^f, ε_v^h in the plane (a_ϕ, ε_v), we can determine the local indeterminacy regions corresponding to the grey areas in Fig. 10.1.

Proposition 3 provides general conditions for local dynamics. To clarify their significance, we start by considering the case where a Taylor rule is not implemented ($a_\phi = 0$). This allows us to understand under which conditions expectation-driven fluctuations occur. In a second step, we will discuss the stabilizing role of a monetary policy managed by a Taylor rule ($a_\phi \in (-\infty, -1) \cup (0, +\infty)$).

From Proposition 3, we can derive the following corollary on the existence of expectation-driven fluctuations when $a_\phi = 0$:

Corollary 1 *Assuming $\varepsilon_v \neq \bar{\varepsilon}_v$ and Assumptions 1–6, the following holds when $a_\phi = 0$:*

1. *When $\varepsilon_u \in \left(\tilde{\varepsilon}_u, \varepsilon_u^s\right)$, the steady state is locally indeterminate for $\varepsilon_v < \varepsilon_v^s < 1$, undergoes a pitchfork bifurcation for $\varepsilon_v = \varepsilon_v^s$, is locally determinate for $\varepsilon_v \in \left(\varepsilon_v^s, \varepsilon_v^f\right)$, undergoes a flip bifurcation for $\varepsilon_v = \varepsilon_v^f$, and is locally indeterminate for $\varepsilon_v > \varepsilon_v^f$.*
2. *When $\varepsilon_u > \varepsilon_u^s$, the steady state is locally determinate for $\varepsilon_v < \varepsilon_v^f$, undergoes a flip bifurcation for $\varepsilon_v = \varepsilon_v^f$, and is locally indeterminate for $\varepsilon_v > \varepsilon_v^f$.*

Proof See Appendix "Proofs of Proposition 3, Corollaries 1 and 2".

Corollary 1 shows the occurrence of persistent endogenous fluctuations around the steady state with a positive bubble under gross substitutability and a not too weak capital-labor substitution when no Taylor rule is implemented. Hence, this result extends Bosi and Seegmuller (2010) and Clain-Chamosset-Yvrard and Seegmuller (2015) to a model with both inside money and capital.[16]

When collateral does not matter ($\eta_1 = 0$), the local stability properties of the model correspond to Corollary 1.2. Endogenous fluctuations and two-period cycles occur only for a significant income effect, i.e. a large degree of concavity ε_v.[17] More interestingly, when collateral matters ($\eta_1 > 0$), local indeterminacy also appears under gross substitutability, i.e. for a small degree of concavity $\varepsilon_v < \varepsilon_v^s < 1$.

The basic mechanism for fluctuations under gross substitutability relies on a portfolio trade-off between the three assets. Because of the difference between the returns on physical and monetary assets, a reallocation between the assets takes place following a modification in agents' expectations.

Economic intuition. If households expect an increase in inflation from period t to $t+1$, the return on bonds becomes less attractive compared to the return on capital. Because of the portfolio effect, households reallocate their savings towards capital. As a consequence, when $\varepsilon_v < \varepsilon_v^s < 1$, the portfolio effect can accelerate capital accumulation. Households consume less by cash (see Eq. (10.24)). The real balances m_{t+1} decrease, entailing a decrease in the return on money. An effective rise in inflation takes place, meaning that the initial expectations are self-fulfilling.

We focus now on the stabilizing role of the monetary policy managed by a Taylor rule. A policy is stabilizing in our framework as soon as it reduces the range of parameter values for which local indeterminacy, and thus expectation-driven fluctuations, emerge. Since the occurrence of fluctuations under gross substitutability is an interesting result, we ensure for the remainder of the paper that $\varepsilon_v^s > 0$, assuming[18]:

[16] In these two papers, the stabilizing role of monetary policies is not addressed in the same way as here. Indeed, the monetary authority directly manages the money growth factor, while it fixes the interest rate in our framework.

[17] Since $\varepsilon_v > \varepsilon_v^f > 1$, income effects dominate substitution effects. Hence, global savings ($\theta_t + k_{t+1}$) are a decreasing function of their return.

[18] ε_u^s is given by Eq. (10.45) in Appendix "Proofs of Proposition 3, Corollaries 1 and 2".

Assumption 7 $\varepsilon_u < \varepsilon_u^s$.

Since ε_u is a free preference parameter, Assumption 7 can always be satisfied and is in accordance with the numerical example provided at the end of Sect. 10.4.2.

We recall that since $a_\phi = \phi/(1-\phi)$, $a_\phi \in (-\infty, -1)$ ($\phi > 1$) means that the monetary rule is active, while $a_\phi \in (0, +\infty)$ ($\phi \in (0, 1)$) means that the rule is passive. Using these notations, we derive the next corollary from Proposition 3 and Fig. 10.1:

Corollary 2 *Assuming $\varepsilon_v \neq \bar{\varepsilon}_v$, $\psi < 1/(1 + i^*\gamma)$ and Assumptions 1–7, the following holds:*

1. *If $a_\phi \in (-\infty, \hat{a}_\phi]$, increasing the responsiveness of the monetary rule ϕ does not affect or reduces the range of parameters for indeterminacy.*
2. *If $a_\phi \in (\hat{a}_\phi, -1)$, increasing the responsiveness of the monetary rule ϕ reduces the range of parameters for local indeterminacy when ε_v is large enough, but raises the range of parameters for local indeterminacy when ε_v is small enough.*
3. *If $a_\phi \in [0, +\infty)$, increasing the responsiveness of the monetary rule ϕ reduces the range of parameters for local indeterminacy when ε_v is large enough, but has no impact on the range of parameters for local indeterminacy when ε_v is small enough.*

Proof See Appendix "Proofs of Proposition 3, Corollaries 1 and 2".

Proposition 3, Corollary 2 and Fig. 10.1 highlight that a monetary policy managed by a Taylor rule has mitigated results concerning the stabilization of expectation-driven fluctuations. Comparing with the results of Corollary 1, a passive rule ($a_\phi \in (0, +\infty)$) stabilizes fluctuations occurring for a large level of ε_v ($\varepsilon_v > \varepsilon_v^f$). However, it has no impact on fluctuations occurring for $\varepsilon_v < \varepsilon_v^s$. On the contrary, an active rule ($a_\phi \in (-\infty, -1)$) could stabilize endogenous fluctuations that occur for $\varepsilon_v < \varepsilon_v^s$, in particular if it is weakly active (a_ϕ sufficiently negative i.e. ϕ close to one). Nevertheless, with respect to the configuration without Taylor rule ($a_\phi = 0$), an active rule destabilizes promoting indeterminacy for new ranges of parameter values. Indeed, under a weakly active rule (a_ϕ sufficiently negative), local indeterminacy occur for all $\varepsilon_v > \varepsilon_v^s$.

To sum, depending on households' preferences (ε_v) and on the responsiveness of the monetary policy with respect to expected inflation (a_ϕ), we may have opposite conclusions concerning the stabilizing role of the monetary policy. This is in contrast with several previous contributions, for instance Bernanke and Woodford (1997), Sorger (2005) or Clain-Chamosset-Yvrard and Seegmuller (2015).

Our explanation of these results is that the monetary authority manipulates the level of the elasticity of the nominal interest rate with respect to the expected inflation (ϕ), which has not a sufficiently significant impact on the interest rate level to dampen or eliminate the impact of the credit market imperfection and the role of collateral. For instance, a weakly active rule has not a huge impact on the nominal interest rate, and therefore does not sufficiently modify the portfolio choices.

10.6 Concluding Remarks

We develop an overlapping generations model with capital accumulation, bonds and money, where the share of consumption purchased on credit depends on a collateral. We show the existence of expectation-driven fluctuations with a positive rational bubble on bonds. In addition, such endogenous fluctuations are in accordance with gross substitutability and a not too weak substitution between inputs. This is explained by a credit market imperfection and the role of a collateral. The basic mechanism for fluctuations relies on a portfolio trade-off between the three assets.

We further analyze the stabilizing role of the monetary policy. To consider an usual policy rule, we focus on a monetary policy fixed according to a Taylor rule on expected inflation. We show that the stabilizing role of such a policy is mitigated. In fact, no clear-cut conclusion on stabilisation is obtained. One reason is that such a policy does not alter sufficiently the portfolio choices. More generally, we think that our results provide an adding argument against the use of standard policy instruments in any circumstances. In this model, these circumstances are the existence of a bubble, a credit market imperfection and the collateral effect.

Appendix

Proof of Proposition 1

A steady state k is a solution of $h(k) = j(k)$, with:

$$h(k) \equiv \frac{u'(c(k))}{\beta v'(d(k))} \tag{10.39a}$$

$$j(k) \equiv \frac{1}{1 + i^* \gamma(k)} \tag{10.39b}$$

where $\quad c(k) \equiv f(k) - k - d(k)$ and $d(k) \equiv \dfrac{k[1 - f'(k)]}{i^* \eta_1(k)[1 - \gamma(k)]}$.

We start by determining the admissible range of values for k. To ensure $d(k) > 0$, we get at the steady state $f'(k) < 1$. Under Assumption 3, $f'(k)$ is a decreasing function of k. Hence, $k > f'^{-1}(1) = \underline{k}$.

Now, we want to determine the range of k such that $c(k) > 0$. The decreasing returns on capital imply $f\left(\underline{k}\right) > \underline{k} f'(\underline{k})$. Since $f'(\underline{k}) = 1$, $c(\underline{k}) = f(\underline{k}) - \underline{k} > 0$. In addition, as $d(k) > 0$, we derive the following inequality:

$$\lim_{k \to +\infty} c(k) < \lim_{k \to +\infty} f(k) - k = -\infty$$

because $f'(k) < 1$ for k large enough. As a result, there exists one value $\overline{k}$ such that $\forall k < \overline{k}$, $c\left(\overline{k}\right) > 0$. By construction, we have $\underline{k} < \overline{k}$, and therefore $(\underline{k}, \overline{k})$ is a nonempty subset.

To prove the existence of a stationary solution k, we use the continuity of $h(k)$ and $j(k)$. Using Eqs. (10.39a) and (10.39b), we determine the boundary values of $h(k)$ and $j(k)$:

$$\lim_{k\to\underline{k}} h(k) = \frac{u'\left(c\left(\underline{k}\right)\right)}{\beta v'(0)} = 0^+ \qquad \lim_{k\to\overline{k}} h(k) = \frac{u'(c(0))}{\beta v'(d(\overline{k}))} = +\infty$$
$$\lim_{k\to\underline{k}} j(k) = \frac{1}{1 + i(\underline{k})\gamma(\underline{k})} \in (0, 1] \; \lim_{k\to\overline{k}} j(k) = \frac{1}{1 + i(\overline{k})\gamma\left(\overline{k}\right)} \leq 1$$

We have $\lim_{k\to\underline{k}} h(k) < \lim_{k\to\underline{k}} j(k)$ and $\lim_{k\to\overline{k}} h(k) > \lim_{k\to\overline{k}} j(k)$. Therefore, there exists at least one value $k^* \in \left(\underline{k}, \overline{k}\right)$ such that $h(k^*) = j(k^*)$. ■

Proof of Example in Section 10.4.2

Let $\underline{\sigma} \equiv 1 - \frac{1}{2+\nu(\eta_1)}$, $\bar{\alpha} \equiv \frac{1}{2+\nu(\eta_1)}$, $\underline{A} \equiv \frac{1+\nu(\eta_1)}{\alpha+\nu(\eta_1)}$ and $\bar{A} \equiv \frac{1/\alpha}{1+\nu(\eta_1)}$. For $a \in (0, 1)$, $c > 1$, $b \in (ac, c)$ and $\epsilon > 0$, Assumption 2 is satisfied at the normalized steady state. Assumption 3 requires $A < 1/\alpha$ and $\sigma > 1 - \alpha$. For $A > \underline{A}$, Assumption 5 is satisfied. Moreover, the bubble is positive at the normalized steady state when $A < \overline{A}$. As a consequence, the set $(\underline{A}, \bar{A})$ must be non-empty. This is true for $\alpha < \bar{\alpha}$ and $\sigma > \underline{\sigma}$.

Proofs of Proposition 3, Corollaries 1 and 2

$$\text{Recall that } \psi = f'(1), \quad \nu(\eta_1) = \eta_1 i^* (1-\gamma) > 0, \quad c^* = f(1) - 1 - \frac{1-\psi}{\nu(\eta_1)} > 0, \tag{10.40}$$
$$a_\phi = \phi/(1-\phi) \in (-\infty, -1[\cup[0, +\infty) \text{ and let } \varepsilon_{dk} = \frac{\psi}{1-\psi}\frac{1-\alpha}{\sigma} + \eta_2 > 0, \text{ and}$$
$$\eta_1^\theta = \frac{1-\psi}{\psi}\frac{1}{i^*(1-\gamma)}.$$

First of all, we suppose for the rest of the proof that $\eta_1 < \eta_1^\theta$, because $\forall\, \eta_1 < \eta_1^\theta$, $\theta > 0$ (see condtion (10.35)).

From Lemma 2, the expressions of $1 - T + D$, $1 + T + D$ and $1 - D$ can be written as follows:

$$1 - T + D = \varepsilon_{dk} \frac{1 - \psi\,[1 + \nu(\eta_1)]}{\xi_1(a_\phi)} \frac{\varepsilon_v - \varepsilon_v^s}{\varepsilon_v - \bar{\varepsilon}_v} \tag{10.41}$$

$$1 + T + D = \frac{\xi_3(a_\phi)}{\xi_1(a_\phi)} \frac{\varepsilon_v - \varepsilon_v^f}{\varepsilon_v - \bar{\varepsilon}_v} \tag{10.42}$$

$$1 - D = \frac{\varepsilon_v - \varepsilon_v^h}{\varepsilon_v - \bar{\varepsilon}_v} \tag{10.43}$$

$$\text{where } \varepsilon_v^s = \frac{1-\psi}{\nu(\eta_1)} \frac{\nu(\eta_1) + \varepsilon_{dk}}{\varepsilon_{dk}} \frac{\varepsilon_u^s - \varepsilon_u}{c^*}, \quad \varepsilon_v^f = \frac{\xi_4(a_\phi)}{\xi_3(a_\phi)} \equiv \varepsilon_v^f(a_\phi),$$
$$\varepsilon_v^h = \frac{\xi_5(a_\phi)}{\xi_1(a_\phi)} \equiv \varepsilon_v^h(a_\phi) \text{ and } \bar{\varepsilon}_v = -\frac{\xi_2(a_\phi)}{\xi_1(a_\phi)} \equiv \bar{\varepsilon}_v(a_\phi) \tag{10.44}$$

$$\text{with } \varepsilon_u^s = c^* \frac{\nu(\eta_1)}{1-\psi} \frac{1}{\nu(\eta_1) + \varepsilon_{dk}} \frac{\nu(\eta_1)}{1 + i^*\gamma}, \tag{10.45}$$

$$\begin{aligned}\xi_1(a_\phi) = &-\psi\nu(\eta_1)\left(1 - \frac{1-\alpha}{\sigma}\right)\frac{1+i^*}{i^*}a_\phi + \{1 - \psi\,[1+\nu(\eta_1)]\}\,\varepsilon_{dk} \\ &- \frac{\psi}{1-\psi}\nu(\eta_1)\left(1 - \frac{1-\alpha}{\sigma}\right),\end{aligned} \tag{10.46}$$

$$\begin{aligned}\xi_2(a_\phi) = &(1-\psi)\,\frac{1+i^*}{i^*}\left\{-\frac{\varepsilon_u}{c^*} + \frac{\nu(\eta_1)}{1+i^*\gamma}\left[1 + i^*\gamma\frac{\psi}{1-\psi}\left(1 - \frac{1-\alpha}{\sigma}\right)\right]\right. \\ &\left.-\varepsilon_{dk}\frac{i^*\gamma}{1+i^*\gamma}\right\}a_\phi - \psi\,[1+\nu(\eta_1)]\,\frac{\varepsilon_u}{c^*} + \nu(\eta_1)^2\frac{\psi}{1+i^*\gamma} + \nu(\eta_1)\psi \\ &\left(1 - \frac{1-\alpha}{\sigma} + \frac{1}{1+i^*\gamma}\right) - (1-\psi)\,\varepsilon_{dk},\end{aligned} \tag{10.47}$$

$$\begin{aligned}\xi_3(a_\phi) = &-\psi\nu(\eta_1)\left(1 - \frac{1-\alpha}{\sigma}\right)\frac{1+i^*}{i^*}a_\phi + \{1 - \psi\,[1+\nu(\eta_1)]\}\,\varepsilon_{dk} \\ &- 2\frac{\psi}{1-\psi}\nu(\eta_1)\left(1 - \frac{1-\alpha}{\sigma}\right),\end{aligned} \tag{10.48}$$

$$\begin{aligned}\xi_4(a_\phi) = &\,2(1-\psi)\frac{1+i^*}{i^*}\left\{\left(1 + \psi\frac{1-\alpha}{\sigma}\right)\frac{\varepsilon_u}{c^*} - \frac{i^*\gamma\nu(\eta_1)}{1+i^*\gamma}\left[1 + \frac{\psi}{1-\psi}\left(1 - \frac{1-\alpha}{\sigma}\right)\right]\right. \\ &\left.+\frac{\varepsilon_{dk}i^*\gamma}{1+i^*\gamma}\right\}a_\phi + \left\{2\psi\left[1 + \nu(\eta_1) + \frac{1-\alpha}{\sigma}\right] + \right. \\ &\left.\frac{1-\psi}{\nu(\eta_1)}\,[\nu(\eta_1) + \varepsilon_{dk}]\,\{1 - \psi\,[1+\nu(\eta_1)]\}\right\}\frac{\varepsilon_u}{c^*} - \frac{\nu(\eta_1)^2\psi}{1+i^*\gamma} \\ &- 2\nu(\eta_1)\psi\left(1 - \frac{1-\alpha}{\sigma}\right) - \nu(\eta_1)\frac{1+\psi}{1+i^*\gamma} + 2(1-\psi)\varepsilon_{dk},\end{aligned} \tag{10.49}$$

$$\xi_5(a_\phi) = (1-\psi)\,\frac{1+i^*}{i^*}\left\{\left(1-\psi\frac{1-\alpha}{\sigma}\right)\frac{\varepsilon_u}{c^*} - \frac{\nu(\eta_1)}{1+i^*\gamma}\right.$$
$$\left.\left[1+\frac{i^*\gamma\psi}{1-\psi}\left(1-\frac{1-\alpha}{\sigma}\right)\right] + \frac{\varepsilon_{dk} i^*\gamma}{1+i^*\gamma}\right\} a_\phi + \psi\left[1+\nu(\eta_1)-\frac{1-\alpha}{\sigma}\right]\frac{\varepsilon_u}{c^*} - \frac{\nu(\eta_1)^2\psi}{1+i^*\gamma}$$
$$- \nu(\eta_1)\psi\left(1-\frac{1-\alpha}{\sigma}+\frac{1}{1+i^*\gamma}\right) + (1-\psi)\varepsilon_{dk} \tag{10.50}$$

We aim to determine the range of parameter values (a_ϕ and ε_v) for which local indeterminacy conditions (i)-(iii) are satisfied. To do this, we must analyze the functions ε_v^f, ε_v^h, $\bar{\varepsilon}_v$ and ξ_i with $i = \{1, 2, 3, 4, 5\}$, then draw ε_v^f, ε_v^h and $\bar{\varepsilon}_v$ in the plane (a_ϕ, ε_v).

We observe that ξ_i with $i = \{1, 2, 3, 4, 5\}$ are linear functions of a_ϕ, i.e. $\xi_i^a a_\phi + \xi_i^b$. Note that $\xi_1^a < 0$ and $\xi_3^a < 0$. Furthermore, there exist $\eta_1^{\xi_1^b} > 0$, $\eta_1^{\xi_2^a} > 0$, $\eta_1^{\xi_2^b} > 0$, $\eta_1^{\xi_3^b} > 0$, $\eta_1^{\xi_4^a} > 0$, $\eta_1^{\xi_4^b} > 0$, $\eta_1^{\xi_5^a} > 0$ and $\eta_1^{\xi_5^b} > 0$ such that $\forall\ \eta_1 < min\{\eta_1^{\xi_1^b}, \eta_1^{\xi_2^a}, \eta_1^{\xi_2^b}, \eta_1^{\xi_3^b}, \eta_1^{\xi_4^a}, \eta_1^{\xi_4^b}, \eta_1^{\xi_5^a}, \eta_1^{\xi_5^b}\} \equiv \tilde{\eta}_1$, one has $\xi_1^b > 0$, $\xi_2^a < 0$, $\xi_2^b < 0$, $\xi_3^b > 0$, $\xi_4^a > 0$, $\xi_4^b > 0$, $\xi_5^a > 0$ and $\xi_5^b > 0$. Therefore, we deduce that $\xi_1(a_\phi) \geq 0$ when $a_\phi \leq \frac{\xi_1^b}{\xi_1^a}$ and $\xi_1(a_\phi) < 0$ otherwise. $\xi_2(a_\phi) \geq 0$ when $a_\phi \leq \frac{\xi_2^b}{\xi_2^a} < 0$ and $\xi_2(a_\phi) < 0$ otherwise. $\xi_3(a_\phi) \geq 0$ when $a_\phi \leq \frac{\xi_3^b}{\xi_3^a}$ and $\xi_3(a_\phi) < 0$ otherwise. $\xi_4(a_\phi) \leq 0$ when $a_\phi \leq \frac{\xi_4^b}{\xi_4^a} < 0$ and $\xi_4(a_\phi) > 0$ otherwise. $\xi_5(a_\phi) \leq 0$ when $a_\phi \leq \frac{\xi_5^b}{\xi_5^a} < 0$ and $\xi_5(a_\phi) > 0$ otherwise.

We analyze now ε_v^f, ε_v^h, ε_v^s and $\bar{\varepsilon}_v$. Suppose that $\eta_1 < min\{\eta_1^\theta, \tilde{\eta}_1\}$.

First, ε_v^s does not depend on a_ϕ. Second, the different critical and bifurcation values $(\bar{\varepsilon}_v, \varepsilon_v^f, \varepsilon_v^h)$ are homographic functions of a_ϕ. ε_v^f has a vertical asymptote at $a_\phi = \frac{\xi_3^b}{\xi_3^a} \equiv \tilde{a}_\phi > 0$. $\bar{\varepsilon}_v$ and ε_v^h have the same vertical asymptote at $a_\phi = \frac{\xi_1^b}{\xi_1^a} > \tilde{a}_\phi$. The first derivatives of ε_v^f, ε_v^h and $\bar{\varepsilon}_v$ with respect to a_ϕ are given by:

$$\frac{\partial \varepsilon_v^f}{\partial a_\phi} = \frac{\xi_4^a\xi_3^b - \xi_4^b\xi_3^a}{\xi_3(a_\phi)^2} > 0,\ \frac{\partial \varepsilon_v^h}{\partial a_\phi} = \frac{\xi_5^a\xi_1^b - \xi_5^b\xi_1^a}{\xi_1(a_\phi)^2} > 0 \text{ and } \frac{\partial \bar{\varepsilon}_v}{\partial a_\phi} = -\frac{\xi_2^a\xi_1^b - \xi_2^b\xi_1^a}{\xi_1(a_\phi)^2} > 0.$$

It would be useful to locate the different bifurcation and critical values (ε_v^f, ε_v^h, ε_v^s and $\bar{\varepsilon}_v$) when $a_\phi = 0$. We can note that $\varepsilon_v^f > 0$, $\varepsilon_v^h > 0$ and $\bar{\varepsilon}_v > 0$ when $a_\phi = 0$. Moreover, $\varepsilon_v^s > 0$ if and only if $\varepsilon_u \in (\tilde{\varepsilon}_u, \varepsilon_u^s)$, where $\varepsilon_u > \tilde{\varepsilon}_u$ is required for the second-order conditions, with:

$$\tilde{\varepsilon}_u \equiv c^* \frac{\nu(\eta_1)^2}{\eta_2\,(1-\psi)\,(1+i^*\gamma)^2} \tag{10.51}$$

The set $(\tilde{\varepsilon}_u, \varepsilon_u^s)$ is nonempty if and only if

$$\eta_2(1-\psi)i^*\gamma > \nu(\eta_1)\,(1-\psi) + \psi\frac{1-\alpha}{\sigma} \tag{10.52}$$

As $\nu(\eta_1) = i^* \eta_1 (1-\gamma)$, the condition (10.52) holds if

$$\eta_1 < \frac{1}{i^*(1-\gamma)} \frac{\eta_2(1-\psi)i^*\gamma - \psi(1-\alpha)/\sigma}{1-\psi} \equiv \bar{\bar{\eta}}_1 \text{ and } \eta_2 > \frac{\psi}{1-\psi}\frac{(1-\alpha)/\sigma}{i^*\gamma} \equiv \bar{\eta}_2 \quad (10.53)$$

Therefore, $\forall\ \eta_1 < \bar{\bar{\eta}}_1$ and $\eta_2 > \bar{\eta}_2$, we have $\tilde{\varepsilon}_u < \varepsilon_u^s$. We suppose now that $\eta_1 < \min\{\eta_1^\theta, \tilde{\eta}_1, \bar{\bar{\eta}}_1\}$ and $\eta_2 > \bar{\eta}_2$. Note that using the function $\gamma(k)$ given by Eq. (10.7), $\eta_2 > \bar{\eta}_2$ is equivalent to:

$$\epsilon > \frac{1+c}{c-1}\left(\frac{\psi}{1-\psi}\frac{1-\alpha}{\sigma i \gamma} - 1\right) \equiv \bar{\epsilon} \quad (10.54)$$

where $\gamma = 1-(a+b)/(1+c)$.

We can show that $\varepsilon_v^s < 1$. Since $\varepsilon_{dk} = \frac{\psi}{1-\psi}\frac{1-\alpha}{\sigma} + \eta_2$ and the condition (10.52) is satisfied, we have $\nu(\eta_1) < (1+i^*\gamma)\,\varepsilon_{dk}$. Therefore, $\varepsilon_v^s = \frac{\nu(\eta_1)}{1+i^*\gamma}\frac{1}{\varepsilon_{dk}} - \frac{\varepsilon_u}{c^*}\frac{\nu(\eta_1)+\varepsilon_{dk}}{\varepsilon_{dk}}\frac{1-\psi}{\nu(\eta_1)} < 1$.

Furthermore, $\varepsilon_v^h < \bar{\varepsilon}_v$ and $\varepsilon_v^h > 1$ for a sufficiently small η_1. Indeed $\varepsilon_v^h > 1$ is equivalent to $\psi Q(\nu(\eta_1)) > 0$, where $Q(\nu(\eta_1))$ is a quadratic polynomial defined on $\mathbb{R}_+$ such that:

$$Q(\nu(\eta_1)) = -\frac{\nu(\eta_1)^2}{1+i^*\gamma} + \nu(\eta_1)\left[\frac{\varepsilon_u}{c^*} - \left(1 - \frac{1-\alpha}{\sigma} + \frac{1}{1+i\gamma}\right)\right.$$
$$\left. + \varepsilon_{dk} + \frac{1-(1-\alpha)/\sigma}{1-\psi}\right] + \left(1 - \frac{1-\alpha}{\sigma}\right)\frac{\varepsilon_u}{c^*}$$

$Q(\nu(\eta_1))$ is a concave function with $Q(\nu(0)) > 0$. As a consequence, there is a threshold $\hat{\eta}_1 > 0$ such that $\forall \eta_1 < \hat{\eta}_1$, $Q(\nu(\eta_1)) > 0$.

Concerning ε_v^f, we can show that $\bar{\varepsilon}_v < \varepsilon_v^f$ for η_1 small enough. Note that $\varepsilon_v^f > \xi_4^b/\xi_1^b$. $\xi_4^b/\xi_1^b > \bar{\varepsilon}_v$ is satisfied if $-\nu(\eta_1)\left[\psi\left(1-\frac{1-\alpha}{\sigma}\right) + \frac{1}{1+i^*\gamma}\right] + (1-\psi)\,\varepsilon_{dk} > 0$. Therefore, there exists $\underline{\eta}_1 > 0$ such that $\forall\ \eta_1 < \underline{\eta}_1$, this inequality is satisfied. Hence, $\forall\ \eta_1 < \underline{\eta}_1$, $\bar{\varepsilon}_v < \varepsilon_v^f$.

Therefore, $\forall\ \eta_1 < \min\{\eta_1^\theta, \tilde{\eta}_1, \bar{\bar{\eta}}_1, \hat{\eta}_1, \underline{\eta}_1\}$, one has $\varepsilon_v^s < 1 < \varepsilon_v^h < \bar{\varepsilon}_v < \varepsilon_v^f$ when $a_\phi = 0$. Moreover, we can show that there exists $\eta_1' > 0$ such that $\forall\ \eta_1 < \eta_1'$, $\varepsilon_v^s < \varepsilon_v^h < \bar{\varepsilon}_v < \varepsilon_v^f$ when $a_\phi = -1$. For the rest of the proof, we assume that $\eta_1 < \min\{\eta_1^\theta, \tilde{\eta}_1, \bar{\bar{\eta}}_1, \hat{\eta}_1, \underline{\eta}_1, \eta_1'\}$.

Recall that a_ϕ is defined on $(-\infty, -1) \cup [0, +\infty)$. At this stage of the proof, we can state by analyzing $1-T+D$, $1+T+D$ and $1-D$ given by Eqs. (10.41)–(10.43) that if $a_\phi \in (-\infty, -1)$, local indeterminacy occurs when $\varepsilon_v < min\{\bar{\varepsilon}_v, \varepsilon_v^f, \varepsilon_v^h, \varepsilon_v^s\}$ or when $\varepsilon_v > max\{\bar{\varepsilon}_v, \varepsilon_v^f, \varepsilon_v^h, \varepsilon_v^s\}$. From the previous results, we deduce that if $a_\phi \in [0, \tilde{a}_\phi)$, local indeterminacy occurs when $\varepsilon_v < \varepsilon_v^s$ or when $\varepsilon_v > \varepsilon_v^f$.

Finally, if $a_\phi > \tilde{a}_\phi$, local indeterminacy occurs when $\varepsilon < \varepsilon_v^s$. For the case $a_\phi \in (-\infty, -1)$, we should determine the location of ε_v^f, ε_v^h, ε_v^s and $\bar{\varepsilon}_v$ in the plane (a_ϕ, ε_v).

The functions ε_v^f, ε_v^h and $\bar{\varepsilon}_v$ are continuous and monotone increasing on $a_\phi \in (-\infty, -1)$. We can show that the graph of these functions cross the horizonntal axis on $a_\phi \in (-\infty, -1)$. Let introduce the different following points $a_\phi^{\xi_2}$, $a_\phi^{\xi_4}$ and $a_\phi^{\xi_5}$, which corresponds to the points at which $\bar{\varepsilon}_v$, ε_v^f and ε_v^h cross the horizontal axis. $a_\phi^{\xi_2}$ is defined by $\bar{\varepsilon}_v = 0$ such that $a_\phi^{\xi_2} = -\frac{\xi_2^b}{\xi_2^a} < 0$. $a_\phi^{\xi_4}$ is defined by $\varepsilon_v^f = 0$ such that $a_\phi^{\xi_4} = -\frac{\xi_4^b}{\xi_4^a} < 0$. $a_\phi^{\xi_5}$ is defined by $\varepsilon_v^h = 0$ such that $a_\phi^{\xi_5} = -\frac{\xi_5^b}{\xi_5^a} < 0$.

After some algebra, we can show that since $\psi < 1/(1 + i^*\gamma)$, there exists $\eta_1^a > 0$ such that $\forall\, \eta_1 < \eta_1^a, a_\phi^{\xi_5} < a_\phi^{\xi_2}$. Furthermore, either $a_\phi^{\xi_2} < a_\phi^{\xi_4}\ \forall\, \eta_1 > 0$ or there exists η_1^b such that $\forall\, \eta_1 < \eta_1^b > 0, a_\phi^{\xi_2} < a_\phi^{\xi_4}$. Hence, if $\eta_1 < min\{\eta_1^a, \eta_1^b, \eta_1^\theta, \tilde{\eta}_1, \bar{\bar{\eta}}_1, \hat{\eta}_1, \underline{\eta}_1, \eta_1'\}$, one has $a_\phi^{\xi_5} < a_\phi^{\xi_2} < a_\phi^{\xi_4} < 0$.

Let $\eta_1'' = min\{\eta_1^a, \eta_1^b, \eta_1^\theta, \tilde{\eta}_1, \bar{\bar{\eta}}_1, \hat{\eta}_1, \underline{\eta}_1, \eta_1'\}$. For the rest of the proof, we suppose that $\eta_1 < \eta_1''$.

Since the functions ε_v^f, ε_v^h and $\bar{\varepsilon}_v$ are continuous and monotone increasing on $(-\infty, -1) \cup [0, \tilde{a}_\phi)$, we can now locate ε_v^f, ε_v^h and $\bar{\varepsilon}_v$ in the plane (a_ϕ, ε_v).

Let $\hat{a}_\phi \equiv a_\phi^{\xi_4}$. Since $\psi < 1/(1 + i^*\gamma)$, $a_\phi^{\xi^5} < a_\phi^{\xi_2} < \hat{a}_\phi$. Using the expressions of $1 - T + D$, $1 + T + D$ and $1 - D$ given by Eqs. (10.41)–(10.43), we state that if $a_\phi \in (-\infty, \hat{a}_\phi)$, local indeterminacy occurs when $\varepsilon_v > max\{\varepsilon_v^f, \varepsilon_v^h, \varepsilon_v^s\}$. If $a_\phi \in [\hat{a}_\phi, -1) \cup [0, \tilde{a}_\phi)$, local indeterminacy occurs when $\varepsilon_v < min\{\varepsilon_v^f, \varepsilon_v^s\}$ or when $\varepsilon_v > max\{\varepsilon_v^f, \varepsilon_v^h, \varepsilon_v^s\}$.

We have shown that ε_v^s can be positive. In such a case, it would be useful to determine when ε_v^f, ε_v^h and $\bar{\varepsilon}_v$ cross ε_v^s. Suppose that $\varepsilon_v^s > 0$. $\varepsilon_v^h = \varepsilon_v^s$ when $a_\phi = -\left(\varepsilon_v^s\xi_1^b - \xi_5^b\right)/(\varepsilon_v^s\xi_1^a - \xi_5^a) \equiv a_\phi^1 < 0, \bar{\varepsilon}_v = \varepsilon_v^s$ when $a_\phi = -\left(\varepsilon_v^s\xi_1^b + \xi_2^b\right)/(\varepsilon_v^s\xi_1^a + \xi_2^a) \equiv a_\phi^2 < 0$, and $\varepsilon_v^f = \varepsilon_v^s$ when $a_\phi = -\left(\varepsilon_v^s\xi_3^b - \xi_4^b\right)/(\varepsilon_v^s\xi_3^a - \xi_4^a) \equiv a_\phi^3 < 0$.

Because $\psi < 1/(1 + i^*\gamma)$, we can show after some algebra that either $a_\phi^1 < a_\phi^2\ \forall\, \eta_1 > 0$ or there exists $\eta_1^e > 0$ such that $\forall\, \eta_1 < \eta_1^e, a_\phi^1 < a_\phi^2$. It is difficult to determine the location of a_ϕ^3. Nevertheless, if $a_\phi^1 < a_\phi^3 < a_\phi^2 < 0$ or if $a_\phi^3 < a_\phi^1 < a_\phi^2 < 0$, we get $1 - T + D < 0$, $1 + T + D < 0$ and $D > 1$ for some values of ε_v. Since this is not feasible, we can eliminate these configurations. Therefore, if $\eta_1 < min\{\eta_1^e, \eta_1''\}$, one has $a_\phi^1 < a_\phi^2 < a_\phi^3 < 0$.

Let $\bar{\eta}_1 = min\{\eta_1^e, \eta_1''\}$. All conditions on η_1 required in this proof are satisfied when $\eta_1 < \bar{\eta}_1$. We can now derive the dynamic properties of the model. The properties of local dynamics are depicted by Fig. 10.1.[19] Grey areas in Fig. 10.1 correspond to the different regions in which the steady state is a sink, in other words to the indeterminacy regions.

[19]Figure 10.1 depicts the dynamic properties of the model with $\varepsilon_u \in (\tilde{\varepsilon}_u, \varepsilon_u^s)$. The configuration with $\varepsilon_u > \varepsilon_u^s$ can be easily deduce from the former one.

We deduce Proposition 3 and Corollary 1 from Fig. 10.1. Since $a_\phi = \phi/(1-\phi)$ is increasing with ϕ, we can derive Corollary 2 from Proposition 3 and Fig. 10.1. ■

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