Chapter 6
Anisotropic Damage in Elasto-plastic Materials with Structural Defects

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6.1 Introduction

We propose here a mathematical formalism, developed within the continuum damage mechanics that allows us to describe the macroscopic behaviour of elasto-plastic material with damaged microstructure. The damage of the material at the microscopic level means the existence of the microcracks or microvoids, that will be modeled by the presence of certain internal state variables, called the damage tensor or scalar damage parameters, which evolve during the irreversible processes.

The continuum damage mechanics investigates from continuum mechanics point of view the internal microstructural changes, concerning the mechanical modeling of the distributed cavities and cracks, which induce the initiation of the macro cracks. The failure is characterized by dominant macro cracks, which are generated as an ultimate stage during the damage (microstructural) process of the material.

The continuum damage mechanics formulates mathematically the mechanical behaviour of the materials deteriorate by the existence of the microcavities and microcracks. Within the continuum damage mechanics two types of problems arise when describing the state of damaged material. The first type is related to the physical nature and the mathematical description of the damage variables, while the second type concerns the elaboration of the constitutive framework, which allows a coherent description of the behaviour of materials with damaged microstructure.

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The damage state can be described mathematically by using scalar and tensorial variables, referring to *isotropic damage* and *anisotropic damage*, which are described by the appropriate evolution equations. The **scalar damage variables** are adequate for the isotropic damage, when a random distribution of microvoids and microcracks characterizes the damaged structure. The scalar damage variables have been extensively used in continuum damage mechanics.

Murakami (1983) discusses mechanical modeling and the damage variables used to describe the damage state and appreciated the primary notions introduced by Kachanov (1986), Rabotnov (1969) as basic for the development of continuum damage mechanics. Murakami (1983, 1988) refers to the existence of the distributed microvoids, which imply microcavities and microcracks, as *damage*, and call the nucleation and the growth of the voids as their *evolution*.

In the anisotropic damage the void growth and micro-shear crack mechanism are active simultaneously. Brünig (2003), Brünig and Ricci (2005) provide a finite strain framework for ductile anisotropic continuum damage based on thermodynamic law for isothermic processes and coupled with plasticity and damage, and the extension to nonlocal plasticity and nonlocal damage can be found in Brünig and Ricci (2005) and Brünig et al. (2013).

In Sect. 6.2, we exemplify some scalar variables, like the *void volume fraction* and *effective area reduction*. We make reference to the effect of *triaxiality* on the ductile damage, and we recall the initial concept of the undamaged configuration in correlation with the anisotropic damage.

In Sect. 6.3, we present the models proposed by Brünig (2003), Brünig and Ricci (2005), which are using the *multiple undamaged (fictitious) configurations* and the specific metric coefficients to describe measures of damage. The macroscopic background is the same for the two above-mentioned papers. We also present the damage model by Ekh et al. (2004) proposed within the crystal plasticity formalism, when the evolution rule for the damage is formulated with respect to the crystalline slip system. The model is based on the fictitious configuration and the equivalence principle of the free energy in the fictitious undamaged configuration and the intermediate configuration, used in the multiplicative decomposition of the deformation gradient.

In Sect. 6.3, we also briefly presented the Chaboche and Lemaitre models, (Chaboche and Lemaitre 1990; Lemaitre 1992), in the compact formulation of damage laws as it was reviewed, presented and numerically implemented by de Souza Neto et al. (2008). Although there is a model developed within the small elasto-plastic formalism and it is based on one scalar damage variable only, and our aims is to discuss the finite elasto-plastic models coupled with the anisotropic damage, we included this model in our presentation due to the large number of extensions. We mention here the paper by Lämmer and Tsakmakis (2000), Malcher et al. (2012).

Two types of constitutive models have been proposed in this chapter, in Sects. 6.4 and 6.5. The first model, discussed in Sect. 6.4, is based on the existence of an undamaged (fictitious) configuration; the anisotropic damage is described in terms of the (second order) damage tensor, $\mathbf{F}^d$, which is a deformation like tensorial variable. The damage tensor $\mathbf{F}^d$, characterizes the passage from a certain plastically deformed configuration (in our case considered to be also stress free configuration) to an undamaged (fictitious) configuration and depicted a measure of anisotropic damage. $\mathbf{F}^d$ is involved in the multiplicative decomposition of the deformation gradient $\mathbf{F}$ into its elastic (reversible), $\mathbf{F}^e$, damaged $\mathbf{F}^d$ and plastic, $\mathbf{F}^p$, components, namely $\mathbf{F} = \mathbf{F}^e\mathbf{F}^d\mathbf{F}^p$. In the proposed framework we describe the material behaviour with respect to the stress free (fictitious) undamaged configuration; the model is compatible with the second law of thermomechanics, expressed as the Clausius-Duhem dissipation inequality. The case of isotropic damage when a scalar field replaces the tensorial damage variable, and the multiplicative decomposition of the deformation gradient is reduced to the initial one, $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, is also considered as a special case.

The second model is presented in Sect. 6.5, and it is developed within the constitutive framework of second order finite elasto-plasticity, formulated by Cleja-Ţigoiu (2007, 2010). The presence of the second order damage tensor is related to the measure of non-metricity of the so-called plastic connection. The model is described within the second order plasticity, based on the multiplicative decomposition of the deformation gradient $\mathbf{F} = \nabla\chi$ (where the function χ describes the motion of the body) into its elastic and plastic components $\mathbf{F}^e, \mathbf{F}^p$, called *distortions*

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \tag{6.1}$$

as well as on the rule of the motion connection decomposition $\boldsymbol{\Gamma} = (\mathbf{F})^{-1}\nabla\,\mathbf{F}$ into its elastic and plastic counterparts.

The behaviour of elasto-plastic materials with damaged microstructure is described in terms of specific differential geometry elements which characterize the internal mechanical state, following Kröner (1992), de Wit (1981). In the proposed elasto-plastic models the defects of lattice structure, like dislocations and disclinations, can be involved through the Cartan torsion of the so-called plastic connection, see Cleja-Ţigoiu (2010, 2002), while the point defects, microvoids and microcracks, in the damaged zone are modeled in terms of the non-metric tensor which belongs to the plastic connection, apart from Cleja-Ţigoiu and Ţigoiu (2011), where the gradient of the elastic strain measures the damage. The non-metric property of the plastic connection is described in terms of a symmetric second order tensor, **h,** which is potential for the non-metric (extra-matter) tensor **Q.**

The continuum damage mechanics also deals with the constitutive and evolution equations which describe the damage and plastic behaviour.

Energetic arguments, like dissipation inequality, along the isothermal deformation processes and power conjugated variables, will be used in order to complete the models. The dissipative nature for the irreversible behavior is modeled by the requirement to satisfy the principle of the free energy imbalance for the isothermal processes. The free energy imbalance principle reformulates the classical second law of thermodynamics within the second order finite elasto-plasticity, following

Gurtin's idea presented in Gurtin (2002), Gurtin et al. (2010), see Cleja-Ţigoiu (2007, 2010). The constitutive and evolution equations are derived to be compatible with free energy imbalance. The resulting models are strongly dependent on the postulated expressions for the free energy and the internal power.

Our exposure in Sect. 6.2 constitutes a concise and critical presentation of the contributions and results which led to basic ideas for the development of elasto-plastic anisotropic damaged materials. We shortly recall the meaning of extensively used scalar damage variables, with reference to the volume void fraction, and to the first micromechanical model for ductile fracture, the Gurson (1977) model, which introduces a strong coupling between the plastic deformation and damage. Modifications of the Gurson model for shear have been proposed and experimentally validated by Nahshon and Hutchinson (2008), Xue (2008), the key point being the extension of the evolution equation for the void volume fraction. Lassance et al. (2007) consider the Gurson model to be representative of the void growth only. The authors introduce and validate an extended version of the Gurson (1977) model, which involves also many other recent improvements of the aforementioned model. The paper applies the micromechanics-based methodology to investigate the damage resistance of certain Al-alloys.

We expose certain ideas, as fictitious damaged and undamaged configurations, effective stress, and so on, which have been fruitfully utilized in modeling the anisotropic damage.

Section 6.3 is devoted to the constitutive models for elasto-plastic materials with microstructural defects (like microcracks and microcavities), which describe the inelastic deformations, including the anisotropic damage and based on the fictitious configurations. We refer to the models described by Murakami (1988), Brünig (2003), Ekh et al. (2004), Menzel et al. (2002), and so on.

Different models connecting damage and elasto-plasticity are based on deformation type damage variables, see the models proposed by Brünig (2003), Brünig and Ricci (2005), Brünig et al. (2008), Menzel et al. (2002), Ekh et al. (2004, 2005) and we also mention de Borst et al. (1999).

Two types of damage (second order) tensors like deformation fields have been introduced in the literature, both of them assuming the hypothesis of the existence of the undamaged (fictitious) configuration. In general the damage deformation tensor, denoted here by $\mathbf{F}^d$, characterizes the passage from an undamaged (fictitious) configuration to a certain plastically deformed configuration, as a measure of anisotropic damage. $\mathbf{F}^d$ is viewed sometimes like a purely internal state variable, see Menzel et al. (2002), Ekh et al. (2004, 2005) which does not influence the multiplicative decomposition of the deformation gradient into its component, apart from the models proposed by Brünig (2003), Brünig and Ricci (2005).

To avoid the confusions which appear, when this mention is missing we pay attention to the configurations on which the tensor fields are defined. We tried to use our notations only, as much as possible, with the aim to unify the notations from different papers, in order to make evident the differences between the models and field definitions, thus facilitating the comparison of the various presented models.

When we refer to the finite elasto-plasticity based on the deformation gradient multiplicative decomposition into elastic and plastic components, we have in mind the concept of the so-called local relaxed (or stress free) configuration, physically motivated by the mechanism of plastic deformations within the crystalline materials, see Cleja-Ţigoiu and Soós (1990). The global stress free configuration does not exist for elasto-plastic materials with crystalline structure. We assume that the local stress free configurations can be uniquely associated to any material point, apart from the orthogonal transformation that can be an element of the material symmetry group. That is why we reconsidered the figures from the papers by Brünig (2003), Murakami (1988), Ekh et al. (2004). The indeterminacy in choosing the stress free configuration has been solved by considering the same crystallographic orientation for the appropriate material neighborhoods, in the initial and relaxed configurations, i.e. the so-called *isoclinic configuration.*

We tacitly used the same idea representing graphs of the undamaged configurations.

Another important fact is related to the *objectivity assumptions*, see Cleja-Ţigoiu (1990), Cleja-Ţigoiu and Soós (1990), which states that if the two motions of the body differ locally by a superposed rigid motion the set of the associated local relaxed configurations can be the same, and moreover the associated internal state variables have equal values. Let us remark that the elastic type constitutive equation in terms of the Cauchy stress tensor has to be *objective*, namely relative to the change of frame in the actual configuration, characterized by an orthogonal mapping $\mathbf{Q}$. The tensor $\mathbf{F}^e$ sustains the transformation, i.e. $\mathbf{F}^{*e} = \mathbf{Q}\mathbf{F}^e$, and $\mathbf{F}^{*d} = \mathbf{F}^d$ and $\mathbf{F}^{*p} = \mathbf{F}^p$. On the other hand in order to a certain elastic type constitutive equation satisfies *the stress free condition* it is necessary for the Cauchy stress to be zero, $\mathbf{T} = 0$, if the elastic strain, say $\mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e$, is the identity tensor.

In our presentation we do not considered the vector damage variables, which were introduced to characterize the effect of the cracks distributed on certain planes. The damage vector is considered to be perpendicular on the plane of the cracks.

6.1.1 List of Notation

Further the following notations will be used:

- $\mathscr{E}$—the three dimensional Euclidean space, with the vector space of translations $\mathscr{V}$;
- *Lin*—the set of the linear mappings from $\mathscr{V}$ to $\mathscr{V}$, *Sym*–i.e. the set of symmetric tensors, $Orth \subset Lin$ the set of all orthogonal second order tensors;
- $\mathbf{u} \cdot \mathbf{v}, \mathbf{u} \otimes \mathbf{v}$ denote scalar and tensorial products of vectors;
- $\mathbf{a} \otimes \mathbf{b}$ and $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$ are defined to be a second order tensor and a third order tensor and are defined by $(\mathbf{a} \otimes \mathbf{b})\mathbf{u} = \mathbf{a}(\mathbf{b} \cdot \mathbf{u})$, $(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})\mathbf{u} = (\mathbf{a} \otimes \mathbf{b})\ (\mathbf{c} \cdot \mathbf{u})$, for all vectors $\mathbf{u}$;

- for $\mathbf{A} \in Lin$—a second order tensor, we introduce the notations: $\{\mathbf{A}\}^S, \{\mathbf{A}\}^a$ for the symmetric and skew-symmetric parts of the tensor and tr$\mathbf{A}$ for the trace of $\mathbf{A} \in Lin$;
- the tensorial product $\mathbf{A} \otimes \mathbf{a}$ for $\mathbf{a} \in \mathscr{V}$, is a third order tensor, with the property $(\mathbf{A} \otimes \mathbf{a})\mathbf{v} = \mathbf{A}(\mathbf{a} \cdot \mathbf{v})$, $\forall \mathbf{v} \in \mathscr{V}$;
- $\mathbf{I}$ is the identity tensor in Lin and $\mathbf{A}^T$ denotes the transpose of $\mathbf{A} \in Lin$;
- for Λ in Lin, the third order field $(\Lambda \times \mathbf{I})$ is defined by $((\Lambda \times \mathbf{I})\mathbf{u})\mathbf{v} = (\Lambda\mathbf{u}) \times \mathbf{v}$, for all vectors $\mathbf{u}$ and $\mathbf{v}$;
- $\partial_{\mathbf{A}}\phi(x)$ denotes the partial differential of the function ϕ with respect to the field $\mathbf{A}$;
- $\nabla\mathbf{A}$ is the derivative (or the gradient) of the field $\mathbf{A}$ in a coordinate system $\{\mathbf{x}^a\}$ (with respect to the reference configuration), $\nabla\mathbf{A} = \frac{\partial A_{ij}}{\partial x^k}\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k$, for $\mathbf{A} = A_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$, namely the calculation follows as the basis is fixed;
- the gradient with respect to the configuration $\mathscr{K}$ is defined by $\nabla_{\mathscr{K}}\mathbf{H} = (\nabla\mathbf{H})(\mathbf{F}^p)^{-1}$ in terms of the gradient with respect to the reference configuration, due to the fact that $\mathbf{F}^p$ denotes the map which put into correspondence the reference and damaged configuration, $\mathscr{K}$;
- the operator $\odot$ associates to the third order tensors $\mathscr{A}, \mathscr{B}$ the second order tensor, denoted $\mathscr{A}\odot\mathscr{B}$ and defined by

$$(\mathscr{A}\odot\mathscr{B}) \cdot \mathbf{L} = \mathscr{A}[\mathbf{I}, \mathbf{L}] \cdot \mathscr{B} = \mathscr{A}_{isk}L_{sn}\mathscr{B}_{ink}, \tag{6.2}$$

  for all second order tensor $\mathbf{L}$;
- the transpose of the third order tensor field $\mathscr{N}$ is given by $\mathscr{N}^T\mathbf{u} = (\mathscr{N}\mathbf{u})^T$, for any $\mathbf{u}$.
- *curl* of a second order tensor field $\mathbf{A}$ is defined by the second order tensor field

$$\begin{aligned}&(\mathrm{curl}\mathbf{A})(\mathbf{u} \times \mathbf{v}) := (\nabla\mathbf{A}(\mathbf{u}))\mathbf{v} - (\nabla\mathbf{A}(\mathbf{v}))\mathbf{u} \quad \forall \mathbf{u}, \mathbf{v} \in \mathscr{V} \text{ and}\\ &(curl\mathbf{A})_{pi} = \varepsilon_{ijk}\frac{\partial A_{pk}}{\partial x^j}\end{aligned} \tag{6.3}$$

  are the component of *curl*$\mathbf{A}$ given in a Cartesian basis. ε_{ijk} denotes the components of Ricci permutation tensor.
- $\mathcal{H}$ denotes the Heaviside function, defined by $\mathcal{H}(x) = 0\ \forall\, x < 0$, and $\mathcal{H}(x) = 1\ \forall\, x \geq 0$

6.2 Damage State

The damage state is described by scalar and tensorial variables. We exemplify the scalar variables, like the *void volume fraction* and *effective area reduction*. We make reference to the effect of *triaxiality* on the ductile damage, and we recall the initial formulation of the undamaged configuration concept in correlation with the anisotropic damage.

6.2.1 Isotropic Damage

Kachanov (1986) introduced the scalar damage variable ω, $(0<\omega<1)$. The damage variable can be interpreted as being the *effective area (net area) reduction* caused by the microcracks and microcavities.

Consider a damaged solid and a volume element of a sufficiently large size with respect to the inhomogeneity and sufficiently small size to be viewed as a material neighborhood of a given material point.

Let us denote by $A(n)$ the area of the section of the volume element identified by the normal **n**, and the effective area of resistance by $\bar{A}(n)$, i.e. the remaining area after eliminating the microcavities and microcracks, assuming $\bar{A}(n)<A(n)$. Thus the *effective area (net area) reduction* ω is the ratio between the net current area, $\bar{A}(n)$, and the area of the given section, $A(n)$, i.e.

$$\omega = \frac{\bar{A}(n)}{A(n)}. \tag{6.4}$$

From a physical point of view the so-called damage variable, $d = 1 - \omega$, is the relative (or corrected) area of the cracks and cavities cut by a plane normal to the direction **n**.

In the uniaxial tension the applied force on a section of the representative element is $\sigma A(n)$, and the *effective stress*, denoted by $\bar{\sigma}$, is defined in terms of Cauchy stress σ by $\bar{\sigma} = \frac{\sigma}{1-d}$, as a consequence of the assumed equality

$$\sigma A(n) = \bar{\sigma}\bar{A}(n). \tag{6.5}$$

In the one dimensional case $\bar{A}(n)$ appears to be the effective load-carried area of the current damaged state. The *fictitious undamaged bar* with the cross-section area $\bar{A}(n)$ and subjected to the same applied force has been considered *mechanically equivalent* and it is called *fictitious undamaged state* (Murakami 1988).

Lemaitre and Chaboche (1978), Lemaitre (1985) characterized the damage state by the change of the elastic constants of the materials. The authors assumed the *hypothesis of elastic strain equivalence*, namely the elastic type constitutive equation of the *damaged material* is derived by the constitutive equation for the *elastic undamaged material*, by replacing the Cauchy stress tensor, σ, by the *effective stress* $\bar{\sigma}$.

$$\begin{aligned} \varepsilon^e &= \frac{\sigma}{\bar{E}} = \frac{\bar{\sigma}}{E} = \frac{\sigma}{E(1-d)} \\ d &= 1 - \frac{\bar{E}}{E}, \quad \bar{\sigma} = \frac{\sigma}{1-d} = E\frac{\sigma}{\bar{E}}, \end{aligned} \tag{6.6}$$

where $\bar{E}$ and E are elastic constants of the damaged and undamaged materials.

6.2.2 Void Volume Fraction

Another scalar damage variable, namely the void volume fraction, f, has been extensively considered in various micromechanical models for ductile fraction. This parameter is the ratio between the volume of microvoids, f_{voids}, and the representative volume element, f_{rve}, i.e. $f = \frac{f_{voids}}{f_{rve}}$. The Gurson model, (Gurson 1977), is the first micromechanical model for ductile fracture, which introduces a strong coupling between the plastic deformation and damage. The main result of the Gurson model estimates the yield function for the porous metal, which is given by

$$\Phi = \frac{\sigma_{eq}^2}{\sigma_y^2} + 2f\cosh\left(\frac{1}{2}\frac{\mathrm{tr}\boldsymbol{\sigma}}{\sigma_y}\right) - 1 - f, \tag{6.7}$$

where the hardening behaviour is described by $\sigma_y = h(\bar{\varepsilon}^p)$ related through the energy balance

$$\sigma_y \dot{\bar{\varepsilon}}^p (1 - f) = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p. \tag{6.8}$$

The evolution equations for the plastic strain, void volume fraction and Cauchy stress, respectively, $(\boldsymbol{\varepsilon}^p, f, \boldsymbol{\sigma})$, are given by:

The associate flow rule is characterized by

$$\dot{\varepsilon}^p = \lambda \frac{\partial \Phi}{\partial \sigma}, \tag{6.9}$$

with λ defined by the Kuhn-Tucker condition $\lambda \geq 0, \quad \Phi \leq 0, \quad \lambda\, \Phi = 0,$ and consistency condition $\lambda\, \dot{\Phi} = 0$.

The law of the variation of the void volume fraction, caused by the accumulation of plastic deformation, is given by

$$\dot{f} = (1 - f)\mathrm{tr}\, \dot{\varepsilon}^p, \tag{6.10}$$

as the voids started to nucleate, and the rate type elastic constitutive equation is described by

$$\dot{\boldsymbol{\sigma}} = \mathscr{E}(\dot{\boldsymbol{\varepsilon}}^p - \dot{\boldsymbol{\varepsilon}}), \tag{6.11}$$

in terms of the Cauchy stress tensor.

The Gurson model has been extended to include void shape dependences and so on, see Siruguet and Leblond (2004) and the reference that can be found, for instance in Lassance et al. (2007), Nahshon and Hutchinson (2008). These extensions are based on "the solutions for the voids subjected to axisymmetric stress and exclude the possibility of *shear localization* and fracture under the low triaxiality, if void nucleation is not invoked", as remarked Nahshon and Hutchinson (2008). We make a special reference to the paper by Lassance et al. (2007), as the attention is focused on the non-symmetric microstructural defects. The authors considered that "the presence of coarse, elongated particles is the key microstructural feature behaviour" of Al-alloy. The authors evidenced that the *elongated* β-type particles are transformed into *rounded* α-type particles, by heat treatment. "At the ambient temperature the α particles and the β particles oriented with the long axis perpendicular to the loading direction undergo interface decohesion, while the β particles oriented perpendicular to the loading direction break into several fragments." They concluded that "the ductility increases with decreasing amount of β particles, increasing temperature and strain rates, and decreasing stress triaxiality." The review performed in the aforementioned paper contains well structured references.

6.2.3 Effect of Stress Triaxiality

The effect of stress triaxiality on ductile fracture and the evolution of the fracture ductility is put experimentally into evidence and discussed by Bao and Wierzbicki (2004, 2005), Brünig et al. (2013, 2008), Nahnshon and Hutchinson (2008), see also the references in the aforementioned papers. "Fracture ductility is understood as the ability of a material to accept large amount of deformation without fracture. Equivalent strain to fracture is good measurement of fracture ductility," see Bao and Wierzbicki (2004).

The *stress triaxiality* is defined by the ratio η

$$\eta = \frac{\sigma_H}{\sigma_e}, \qquad \text{where}$$
$$\sigma_H = \tfrac{1}{3}\,\mathrm{tr}\mathbf{T}, \quad \sigma_{eq} = \sqrt{\tfrac{3}{2}\,\mathrm{Dev}\mathbf{T}\cdot\mathrm{Dev}\mathbf{T}}, \quad \mathrm{Dev}\mathbf{T} = \mathbf{T} - \tfrac{1}{3}\,\mathrm{tr}\mathbf{T}\mathbf{I}. \tag{6.12}$$

σ_H is the mean stress and σ_{eq} is the second invariant of the stress deviator Dev**T**.

Based on the experimental and numerical results Bao and Wierzbicki (2004, 2005) concluded that the equivalent strain to fracture, denoted by $\bar{\varepsilon}_f$, can be represented as a function of stress triaxiality. The relations between the effective plastic strain at fracture and triaxiality is not monotonous. Three branches have been put into evidence, being governed by shear mode for negative triaxiality, by void growth dominant failure for large triaxiality and by a combination of shear and voids growth mode for the stress triaxiality between the two regimes mentioned above. $\bar{\varepsilon}_f$ is supposed to be analytically represented in terms of the stress triaxiality, i.e. $\bar{\varepsilon}_f = f(\eta)$ which is specific for a given material. Finally the best fit of the experimental data have been presented as average stress triaxiality versus equivalent strain to fracture, i.e. the fracture locus has been defined. The authors mentioned that the displacement to the fracture has been determined during the experiments and by the force displacement response. The significant drop in loading has been taken to be the point of the initiation of the fracture. It is observed that after its initiation the crack grows very rapidly during the test.

To capture the effect of stress state on the ductile damage and failure, Brünig et al. (2013, 2008) introduced the damage potential functions and damage criteria which are expressed in terms of stress intensity, stress triaxiality and Lode parameter. The damage rule takes into account the isotropic and anisotropic parts corresponding to isotropic growth of voids and anisotropic evolution of micro-shear-cracks, respectively. The parameters can be identify by experiments or by numerical simulations on microscale.

Malcher et al. (2012) considered three isotropic hardening models, which include stress triaxiality and Lode angles, (as a measure of the third invariant of the stress): the extension of the Gurson model, proposed by Tveergard and Needleman (1984), the Lemaitre model (1985), and Bai and Wierzbicki model (2008). Due to the fact that Bai and Wierzbicki (2008) did not include in the model a damage variable, but included the stress triaxiality and Lode angle, Malcher et al. (2012) considered a modified model, by introducing the fracture indicator (a post-processed variable). In the numerical simulations, the specimens with different geometries have been employed in order to generate various stress and strain states, which covered a wide range of triaxiality and Lode angles. The authors concluded that for higher level of stress triaxiality the model proposed by Bai and Wierzbicki (2008) combined with the fracture indicator is more in agreement with the experimental results. Contrary, for a low level of the triaxiality the modified Gurtin model (Tveergard and Needleman 1984) is in agreement with experiments with reference to the equivalent plastic strain. The final conclusion in Malcher et al. (2012): the analyzed models need to be improved, as the models have limitations on the values of the displacement to fracture, the equivalent plastic strain to fracture or in term of fracture localization, under combined loading conditions.

6.2.4 Undamaged Configuration

The second and higher order tensors are introduced to characterize the complex three-dimensional distribution and evolution of the microvoids and microcracks, i.e. the material anisotropic damage. Murakami (1983), Murakami and Ohno (1980, 1981), described the anisotropic damage by a second order symmetric tensor, $\mathbf{D}$,

$$\mathbf{D} = \sum_{i=1}^{i=3} D_i \mathbf{n}_i \otimes \mathbf{n}_i, \tag{6.13}$$

where D_i and $\mathbf{n}_i$ are the principal values and directions. D_i can be interpreted as the void area density in the plane perpendicular to direction of the damage $\mathbf{n}_i$.

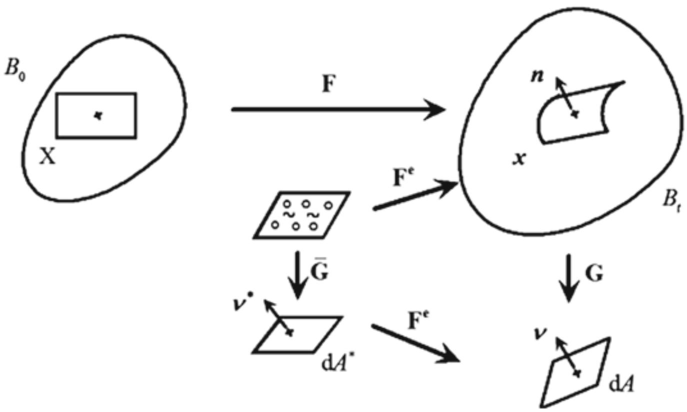

Fig. 6.1 $\mathbf{F}^e$ the elastic part of the deformation gradient; $\mathbf{G}$ and $\bar{\mathbf{G}}$ linear transformations from the current deformed body $\mathscr{B}_t$, and from the stress free and damaged configuration, respectively, to the fictitious associated configurations

By assuming that the principal effect of the material damage consists of the net area decrease due to the three-dimensional distribution of micro defects, Murakami (1988) considered an area vector element in the current (actual) damaged configuration, say $\boldsymbol{v}\, dA$ and postulated that there exists a *fictitious undamaged configuration*, and the equivalent load-carrying area vector is denoted by $\boldsymbol{v}^*\, dA^*$. Here $\boldsymbol{v}$ and $\boldsymbol{v}^*$ are unit normals to the appropriate areas, see Fig. 6.1.

If $\mathbf{G}$ denotes the tensor which characterizes the passage from the current deformed damaged configuration to fictitious undamaged configuration associated with the previous one, then by applying the Nanson formula we obtain

$$\begin{aligned} &\boldsymbol{v}\, dA = (\det\mathbf{G})\mathbf{G}^{-T}\mathbf{n}\, da \quad \text{or} \\ &\boldsymbol{v}\, dA = (\mathbf{I}-\mathbf{D})\mathbf{n}\, da, \quad \text{where } \mathbf{I}-\mathbf{D} = (\det\mathbf{G})\mathbf{G}^{-T}. \end{aligned} \tag{6.14}$$

Here $\boldsymbol{v}\, dA$ denotes the associated area vector in the undamaged configuration associated to $\mathbf{n}\, da$, the vector area in deformed damaged configuration, $\mathscr{B}_t$. The definition (6.14) introduces a *fictitious deformation* from the current damaged configuration to the so called *fictitious undamaged* configuration, $\mathbf{G}$. The tensor $\mathbf{D}$ (and $\mathbf{G}$) depends on the current state of deformation, as Murakami (1988) observed. Due to the fact that only the *irreversible change of the structure* is responsible for the damage, Murakami (1988) associated the undamaged (fictitious) configuration with the deformed stress free configuration. Consequently, a similar formula to (6.14) is derived

$$\boldsymbol{v}^*\, dA^* = (\mathbf{I}-\overline{\mathbf{D}})\bar{\mathbf{n}}\, d\bar{a}, \tag{6.15}$$

where $\mathbf{I} - \overline{\mathbf{D}}$ is associated with $\overline{\mathbf{G}}$ and $\boldsymbol{\nu}^* \, dA^*$ is the area vector in the undamaged (fictitious) stress-free configuration, while $\bar{\mathbf{n}} \, d\bar{a}$ denotes the associated area vector in the damaged stress free configuration. The following formula is derived in Murakami (1988)

$$\begin{aligned} &(\det\overline{\mathbf{G}})\overline{\mathbf{G}}^{-T} = (\mathbf{F}^e)^T(\det\mathbf{G})\mathbf{G}^{-T}(\mathbf{F}^e)^{-T}, \\ &\overline{\mathbf{D}} = (\mathbf{F}^e)^T\mathbf{D}(\mathbf{F}^e)^{-T}, \end{aligned} \tag{6.16}$$

in terms of $\mathbf{F}^e$, which realizes the passage from the stress free configuration to the current damaged configuration.

Remark Let us remark that the formulae (6.16) hold only under the assumption that just $\mathbf{F}^e$ realizes the passage from the undamaged and stress free configuration and fictitious undamaged configuration (associated with the current deformed configuration). Moreover, the second order field $\overline{\mathbf{D}}$ and consequently the damage transformation $\overline{\mathbf{G}}$ are symmetric.

The effect of the Cauchy stress, say $\boldsymbol{\sigma}$, acting on the body is given by the *effective stress tensor*

$$\bar{\boldsymbol{\sigma}} = \frac{1}{2}((\mathbf{I} - \mathbf{D})^{-1}\boldsymbol{\sigma} + \boldsymbol{\sigma}(\mathbf{I} - \mathbf{D})^{-1}), \tag{6.17}$$

introduced by Murakami and Ohno (1981). The tensor $(\mathbf{I} - \mathbf{D})^{-1}$ represents the stress effect increase due to damage.

Due to the hypothesis concerning the symmetry of the damage $\mathbf{D}$, Murakami expressed the idea that this damage state should correspond to the orthotropic symmetry only, see Murakami (1988).

Remark The formula (6.17) can be rewritten as

$$\bar{\boldsymbol{\sigma}} = \frac{1}{2}\sum_{i=1}^{i=3}\frac{1}{1 - D_i}(\mathbf{n}_i \otimes (\boldsymbol{\sigma}\mathbf{n}_i) + \boldsymbol{\sigma}\mathbf{n}_i \otimes \mathbf{n}_i), \text{ where } \frac{1}{1 - D_i} = \frac{\bar{A}_i}{A_i} \tag{6.18}$$

Consequently, if $\mathbf{n}_i$ is a proper vector for $\boldsymbol{\sigma}$, the formula (6.18) could be considered as an extension to the anisotropic damage of the uniaxial formula $\bar{\sigma}_i = \frac{\sigma_i}{1-D_i}$, (see the formulae (6.4) and (6.5)).

The evolution equation of the damage is expressed, following Murakami and Ohno (1980, 1981) by

$$\dot{\mathbf{D}} = \mathbf{H}(\bar{\boldsymbol{\sigma}}, (\mathbf{I} - \mathbf{D})^{-1}, \kappa), \tag{6.19}$$

where κ denotes a hardening parameter.

In Chap. 5 of the book by Voyiadjis and Kattan (2005), the *fourth-order anisotropic damage effect tensor* $\mathbf{M}$ is the key point in describing the anisotropic damage. **M** expresses the linear transformation giving rise to the effective stress tensor $\bar{\boldsymbol{\sigma}}$ in terms of the Cauchy stress tensor $\boldsymbol{\sigma}$, as in Murakami and Ohno (1981), namely

$$\bar{\boldsymbol{\sigma}} = \mathbf{M}\boldsymbol{\sigma}, \quad \text{where} \quad \mathbf{M}\boldsymbol{\sigma} = \frac{1}{2}\left(\boldsymbol{\sigma}(\mathbf{I}-\mathbf{D})^{-1} + (\mathbf{I}-\mathbf{D})^{-1}\boldsymbol{\sigma}\right). \tag{6.20}$$

The explicit representation of the fourth-order damage tensor **M** using the second order damage tensor **D**[1] is important in implementation of the constitutive models of damage. The representation of **M**, as a matrix (6,6) is given in terms of the six components of **D**, or using the proper values **D**, but this time in the tensorial representation with respect to the proper vector of **D**. In Chap. 7 of the book by Voyiadjis and Kattan (2005), the *fourth-order anisotropic damage effect tensor* **M** remains a general one, without any correlation with certain second order damage tensor, and a general elasto-plastic model connected with damage is proposed in an Eulerian formalism. A modified elasto-plastic stiffness tensor includes the effect of damage through the use of the undamaged stress configuration and the hypothesis of elastic energy equivalence.

The fourth order damage tensors have been also introduced, see for instance Murakami and Imaizumi (1982), Lubarda and Krajcinovic (1995), Voyiadjis and Park (1996), to take into account the damage induced material anisotropy.

6.3 Models with Damage State Variables

Two constitutive models for ductile anisotropic continuum damage, based on thermodynamic law for isothermic processes and connected with plasticity and damage, to capture the dissipative nature of the inelastic deformation are presented in Sect. 6.3.1 following Brünig (2003), Brünig and Ricci (2005), and in Sect. 6.3.2 following Ekh et al. (2004). Section 6.3.3 makes references to Lemaitre and Chaboche model (1990).

6.3.1 Model with Multiple Undamaged Configurations

Brünig (2003), Brünig and Ricci (2005) provide a finite strain framework, using the *multiple undamaged (fictitious) configurations* and specific metric coefficients to describe measures of damage. The extension to nonlocal plasticity and nonlocal damage can be found in Brünig and Ricci (2005), the macroscopic background being the same in all the aforementioned papers. Three types of undamaged

[1] **D** is denoted by Φ in Voyiadjis and Kattan (2005).

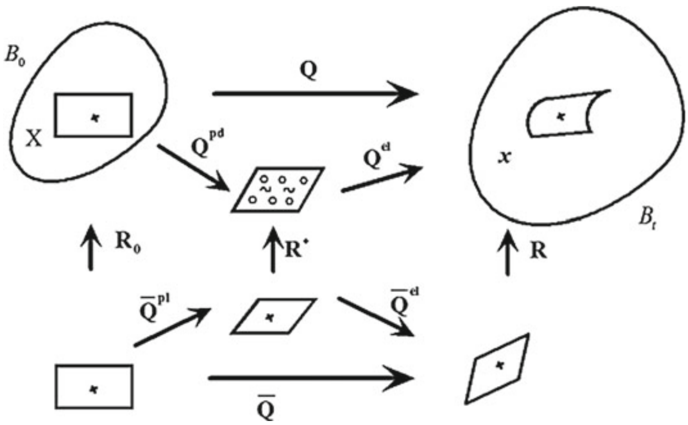

Fig. 6.2 The framework considered by Brünig (2003): $\mathbf{R}_0, \mathbf{R}^*$ and $\mathbf{R}$ are defined on the appropriate undamaged configurations obtained by fictitious removing the defects of the initial, stress-free intermediar and actual configurations: $\mathbf{Q}^{pd}, \mathbf{Q}^{el}$, the inelastic (plastic and damage) and elastic parts of the metric transformation tensor $\bar{\mathbf{Q}}^{pl}$, $\bar{\mathbf{Q}}^{el}$, the effective plastic and elastic parts of the effective metric transformation $\bar{\mathbf{Q}}$

configurations have been introduced, namely initial, $\mathscr{E}_0$, intermediate, $\mathscr{E}^*$, and current, $\mathscr{E}$, undamaged configurations, respectively, see Fig. 6.2. The current undamaged configuration $\mathscr{E}$ and the initial undamaged configuration of the body, $\mathscr{E}_0$, are obtained from the current configuration (denoted by $\mathscr{B}_t$ in Fig 6.2) and initial configuration, $\mathscr{B}_0$, by "fictitious removing all the damage" of the deformed body and initial body, respectively. The elastic unloaded configuration, $\mathscr{B}^*$, is associated to the deformed body $\mathscr{B}$ and the corresponding fictitious elastically unloaded and undamaged configuration is denoted by $\mathscr{E}^*$. We remark that all these configurations are local and only the initial and the deformed configurations are global. The set of undamaged configurations is similar to those provided by Murakami (1988), except the unloading initial configuration.

In Fig. 6.2, we represented here the locally appropriate neighborhoods associated to a given material point in the body, **X**, only.

We pointed out certain specific key points in formulating the background of the model.

i. The multiplicative decomposition of the so-called *metric transformation tensor*, $\mathbf{Q}$, into its inelastic (plastic and damage) part, $\mathbf{Q}^{pd}$, and elastic part, $\mathbf{Q}^{el}$, is considered, namely $\mathbf{Q} = \mathbf{Q}^{pd}\mathbf{Q}^{el}$.[2] A similar multiplicative decomposition is introduced, this time with reference to the undamaged configurations. The

[2]The correct written form of the above decomposition and which corresponds to the mentioned figures in the papers (Brünig 2003; Brünig and Ricci 2005) is $\mathbf{Q} = \mathbf{Q}^{el}\mathbf{Q}^{pd}$.

definitions of the elastic strain and damage strain tensors, are differently introduced

$$\mathbf{A}^{el} = \frac{1}{2}\ln \mathbf{Q}^{el}, \quad \mathbf{A}^{da} = \frac{1}{2}\ln \mathbf{R}^{*}, \quad \text{defined in Brünig (2003)}$$
$$\mathbf{A}^{el} = \frac{1}{2}(\mathbf{I} - (\mathbf{Q}^{el})^{-1}), \quad \mathbf{A}^{da} = \frac{1}{2}(\mathbf{I} - (\mathbf{R}^{*})^{-1}), \quad \text{in Brünig and Ricci (2005).} \tag{6.21}$$

These tensors are defined on the appropriate vector spaces associated with $\mathscr{B}^*$ and $\mathscr{E}^*$, respectively in the first definitions and with $\mathscr{B}$ and $\mathscr{B}^*$, respectively in the second definition. Consequently, these tensor fields are not referring to the same configurations and their composition is generally unjustified.

ii. The appropriate strain rate tensors have been introduced in Brünig (2003), Brünig and Ricci (2005), Brünig et al. (2013), and an additive decomposition of the strain rate tensor defined by $\dot{\mathbf{H}} = \frac{1}{2}(\mathbf{Q})^{-1}\dot{\mathbf{Q}}$, into the elastic and inelastic strain rates have been derived. We also remark that the strain rates $\dot{\mathbf{H}}^{el} = \frac{1}{2}(\mathbf{Q}^{el})^{-1}\dot{\mathbf{Q}}^{el}$ and $\dot{\mathbf{H}}^{da} = \frac{1}{2}(\mathbf{R}^{*})^{-1})\dot{\mathbf{R}}^{*}$ are associated with the configurations $\mathscr{B}^*$ and $\mathscr{E}^*$, respectively, if we look at their written expressions.[3]

Comments. Generally strong restrictions have to be imposed on the considered tensor fields in order to provide the imposed algebraic symmetry. For instance, although $(\mathbf{Q})^{-1}$ and $\dot{\mathbf{Q}}$ are symmetric tensors if $\mathbf{Q}$ is symmetric, the tensors $(\mathbf{Q})^{-1}\dot{\mathbf{Q}}$ and $\dot{\mathbf{Q}}(\mathbf{Q})^{-1}$ could not be symmetric. In order to avoid these unjustified issues, the linear and invertible transformations should be introduced in order to define the passage between various configurations, say for instance $\mathbf{F}^e$ instead of $\mathbf{Q}^{el}$. Consequently the symmetric and positive definite tensors which characterize the corresponding metric tensors can be naturally provided, but they do not enter the multiplicative decomposition.

iii. The elastic type constitutive equation, presented by Brünig (2003) formula (6.82), and by Brünig and Ricci (2005) formula (6.78), characterizes the Kirchhoff tensor $\mathbf{T}$ in terms of elastic strain, $\mathbf{A}^{el}$ (see the definitions given by (6.21)),

$$\begin{aligned}\mathbf{T} = 2(G + \eta_2\,\mathrm{tr}\mathbf{A}^{da})\mathbf{A}^{el} + [(K - \frac{2}{3}G + 2\eta_1\,\mathrm{tr}\mathbf{A}^{da})\mathrm{tr}\mathbf{A}^{el} + \\ + \eta_3(\mathbf{A}^{da}\cdot\mathbf{A}^{el})]\mathbf{I} + \eta_3(\mathrm{tr}\mathbf{A}^{el})\mathbf{A}^{da} + \eta_4(\mathbf{A}^{da}\mathbf{A}^{el} + \mathbf{A}^{el}\mathbf{A}^{da}).\end{aligned} \tag{6.22}$$

and containing the damage strain measure, $\mathbf{A}^{da}$.

[3] The correct definition for $\dot{\mathbf{H}}^{el}$ ought to be $\dot{\mathbf{H}}^{el} = \frac{1}{2}((\mathbf{Q}^{el})^{-1}\dot{\mathbf{Q}}^{el} + \dot{\mathbf{Q}}^{el}(\mathbf{Q}^{el})^{-1})$.

Remark The constitutive Eq. (6.22), say together with $(6.21)_2$ which characterizes an elastic behaviour contains two measure of deformations with respect to different configurations, the elastic strain $\mathbf{A}^{el}$, with respect to the deformed configuration, while $\mathbf{A}^{da}$ is defined on the stress free and damaged configuration. Moreover, the Kirchhoff tensor $\mathbf{T}$ and $\mathbf{A}^{el}$ are objective fields, while $\mathbf{A}^{da}$ is not.

Remark Due to the wrong writing in the composed tensor fields, which do not correspond to their images plotted in Fig. 6.1 from Brünig (2003), Brünig and Ricci (2005), further we do not make reference to the appropriate formulae presented in the aforementioned papers. We underline now some principal ideas that follow from the papers (Brünig 2003; Brünig and Ricci 2005), and that are fruitful in describing anisotropic damage.

In the damage-coupled elasto-plastic models, these dissipative processes, namely plastic flow and damage, are treated by the constitutive models proposed in Brünig (2003), Brünig and Ricci (2005), Brünig et al. (2013), as different in their nature and effects on mechanical properties of the materials and structures. Brünig (2003) motivated the differences by the fact that "The pure plastic flow develops by dislocation motion and sliding phenomena along the some preferential crystallographic planes, whereas damage-related irreversible deformations are due to residual opening of micro defects after unloading." The free energy functions are introduced separately with respect to the fictitious undamaged configuration, $\mathscr{E}^*$, and to the current damaged configuration $\mathscr{B}^*$. The plastic strain rate tensor is determined via a non-associative plastic flow rule. The damaged surface is characterized in terms of the stress tensor with respect to stress free damaged configuration, $\mathscr{B}^*$.

The energies involved in plastic flow and damage processes are postulated to be independent. The free energy function of the damaged elasto-plastic material, see formula (6.61) by Brünig (2003), is considered to be represented in terms of three functions

$$\Phi = \Phi^{el}(\mathbf{A}^{el}, \mathbf{A}^{da}) + \Phi^{pl}(\gamma) + \Phi^{da}(\mu), \tag{6.23}$$

where Φ^{el} is dependent on the elastic strain $\mathbf{A}^{el}$ and damage strain tensor $\mathbf{A}^{da}$, the plastic and damage parts, Φ^{pl} and Φ^{da}, are dependent on the plastic and damage scalars, internal variables, γ and μ, respectively. The effective specific free energy $\bar{\Phi}$ of the fictitious undamaged configuration, see formula (6.50) by Brünig (2003), is decomposed into two parts, an effective elastic one and an effective plastic part, respectively,

$$\bar{\Phi} = \bar{\Phi}^{el}(\overline{\mathbf{A}}^{el}) + \bar{\Phi}^{pl}(\gamma). \tag{6.24}$$

Brünig (2003) states that the model "does not need strain equivalence, stress equivalence or strain energy approaches often used in continuum damage theory," but the equality of the appropriate elastic type metric transformations, $\mathbf{Q}^{el}$ and $\overline{\mathbf{Q}}^{el}$, is introduced. Thus the equivalence of the elastic strain tensors, $\mathbf{A}^{el} = \overline{\mathbf{A}}^{el}$, is accepted. Moreover, two types of the dissipation principles, one related to the plastically deformed body coupled with anisotropic damage and the other one concerning the undamaged fictitious configurations are considered. The correlation between these dissipative principles is realized by the equality $\mathbf{A}^{el} = \overline{\mathbf{A}}^{el}$.

In the **effective undamaged configuration**, $\mathscr{E}^*$, the plastic yield condition is described in terms of the effective stress tensor $\overline{\mathbf{T}}$ by

$$f^{pl}(\overline{\mathbf{T}}, c) = 0, \tag{6.25}$$

where c denotes the so-called *strength coefficient of the matrix material.* As a specific form, the linear influence of the hydrostatic stress is considered in the expression for the yield condition given by

$$f^{pl}(\bar{I}_1, \bar{J}_2, c) = \sqrt{\bar{J}_2} - c\Big(1 - \frac{a}{c}\,\bar{I}_1\Big) = 0, \tag{6.26}$$

where $\bar{I}_1 = \mathrm{tr}\overline{\mathbf{T}}, \quad \bar{J}_2 = \frac{1}{2}\mathrm{dev}\overline{\mathbf{T}} \cdot \mathrm{dev}\overline{\mathbf{T}}$. A non-associative flow rule is defined using the plastic potential function, say $g^{pl} = \sqrt{\bar{J}_2}$.

Brünig and Ricci (2005) proposed a non-local continuum theory of anisotropic damage, which incorporates a non-local yield condition

$$f^{pl}(\overline{\mathbf{T}}, c) \equiv f^{pl}(\bar{I}_1, \bar{J}_2, c) = (1 - \frac{a}{c}\bar{I}_1)^{-1}\sqrt{\bar{J}_2} - c(\gamma, \nabla^2\gamma) = 0, \tag{6.27}$$

where γ is the scalar internal variable and c denotes the strength coefficient of the material.

The **anisotropically damaged configurations** are used by Brünig (2003), Brünig et al. (2008) to describe the behaviour of the damaged materials, with reference to the damage. The damage dissipation potential is introduced as a function dependent on the stress tensor with respect to the configuration $\mathscr{B}^*$, $\widetilde{\mathbf{T}}$, and the appropriate damage criterion is given by

$$f^{da}(\widetilde{\mathbf{T}}, \tilde{\sigma}) = 0, \tag{6.28}$$

where $\tilde{\sigma}$ denotes the damage threshold. The damage strain rate is prescribed by the damage potential, denoted g^{da}, which is defined in terms of the same stress measure $\widetilde{\mathbf{T}}$, as

$$\dot{\mathbf{H}}^{da} = \mu \partial_{\widetilde{\mathbf{T}}} g^{da}. \tag{6.29}$$

Analyzing experimental results, the following damage criterion has been considered to be adequate for describing the damage behaviour in ductile materials, see Brünig (2003),

$$f^{da}(\tilde{I}_1, \tilde{J}_2, \tilde{\sigma}) = \tilde{I}_1 + \tilde{\beta}\sqrt{\tilde{J}_2} - \tilde{\sigma} = 0 \tag{6.30}$$

where $\tilde{\sigma}$ is dependent on the scalar internal variable μ, and its gradient $\nabla\mu$, which is involved in a non-local theory. The scalar function $\tilde{\beta}$ describes the influence of the deviatoric part of the stress on damage. In order to define the damage evolution equation, the damage potential has been introduced by

$$g^{da}(\widetilde{\mathbf{T}}) = \alpha\tilde{I}_1 + \beta\sqrt{\tilde{J}_2}, \tag{6.31}$$

where α and β are damage parameters. The non-associated damage rule is derived in Brünig and Ricci (2005) under the form

$$\dot{\mathbf{H}}^{ad} = \tilde{\mu}(\alpha\mathbf{I} + \beta\frac{1}{\sqrt{2\tilde{J}_2}}\,\mathrm{dev}\,\widetilde{\mathbf{T}}), \tag{6.32}$$

with the remark that the first term is related to the growth of microvoids, while the second term considers the "dependence of the evolution of the size, shape and orientation of the micro defects."

Remark The rate independent models have been adopted in the papers (Brünig 2003; Brünig and Ricci 2005), and the necessity to introduce the consistency conditions is considered, but without any references to the correlations between the damage and yield functions.

Remark The applicability of the models proposed in Brünig (2003), Brünig and Ricci (2005) have been proved by the numerical simulations performed and analyzed in the above mentioned papers.

6.3.2 Crystal Plasticity Model Coupled with Anisotropic Damage

Menzel et al. (2002) developed a framework of continuum damage based on the fictitious configuration and the equivalence principle of the free energy in the fictitious configuration and the intermediate configuration, see Fig. 6.3. The intermediate configuration (which is called the local relaxed configuration in our description (Cleja-Ţigoiu and Soós 1990)) is associated with the multiplicative decomposition of the deformation gradient into its elastic and plastic parts. The second order tensor $\mathbf{F}^d$, called the *integrity tensor*, characterizes the passage from an *undamaged* (*fictitious*) *configuration* to the intermediate configuration and it is not involved in the multiplicative decomposition of the deformation gradient. The damage model proposed by Ekh et al. (2004) appeals to the crystal plasticity model and the evolution rule for the damage is formulated with respect to the crystalline

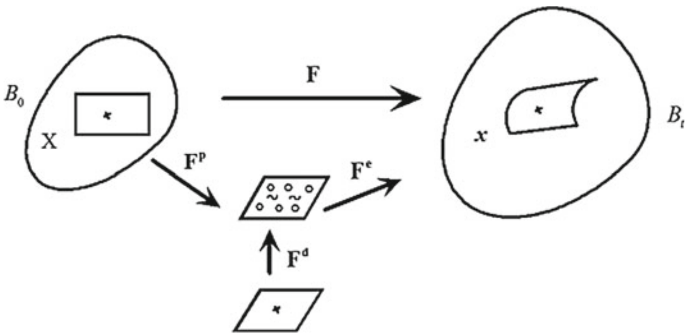

Fig. 6.3 The elastic and plastic parts, $\mathbf{F}^e$ and $\mathbf{F}^p$, of the deformation gradient. $\mathbf{F}^d$ the damage tensor defined on the undamaged and stress-free local configuration

slip systems. Not only the damage tensor $\mathbf{F}^d$ but also scalar integrity measure b_α, which are a set of scalar damage variables are involved in the expression of the *effective Schmid stress*, $\bar{\tau}_\alpha$, which is associated with the α-slip system. The α-slip system is denoted by $(\bar{\mathbf{s}}_\alpha, \bar{\mathbf{m}}^\alpha)$, when we refer to the intermediate configuration and by $(\mathbf{s}_\alpha \equiv \mathbf{F}^e \bar{\mathbf{s}}_\alpha, \mathbf{m}^\alpha \equiv (\mathbf{F}^e)^{-T} \bar{\mathbf{m}}_\alpha)$ with respect to the actual configuration.

The reference, local intermediate and actual configurations, as well as the (local) undamaged stress free configuration are represented in Fig. 6.3. We introduce the tensor field, denoted by $\widehat{\mathbf{F}}$, which realizes the passage from the undamaged to the actual (deformed) configuration in Fig. 6.4.

- The free energy with respect to the stress free and damaged configuration is dependent on the elastic strain (elastic right Cauchy- Green tensor), the damage tensor, internal scalar variables denoted by $\{k_\alpha\}$, representing the hardening variables on each slip systems,

$$\varphi = \varphi(\mathbf{C}^{\mathbf{e}}, \mathbf{F}^d, \{k_\alpha\}). \tag{6.33}$$

The free energy is additively represented by the elastic φ^e and hardening part $\varphi^{(h)}$

$$\varphi = \varphi^e(\mathbf{C}^e, \mathbf{b}^d) + \varphi^{(h)}(\{k_\alpha\}), \quad \text{where} \quad \mathbf{b}^d = \mathbf{F}^d(\mathbf{F}^d)^T \tag{6.34}$$

with the damage influence on the elastic part of free energy. The following **assumption** motivated by the principle of the *elastic* strain energy equivalence has been introduced by Menzel and Steinmann (2003), the *elastic* free energy with respect to the stress free and damaged configuration and to the effective configuration, respectively, have equal values, i.e.

$$\varphi^e(\mathbf{C}^e, \mathbf{b}^d) = \hat{\varphi}^e(\widehat{\mathbf{C}}), \quad \text{where} \quad \widehat{\mathbf{C}} = (\mathbf{F}^d)^T \mathbf{C}^e \mathbf{F}^d. \tag{6.35}$$

Under the **supplementary condition** stipulating that the $\hat{\varphi}^e$ is *isotropic* with respect to its argument, $\widehat{\mathbf{C}}$, the following representation follows

$$\varphi^e(\mathbf{C}^e, \mathbf{b}^d) = \hat{\varphi}(j_k(\widehat{\mathbf{C}})), \tag{6.36}$$

where $j_k(\widehat{\mathbf{C}}) = \mathrm{tr}((\mathbf{C}\mathbf{b}^d)^k)$, $k = 1, 2, 3$, i.e. the invariants of the mentioned tensor.

As a consequence of the thermodynamic restrictions imposed by the Clausius-Duhem inequality

$$\mathbf{T} \cdot \dot{\mathbf{F}}(\mathbf{F})^{-1} - \dot{\varphi} \geq 0, \tag{6.37}$$

written with respect to the actual configuration, the free energy density is potential for the stress tensor. The symmetric Piola-Kirchhoff stress tensor with respect to the intermediate configuration can be expressed as

$$\frac{\overline{\mathbf{T}}}{\tilde{\rho}} = 2\partial_{\mathbf{C}^e} \varphi^e(\mathbf{C}^e, \mathbf{b}^d). \tag{6.38}$$

Thus a thermodynamic stress which is power conjugated to the rate of damage in a slip system is associated with the damage tensor $\mathbf{b}^d$ via the relationship

$$\boldsymbol{\beta}^d = -2\partial_{\mathbf{b}^d} \varphi^e(\mathbf{C}^e, \mathbf{b}^d) \tag{6.39}$$

being defined by a similar procedure as that used to define the symmetric Piola-Kirchhoff stress tensor, see (6.38).

The thermodynamic stresses κ_α are associated with the hardening variables k_α and are defined as in the standard materials by

$$\boldsymbol{\kappa}_\alpha = -\partial_{k_\alpha} \varphi^h(\{k_\alpha\}). \tag{6.40}$$

The dissipation inequality (6.37) together with (6.38)–(6.40) is reduced to the inequality

$$\mathbf{C}^e \frac{\widetilde{\mathbf{T}}}{\tilde{\rho}} \cdot \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} + \boldsymbol{\beta}^d \mathbf{b}^d \cdot \dot{\mathbf{F}}^d(\mathbf{F}^d)^{-1} + \sum_\alpha \mu_\alpha \kappa_\alpha \dot{k}^\alpha \geq 0. \tag{6.41}$$

- The yield function is assumed to be dependent on damage and thermodynamic stresses, and is defined in terms of *effective resolved shear stress*, denoted by $\tilde{\tau}_\alpha$,

$$\Phi_\alpha = \tilde{\tau}_\alpha - [Y_\alpha + \kappa_\alpha], \quad \tilde{\tau}_\alpha = \frac{1}{b_\alpha}\tau_\alpha, \quad \tau_\alpha = \bar{\mathbf{s}}_\alpha \cdot \mathbf{C}^e \frac{\widetilde{\mathbf{T}}}{\tilde{\rho}} \bar{\mathbf{m}}^\alpha. \tag{6.42}$$

τ_α is called the *resolved shear stress* and b_α are scalar parameters which characterize the evolution of damage.

The flow rule is of the associative type and is formulated for $\mathbf{F}^p$ and for $\mathbf{F}^d$.

The rate of plastic part of deformation gradient is associated to the yield function (6.42) as

$$\dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} = \sum_\alpha \mu_\alpha \frac{\partial \Phi_\alpha}{\partial \overline{\mathbf{M}}}, \quad \overline{\mathbf{M}} = \mathbf{C}^e \frac{\widetilde{\mathbf{T}}}{\tilde{\rho}}. \tag{6.43}$$

In order to define the *damage rule*, the authors introduced in Ekh et al. (2004) the *integrity resolved shear*, similarly to the resolved shear stress (6.42), namely

$$\bar{\beta}_\alpha = \bar{\mathbf{s}}_\alpha \cdot \mathbf{M}^d \bar{\mathbf{m}}_\alpha, \quad \mathbf{M}^d = \boldsymbol{\beta}^d \mathbf{b}^d. \tag{6.44}$$

- There exists a damage potential $\Gamma_\alpha(\bar{\beta}_\alpha, \mathbf{b}_\alpha)$ associated with each slip system α, such that

$$\dot{\mathbf{F}}^d (\mathbf{F}^d)^{-1} = \sum_\alpha \mu_\alpha \frac{\partial \Gamma_\alpha}{\partial \mathbf{M}^d} = \sum_\alpha \mu_\alpha \frac{\partial \Gamma_\alpha}{\partial \bar{\beta}_\alpha} (\bar{\mathbf{s}}_\alpha \otimes \bar{\mathbf{m}}_\alpha) = \sum_\alpha \dot{b}_\alpha (\bar{\mathbf{s}}_\alpha \otimes \bar{\mathbf{m}}_\alpha). \tag{6.45}$$

- The stress-type hardening parameters, the so-called drag-stress corresponding to the isotropic hardening variables $\{k_\alpha\}$ are defined by the appropriate evolution equations in α-slip system

$$\dot{k}_\alpha = \mu_\alpha \frac{\partial \Phi_\alpha}{\partial \kappa_\alpha} \equiv -\mu_\alpha. \tag{6.46}$$

- The scalar parameters which characterize the evolution of damage are described by the appropriate evolution equations given by

$$\dot{b}_\alpha = \sum_\alpha \mu_\alpha \frac{\partial \Gamma_\alpha}{\partial \bar{\beta}_\alpha}. \tag{6.47}$$

Finally the dissipation inequality is expressed as follows:

$$\sum_\alpha \mu_\alpha \left[\Phi_\alpha + Y_\alpha + \bar{\beta}_\alpha \frac{\partial \Gamma_\alpha}{\partial \bar{\beta}_\alpha} \right] \geq 0. \tag{6.48}$$

In the case of the rate-dependent plasticity μ_α is defined in terms of non-negative and monotonically increasing overstress functions $\eta_\alpha(\Phi_\alpha)$

$$\mu_\alpha = \frac{1}{t_{*\alpha}} \langle \eta_\alpha(\Phi_\alpha) \rangle. \tag{6.49}$$

The functions $\eta_\alpha(\Phi_\alpha)$ have the properties $\eta_\alpha(\Phi_\alpha) = 0$ if $\Phi_\alpha \leq 0$, and $\eta_\alpha(\Phi_\alpha) > 0$ if $\Phi_\alpha > 0$, and $t_{*\alpha}$ is the relaxation time.

Here the function $\langle x \rangle = \frac{1}{2}(x + |x|)$ is defined for all x real numbers.

Remark In the numerical application, given by Ekh et al. (2004) the small deformation strain model is considered, and the scalar damage parameters have been chosen $b_\alpha = 1 - d_\alpha$, and μ_α have been introduced corresponding to rate dependent (viscoplastic) models.

Comments. We refer now to a certain physical meaning that can be assigned to the damage variable within the crystal plasticity framework. In the viscoplastic model considered by Cleja-Ţigoiu and Paşcan (2014) the evolution in time of the plastic distortion is described by multislip in an appropriate crystallographic system, with hardening laws dependent on the scalar dislocation densities, denoted by ρ_α in α-slip system. The evolution in time of the scalar dislocation densities is described by non-local (i.e. diffusion-like) evolution equations, which can be reduced to differential ones when the diffusion parameter, k, is vanishing. The problems concerning the deformation of the sheet made up from such viscoplastic crystalline material, which is generated by different slip systems that could be simultaneously activated, were numerically solved. In compression problem, for the boundary impenetrable to dislocations all eight activated slip systems were considered together with the activation condition. The large band-zones of relative minimum and maximum values of the total dislocation densities, denoted by ρ_{tot}, can be seen for k = 0.

The non-homogeneous band zones with the alternating maximum and minimum values of plastic distortion components, as well as for stress components, follow the localized zones of ρ_{tot}. Analyzing the numerical solutions for the boundary value problem we conclude that the total dislocation density accumulated during the elasto-plastic process can be interpreted as a scalar damage variable. The damage is essentially anisotropic, due to the presence of different slip systems activated, the damage variable as the total dislocation density is well defined from the physical point of view.

6.3.3 Lemaitre and Chaboche Models

We present now the models of coupled elasto-plasticity and damage constitutive equations for small deformations, with only scalar damage variables, namely the unified formulation of damage laws, following the exposure that can be found in Sect. 6.3.1 (Malcher et al. 2012; Lemaitre and Chaboche 1990; Lemaitre 1992, and so on). The models proposed by Lemaitre and Chaboche are based on the concept of *effective stress* and the hypothesis of *strain equivalence* and are largely applied and extended in the literature of the continuum damage field.

In Chap. 12 of the book (de Souza Neto et al. 2008), the authors reviewed and discussed some elasto-plastic damage models and their numerical implementation.

We listed the principal hypotheses adopted in the models.

i. The *existence of the free energy* density, φ, as function of the state variables $(\boldsymbol{\varepsilon}^e, R, \mathbf{X}, D)$, where $\boldsymbol{\varepsilon}^e$ is the elastic strain, R and D are scalar hardening and scalar damage variables, and $\mathbf{X}$ denotes the second order tensor, describing the kinematic hardening. The free energy function is described in terms of both elastic part φ^e, dependent on damage and irreversible part, φ^p, i.e.

$$\begin{aligned}\varphi &= \varphi(\boldsymbol{\varepsilon}^e, R, \mathbf{X}, D),\\ \varphi &= \varphi^e(\boldsymbol{\varepsilon}^e, D) + \varphi^p(R, \mathbf{X}).\end{aligned} \tag{6.50}$$

ii. Under the assumption that the elastic part of the free energy is given by

$$\varphi^e = \frac{1}{2}(1-D)\mathscr{E}\boldsymbol{\varepsilon}^e \cdot \boldsymbol{\varepsilon}^e, \tag{6.51}$$

the Cauchy stress is derived from the free energy, viewed as thermodynamic potential,

$$\boldsymbol{\sigma} = \rho\frac{\partial\varphi}{\partial\boldsymbol{\varepsilon}^e} = \rho(1-D)\mathscr{E}\boldsymbol{\varepsilon}^e, \tag{6.52}$$

where ρ is the mass density.

Equivalently the damage elastic law can be written in terms of *effective stress*, as

$$\begin{aligned}\boldsymbol{\sigma}_{eff} &= \frac{1}{\rho(1-D)}\boldsymbol{\sigma},\\ \boldsymbol{\sigma}_{eff} &= \mathscr{E}\boldsymbol{\varepsilon}^e.\end{aligned} \tag{6.53}$$

The thermodynamic force conjugated to the scalar damage variable, D, is defined by

$$Y = -\rho\frac{\partial\varphi}{\partial D}. \tag{6.54}$$

Using the invertibility of the elastic stiffness tensor $\mathcal{E}$, in the case of isotropic elastic behaviour, the expression of Y leads to another important feature of the damage, the influence of the triaxiality. Y is dependent on the *triaxiality by the factor* R_v,

$$\begin{aligned} Y &= \frac{1}{2\rho(1-D)^2}\boldsymbol{\sigma}\cdot\mathscr{E}^{-1}\boldsymbol{\sigma} \equiv -\frac{\sigma_{eq}R_v}{2E\rho(1-D)^2},\\ R_v &= \frac{2}{3}(1+v)+3(1-2v)\left(\frac{\sigma_H}{\sigma_{eq}}\right). \end{aligned} \tag{6.55}$$

Y corresponds to the variation of internal energy density due to damage growth at constant stress.

iii. The plastic part of the free energy function is defined by

$$\varphi^p(R,\mathbf{X}) = \varphi^I(R) + \frac{a}{2}\mathbf{X}\cdot\mathbf{X}, \tag{6.56}$$

where a is material constant. The thermodynamic forces associated with isotropic hardening and kinematic hardening

$$\kappa = \rho\frac{\partial\varphi^p}{\partial R} = \kappa(R), \quad \boldsymbol{\alpha} = \rho\frac{\partial\varphi^p}{\partial\mathbf{X}}, \tag{6.57}$$

$\boldsymbol{\alpha}$ is called the *back stress*.

The *yield function* ϕ is defined by

$$\phi(\boldsymbol{\sigma},\kappa,\boldsymbol{\alpha},D) = \tfrac{1}{1-D}\left(Dev\boldsymbol{\sigma}-\boldsymbol{\alpha}\right)_{eq} - \sigma_Y - \kappa, \tag{6.58}$$

where σ_Y is the uniaxial yield stress.

The *potential of dissipation* is given by

$$\begin{aligned} \varphi &= \phi + \tfrac{b}{2a}\tfrac{1}{1-D}\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} + F_D(Y),\\ F_D(Y) &= \tfrac{r}{(1-D)(s+1)}\left(\tfrac{Y}{r}\right)^{s+1}H(\bar{\varepsilon}^p - p_D). \end{aligned} \tag{6.59}$$

a, b are constants which characterize the so-called Armstrong-Frederick hardening law. In order to have similarity between the terms containing $\boldsymbol{\alpha}$ in the expression of the potential of dissipation we introduced here $\frac{1}{1-D}$.

The function F_D is the key point in representing the damage evolution, r and s are material constants and p_D is a material constant, which represents the *damage threshold*.

The plastic behaviour of the material is described using the potential of dissipation by

$$\begin{aligned}
&\dot{\boldsymbol{\varepsilon}}^p = \lambda \tfrac{\partial \varphi}{\partial \boldsymbol{\sigma}} \equiv \lambda \tfrac{1}{1-D}\mathbf{N},\\
&\dot{\boldsymbol{\alpha}} = \lambda \tfrac{\partial \varphi}{\partial \boldsymbol{\alpha}} \equiv \lambda \tfrac{1}{1-D}(a\mathbf{N} - b\boldsymbol{\alpha}),\\
&\dot{D} = \lambda \tfrac{\partial F_D}{\partial Y} \mathscr{H}(\bar{\boldsymbol{\varepsilon}}^p - p_D) \equiv \lambda \tfrac{1}{1-D}\left(\tfrac{Y}{r}\right)^s H(\bar{\varepsilon} - p_D),\\
&\dot{R} = \lambda,
\end{aligned} \tag{6.60}$$

where $\bar{\varepsilon}^p$ is the equivalent plastic strain, and p_D is a material constant, which represents the *damage threshold*. Here the function $\mathbf{N}$ characterizes the direction of the plastic strain rate given by

$$\mathbf{N} = \tfrac{3}{2}\tfrac{Dev\boldsymbol{\sigma} - \boldsymbol{\alpha}}{(Dev\sigma - \boldsymbol{\alpha})_{eq}}. \tag{6.61}$$

Damage Thresholds. In the pure tension case there exists a certain value of the plastic strain, ε_D^p below for which no damage caused by microcracks occurs, namely if $\varepsilon_p < \varepsilon_{pD}$ then $D = 0$. On the other hand there exists a value of damage, $D = D_c$ which marks the macro crack initiation.

Damage is always related to some irreversible strain either at the microlevel or the mesolevel, this property is considered by the presence of λ in the evolution equation for D, which is written in (6.60). The damage remains equal to zero if $\bar{\boldsymbol{\varepsilon}}^p < p_D$, and the evolution occurs if $\bar{\boldsymbol{\varepsilon}}^p \geq p_D$. p_D is a function of the applied stress and $\bar{\boldsymbol{\varepsilon}}^p$ is the equivalent plastic strain. In the evolution equation of damage, (6.60), the Heaviside function has been introduced to emphasize the role of the *damage threshold.*

As **an extension** of the Lemaitre and Chaboche models, we mention that in Lämmer and Tsakmakis (2000) proposed the elasto-plastic models coupled with damage (described in terms of scalar damage variable) for small and finite deformations. In the finite strain models the strain measure on the intermediate configuration has been defined as

$$\hat{\boldsymbol{\Gamma}} = \frac{1}{2}(\mathbf{C}^e - \mathbf{c}^p), \text{ with } \mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e \text{ and } \mathbf{c}^p = (\mathbf{F}^p)^{-T}(\mathbf{F}^p)^{-1} \text{ (in our notation).}$$

The additive decomposition of the appropriate strain measure into its elastic and plastic part is introduced by

$$\begin{aligned}
&\hat{\boldsymbol{\Gamma}} = \hat{\boldsymbol{\Gamma}}^e + \hat{\boldsymbol{\Gamma}}^p, \quad \text{where}\\
&\hat{\boldsymbol{\Gamma}}^e = \tfrac{1}{2}(\mathbf{C}^e - \mathbf{I}), \quad \hat{\boldsymbol{\Gamma}}^p = \tfrac{1}{2}(\mathbf{I} - \mathbf{c}^p)
\end{aligned} \tag{6.62}$$

The Oldroyd derivative is given in terms of $\hat{\mathbf{L}}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}$, in order to have the equality written below

$$\overset{(\Delta)}{\hat{\boldsymbol{\Gamma}}^p} = \hat{\mathbf{D}}^p, \quad \hat{\mathbf{D}}^p = \tfrac{1}{2}(\hat{\mathbf{L}}^p + (\hat{\mathbf{L}}^p)^T), \quad \text{where} \\ \overset{(\Delta)}{\hat{\boldsymbol{\Gamma}}^p} = \frac{d}{dt}\hat{\boldsymbol{\Gamma}}^p + (\hat{\mathbf{L}}^p)^T\hat{\boldsymbol{\Gamma}}^p + \hat{\boldsymbol{\Gamma}}^p\hat{\mathbf{L}}^p \tag{6.63}$$

The three models have been developed by Lämmer and Tsakmakis (2000). These models differ in the definitions of the yield function and the law describing the hardening effects. In models A only the stress tensor is replaced by the effective stress, in B the stress tensor and the back stress are replaced by the appropriate effective fields, while in C the scalar hardening variables is also replaced by its effective associate field. The models are developed within the thermomechanical framework and the influence of triaxiality is involved in the models, using a similar arguments as in the Lemaitre and Chaboche model, see the formula (6.55), in Sect. 6.3.3.

6.4 Model with Stress-Free Undamaged Configuration and Deformation-like Damage Tensor F^d

We present here some results partially published by Cleja-Ţigoiu (2011), concerning the elasto-plastic models with second order defect density tensor, under the hypothesis of large deformation. In the model proposed here we assume the existence of the stress-free, undamaged configuration. We introduced simplifications in the succession of the damaged and the undamaged configurations, that has been presented in the above mentioned papers by Brünig (2003), Brünig and Ricci (2005).

Remark We consider only one undamaged configuration, associated with the stress free (intermediate) local configuration, namely $\mathbf{R}^* = \mathbf{F}^d$, we refer to Fig. 6.2. The deformation like damage tensor $\mathbf{F}^d$ is an invertible one, and it is not apriori a symmetric tensor. The initial configuration of the body does not contain microvoids and microcracks (more precisely these initial micro defects can be neglected), which means that $\mathbf{R}^0 = \mathbf{I}$. In Figs. 6.2 and 6.4 all these elements can be seen. We remark the differences between the considered configurations plotted in Figs. 6.4 and 6.3, where the plastic part of deformation is viewed like in the deformation gradient multiplicative decomposition. Contrary, in this section the damage tensor is involved into the multiplicative decomposition, establishing a similarity with the models briefly presented in Sect. 6.3.1. We mention that $\bar{\mathbf{Q}}^{pl}$, which is considered to be symmetric and positive definite in Brünig (2003), Brünig and Ricci (2005), is replaced by an invertible tensor $\mathbf{F}^p$, which is called plastic distortion in our model.

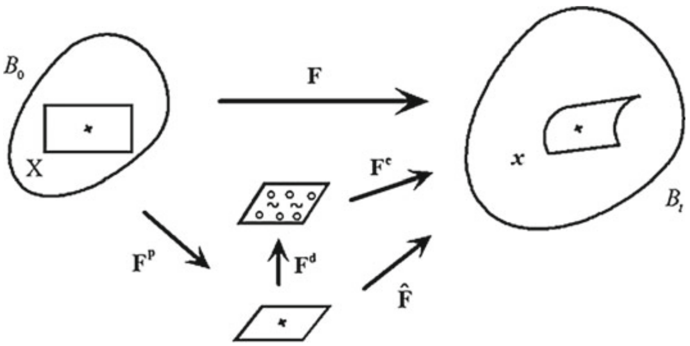

Fig. 6.4 Elastic, plastic and damage tensors as parts of the deformation gradient $\mathbf{F}$, $\mathbf{F} = \mathbf{F}^e\mathbf{F}^d, \mathbf{F}^p$, with $\mathbf{F}^d$ the transformation from the undamaged and stress free configuration to the damaged and stress free configuration

Let us consider k the reference configuration and the actual (deformed) configuration $\chi(\cdot, t)$ of the body $\mathscr{B}$, where χ represents a motion of the body.

Ax. 1. We **assume** that at any time t, for any $\mathbf{X} \in \mathscr{B}$ there exist:

- $\tilde{\mathscr{K}}$ a *stress free, damaged configuration* and
- $\mathscr{K}$ a stress free, undamaged configuration.

Starting from these assumptions, we define the local deformations: $\mathbf{F}^e$ the elastic component, which characterizes the passage from $\tilde{\mathscr{K}}$ to $\chi(\cdot, t)$, $\mathbf{F}^p$ the plastic component, which characterizes the passage from the reference configuration to $\mathscr{K}$ and $\mathbf{F}^d$ the damage deformation tensor, which characterizes the passage from the stress free, undamaged (fictitious) configuration $\mathscr{K}$ to the damaged one, $\tilde{\mathscr{K}}$.

Mass densities ρ^d, ρ^p, ρ are written in *stress free damaged and undamaged configurations, respectively, and in actual configuration* and are related by the following relationships

$$\rho\,\mathrm{det}\mathbf{F}^e = \rho^d, \quad \rho^d\mathrm{det}\mathbf{F}^d = \rho^p. \tag{6.64}$$

Ax. 2. For any motion χ, $\forall\mathbf{X}, \forall t$, the deformation gradient $\mathbf{F} := \nabla\chi(\mathbf{X}, t)$ is multiplicatively decomposed into its $\mathbf{F}^p$ plastic, $\mathbf{F}^d$ damage and $\mathbf{F}^e$-elastic parts

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^d\mathbf{F}^p, \quad \hat{\mathbf{F}} = \mathbf{F}^e\mathbf{F}^d. \tag{6.65}$$

All the tensor fields are invertible.

6.4.1 Elastic Type Response Dependent on Damage

In describing the behaviour of elasto-plastic body with damaged structure the following stress tensors are introduced with respect to the appropriate configurations

$\mathbf{T}(\mathbf{x},t)$—the Cauchy stress in the actual configuration $\chi(\cdot,t)$, where $\mathbf{x}=\chi(\mathbf{X},t)$;
$\tilde{\mathbf{T}}(\mathbf{x},t)$—the Piola-Kirchhoff stress in the stress free and damaged configuration, denoted by $\tilde{\mathscr{K}}$;
$\bar{\mathbf{T}}(\mathbf{x},t)$—the Piola-Kirchhoff stress in the stress free and undamaged configuration, denoted by $\mathscr{K}$, the so-called *effective stress*.

These stress measures are related by the following relationships

$$\begin{aligned}\tilde{\mathbf{T}} &= \det(\mathbf{F}^e)(\mathbf{F}^e)^{-1}\mathbf{T}(\mathbf{F}^e)^{-T},\\ \bar{\mathbf{T}} &= (\det\mathbf{F}^d)(\mathbf{F}^d)^{-1}\tilde{\mathbf{T}}(\mathbf{F}^d)^{-T},\\ \bar{\mathbf{T}} &= (\det\hat{\mathbf{F}})(\hat{\mathbf{F}})^{-1}\mathbf{T}(\hat{\mathbf{F}})^{-T}.\end{aligned} \tag{6.66}$$

The Mandel type stress tensors are defined with respect to the configurations $\bar{\mathcal{K}}$ and $\mathcal{K}$ by

$$\frac{1}{\rho^d}\tilde{\boldsymbol{\Sigma}} = \frac{1}{\rho\,\mathrm{det}\mathbf{F}^e}(\mathbf{F}^e)^T\mathbf{F}^e\tilde{\mathbf{T}}, \quad \frac{1}{\rho^p}\bar{\boldsymbol{\Sigma}} = \frac{1}{\rho\,\mathrm{det}\hat{\mathbf{F}}}(\hat{\mathbf{F}})^T\hat{\mathbf{F}}\bar{\mathbf{T}}. \tag{6.67}$$

We omitted the presence of t in the notations concerning the damaged configurations.

Ax. 3. The behaviour of the material is elastic with respect to stress free and damaged configuration, in terms of the Piola-Kirchhoff stress tensor, $\tilde{\mathbf{T}}$,

$$\begin{aligned}&\tilde{\mathbf{T}}(\mathbf{x},t) = \rho_d\mathbf{h}_{\tilde{\mathscr{K}}}(\mathit{\Delta}^e,\alpha),\\ &\quad\text{where}\quad \mathit{\Delta}^e(\mathbf{X},t) = \frac{1}{2}(\mathbf{C}^e-\mathbf{I}), \quad \mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e\end{aligned} \tag{6.68}$$

or equivalently in terms of the Cauchy stress tensor

$$\mathbf{T} = \rho\mathbf{F}^e\mathbf{h}_{\tilde{\mathscr{K}}}(\mathit{\Delta}^e,\alpha)(\mathbf{F}^e)^T. \tag{6.69}$$

The strain tensors which appear in the relationships defined herein are defined by

$$\hat{\mathbf{F}} = \mathbf{F}^e\mathbf{F}^d, \quad \hat{\mathbf{C}} := \hat{\mathbf{F}}^T\hat{\mathbf{F}}, \quad \mathbf{C}^d = (\mathbf{F}^d)^T\mathbf{F}^d. \tag{6.70}$$

Let us remark that as a consequence of (6.70)

$$\mathbf{F} = \hat{\mathbf{F}}\mathbf{F}^p, \quad \hat{\mathbf{C}} := (\mathbf{F}^p)^{-T}\mathbf{C}(\mathbf{F}^p)^{-1}, \quad \text{where} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{6.71}$$

Let us calculate the elastic strain measure which has been introduced in (6.68), via the relationship (6.70)

$$\begin{gathered} \varDelta^e = \frac{1}{2}(\mathbf{F}^d)^{-T}(\hat{\mathbf{C}} - (\mathbf{F}^d)^T\mathbf{F}^d)(\mathbf{F}^d)^{-1} \\ \text{or} \quad \hat{\mathbf{C}} - \mathbf{C}^d = 2(\mathbf{F}^d)^T(\varDelta^e)\mathbf{F}^d. \end{gathered} \tag{6.72}$$

Ax. 4. *The elastic constitutive equation* in stress free and undamaged configuration in terms of the effective stress is expressed in relation to the strain through

$$\bar{\mathbf{T}}(\mathbf{x}, t) = \rho^p \hat{\mathbf{h}}_{\mathscr{K}}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha}). \tag{6.73}$$

Remark The new elastic type constitutive function introduced in (6.73), $\hat{\mathbf{h}}_{\mathscr{K}}$, is related to the old one given by (6.68), $\mathbf{h}_{\tilde{\mathscr{K}}}$, through the relationship

$$\mathbf{h}_{\tilde{\mathscr{K}}}(\varDelta^e, \boldsymbol{\alpha}) := \mathbf{F}^d\hat{\mathbf{h}}_{\mathscr{K}}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha})(\mathbf{F}^d)^T, \tag{6.74}$$

together with (6.72).

In other words, the dependence of the constitutive function on the configuration $\tilde{\mathscr{K}}$ has been postulated in terms of dependence on the damage tensor $\mathbf{F}^d$, which makes the passage from the stress free and damaged configuration $\tilde{\mathscr{K}}$ to the stress free and undamaged configuration, $\mathscr{K}$.

As a consequence of the stipulated definitions and properties, *the elastic type constitutive equation characterizes Cauchy stress, with respect to the stress free and undamaged configuration*, via $(6.66)_3$, as it follows

$$\mathbf{T} = \rho\hat{\mathbf{F}}\mathbf{h}_{\mathscr{K}}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha})(\hat{\mathbf{F}})^T, \tag{6.75}$$

$\mathbf{F}^d$, being involved like an internal variable.

The stress free (*or relaxation*) *restriction is formulated*, following our development given in Cleja-Ţigoiu and Soós (1990), under the form

$$\hat{\mathbf{h}}_{\mathscr{K}}(\mathbf{S}, \boldsymbol{\alpha}) = 0, \text{for} \quad \mathbf{S} \in Sym \quad \text{if and only if} \quad \mathbf{S} = 0. \tag{6.76}$$

Here in the considered case, the relaxation restriction takes place if and only if $\hat{\mathbf{C}} := (\mathbf{F}^d)^T\mathbf{F}^d \equiv \mathbf{C}^d$, or if and only if $\mathbf{C}^e := (\mathbf{F}^e)^T\mathbf{F}^e = \mathbf{I}$.

6.4.2 Equations for Damage and Plasticity

We adopt the point of view formulated by Brünig (2003), Brünig and Ricci (2005), saying that *by combining plasticity and damage it seems to be natural that plasticity can only affect the undamaged material skeleton*.

Following the constitutive framework of finite elasto-plasticity, as it has been postulated by Cleja-Tigoiu and Sóos (1990, 1990), the evolution equation for $\mathbf{F}^p$, as well as for $\mathbf{F}^d$, which appears to be like an internal variable, will be written with respect to the stress free configuration. Here we choose the *stress free and undamaged* configuration.

(**Ev.1**). The rate of *plastic part of deformation* is described in terms of the Piola-Kirchhoff type stress measure, $\bar{\mathbf{T}}$,

$$\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} = \mu_1 \mathscr{B}_{\mathscr{K}}(\bar{\mathbf{T}}, \boldsymbol{\alpha}), \tag{6.77}$$

associated with the yield conditions

$$\bar{f}(\bar{\mathbf{T}}, \boldsymbol{\alpha}) \leq 0 \quad \mu_1 \geq 0, \quad \mu_1 \bar{f}(\bar{\mathbf{T}}, \boldsymbol{\alpha}) = 0, \quad \mu_1 \dot{\bar{f}}(\bar{\mathbf{T}}, \boldsymbol{\alpha}) = 0. \tag{6.78}$$

Let us remark that the rate of damage tensor $\mathbf{F}^d$ can be expressed by $\mathbf{L}^d := \dot{\mathbf{F}}^d(\mathbf{F}^d)^{-1}$ with respect to $\widetilde{\mathscr{K}}$ and by $\mathbf{l}^d$ with respect to the stress free and undamaged configuration $\mathscr{K}$. Here the two rates of damage tensor $\mathbf{F}^d$ are related through

$$\mathbf{l}^d = (\mathbf{F}^d)^{-1}\mathbf{L}^d\mathbf{F}^d, \quad \text{where} \quad \mathbf{l}^d := (\mathbf{F}^d)^{-1}\dot{\mathbf{F}}^d. \tag{6.79}$$

(**Ev.2**). The evolution equation for damage tensorial variable $\mathbf{F}^d$, will be written in terms of the stress measure $\bar{\mathbf{T}}$ and $\mathbf{F}^d$,

$$(\mathbf{F}^d)^{-1}\dot{\mathbf{F}}^d = \mu_2 \mathscr{D}_{\mathscr{K}}(\bar{\mathbf{T}}, \mathbf{F}^d). \tag{6.80}$$

We add two hypothesis concerning the evolution of the damage:

(**Ev.3**). The evolution equation is associated with the *damage criterion*

$$g_{\mathscr{K}}(\bar{\mathbf{T}}, \mathbf{F}^d) \geq 0. \tag{6.81}$$

For $\mathbf{F}^d = \mathbf{I}$, $g_{\mathscr{K}}(\bar{\mathbf{T}}, \mathbf{I}) \geq 0$ together with the elastic type constitutive Eq. (6.73) characterizes the activation condition for the damage.

Remark In the model the initial value of $\hat{\mathbf{C}}$ at which the damage may initiate satisfies the condition

$$g_{\mathscr{K}}(\hat{\mathbf{h}}_0(\hat{\mathbf{C}} - \mathbf{I}, 0), \mathbf{I}) = 0, \tag{6.82}$$

Remark. In our model we suppose that the damage can occur only if a certain threshold in the stress space (which also means a certain criterion in (elastic) strain space due to the possible composition with the constitutive Eq. (6.73)) is reached or is exceeded.

(Ev. 4). No evolution of the damage is produced if there is no variation of the plastic part of deformation, which will be formalized by the condition to have the same plastic multiplier, $\mu_1 = \mu_2 \equiv \mu$

6.4.3 Dissipative Nature of the Irreversible Behaviour

We introduce the **assumption**: The elasto-plastic behaviour of the material with damaged structure is restricted to satisfy the **free energy imbalance** in $\mathscr{K}$, i.e. in the *stress free and undamaged configuration*,

$$-\dot{\varphi}_{\mathscr{K}} + \mathscr{P}_{int} \geq 0, \tag{6.83}$$

where $\varphi_{\mathscr{K}}$ is the given free energy density and $\mathscr{P}_{int}$ denotes the internal power expanded during the elasto-plastic process.

The Clausius-Duhem type inequality is reformulated as *free energy imbalance* principle in $\mathscr{K}$, and is considered to be written for any virtual (isothermal) processes. The thermomechanical restrictions on the constitutive framework are derived based on the formulated principle of dissipation.

The free energy with respect to the stress free and damaged configuration is dependent on the elastic strain (elastic right Cauchy- Green tensor), $\mathbf{C}^e$, the damage tensor, $\mathbf{F}^d$, internal variables denoted by $\boldsymbol{\alpha}$,

$$\varphi_{\mathscr{K}} = \varphi(\mathbf{C}^e, \mathbf{F}^d, \boldsymbol{\alpha}, (\mathbf{F}^p)^{-1}), \tag{6.84}$$

as well as being dependent on the configuration relative to which the constitutive representation has done, namely on $(\mathbf{F}^p)^{-1}$.

In finite elasto-plasticity it is supposed that the free energy density can be additively represented by the elastic and irreversible part

$$\varphi_{\mathscr{K}} = \varphi^e(\mathbf{C}^e, \mathbf{F}^d) + \varphi^{(iv)}(\mathbf{F}^d, (\mathbf{F}^p)^{-1}, \boldsymbol{\alpha}), \tag{6.85}$$

with the damage influence on the elastic part of free energy. Motivated by the principle of the elastic free energy equivalence, see Menzel et al. (2002), the free energy with respect to the effective configuration is postulated here under the form

$$\varphi_{\mathscr{K}} = \hat{\varphi}^e(\hat{\mathbf{C}} - \mathbf{C}^d) + \varphi^{(iv)}(\mathbf{F}^d, (\mathbf{F}^p)^{-1}, \boldsymbol{\alpha}), \tag{6.86}$$

In the expression of the elastic part of the free energy function, written in (6.85), the relative strain measure $\hat{\mathbf{C}} - \mathbf{C}^d$ is introduced by (6.72).

The internal power is calculated in terms of the fields expressed with respect to the deformed configuration, namely the Cauchy stress tensor $\mathbf{T}$ and gradient of the velocity vector $\mathbf{v}$, by

$$\mathscr{P}_{int} = \frac{1}{\rho}\mathbf{T}\cdot\{\mathbf{L}\}^S, \text{ with } \quad \mathbf{L} = \nabla\mathbf{v} \equiv \dot{\mathbf{F}}(\mathbf{F})^{-1}, \text{ and } \{\mathbf{L}\}^S = \frac{1}{2}(\mathbf{L}+\mathbf{L}^T). \quad (6.87)$$

The kinematical relationships are derived from (6.65),

$$\begin{aligned} &\mathbf{L} = \nabla\mathbf{v} \equiv \mathbf{L}^e + \mathbf{F}^e\mathbf{L}^d(\mathbf{F}^e)^{-1} + \hat{\mathbf{F}}\mathbf{L}^p(\hat{\mathbf{F}})^{-1}, \quad \hat{\mathbf{F}} = \mathbf{F}^e\mathbf{F}^d \quad \text{with} \\ &\mathbf{L}^e = \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1}, \quad \mathbf{L}^d = \dot{\mathbf{F}}^d(\mathbf{F}^d)^{-1}, \quad \mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}. \end{aligned} \quad (6.88)$$

Proposition *The internal power is expressed in terms of the elastic, plastic and damage power, represented here by the scalar product of the appropriate rates with the power conjugate stress measures, respectively,*

$$\frac{1}{\rho}\mathbf{T}\cdot\mathbf{L} = \frac{1}{\rho^d}\tilde{\boldsymbol{\Sigma}}\cdot\dot{\mathbf{F}}^d(\mathbf{F}^d)^{-1} + \frac{1}{\rho^p}\bar{\boldsymbol{\Sigma}}\cdot\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} + \frac{\mathbf{T}}{\rho}\cdot\dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1}. \quad (6.89)$$

where $\tilde{\boldsymbol{\Sigma}}$ *and* $\bar{\boldsymbol{\Sigma}}$ *are the Mandel type stresses, which are introduced by* (6.67).

We prove the above relationships. We pay attention to the first and second terms written in (6.89).

When we take the scalar product written below we get

$$\frac{1}{\rho}\mathbf{T}\cdot\mathbf{F}^e\mathbf{L}^d(\mathbf{F}^e)^{-1} = \frac{1}{\rho}(\mathbf{F}^e)^T\mathbf{T}(\mathbf{F}^e)^{-T}\cdot\mathbf{L}^d = \frac{1}{\rho^d}(\mathbf{F}^e)^T\mathbf{F}^e\tilde{\mathbf{T}}\cdot\mathbf{L}^d = \frac{1}{\rho^d}\tilde{\boldsymbol{\Sigma}}\cdot\mathbf{L}^d \quad (6.90)$$

and

$$\frac{1}{\rho}\mathbf{T}\cdot\hat{\mathbf{F}}\mathbf{L}^p(\hat{\mathbf{F}})^{-1} = (\hat{\mathbf{F}})^T\frac{1}{\rho}\mathbf{T}(\hat{\mathbf{F}})^{-T}\cdot\mathbf{L}^p = (\hat{\mathbf{F}})^T\hat{\mathbf{F}}\frac{1}{\rho\det\hat{\mathbf{F}}}\bar{\mathbf{T}}\cdot\mathbf{L}^p = \frac{1}{\rho^p}\bar{\boldsymbol{\Sigma}}\cdot\mathbf{L}^p. \quad (6.91)$$

as a consequence of the defined stress measures by (6.66) and (6.67).

The rate of free energy density written with respect to the stress free and undamaged configuration can be calculated starting from (6.86)

$$\begin{aligned} \dot{\varphi}_{\mathscr{K}} = {} & \partial_{\overline{\mathbf{C}}}\varphi^{(e)}\cdot(\dot{\hat{\mathbf{C}}} - \dot{\mathbf{C}}^{\mathbf{d}}) + \partial_{\mathbf{F}^{\mathbf{d}}}\varphi^{(\mathbf{iv})}\cdot\dot{\mathbf{F}}^{\mathbf{d}} + \\ & + \partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)}\cdot(-(\mathbf{F}^p)^{-1}\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}) + \partial_{\boldsymbol{\alpha}}\varphi^{(iv)}\cdot\dot{\boldsymbol{\alpha}} \end{aligned} \quad (6.92)$$

The time derivatives of the following fields can be derived

$$\begin{aligned} &\dot{\hat{\mathbf{F}}}(\hat{\mathbf{F}})^{-1} = \mathbf{L} - \hat{\mathbf{F}}\mathbf{L}^p(\hat{\mathbf{F}})^{-1}, \\ &\dot{\hat{\mathbf{C}}} = (\dot{\hat{\mathbf{F}}})^{\mathbf{T}}\hat{\mathbf{F}} + (\hat{\mathbf{F}})^{\mathbf{T}}\dot{\hat{\mathbf{F}}} = 2\hat{\mathbf{F}}^{\mathbf{T}}\mathbf{D}\hat{\mathbf{F}} - ((\mathbf{L}^p)^{\mathbf{T}}\hat{\mathbf{C}} + \hat{\mathbf{C}}\mathbf{L}^p) \end{aligned} \quad (6.93)$$

as a consequence of the formulae (6.71) together with (6.88). From $(6.70)_3$ we get

$$\dot{\mathbf{C}}^d = 2(\mathbf{F}^d)^T\{\mathbf{L}^d\}^S\mathbf{F}^d. \tag{6.94}$$

Proposition *Within the constitutive framework formulated above the following formulation for the free energy imbalance has been derived*

$$\left\{\frac{\mathbf{T}}{\rho} - 2\,\hat{\mathbf{F}}\partial_{\bar{\mathbf{C}}}\varphi^{(e)}(\hat{\mathbf{F}})^T\right\}\cdot\{\tilde{\mathbf{L}}\}^S + \{2\,\mathbf{F}^d\partial_{\bar{\mathbf{C}}}\varphi^{(e)}(\mathbf{F}^d)^T - \partial_{\mathbf{F}^d}\varphi^{(iv)}(\mathbf{F}^d)^T\}\cdot\tilde{\mathbf{L}}^d + \\ + \{2\hat{\mathbf{C}}\partial_{\bar{\mathbf{C}}}\varphi^{(e)} + \mathbf{F}^{p-T}\partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)}\}\cdot\tilde{\mathbf{L}}^p - \partial_{\boldsymbol{\alpha}}\varphi^{(iv)}\cdot\underline{\boldsymbol{\alpha}}\geq 0, \tag{6.95}$$

$\forall \quad \tilde{\mathbf{L}}, \tilde{\mathbf{L}}^p, \quad \tilde{\mathbf{L}}^d$, *and* $\underline{\boldsymbol{\alpha}}$

In order **to prove** the above formula we replace the internal power defined by (6.87) and the derivative with respect to time of the free energy density calculated in (6.92) together with (6.93) and (6.94) in the expression of the free energy imbalance (6.83). Thus

$$\frac{\mathbf{T}}{\rho}\cdot\{\mathbf{L}\}^S - 2\partial_{\bar{\mathbf{C}}}\varphi^{(e)}\cdot(\hat{\mathbf{F}})^T\{\mathbf{L}\}^S\mathbf{F}^e + \\ + 2\partial_{\bar{\mathbf{C}}}\varphi^{(e)}\cdot\{\hat{\mathbf{C}}\mathbf{L}^p\}^S + 2\partial_{\bar{\mathbf{C}}}\varphi^{(e)}\cdot(\mathbf{F}^d)^T\{\mathbf{L}^d\}^S\mathbf{F}^d - \\ - \partial_{\mathbf{F}^d}\varphi^{(iv)}\cdot\mathbf{L}^d(\mathbf{F}^d) + \mathbf{F}^{p-T}\partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)}\cdot\mathbf{L}^p - \partial_{\boldsymbol{\alpha}}\varphi^{(iv)}\cdot\dot{\boldsymbol{\alpha}}\geq 0. \tag{6.96}$$

Here we replaced $\dot{\mathbf{F}}^p$ and $\dot{\mathbf{F}}^d$ by $\mathbf{L}^p\mathbf{F}^p$ and $\mathbf{L}^d\mathbf{F}^d$, respectively.

If the virtual rate of appropriate fields has been also introduced, the formula (6.95) follows from (6.96).

Theorem *The following thermodynamic restrictions are provided from the free energy imbalance:*

I. *The free energy density is potential for the Cauchy stress tensor*

$$\frac{\mathbf{T}}{\rho} = 2\hat{\mathbf{F}}\partial_{\bar{\mathbf{C}}}\varphi^{(e)}(\hat{\mathbf{F}})^T \quad \text{or} \quad \frac{\bar{\mathbf{T}}}{\rho^p} = 2\,\partial_{\bar{\mathbf{C}}}\hat{\varphi}^e, \tag{6.97}$$

with the notation $\bar{\mathbf{C}} \equiv \hat{\mathbf{C}} - \mathbf{C}^d$, *if the free energy density is written under the form* (6.86).

II. *The residual dissipation inequality becomes*

$$\begin{aligned}&\left\{\mathbf{C}^d\frac{\bar{\mathbf{T}}}{\rho^p}-(\mathbf{F}^d)^T\partial_{\mathbf{F}^d}\varphi^{(iv)}\right\}\cdot\tilde{\mathbf{l}}^d+\\&+\left\{\hat{\mathbf{C}}\frac{\bar{\mathbf{T}}}{\rho^p}+\mathbf{F}^{p-T}\partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)}\right\}\cdot\tilde{\mathbf{L}}^p-\partial_{\boldsymbol{\alpha}}\varphi^{(iv)}\cdot\dot{\boldsymbol{\alpha}}\geq 0.\end{aligned} \tag{6.98}$$

Here $\mathbf{l}^d=(\mathbf{F}^d)^{-1}\dot{\mathbf{F}}^d$ *is the rate of damage tensor relative to the stress free and undamaged configuration.*

Proof Let us consider that during the deformation process with arbitrarily given $\tilde{\mathbf{L}}$, no evolution of irreversible behaviour occurs, i.e. $\tilde{\mathbf{L}}^p=0, \quad \tilde{\mathbf{L}}^d=0$ and $\dot{\tilde{\boldsymbol{\alpha}}}=0$. Then the elastic type restriction (6.97) on the constitutive function follows from (6.96). When we replace (6.97) and the rate of damage $\mathbf{l}^d$ in (6.95) the dissipation inequality (6.98) follows.

We introduce the assumption that the viscoplastic type constitutive equations characterize the irreversible behaviour of the elasto-plastic material coupled with damage. The expressions of viscoplastic constitutive equations are suggested by the reduced dissipation inequality (6.98).

Ax. 5. *The evolution equations for plastic part of deformation and damage* are postulated to be given by

$$\begin{aligned}\lambda_d\,\tilde{\mathbf{l}}^d&=\mathbf{C}^d\frac{\bar{\mathbf{T}}}{\rho^p}-(\mathbf{F}^d)^T\partial_{\mathbf{F}^d}\varphi^{(iv)},\\\lambda_p\,\tilde{\mathbf{L}}^p&=\hat{\mathbf{C}}\frac{\bar{\mathbf{T}}}{\rho^p}+\mathbf{F}^{p-T}\partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)},\\\lambda_a\,\dot{\boldsymbol{\alpha}}&=-\partial_{\boldsymbol{\alpha}}\varphi^{(iv)}.\end{aligned} \tag{6.99}$$

Ax. 6. The evolution Eqs. (6.99) are compatible with the reduced dissipative inequality, namely the constitutive functions λ_d, λ_p and λ_a are given to satisfy the inequality

$$\lambda_d\mathbf{L}^d\cdot\mathbf{L}^d+\lambda_p\mathbf{L}^p\cdot\mathbf{L}^p+\lambda_a\,\dot{\boldsymbol{\alpha}}\cdot\dot{\boldsymbol{\alpha}}\geq 0. \tag{6.100}$$

6.4.4 Constitutive Models

In this model $\mathbf{F}^d$ is a second order invertible tensor, which characterizes the passage from the stress free and undamaged configuration, i.e. a fictitious configuration, which is denoted by $\mathscr{K}$ to the stress free and damaged configuration, say $\tilde{\mathscr{K}}$.

- The elastic type constitutive equation gives rise either to the Cauchy stress tensor or to the Piola-Kirchhoff stress tensor (effective stress) by

$$\mathbf{T} = \rho\hat{\mathbf{F}}\mathbf{h}_{\mathscr{K}}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha})(\hat{\mathbf{F}})^T, \Leftrightarrow \quad \bar{\mathbf{T}} = \rho^p\mathbf{h}_{\mathscr{K}}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha}) \tag{6.101}$$

The elastic type constitutive function can be expressed in terms of the free energy density, following (6.97), by

$$\mathbf{h}_{\mathscr{K}}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha}) \equiv \partial_{\hat{\mathbf{C}}}\varphi^{(e)}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha}). \tag{6.102}$$

- The evolution equation for the plastic part of deformation, written in (6.77) together with the $(6.99)_2$ is characterized by

$$\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} = \mu_1\left(\hat{\mathbf{C}}\frac{\bar{\mathbf{T}}}{\rho^p} + \mathbf{F}^{p-T}\partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)}\right), \quad \mu_1 = \frac{1}{\lambda_p} \tag{6.103}$$

- The evolution equation for the damage tensor, written in (6.80) together with $(6.99)_1$ is characterized by

$$(\mathbf{F}^d)^{-1}\dot{\mathbf{F}}^d = \mu_2\left(\mathbf{C}^d\frac{\bar{\mathbf{T}}}{\rho^p} - (\mathbf{F}^d)^T\partial_{\mathbf{F}^d}\varphi^{(iv)}\right), \quad \mu_2 = \frac{1}{\lambda_d}. \tag{6.104}$$

- The evolution equation for hardening variables $\boldsymbol{\alpha}$ is given by (6.99)

$$\dot{\boldsymbol{\alpha}} = -\mu_3\partial_{\boldsymbol{\alpha}}\varphi^{(iv)}, \quad \mu_3 = \frac{1}{\lambda_a} \tag{6.105}$$

Finally, we consider the **model with isotropic damage**, as a particular case of the previously presented model. The damage tensor is reduced to

$$\mathbf{F}^d = \kappa\mathbf{R}^d, \quad \mathbf{R}^d \in Orth, \quad \kappa = 1 - d. \tag{6.106}$$

The multiplicative decomposition of the deformation gradient is reduced to $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$. The scalar damage variable is viewed as a scalar internal variable, that variation in time being described by the specific evolution equation. The tensors $\mathbf{C}^d$ and $\hat{\mathbf{F}}$, defined by $(6.70)_3$ and $(6.65)_2$, result

$$\mathbf{C}^d = \kappa^2\mathbf{I} \quad , \quad \hat{\mathbf{F}} = \kappa\mathbf{F}^e(\mathbf{R}^d). \tag{6.107}$$

We use the polar decomposition of the elastic part of the deformation gradient, $\mathbf{F}^e = \mathbf{V}^e\,\mathbf{R}^e$, where $\mathbf{R}^e \in Orth$.

Proposition *There exists a stress free and undamaged configuration associated to* $\tilde{\mathcal{K}}$, *which can be characterized in term of* $\hat{\mathbf{F}} = \kappa \mathbf{V}^e$, *where* $\mathbf{V}^e \in Sym$, *representing the left hand side, elastic stretch Cauchy-Green tensor.*

- The elastic type constitutive equation in terms either of the Cauchy stress tensor or of the Piola-Kirchoff stress tensor (effective stress) is expressed by the free energy density, following (6.97)

$$\begin{aligned} &\mathbf{T} = \rho \hat{\mathbf{F}} \partial_{\mathbf{C}^e} \varphi^{(e)}(\kappa^2(\mathbf{C}^e - \mathbf{I}), \boldsymbol{\alpha})(\hat{\mathbf{F}})^T, \quad \text{equivalently} \\ &\bar{\mathbf{T}} = \rho^p \partial_{\mathbf{C}^e} \varphi^{(e)}(\kappa^2(\mathbf{C}^e - \mathbf{I}), \boldsymbol{\alpha}) \end{aligned} \tag{6.108}$$

- The evolution equation for the plastic part of deformation, written in (6.77) is characterized by

$$\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} = \mu_1 \partial_{\mathbf{C}^e} \varphi^{(e)} \left(\mathbf{C}^e \frac{\bar{\mathbf{T}}}{\rho^p} + \mathbf{F}^{p-T} \partial_{(\mathbf{F}^p)^{-1}} \varphi^{(iv)} \right). \tag{6.109}$$

- The evolution equation for the damage tensor, written in (6.80) is characterized by

$$\dot{\kappa} = \mu_2 \kappa^2 \left(\kappa\, tr \left(\frac{\bar{\mathbf{T}}}{\rho^p} \right) - tr(\partial_{\mathbf{F}^d} \varphi^{(iv)} \right). \tag{6.110}$$

- The evolution equation for hardening variables is given by

$$\dot{\boldsymbol{\alpha}} = -\mu_3 \partial_{\boldsymbol{\alpha}} \varphi^{(iv)}. \tag{6.111}$$

6.5 Models with Non-metric Property

We present here some ideas that can be found in the paper by Cleja-Ţigoiu and Ţigoiu (2013), that require further development. The behaviour of elasto-plastic materials with damaged microstructure is described in terms of specific differential geometry elements which characterize the internal mechanical state, following Kröner (1992), de Wit (1981). In the proposed elasto-plastic models the defects of lattice structure, like dislocations and disclinations, can be involved through the Cartan torsion of the so-called plastic connection, see Cleja-Ţigoiu (2007, 2010, 2014).

The point defects, microvoids and microcracks in the damaged zone are modeled in terms of the *non-metric tensor which belongs to the plastic connection*, apart from Cleja-Ţigoiu and Ţigoiu (2011) where the gradient of the elastic strain measures the damage. The non-metric property of the plastic connection is described in terms of a symmetric second order tensor, $\mathbf{h}^d$, which is potential for the non-metric (extra-matter) tensor **Q.** The symmetric second order tensor $\mathbf{h}^d$, is called here the *damage tensor.*

6.5.1 Constitutive Hypotheses

We present here the basic ideas developed within the finite elasto-plasticity with second order deformations provided by Cleja-Ţigoiu (2007, 2010), Cleja-Ţigoiu and Ţigoiu (2013).

Let us consider the function χ which defines the motion of the body, $\mathscr{B}$. The deformation gradient associated with the motion is defined by $\mathbf{F} = \nabla\chi$ and the expression of the second order gradient of the motion χ, pulled back to the reference configuration is given by $(\mathbf{F})^{-1}\nabla\mathbf{F}$, and is denoted by $\boldsymbol{\Gamma}$, namely $\boldsymbol{\Gamma} = (\mathbf{F})^{-1}\nabla\,\mathbf{F}$. Here $\nabla\mathbf{F}$ and $\boldsymbol{\Gamma}$ are represented as third order fields in a certain coordinate system.

Hypotheses The plastic behaviour is characterized in terms of the pair $(\mathbf{F}^p, \overset{(p)}{\boldsymbol{\Gamma}})$, whose components *are incompatible.*

The second order tensor field $\mathbf{F}^p$, which is called plastic distortion, or the plastic part of the deformation gradient, and $\overset{(p)}{\boldsymbol{\Gamma}}$ is characterized by a third order field in a curvilinear coordinate system and represents the Christoffel-Riemann coefficient of a connection, called here plastic connection.

Assumptions The plastic distortion does not satisfy the first integrability condition, i.e. *the plastic distorstion is incompatible.* The plastic connection $\overset{(p)}{\boldsymbol{\Gamma}}$ does not satisfy the second integrability condition, i.e. i.e. the plastic connection is incompatible.

We recall the classical results concerning the theorems (in the smooth case).

(First Integrability Theorem) Let $\mathscr{U}$ be a simply connected domain in R^3 and $\mathbf{F} : \mathscr{U} \rightarrow Lin$. The following three assertions are equivalent

$$\begin{array}{ll} a. & \mathbf{F} \quad \text{is a gradient, which means the existence of a vector field} \quad \mathbf{Z} \\ & \qquad \text{such that}\, \mathbf{F} = \nabla\mathbf{Z}; \\ b. & (\nabla\mathbf{F}(\mathbf{x})(\mathbf{u}))\mathbf{v} - (\nabla\mathbf{F}(\mathbf{x})\mathbf{v})\mathbf{u} = 0, \quad \forall\,\mathbf{x} \in \mathscr{U},\, \forall\mathbf{u},\mathbf{v} \in \mathscr{V} \\ c. & (\operatorname{curl}\mathbf{F}(\mathbf{x}))(\mathbf{u}\times\mathbf{v}) = \mathbf{0}, \quad \forall \quad \mathbf{x} \in \mathscr{U}, \forall\mathbf{u},\mathbf{v}. \end{array} \tag{6.112}$$

Definition A connection $\boldsymbol{\Gamma}$ is integrable if there exists a tensor field $\mathbf{F}$ such that the partial differential equation (written in a local representation) is satisfied

$$\boldsymbol{\Gamma} = \mathbf{F}^{-1}\nabla\,\mathbf{F}, \quad \forall \quad \mathbf{x} \in \mathscr{U}, \tag{6.113}$$

Definition The fourth order Riemann-curvature tensor $\mathscr{R}$, attached to $\boldsymbol{\Gamma}$, is defined by

$$\mathscr{R}(\mathbf{u},\mathbf{v}) = ((\nabla\boldsymbol{\Gamma})\mathbf{u})\mathbf{v} - ((\nabla\boldsymbol{\Gamma})\mathbf{v})\mathbf{u} + (\boldsymbol{\Gamma}\mathbf{u})(\boldsymbol{\Gamma}\mathbf{v}) - (\boldsymbol{\Gamma}\mathbf{v})(\boldsymbol{\Gamma}\mathbf{u}). \qquad (6.114)$$

The equation written in definition (6.113) is known as the **second integrability condition**. The following theorem states a relationship between the two definitions.

Theorem *The second integrability condition takes place if the Riemann-curvature tensor $\mathscr{R}$ belonging to $\boldsymbol{\Gamma}$ is vanishing, which means the Frobenius condition holds.*

Three type of configurations are used in the models, namely the reference and the deformed configurations at time t, $\chi(\cdot,t)$, as well as the **so-called damaged (anholonomic) configuration**, generically denoted by $\mathscr{K}$, and which is viewed as the pair $(\mathbf{F}^p, \overset{(p)}{\boldsymbol{\Gamma}})$.

The model is described within the *second order plasticity*, based on the multiplicative decomposition of the deformation gradient $\mathbf{F} = \nabla\chi$ (where the function χ describes the motion of the body) into its elastic and plastic components $\mathbf{F}^e, \mathbf{F}^p$, called *distortions*

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \qquad (6.115)$$

as well as on the rule of the $\boldsymbol{\Gamma} = (\mathbf{F})^{-1}\nabla\mathbf{F}$ motion connection decomposition into its elastic and plastic counterparts, which are defined as it follows

$$\boldsymbol{\Gamma} = (\mathbf{F}^p)^{-1}\overset{(e)}{\boldsymbol{\Gamma}}_{\mathscr{K}}[\mathbf{F}^p,\mathbf{F}^p] + \overset{(p)}{\boldsymbol{\Gamma}}. \qquad (6.116)$$

For any third order tensor $\boldsymbol{\Gamma}$, and for any second order tensors, $\mathbf{F}_1, \mathbf{F}_2$, the third order tensor $\boldsymbol{\Gamma}[\mathbf{F}_1,\mathbf{F}_2]$ is defined by

$$(\boldsymbol{\Gamma}[\mathbf{F}_1,\mathbf{F}_2]\mathbf{u})\mathbf{v} = (\boldsymbol{\Gamma}(\mathbf{F}_1\mathbf{u}))\,\mathbf{F}_2\mathbf{v}, \qquad (6.117)$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$.

In the formula (6.116) giving rise to the *decomposition of the second order deformation*, the elastic connection with respect to the damaged configuration has been introduced, as a direct consequences of the appropriate relationships between the three order fields, when we pass from the reference configuration to the damaged configuration $\mathscr{K}$ by the plastic distortion $\mathbf{F}^p$,

$$\begin{aligned} &\overset{(e)}{\boldsymbol{\Gamma}}_{\mathscr{K}} = \mathbf{F}^p\,\overset{(e)}{\boldsymbol{\Gamma}}[(\mathbf{F}^p)^{-1},(\mathbf{F}^p)^{-1}], \text{where} \\ &\overset{(e)}{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} - \overset{(p)}{\boldsymbol{\Gamma}}. \end{aligned} \qquad (6.118)$$

Remark We shortly justify the rationale put down at the composition rule involved in the above formula, following Cross (1973), Wang (1973). The formula (6.116) is a relationships between the second order derivatives for a composition rule written for two applications.

We **assume** that the plastic connection $\overset{(p)}{\boldsymbol{\Gamma}}$ has *non-metric property with respect to the metric tensor* $\mathbf{C}^p = (\mathbf{F}^p)^T\mathbf{F}^p$, apart from the hypothesis adopted by Cleja-Ţigoiu (2007, 2010).

Consequently *there exists* a third order tensor $\mathbf{Q}$, such that $\mathbf{Qu} \in Sym$ and

$$-(\nabla\mathbf{C}^p)\mathbf{u} + (\mathbf{C}^p\overset{(p)}{\boldsymbol{\Gamma}}\mathbf{u})^T + \mathbf{C}^p(\overset{(p)}{\boldsymbol{\Gamma}}\mathbf{u}) = \mathbf{Qu}, \tag{6.119}$$

hold for all vectors $\mathbf{u}$.

The following representation for the plastic connection can be derived, see Cleja-Ţigoiu and Ţigoiu (2013),

$$\overset{(p)}{\boldsymbol{\Gamma}} = \overset{(p)}{\mathscr{A}} + (\mathbf{C}^p)^{-1}(\boldsymbol{\Lambda}\times\mathbf{I} + \tfrac{1}{2}\mathbf{Q}), \quad \overset{(p)}{\mathscr{A}} = (\mathbf{F}^p)^{-1}\nabla\mathbf{F}^p, \tag{6.120}$$

Here the third order tensor field $\boldsymbol{\Lambda}$ is a measure of *disclination*, being defined in such a way to have the equality $((\boldsymbol{\Lambda}\times\mathbf{I})\mathbf{u})\mathbf{v} = (\boldsymbol{\Lambda}\mathbf{u})\times\mathbf{v}$.

$\overset{(p)}{\mathscr{A}}$ defines the so-called *Bilby type plastic connection*.

Following Kröner (1992) we assume the existence of a second order tensor, $\mathbf{h}^d$, which is a potential for the non-metric (extra-matter) tensor $\mathbf{Q}$, namely $\mathbf{Q} = \nabla\mathbf{h}^d$. As a direct property of the above introduced definitions, the plastic metric tensor $\mathbf{C}^p$ is corrected by $\mathbf{h}^d$, to restore the metric property of the plastic connection, i.e.

$$-\nabla(\mathbf{C}^p + \mathbf{h}^d)\mathbf{u} + (\mathbf{C}^p\,\overset{(p)}{\boldsymbol{\Gamma}}\,\mathbf{u})^T + \mathbf{C}^p(\overset{(p)}{\boldsymbol{\Gamma}}\,\mathbf{u}) = 0, \quad \forall \quad \mathbf{u} \in \mathscr{V}. \tag{6.121}$$

Remark The plastic distortion $\mathbf{F}^p$ and the tensorial damage variable $\mathbf{h}^d$ are *incompatible*, which means that they are not the derivative of certain vector fields, see de Wit (1981). The second order torsion tensor, $\mathscr{N}^p$, related to the third order Cartan torsion $\mathbf{S}^p$,

$$\begin{aligned}&(\mathbf{S}^p\tilde{\mathbf{u}})\tilde{\mathbf{v}} = \mathscr{N}^p(\tilde{\mathbf{u}}\times\tilde{\mathbf{v}}), \quad \forall \quad \tilde{\mathbf{u}}, \tilde{\mathbf{v}};\\ &\mathscr{N}^p = (\mathbf{F}^p)^{-1}\mathrm{curl}\mathbf{F}^p + (\mathbf{C}^p)^{-1}(\mathrm{curl}\mathbf{h}^d + (\mathrm{tr}\,\boldsymbol{\Lambda})\mathbf{I} - (\boldsymbol{\Lambda})^T)\end{aligned} \tag{6.122}$$

The following *defect fields* have been introduced

$$\begin{array}{lll} \boldsymbol{\alpha} & := (\mathbf{F}^p)^{-1}\mathrm{curl}(\mathbf{F}^p) & \text{dislocation density} \\ \boldsymbol{\alpha}^d & = (\mathbf{C}^p)^{-1}\mathrm{curl}\mathbf{h}^d & \text{damage defect density} \\ \boldsymbol{\alpha}^{\Lambda} & := \mathrm{tr}\,\boldsymbol{\Lambda}\mathbf{I} - (\boldsymbol{\Lambda})^T & \text{disclination density,} \end{array} \tag{6.123}$$

which characterize the incompatibilities existing in the materials, following for instance Kröner (1992) and de Wit (1981).

Remark The *damage defect density* $\boldsymbol{\alpha}^d = (\mathbf{C}^p)^{-1}\mathrm{curl}\mathbf{h}^d$ is not symmetric and contains the plastic metric tensor and the damage tensor $\mathbf{h}^d$. Thus there is a measure of damage explicitly dependent on the plastic distortion.

For the shake of simplicity we do not consider here the disclination among the lattice defect, apart from Cleja-Ţigoiu (2010), Cleja-Ţigoiu (2014), where the influence of the lattice defect modeled by $\boldsymbol{\Lambda}$ has been emphasized.

In the model the damage variable $\mathbf{h}^d$ was defined on the reference configuration, and we introduce the tensorial damage variable $\mathbf{H}$ which is pushed forward to the damaged configuration of $\mathbf{h}^d$. We also define the appropriate gradients of the aforementioned fields, as follows

$$\begin{aligned} &\mathbf{H} = (\mathbf{F}^p)^{-T}\mathbf{h}^d(\mathbf{F}^p)^{-1}, \\ &\nabla_{\mathcal{K}}\mathbf{H} = (\nabla\mathbf{H})(\mathbf{F}^p)^{-1}. \end{aligned} \tag{6.124}$$

6.5.2 Dissipation Postulate

The models are *dissipative* and the constitutive equations for the macro forces as well as the appropriate evolution laws which involves the micro forces are derived to be compatible with the free energy imbalance principle, formulated in Gurtin et al. (2010) and adapted here to involve the internal expanded power during the irreversible (plastic) process coupled with damage, as in Cleja-Ţigoiu (2007), Cleja-Ţigoiu (2010).

Let us denote by $\varphi_{\mathcal{K}}$ the expression of the free energy function with respect to the damaged configuration and by $(\mathcal{P}_{int})_{\mathcal{K}}$ the density of the internal expended power.

Ax. 1. The elasto-plastic behavior of the material is restricted to satisfy in *damaged configuration* the imbalanced free energy condition

$$-\dot{\varphi}_{\mathcal{K}} + (\mathcal{P}_{int})_{\mathcal{K}} \geq 0 \quad \text{for any virtual (isothermic) processes.} \tag{6.125}$$

The model is strongly dependent on the postulated expression for the free energy density, as well as of the postulated form for the virtual internal power in the damaged configuration.

Ax. 2. The *free energy density* function in the damaged configuration is postulated to be dependent on the second order elastic deformation, in terms of $\mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e$, and to be dependent on the damaged configuration, through the part of second order plastic deformation $((\mathbf{F}^p)^{-1}, \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}})$

$$\varphi = \varphi_{\mathscr{K}}(\mathbf{C}^e, (\mathbf{F}^p)^{-1}, \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}, \mathbf{H}, \nabla_{\mathscr{K}}\mathbf{H}), \tag{6.126}$$

As the tensorial damage variable and its gradient have been introduced among the independent variables in the expression of the free energy density, the power conjugated variables with their rates should be introduced in the expression postulated for the virtual internal power.

Within the constitutive framework developed by Cleja-Ţigoiu (2010) and adapted to the problem which we discuss here and which refers to damage, the free energy imbalance principle can be reformulated taking into account the expression of the *virtual internal power* in $\mathscr{K}$.

$$\begin{aligned}(\mathrm{virt}\mathscr{P}_{int})_{\mathscr{K}} = &\frac{1}{2}\frac{\boldsymbol{\pi}}{\tilde{\rho}} \cdot \delta\mathbf{C}^e + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}} \cdot ((\mathbf{F}^e)^{-1}(\nabla_{\chi}\mathbf{L}[\mathbf{F}^e, \mathbf{F}^e]) - \nabla_{\mathscr{K}}\mathbf{L}^p)\\ &+ \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p \cdot \tilde{\mathbf{L}}^p + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p \cdot \nabla_{\mathscr{K}}\tilde{\mathbf{L}}^p + \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^d \cdot \delta\mathbf{H} + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^d \cdot \nabla_{\mathscr{K}}\delta\mathbf{H}.\end{aligned} \tag{6.127}$$

$(\boldsymbol{\pi}, \boldsymbol{\mu}_{\mathscr{K}})$ are the macroforces in $\mathscr{K}$, namely Piola-Kirchhoff stress tensor and stress momentum pulled back to the configuration with torsion, $\mathscr{K}$, see Cleja-Ţigoiu (2007). The macroforces in $\mathscr{K}$ are power conjugated to $\dot{\mathbf{C}}^e$ and to the gradient of the velocity gradient in the actual configuration, $\nabla_{\chi}\mathbf{L}$, pulled back to the configuration $\mathscr{K}$.

$\boldsymbol{\mu}^p, \boldsymbol{\mu}^d$ are *micro stress momenta* (third order tensors) which are conjugated to the gradients of the rate of plastic distortion $\mathbf{L}^p$ and of the rate of $\mathbf{H}$, respectively. The internal virtual power (6.127) is written for any virtual rate of plastic distortion $\tilde{\mathbf{L}}^p$, and any virtual variation of damage tensor, $\delta\mathbf{H}$, and for their gradients $\nabla_{\mathscr{K}}\tilde{\mathbf{L}}^p$ and $\nabla_{\mathscr{K}}\delta\mathbf{H}$, respectively. Based on the following kinematic relationships

$$\dot{\mathbf{C}}^e = 2\,(\mathbf{F}^e)^T\{\mathbf{L}\}^s\mathbf{F}^e - 2\,\{\mathbf{C}^e\mathbf{L}^p\}^s, \quad \text{as} \quad \mathbf{C}^e = (\mathbf{F}^p)^{-T}\mathbf{C}(\mathbf{F}^p)^{-1}, \tag{6.128}$$

which are written in terms of $\mathbf{L}$ and $\mathbf{L}^p$, the virtual variation $\delta\,\mathbf{C}^e$ is derived. To do that, $\mathbf{L}$ and $\mathbf{L}^p$ are replaced by their virtual expression, $\tilde{\mathbf{L}}$ and $\tilde{\mathbf{L}}^p$.

Assuming that during the elasto-plastic process no evolution of plastic distortion and damage is produced when an elastic process is considered, the following statement can be proved.

Proposition *A fist consequence follows from the principle of the free energy imbalance, namely the free energy is potential for the macro force, namely the Cauchy stress is expressed by*

$$\mathbf{T}(\mathbf{x},t) = 2\rho\mathbf{F}^e\partial_{\mathbf{F}^e}\varphi(\mathbf{F}^e)^T. \tag{6.129}$$

Balance Equations for Micro Forces. We mention that the micro forces are power conjugated to the rate of kinematic variables and of their gradients, in the plastic and damage mechanism. They satisfy their own *micro balance equations* in the damaged configuration, $\mathscr{K}$, which are postulated (see Cleja-Ţigoiu (2007), Cleja-Ţigoiu (2010)) to be given by

$$\boldsymbol{\Upsilon}^p = div_{\mathscr{K}}\,(\boldsymbol{\mu}^p - \boldsymbol{\mu}_{\mathscr{K}}) + \tilde{\rho}\,\mathbf{B}^p_m, \quad \boldsymbol{\Upsilon}^d = div_{\mathscr{K}}\,\boldsymbol{\mu}^d + \tilde{\rho}\,\mathbf{B}^d_m \tag{6.130}$$

with the appropriate boundary conditions on $\partial\mathscr{K}(\mathscr{P},t)$. When we pass to the reference configuration the micro balance (6.130) can be written in the following form

$$\begin{aligned} J^p\,\boldsymbol{\Upsilon}^p_{\mathscr{K}} &= div\,(J^p\,(\boldsymbol{\mu}^p - \boldsymbol{\mu}_{\mathscr{K}})(\mathbf{F}^p)^{-T}) + \rho_0\,\mathbf{B}^p_m, \\ J^p\boldsymbol{\Upsilon}^d &= div\,(J^p\,\boldsymbol{\mu}^d(\mathbf{F}^p)^{-T})) + \rho_0\,\mathbf{B}^d_m, \quad J^p = |\det\mathbf{F}^p|. \end{aligned} \tag{6.131}$$

Here $\mathbf{B}^p_m$ and $\mathbf{B}^d_m$ are the mass density forces associated with the plastic and damage mechanism.

The balance equation *for macro force, i.e. the Cauchy stress tensor*, is reduced to the classical balance equation $\operatorname{div}\mathbf{T} + \rho\mathbf{b} = 0$.

6.5.3 Constitutive and Evolution Equations with Respect to the Reference Configuration

In order to describe the behaviour of the elasto-plastic material with damaged structure, modeled by the tensorial variable $\mathbf{h}^d$ and its gradient $\nabla\mathbf{h}^d$, the form of the free energy density could be postulated directly with respect to the reference configuration,

The free energy in $\mathscr{K}$ can be expressed in a *pulled back* to the reference configuration form

$$\varphi = \bar{\varphi}(\mathbf{C}, \mathbf{F}^p, \overset{(\mathrm{p})}{\mathscr{A}}, \mathbf{h}^d, \nabla\mathbf{h}^d) \tag{6.132}$$

taking into account the appropriate relationships between the fields when we pass from the damaged configuration $\mathscr{K}$ to the reference configuration by $(\mathbf{F}^p)^{-1}$.

The free energy density is assumed to be dependent on $\mathbf{h}^d$, and its gradient, namely the non-metric (extra-matter) tensor $\mathbf{Q}$.

We develop the kinematic of the process which leads to the evolution equations which prescribe $\dot{\mathbf{h}}^d$, and

$$\mathbf{l}^p = \left(\frac{d}{dt}(\mathbf{F}^p)^{-1}\right)\mathbf{F}^p = -(\mathbf{F}^p)^{-1}\mathbf{L}^p\mathbf{F}^p \tag{6.133}$$

when the dissipation inequality is also revised.

First we proceed to directly reformulate the imbalanced form of the free energy principle with respect to the reference configuration, taking into account the following expression of the internal power, namely

$$\begin{aligned}(\mathscr{P}_{int})_{\mathscr{K}} = &\frac{1}{2}\frac{\boldsymbol{\pi}}{\tilde{\rho}}\cdot\dot{\mathbf{C}}^e + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}\cdot((\mathbf{F}^e)^{-1}(\nabla_\chi\mathbf{L}[\mathbf{F}^e,\mathbf{F}^e]) - \nabla_{\mathscr{K}}\mathbf{L}^p) + \\ &+\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p\cdot\mathbf{L}^p + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\cdot\nabla_{\mathscr{K}}\mathbf{L}^p + \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^d\cdot\frac{D}{Dt}\mathbf{H} + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^d\cdot\nabla_{\mathscr{K}}\frac{D}{Dt}\mathbf{H}.\end{aligned} \tag{6.134}$$

Second, we compute the time derivative of the free energy density function taking into account the derivative formulae for the appropriate fields.

- $\delta\mathbf{H}$ which is involved in the expression of the virtual internal power, (6.127), is defined to be the rate of $\mathbf{h}^d$ pushed away to the configuration $\mathscr{K}$, namely

$$\tfrac{D}{Dt}(\mathbf{H}) = (\mathbf{F}^p)^{-T}\,\dot{\mathbf{h}}^d(\mathbf{F}^p)^{-1}, \tag{6.135}$$

- The gradient with respect to $\mathscr{K}$ applied to the previous rate, i.e. $\nabla_{\mathscr{K}}\delta\mathbf{H}$, leads to

$$\nabla_{\mathscr{K}}\big(\tfrac{D}{Dt}(\mathbf{H})\big) = (\mathbf{F}^p)^{-T}\,\{\nabla(\dot{\mathbf{h}}^d) - (\dot{\mathbf{h}}^d\overset{(\mathrm{p})}{\mathscr{A}}) - (\dot{\mathbf{h}}^d\overset{(\mathrm{p})}{\mathscr{A}})^T\}[(\mathbf{F}^p)^{-1},(\mathbf{F}^p)^{-1}], \tag{6.136}$$

  where the transpose of the third order tensor field $\mathscr{N}$ is given by $\mathscr{N}^T\mathbf{u} = (\mathscr{N}\mathbf{u})^T$, for any $\mathbf{u}$.
- The rate of the appropriate fields which enter the expression of the internal power associated to the processes is calculated in terms of $\mathbf{l}^p$ as given by the formulae

$$
\begin{aligned}
&\dot{\mathbf{C}}^{e} = (\mathbf{F}^{p})^{-T}\{\dot{\mathbf{C}} + \mathbf{C}\mathbf{l}^{p} + (\mathbf{l}^{p})^{T}\mathbf{C}\}(\mathbf{F}^{p})^{-1}\\
&\nabla_{\mathscr{K}}\mathbf{L}^{p} = -\mathbf{F}^{p}\{\nabla\mathbf{l}^{p} + \overset{(\mathrm{p})}{\mathscr{A}}[\mathbf{I},\mathbf{l}^{p}] - \mathbf{l}^{p}\,\overset{(\mathrm{p})}{\mathscr{A}}\}[(\mathbf{F}^{p})^{-1},(\mathbf{F}^{p})^{-1}],\\
&\left(\frac{d}{dt}(\overset{(\mathrm{p})}{\mathscr{A}})\right) = -\nabla\,\mathbf{l}^{p} + \mathbf{l}^{p}\,\overset{(\mathrm{p})}{\mathscr{A}} - \overset{(\mathrm{p})}{\mathscr{A}}[\mathbf{I},\mathbf{l}^{p}].
\end{aligned}
\tag{6.137}
$$

Second, we introduce the new appropriate measures for the forces which enter the reformulated expression for the principle of the imbalanced free energy when we passed from the damaged configuration to the reference one, namely

$$
\begin{aligned}
&\frac{\mathbf{m}_0^{d}}{\rho_0} = \frac{1}{\tilde{\rho}}(\mathbf{F}^{p})^{-1}\boldsymbol{\mu}^{d}[(\mathbf{F}^{p})^{-T},(\mathbf{F}^{p})^{-T}], \quad \frac{\mathbf{m}_0^{p}}{\rho_0} = \frac{1}{\tilde{\rho}}(\mathbf{F}^{p})^{T}\boldsymbol{\mu}^{p}[(\mathbf{F}^{p})^{-T},(\mathbf{F}^{p})^{-T}]\\
&\frac{\boldsymbol{\Upsilon}_0^{p}}{\rho_0} = (\mathbf{F}^{p})^{-1}\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^{d}(\mathbf{F}^{p})^{-T}, \quad \frac{\boldsymbol{\Upsilon}_0^{p}}{\rho_0} = (\mathbf{F}^{p})^{T}\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^{p}(\mathbf{F}^{p})^{-T},\\
&\boldsymbol{\Sigma}_0 = \mathbf{C}\boldsymbol{\pi}_0, \quad \frac{\boldsymbol{\pi}_0}{\rho_0} = (\mathbf{F}^{p})^{-1}\frac{1}{\tilde{\rho}}\boldsymbol{\pi}(\mathbf{F}^{p})^{-T},
\end{aligned}
\tag{6.138}
$$

where $\boldsymbol{\Sigma}_0$ and $\boldsymbol{\pi}_0$ denote Mandel stress measure and Piola-Kirchhoff stress tensor, respectively, with respect to the reference configuration.

As a consequence of the formulated dissipated postulate the expression for the macro forces is derived.

- We introduce micro stress momenta associated with the damage and dislocations by the non-dissipative (energetic) constitutive relations, the so-called *energetic micro forces*

$$
\tfrac{1}{\rho_0}\mathbf{m}_0^{d} = \partial_{\nabla\mathbf{h}^{d}}\,\varphi, \quad \tfrac{1}{\rho_0}\mathbf{m}_0^{p} = \partial_{\overset{(\mathrm{p})}{\mathscr{A}}}\varphi.
\tag{6.139}
$$

- The rates of the plastic distorsion and of the quasi-plastic strain, $\mathbf{F}^{p}$ and $\mathbf{h}^{d}$, and the constitutive functions have to be *compatible with the dissipation inequality*, and they are postulated as follows

$$
\begin{aligned}
&\frac{1}{\rho_0}(\boldsymbol{\Sigma}_0 - \boldsymbol{\Upsilon}_0^{p}) + (\mathbf{F}^{p})^{T}\partial_{\mathbf{F}^{p}}\varphi = \xi_1\,\mathbf{l}^{p},\\
&\frac{1}{\rho_0}(\boldsymbol{\Upsilon}_0^{d} - \partial_{\mathbf{h}^{d}}\varphi) - 2\{\partial_{\nabla\mathbf{h}^{d}}\varphi\odot\overset{(\mathrm{p})}{\mathscr{A}}\}^{S} = \xi_2\,\dot{\mathbf{h}}^{d},
\end{aligned}
\tag{6.140}
$$

where the operator $\odot$ associates to the third order tensors $\mathscr{A}$ and $\mathscr{B}$ the second order tensor, denoted $\mathscr{A}\odot\mathscr{B}$ and defined by

$$(\mathscr{A} \odot \mathscr{B}) \cdot \mathbf{L} = \mathscr{A}[\mathbf{I}, \mathbf{L}] \cdot \mathscr{B} = \mathscr{A}_{isk} L_{sn} \mathscr{B}_{ink}, \tag{6.141}$$

for all second order tensor $\mathbf{L}$.

The micro forces are eliminated from the evolution equations for plastic deformation and damage tensor, $\mathbf{h}^d$ via their own balance equations. The non-local evolution equations can be either associated for instance with an appropriate yield function in terms of effective stress and damage back stress tensor, or described as the viscoplastic ones.

The rate independent elasto-plastic model with anisotropic damage can be derived as it follows:

- The scalar constitutive functions ξ_1, ξ_2 are defined in such a way to be compatible with the dissipation inequality

$$\xi_1 \mathbf{l}^p \cdot \mathbf{l}^p + \xi_2 \dot{\mathbf{h}}^d \cdot \dot{\mathbf{h}}^d \geq 0. \tag{6.142}$$

Let us introduce *internal variables like stress*

1. the *back stress*, denoted by $\boldsymbol{\Sigma}_{back}$, which is introduced in order to describe the hardening of the material,

$$\boldsymbol{\Sigma}_{back} := \boldsymbol{\Upsilon}_0^p - \rho_0 (\mathbf{F}^p)^T \partial_{\mathbf{F}^p} \varphi \tag{6.143}$$

2. the *damage stress* variable

$$\mathbf{b} := \boldsymbol{\Upsilon}_0^d - \rho_0 \partial_{\mathbf{h}^d} \varphi - 2\rho_0 \{\partial_{\nabla \mathbf{h}^d} \varphi \odot \overset{(\mathrm{p})}{\mathscr{A}}\}^S. \tag{6.144}$$

When the micro forces are eliminated via the micro balance Eq. (6.131) together with the energetic representation for micro stress momenta (6.139) the following expressions are provided for the back stress and damage stress

$$\begin{aligned} \boldsymbol{\Sigma}_{back} &= \frac{1}{J^p} \mathrm{div}(\rho_0 \mathbf{F}^p \partial_{\overset{(\mathrm{p})}{\mathscr{A}}} \varphi [\mathbf{I}, (\mathbf{F}^p)^T])) - \rho_0 (\mathbf{F}^p)^T \partial_{\mathbf{F}^p} \varphi, \\ \mathbf{b} &:= -\rho_0 \partial_{\mathbf{h}} \varphi + \frac{1}{J^p} \mathrm{div}(\rho_0 \mathbf{F}^p \partial_{\nabla \mathbf{h}} \varphi)[\mathbf{I}, (\mathbf{F}^p)^T]. \end{aligned} \tag{6.145}$$

As a consequence of the micro balance equations together with the energetic definitions for the micro stress momenta the reduced dissipation inequality referring to the *irreversible behavior coupled with damage* can be derived under the form

$$(\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_{back}) \cdot \mathbf{l}^p + \mathbf{b} \cdot \dot{\mathbf{h}}^d \geq 0. \tag{6.146}$$

We introduce the rate-independent model, following the idea proposed by Grudmundson (2004), which is in the sprit of classical plasticity.

In terms of *effective fields* we introduce a *convex function* with respect to its arguments, say for instance like in classical plasticity, namely

$$\hat{\Psi} := \sqrt{|\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_{back}|^2 + |\mathrm{b}|^2}, \tag{6.147}$$

and a yield function

$$\hat{\mathscr{F}} := \hat{\Psi} - R(\zeta), \quad \text{with} \quad R(\zeta) > 0, \quad R'(\zeta) > 0, \quad R(0) = k > 0, \tag{6.148}$$

with R a scalar constitutive function dependent on the scalar hardening variable of the deformation type, say ζ, which has to be introduced by a differential type equation.

The relationships (6.140) will be viewed as evolution equations to describe the rates of plastic distortion, through $\mathbf{l}^p$, and for the scalar dislocation density $\frac{d}{dt}\rho^d$, namely

$$\begin{aligned} \mathbf{l}^p &:= \lambda \frac{\Sigma_0 - \Sigma_{back}}{R(\zeta)} \mathscr{H}(\hat{\mathscr{F}}), \\ \frac{d}{dt}\mathbf{h} &:= \lambda \frac{\mathrm{b}}{R(\zeta)} \mathscr{H}(\hat{\mathscr{F}}), \\ \dot{\zeta} &:= \lambda \mathscr{H}(\hat{\mathscr{F}}), \end{aligned} \tag{6.149}$$

λ has the role of plastic multiplier and satisfies Kuhn-Tucker and consistency condition.

6.5.4 Model Proposed by Aslan et al. (2011)

We make references to the class of anisotropic elasto-viscoplastic micromorphic media which was provided by Aslan et al. (2011), within the constitutive framework of finite deformation, based on the multiplicative decomposition. The degrees of freedom of the proposed model are the displacement vector $\mathbf{u}$ and the *micro deformation variable* $\hat{\chi}^p$, which is generally a *non-symmetric second order tensor*.

The *relative deformation tensor*, denoted by $\mathbf{e}^p$, is defined by

$$\mathbf{e}^p = (\mathbf{F}^p)^{-1}\hat{\chi}^p - \mathbf{I}, \tag{6.150}$$

and measures the departure of the micro deformation from the plastic distortion as

$$\mathbf{e}^p = 0, \quad \text{if and only if} \quad \hat{\chi}^p = \mathbf{F}^p. \tag{6.151}$$

We remark that $\mathbf{e}^p$ is a second order tensor defined on the reference configuration and $\hat{\chi}^p$ is defined on the reference configuration with the value in the intermediate configuration.

The *gradient of the set of degrees of freedom*

$$(\mathbf{F}, \mathbf{K}), \quad \mathbf{F} = \mathbf{I} + \nabla \mathbf{u}, \quad \mathbf{K} = \mathrm{Curl}\hat{\chi}^p. \tag{6.152}$$

The state variables are introduced by the set the following fields

$$\left(\mathbf{E}^e \equiv \frac{1}{2}((\mathbf{F}^e)^T \mathbf{F}^e - \mathbf{I}), \mathbf{e}^p, \mathbf{K}, \boldsymbol{\alpha}\right). \tag{6.153}$$

The free energy density function φ is assumed to be dependent on the state variables

$$\varphi = \varphi(\mathbf{E}^e, \mathbf{e}^p, \mathbf{K}, \boldsymbol{\alpha}). \tag{6.154}$$

The internal power density is introduced by the following expression

$$p^{(i)} = \boldsymbol{\sigma} \cdot \mathbf{L} + \mathrm{s} \cdot \dot{\hat{\chi}}^p + \mathbf{M} \cdot \mathrm{Curl}\, \dot{\hat{\chi}}^p, \tag{6.155}$$

where $(\boldsymbol{\sigma}, \mathbf{s}, \mathbf{M})$ denote stress-like fields which are power conjugated to the velocity gradient, rate of microdeformation, $\dot{\hat{\chi}}^p$ and its curl. The consequences that can be derived from the dissipation principle defined by $p^{(i)} - \rho\dot{\varphi} \geq 0$ were investigated by Aslan et al. (2011).

The balance equation for the Cauchy stress, $\boldsymbol{\sigma}$, div $\sigma = 0$, as well as the appropriate balance equation for micro stresses, $\mathit{Curl}\, \mathbf{M} + \mathbf{s} = 0$, have been introduced to be satisfied by the pair of forces $(\mathbf{s}, \mathbf{M})$, which are power conjugated to $\dot{\hat{\chi}}$ and $\mathit{Curl}\, \dot{\hat{\chi}}$, respectively.

Conclusions. Certain *similarities* can be established between the model (Aslan et al. 2011) and the models with non-metric property.

a. $(\mathbf{F}, \mathbf{K})$ and $(\mathbf{F}^p, \overset{(\mathrm{p})}{\boldsymbol{\Gamma}})$ describe the second order effect;
b. $\mathbf{e}^p$ and $\mathbf{h}^d$ are anisotropic second order damage tensors;
c. The appropriate balance equations have been formulated for *micro forces*;
d. The specific dissipation inequalities characterize the dissipative nature of plastic deformation and damage.

The physical motivation and mathematical description of the damage are completely different. $\mathbf{e}^p$ *measures the discrepancy* between the micro deformation and the plastic distortions, while $\mathbf{h}^d$ *characterizes the lost of the metricity* of the geometrical structure, as a consequence of the existence of microvoids and microcracks. No evolution equation has been introduced in Aslan et al. (2011) for the micro deformation or for the relative deformation tensor. In the model with non-metric property the evolution equation for the tensorial damage variable is coupled with the evolution equation for plastic distortion.

6.6 Conclusion

In the models proposed in this chapter, the key point is the presence of the tensorial variables which describe the anisotropic damage connected to the large plastic deformation. The physical nature and the mathematical description of the damage variables are related to the presence of the microcracks and microvoids, developed at the microstructural level. We pay attention to the configurations on which the tensor fields are defined to avoid the confusions which appear when this mention is missing.

The model presented in Sect. 6.4 is based on the fictitious undamaged and stress free configuration and on the existence of the second order tensor, $\mathbf{F}^d$ which realizes the passage from this configuration to the damaged and stress free configuration. Only one type of undamaged configuration has been necessary to develop the proposed model, like in Ekh et al. (2004). Contrary to Menzel et al. (Ekh et al. 2004) the damage anisotropic tensor $\mathbf{F}^d$ is involved in the gradient deformation multiplicative decomposition, and the plastic distortion $\mathbf{F}^p$ describes the passage from the reference configuration to the undamaged and stress free configuration. The damage tensor field $\mathbf{F}^d$ is not symmetric, as it is considered by Brünig (2003), Murakami (1988). The composed tensor $\mathbf{F}^d\mathbf{F}^p$ characterizes damage and plastic coupled effect, when the passage from the reference configuration to the damaged stress free configuration occurs.

In the model proposed in Sect. 6.5 we defined the damage tensor to be the second order symmetric tensor field $\mathbf{h}^d$, which characterizes a measure of *non-metric property* for the geometry of elasto-plastic material with damaged structure. The rationale of our choice is motivated by the fact that the local metric property of the material with crystalline structure is lost in the material with damaged microstructure. The symmetric tensor field $\mathbf{h}^d$, which is not a metric tensor, restores the metricity of the so-called plastic connection, with respect to the reference configuration. In the two models the evolution equations for the plastic distortion and tensorial damage variable are derived to be compatible with the appropriate dissipation principle, the classical one for the first model and the free energy imbalance principle for the second one.

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