A Differential Game of a Duopoly with Network Externalities

Mario Alberto García-Meza and José Daniel López-Barrientos

Abstract In this work, we develop a differential game of a duopoly where two firms compete for market share in an industry with Network Externalities. Here the evolution of the market share is modeled in such a way that the effects of advertising efforts that both firms make are a function of the share itself. This means that the efficacy of marketing efforts are diminished with low market share and enhanced when it is higher. We show that Network Externalities can influence the decision a firm makes about marketing expenditures. Particularly, when a firm is large enough, the creation of a monopoly is easier when this market structure is present. For this, we obtain the optimal strategies for the firms and test them on a simulation, where we compare the market with and without this kind of externalities. We find that the value of the market share in proportion with the cost of obtaining it by advertising efforts is the key to know the long term equilibrium market share.

Keywords Differential games • Advertising competition • Network externalities

1 Introduction

There are many imperfections that can affect the structure of a market. Among these, *Network Externalities* is a class that cannot be ignored, since many important markets, such as telecommunications and software, are under its influence. Network Externalities (NE) emerge when the user's utility from consuming a certain product is a function of the number of people that use the same brand [5, 13].

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When some industry presents this kind of particularities, we must expect that the strategies from incumbents and entrants differ from those in a standard market, in order to be adapted to that particular environment. Network Externalities (also known as *Network Effects* or *demand-side economies of scale*) are found in a large variety of markets from telecommunications to computer software.

The telecommunications market is just one example of *direct* externalities. In [11], the authors study the existence of NE in the wireless telecommunication market in Europe and North America and the way their governments dealt with them. Whereas Europe used mandated standards, North America opted to let the market determine its own structure, yielding a worse result in terms of market concentration.

Computer software, on the other hand, is an example of an *indirect* type of NE. There is plenty of evidence for this kind of externality in a market [10, 12, 14]. We can find another example of indirect NE in DVD's market (formerly VCR's market) in [3, 16, 17], where standards are shown to affect the market. In video games [2, 18, 20], for instance, it is clearer the fact that two-sided markets can create Network Externalities as well.

The existence of NE in the market, in direct or indirect forms, means that the strategies the firms use must be adapted to be more akin with the reality they face. In particular, we might find that these effects act as barriers of entry that give an advantage to the incumbent and make a more expensive entry for new firms in the market. We represent this situation by means of a variation of the well-known Lanchester model (see, for instance, [4, 21]).

The way the structures make such an effect in the market is reviewed in Sect. 2. Here, we lay the ground needed to state that NE act as barriers of entry, protecting incumbents from new firms and making the permanence in the market more expensive for the latter.

In Sect. 3 we dwell on the details of the model (1) and (2) below, a model of a duopoly under NE, whose market dynamics presents barriers of entry created by NE. These barriers of entry are modeled by making the effectiveness of the marketing efforts dependent on the size of the market share. In this section, we find the analytical solution for the game by means of Pontryagin's Maximum Principle (PMP) to get optimal marketing expenditures and the evolution of the market share under these controls. It is important to acknowledge the fact that, although PMP yields just open-loop controls, we have chosen this technique over, for instance, the Principle of Dynamic Programming because the interpretation and simulation of the formulas is straight-forward.

In Sect. 4 we make simulations of the market, in order to get some insights about the effects of NE in the interaction between entrant and incumbent. Here, we present graphically the motion of the market, plotting it for every initial state.

The conclusions of this work can be found in Sect. 5, along with some ideas for future research.

2 Network Externalities

In microeconomics, a good is believed to provide some utility to whomever buys it and consumes it. The form of the utility can vary, but is generally a function of the consumption of the good itself alone. That may not be the case in reality; we may value the goods we buy in terms of subjective values dependent on factors external to the item at hand. An interesting example is when the utility function of the agents takes into account the number of people that buy the same product. This phenomenon is called *Network Externalities* (NE), since the purchase of an individual yields an externality for other buyers by increasing the utility of the purchase and, therefore, distorting its behaviour.

NE can be found in several markets such as telecommunications, software, video games and banking. There are *direct* NE, when the utility of a good is in direct proportion to the number of users. The telecommunications market is a canonical example; the utility of owning a phone line depends directly on how many consumers are there to communicate with. There are also *indirect* NE, when the utility is derived as a byproduct of the size of the network. Two-sided markets such as software or video games typically present this kind of NE: a large size of the network of users of some hardware's brand gives an incentive to developers to make more applications for that particular brand of console or hardware equipment, which in turn yields a higher utility for the users, enabling them to choose between more and better software applications.

A third (and often neglected) kind of NE is the *post-service* Network Externality. It is a more subtle kind, where the utility from the post-service is the one affected by the size of the network. One example is the purchase of a car: if there are enough cars of the same brand available in a given city, the user might find it easier (and cheaper) to hire the services of a mechanic and the parts of the car would be available more easily.

An important factor in the formation of this kind of structure in the market is the compatibility between products. For example, in the telecommunications market, if all competitors share infrastructure, the cost of communicating between brands is lower than it would be if every firm had its own. The latter case would yield a higher utility for the users of the firm where the network is larger, for it would mean a lower price to communicate with their peers, whereas the former would yield the same result regardless of the chosen brand, as if all firms were a single network.

2.1 How Network Externalities Affect the Market

When NE are present, it is not only important to know how many units a certain brand sells, but also the size of the network associated with it. A key characteristic that determines the size of the network is *compatibility*. An example of this can be found in the smartphones market, where a great variety of mobile devices can share a single operative system or carrier.

Some models have been developed to explain this phenomenon. In [13], the authors explore a model with homogeneous goods where the firms have an expectation of the number of consumers of their brand. In this model, the consumers interact with the firms by making the decision of their purchase, according to the utility the brand gives them. Nonetheless, the consumer's utility comes from two sources: the standalone utility from the consumption of the product, which in perfect competition determines the so-called *hedonic price* (see [9, 15]), and the utility derived from the size of the network.

It is worth mentioning that different firms or brands can share the same network. If the network is big enough, the consumers might be more inclined to choose the brands in a shared network, so long as the sum of the standalone utility and the one provided by the network is bigger than other options she can afford.

In the model provided by [13], the consumers create *expectations* of the size of the network and decide their purchase with imperfect information. The benefits of a firm are directly related to the size of the network by the quantities they sell, and by the added value resultant of being in a large network. If we assume that there are only two networks in the market, then the firms in the largest network will have an advantage with respect to the ones in a smaller one.

The value of a firm to the customer can be reflected in the decision process. If we had a dynamic model instead of a static one, then the agents would need to decide in every moment of time the firm they want to buy from. This would give the firms an incentive to make an effort to increase the size of the network.

We might think that there are two ways to increase the size of a network. One is to make the product compatible with the largest network, but one can make an argument that this is an expensive action, an extreme measure that cannot be done in a flexible, continuous adjustment, unless it is done by creating some kind of coalitions which might be illegal in some countries. The other way to get a larger network is by advertising efforts. Section 3 analyses a market where direct Network Externalities are present. The firms will expend in advertising to get a larger market share. Thus, here we will assume that either the size of the network is equal to the quantity sold, or that all firms in a network act in a tacit collusion.

3 The Model

In this section we develop a dynamic model of a duopoly where the firms compete in a market where NE are present. We do this as a duopoly analysis, but it can also be thought of as a situation where two different standards which concentrate any number of firms.

In this model, our main concern is the way NE affect the market share of the firms over time. Consider a duopoly based on the well-known Lanchester model (see [4, 21]) where each firm wants to maximize

$$J_i = \int_0^T \mathrm{e}^{-rt} \left\{ p_i x_i(t) - \frac{c_i}{2} u_i(t)^2 \right\} \mathrm{d}t + \mathrm{e}^{-rT} S_i(x_i(T)), \tag{1}$$

subject to the state dynamics:

$$\dot{x}_i(t) = u_i(t) x_j(t)^{\alpha_{NE}} - u_j(t) x_i(t)^{\beta_{NE}}, \quad x_i(0) \in [0, 1]; \tag{2}$$

For $i = 1, 2$ and $i \neq j$. Here, $x_i(t)$ is the market share of brand i at time $t \in [0, T]$, a normalized quantity that will be sold by the i-th firm at the (exogenously given) price p_i. The qualifier "normalized" means that $x_i \in [0, 1]$ and the value of the market share of firm j is given by $x_j = 1 - x_i$, i.e. we care about the captive market only: we assume that there are no agents in the market that can opt for not buying. The value function J_i that the firms want to maximize is a function of the discounted sales minus expenditures. Here, r is the discount rate, which for simplicity can be thought of as the risk-free interest rate of the economy.

We assume that there is no fixed cost nor cost of production other than the advertising expenditures. Note that (1) yields that the marginal cost of obtaining a new unit of market share is equal to c_i, which is a constant for each firm, and u_i is the amount of units of advertising purchased. The terminal surplus given at terminal time $t = T$ is given by the function

$$S(x_i(T)) = s_i x_i(T), \tag{3}$$

where s_i is a given constant, times the market share at the end of the horizon.

3.1 The Network Externalities Modeled on a Lanchester Dynamics

The exponents in restriction (2) stand for the so-called *saturation effects*, i.e., the expenditure made by the i-th firm on advertising is pretended to affect the market share of the j-th firm. In the popular approach adopted by, for example, [4, 6, 8], saturation effects are fixed, and therefore, the dynamics takes the form

$$\dot{x}_i(t) = u_i(t) x_j(t)^{\alpha} - u_j(t) x_i(t)^{\beta}, \quad x_i(0) \in [0, 1], \tag{4}$$

where α, β are given constants such that $\alpha + \beta = 1$. If $\alpha = \beta = 1/2$ (see [1, 7]), then both firms' marginal expenditure in advertising would *steal* the same market share from the competition for all moments of time. To see this, note that we can approximate $x_i(t)^{1/2}$ by $x_i(t) + x_i(t)x_j(t)$ for $i = 1, 2$. (This approximation, valid for small values, has also been used in [19, 21] for the analysis of this kind of models.) This way, (4) turns into

$$\dot{x}(t) = u_i(t)x_j(t) - u_j(t)x_i(t) + [u_i(t) - u_j(t)]x_i(t)x_j(t).$$

Here, the first two terms show us the direct effect of the expenditure. The third term shows the interaction of the firms, affected by the difference in expenditures of the i-th and j-th firms. It is easy to see that if the expenditure of the i-th firm is larger than that of j-th firm, the market will favour the i-th firm. Note that this effect depends on the expenditure each firm makes in advertising, because $\alpha = \beta$. Otherwise, if $\alpha > \beta$, the marginal expenditure of the i-th firm would have a greater effect and the j-th firm would be in disadvantage in every moment of time. It is worth to mention that, since we consider only two players, the sum of their markets should always equal one. This means that

$$\dot{x_1}(t) + \dot{x_2}(t) = 0. \tag{5}$$

But, on the aforementioned case, this holds only when $\alpha = \beta = 1/2$. The dynamics used in our model is such that the saturation effects vary with the market share itself. Moreover, our model satisfies (5) as well. We achieve this by considering $\alpha_{NE}(x_i)$ and $\beta_{NE}(x_j)$ as variables that depend on the market share. Thus, we let

$$\alpha_{NE}(x_i) := 1 - x_i = x_j,$$
$$\beta_{NE}(x_j) := 1 - x_j = x_i.$$

Therefore, the restriction in (2) becomes

$$\dot{x}_i(t) = u_i(t)x_j(t)^{x_j(t)} - u_j(t)x_i(t)^{x_i(t)}, \quad x(0) = x_0 \in]0, 1[. \tag{6}$$

This is the main contribution of this work. To the knowledge of the authors, a formulation of Lanchester dynamics whose saturation effects vary in function of the market share in the way stated above has not been analysed before. This assumption complicates the formulation of the dynamics of the market share and, hence, makes more difficult the characterization of the optimal strategies of the differential game at hand. To overcome this problem we will consider open-loop strategies and use the PMP to solve the differential game.

With NE, the effect of advertising gets *stronger* when the firm's market share gets closer to 1 and, conversely, it gets weaker as it approaches 0. This behaviour is our way to model NE in a dynamic game: the efforts made by the i-th firm get an additional effect when it has a large market share. The reason behind this behaviour of the market is supposed to be exogenous, but we consider it to be a side effect of compatibility. This matches a situation where direct Network Externalities are present, since the agents will decide their next purchase by looking at the size of the network and their preferences will be distributed in the same proportion as the market share. In this sense, the saturation effects are a measure of the compatibility of the network.

Figure 1 shows a simulation of the dynamics of x_i with equal marketing efforts for all initial states $x_i(0) \in [0, 1]$. This simulation shows the motion of the market share for a system like (4), without NE. Similarly, Fig. 2 represents a system dynamics

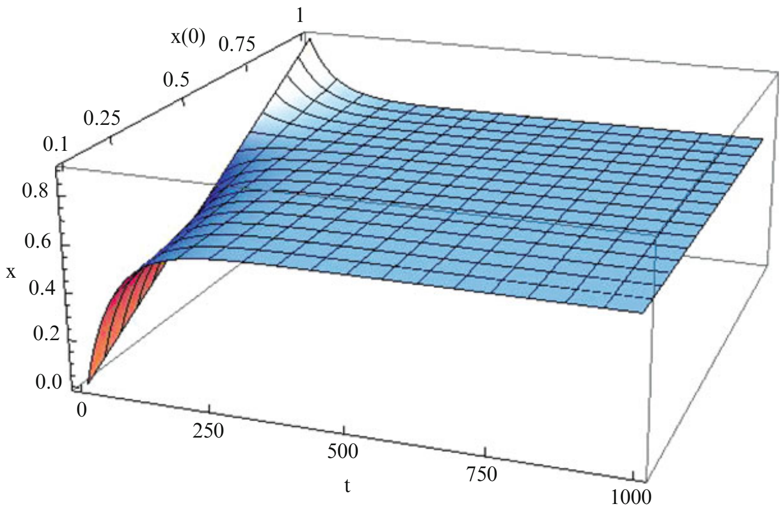

Fig. 1 The market share dynamics for all initial states in $[0, 1]$ when $u_1 = u_2$ without Network Externalities for $T = 1000$

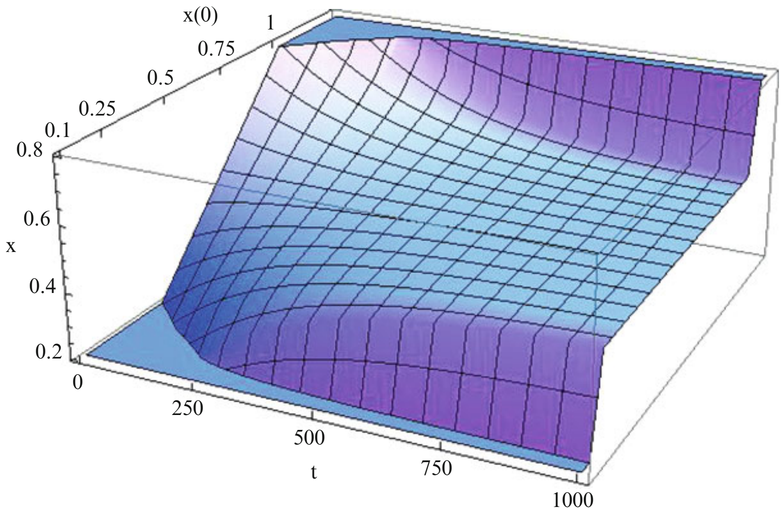

Fig. 2 The market share dynamics for all initial states in $[0, 1]$ when $u_1 = u_2$ with Network Externalities for $T = 1000$

as in (2), with NE present. Note that, for both cases, the only initial state where $\dot{x}_i$ is equal to zero, that is, the market share will remain constant, is in the middle, where $x_i = x_j = 1/2$. In the other initial states, we can see that the market share will move towards this middle point of market shares. An interpretation is that, since consumers are exposed to the same amount of advertising from both firms, they tend in the long run to divide the market. Therefore, any initial state below the halfway point means an increase in the market share for the next moment in time until the market share is divided in half and, analogously, if the initial market share is larger, the value of $\dot{x}$ will be negative until the share is equally divided for both firms.

The difference between a market with Network Externalities [i.e. a market whose dynamics is as in (2)], and one without them [see (4)], is the *speed* of convergence. As we can appreciate from Fig. 2, whereas the market without externalities will always end up in the middle point no matter what the initial value is, if we have that $x_0 = 0$ and $x_0 = 1$, the dynamics for a system with NE will equal zero. This can be interpreted by saying that a monopoly would endure in the economy indefinitely.

Figure 3 presents a system where firm 1 is expending more on advertising than firm 2, i.e., $u_1 > u_2$, in particular, $u_1 = 1.25, u_1 = 1$. Here we can see that the effects of the marketing expenditures favour the firms with larger initial share. If firm 1 kept this proportion of expenditure in advertising for a sufficiently large amount of time, the final market share would be larger if the market was the one with NE. Moreover, if the initial state was larger than the second initial state, firm 1 would be able to become a monopoly. This cannot be done in a market without NE.

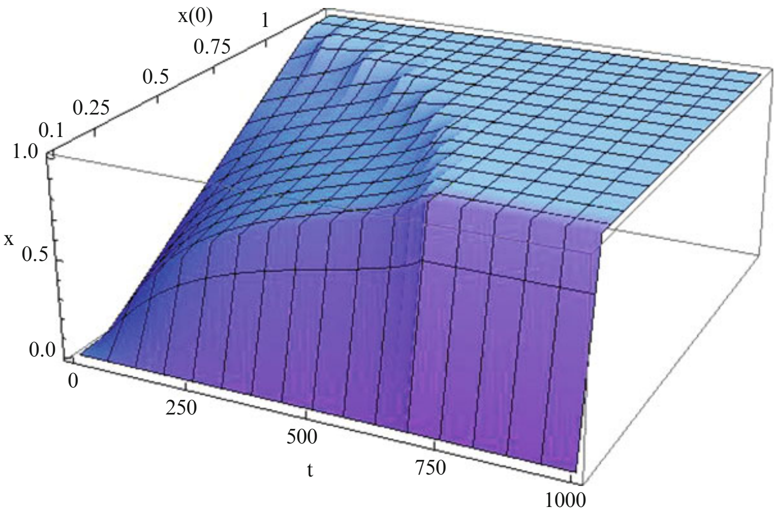

Fig. 3 The market share dynamics for different initial states when $u_1 > u_2$ with and without Network Externalities for $T = 10$

3.2 Optimal Marketing Expenditures

To solve the problems (1)–(6), we will use PMP (see [22]), and find first order conditions to determine the optimal expenditures in advertising for both firms. The Hamiltonian for the i-th firm is given by

$$H^i(x_i, u_i, \lambda_i, t) = p_i x_i - \frac{c_i}{2} u_i(t)^2 + \lambda_i[u_i(t)x_j(t)^{x_j(t)} - u_j(t)x_i(t)^{x_i(t)}].$$

By stating the problem in these terms, we turn a problem with three variables (x, u, t) and one restriction into a four variables problem with no restrictions. A firm might be tempted to make the advertising expenditures equal to zero, but if the other firm's expenditure is positive, then that would mean sacrificing market share in the long run. The co-state variable can be seen as the *shadow price* of the market share. That is, the value assigned to the market share, when it is difficult to know or calculate. Its dynamics, according to PMP, is given by

$$\begin{aligned}
\dot{\lambda}_i(t) &= r_i\lambda_i(t) - \frac{\partial}{\partial x_i} H^{i*}_{u^*}(x(t), \lambda^i(t), t) \\
&= r_i\lambda_i(t) - \frac{\partial}{\partial x_i}\left[p_i x_i - \frac{c_i}{2}u_i^*(t)^2 + \lambda_i[u_i^*(t)x_j(t)^{x_j(t)} - u_j^*(t)x_i(t)^{x_i(t)}]\right] \\
&= r_i\lambda_i(t) - p_i + \lambda_i\left(u_i^* x_j^{x_j}(1 + \ln(x_j)) + u_j^* x_i^{x_i}(1 + \ln(x_i))\right), \qquad (7)
\end{aligned}$$

where u^* stands for the control that optimizes the problem; with transversality condition

$$\lambda_i(T) = S_i'(x_i(T)). \qquad (8)$$

That is, we take the first derivative of the Hamiltonian with respect to the state variable, assuming that $u_i(t)$ is already optimal to know the motion of the market share's value. To find optimal expenditures in advertising, we equal the derivatives of the Hamiltonian (with respect to the controllers) to zero. That is,

$$\frac{\partial}{\partial u_i} H_i(x_i, u_i, \lambda_i, t) = -c_i u_i(t) + \lambda_i x_j(t)^{x_j(t)} = 0.$$

By Theorem 3.2, in [22], this equality gives us as a result that the optimal simultaneous expenditures on advertising of firm $i = 1, 2$ are given by

$$u_i^* = \frac{\lambda_i(t)}{c_i} x_j(t)^{x_j(t)}. \qquad (9)$$

That is, the optimal expenditure will vary in proportion with the market share of the competitor (i.e. the other firm), but its effects will also be affected by it. In addition, the shadow price will also affect the optimal control of the system. Note that (9) displays a division by the marginal cost of advertising c_i. This means that at every moment of time, the firms will evaluate the value of the market share with respect to the amount they have to expend to get an additional unit of it.

If we plug the optimal controllers referred to in (9) into the shadow price motion (7), we have that

$$\dot{\lambda}_i(t) = r_i\lambda_i(t) - p_i + \frac{\lambda_i(t)^2}{c_i} x_j(t)^{2x_j(t)}(1 + \ln(x_j(t))) + \frac{\lambda_i(t)\lambda_j(t)}{c_i} x_i(t)^{2x_i(t)}(1 + \ln(x_i(t))). \tag{10}$$

This expression describes the dynamics of the shadow price. As we can see, it depends almost exclusively on the market share and λ itself. This shadow price influences the optimal behaviour of the firms. To obtain the value of the shadow price in the terminal time, we use the transversality condition (8). For this purpose, we state the terminal payoff function as in (3). This yields a simplified transversality condition:

$$\lambda_i(T) = s_i. \tag{11}$$

Plugging the optimal controls from Eq. (9) in the dynamics described in Eq. (6) we get

$$\dot{x}(t) = \frac{\lambda_i(t)}{c_i} x_j(t)^{2x_j(t)} - \frac{\lambda_j(t)}{c_j} x_i(t)^{2x_i(t)}. \tag{12}$$

This is the behaviour of the market share. Note that the value of $\frac{\lambda_i(t)}{c_i}$ is key to know how the market share would behave. This means that, in an optimal path, the agents make their optimal choices of expenditure on advertising according to how important it is for them to get more market share. Note that if one assumes that the cost remains constant, then the control variables will be highly dependent of $\lambda(t)$.

3.3 Steady State

Finally, we want to find the steady state of our dynamics. That is, the market share of the firms where the dynamic is stable over time. To achieve this, we will state the model in absolute terms, that is, we will turn (4) into

$$\dot{s}_i(t) = u_i(m - s_i)^{\alpha} - u_j s_i^{\beta},$$

where s_i stands for the sales of firm i, and m is the size of the whole market, therefore, $x_i = m/s_i$. This way, for example, when $\alpha = \beta = 1/2$ we have that the steady sales $\bar{s}$ is the level of sales that results from making the dynamics $\dot{s}_i(t)$ equal to zero, that is,

$$\bar{s}_i = \frac{u_i^2 m}{u_i^2 + u_j^2}.$$

In the case of the dynamics described by (6), when we state them in absolute terms they become

$$\dot{s}_i(t) = u_i(t)(m - s_i)^{1-s_i/m} - u_j(t)s_i^{s_i/m}, \quad s_i(0) = 0. \tag{13}$$

Since (13) is an implicit function of s_i, we use Figs. 4, 5, 6, 7 to obtain some insights on the existence of a steady state of the i-th firm.

Figures 4 and 5 show the dynamics of the sales when the sales and one of the controllers vary (while the control of the other player is fixed at 0.5). From these pictures, we may argue the existence of a steady state.

In Figs. 6 and 7, we can look how the state dynamics does reach the plane $\dot{s}_i = 0$ without having to fix any of the controllers. Now, by means of a first order Taylor series around $s = 1$ (with error term $O(s-1)^2$), we approximate an expression for the steady state where sales stabilize ($\dot{s}_i = 0$). Therefore we have

$$\bar{s}_i(t) = \left[m(m-1)^{1/m} \left(u_j(t) - (m-1)^{1-1/m} - \frac{\phi}{m(m-1)^{1/m}} \right) \right] \frac{1}{\phi}. \tag{14}$$

where $\phi = u_i(t)(m-1)(1+\ln(m-1)) + u_j(t)(m-1)^{1/m}$. We now plug (14) into (13) and plot the dynamics $\dot{s}_i$ when the controllers vary. See Fig. 8.

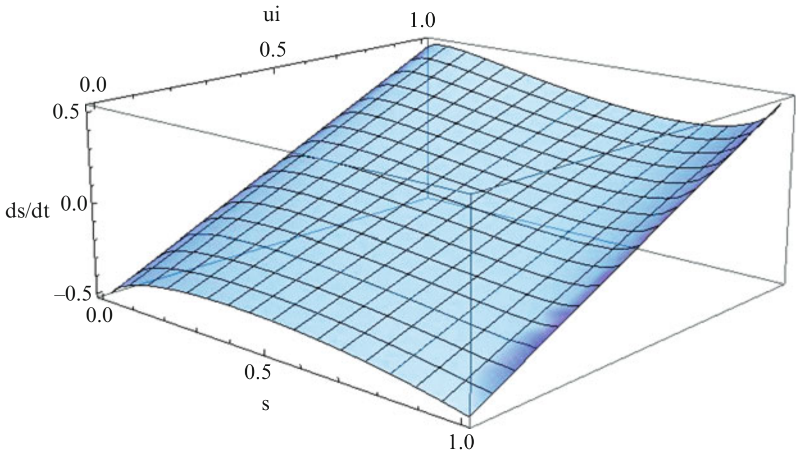

Fig. 4 Sales dynamics response to the state of the game with different units of control u_i

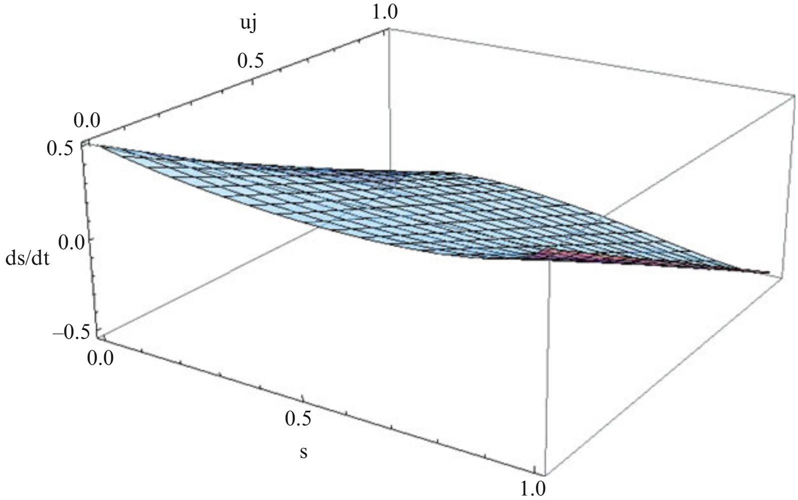

Fig. 5 Sales dynamics response to the state of the game with different units of control u_j

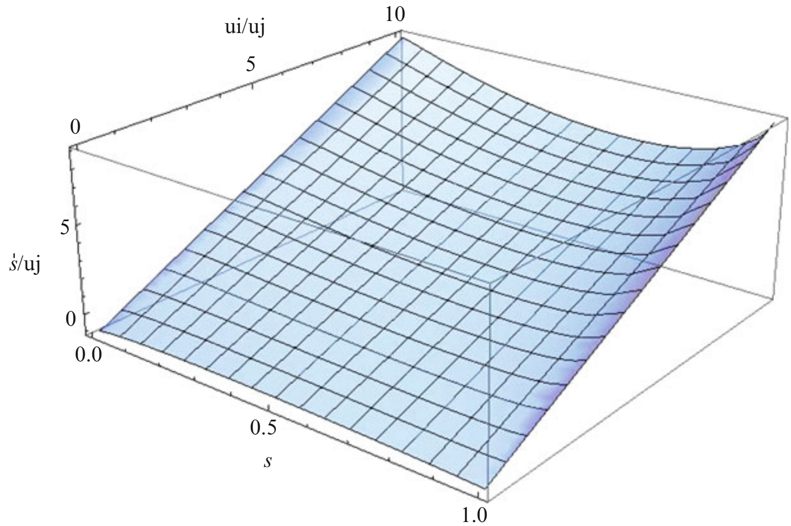

Fig. 6 Sales dynamics response to the state in face of the proportion of advertising efforts u_i/u_j

From Fig. 8 we can infer that, as the controller from the j-th player becomes greater with respect to the controller of the i-th player, the steady state of the latter agent (14) remains stable. This is of particular interest, because it implies that as one player leaves the other act, his own sales will stay stationary. The situation is analogous for the other firm.

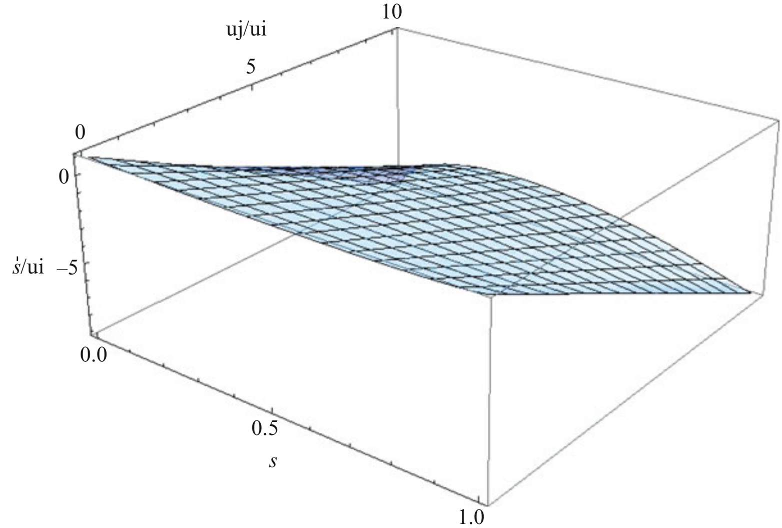

Fig. 7 Sales dynamics response to the state in face of the proportion of advertising efforts u_j/u_i

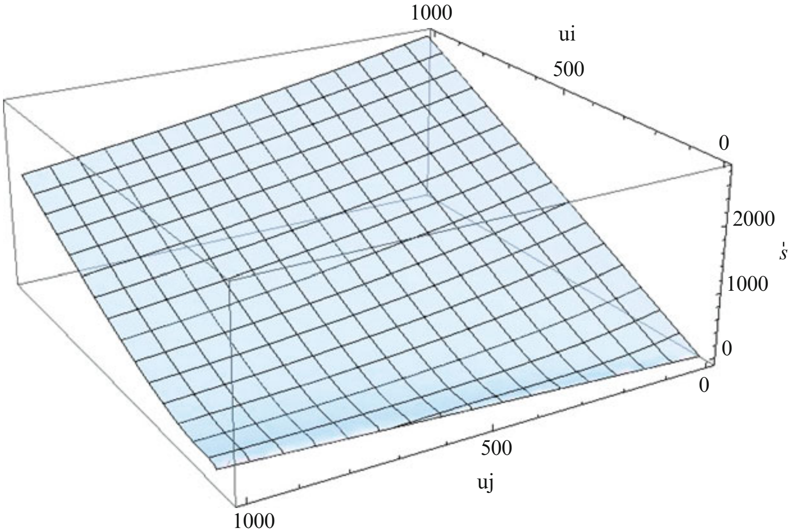

Fig. 8 Sales dynamics response to the state of the game with different units of control u_i

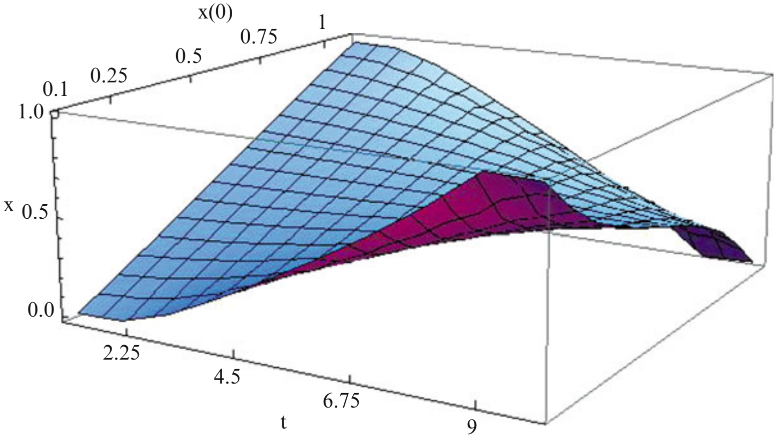

Fig. 9 The market share dynamics for different initial states with Network Externalities and optimal advertising efforts for $T = 10$. High discount rates

4 Simulations

In this section we analyse the dynamics in which the game develops when both players are optimizing, that is, when the game is at equilibrium. For this section, we take the dynamics in (12) and use computer simulations to find the state of the market share of the players in the game over time.

To make a complete analysis, we plot the three-dimensional surface for different initial states $x_0 \in]0, 1[$. Besides, we also plot the surface for the dynamics of the shadow price of market share shown in (10), which is representative of the dynamics of the advertising expenditure.

Figure 9 shows the motion of the market share over time in different initial states. For notational convenience, we assume that x means x_i for $i = 1, 2$. For synoptic purposes, as initial values, we set the same costs for both firms (at a level of 1). We state that the prices are the same for the agents, and started with a discount rate of $r = 1$. Naturally, these variables can be fed with different values more akin to specific situations. Additionally, an initial state of $\lambda_i(0) = 1$ was fed to the system for both firms $i = 1, 2$.

With this settings, we can see that a firm with a very small initial market share can end up owning the market, and analogously, a monopoly might give up on the market over time.

This behaviour can be explained by the choice of a high value for the discount rate. At this rate, a firm with a high market share is interested in exploiting its market power in the present, but is not willing to engage in an attrition war for a long time. The breaking point of the interest that the monopoly has in the market share can be visualized in Fig. 10, that shows the motion of the shadow price.

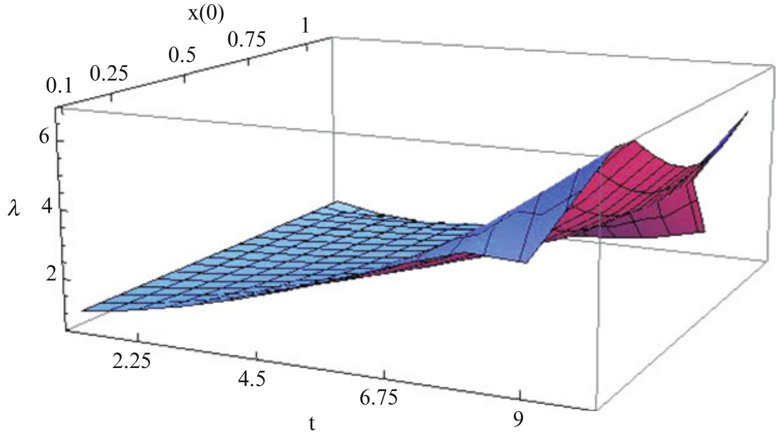

Fig. 10 The dynamics of the shadow price with high discount rates

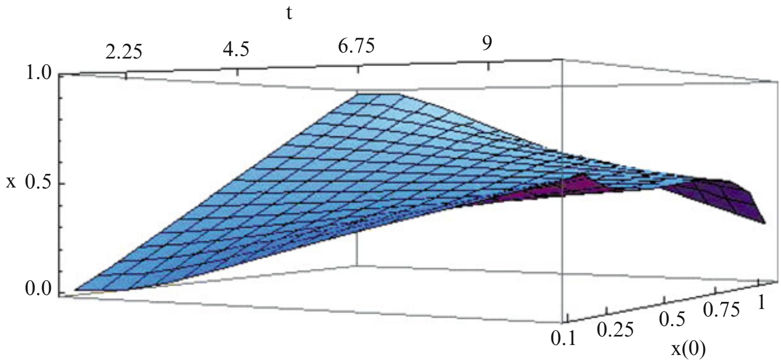

Fig. 11 The dynamics of the market share with low discount rates

In contrast, if we see the same dynamics of Fig. 11, where the discount rate is $r = 0.01$, we can observe that the dynamics, although are not as stable as in Sorger's model (see [21]), have some tendency towards the middle point. More importantly, the shadow price shown in Fig. 12 with a small initial market share reaches a maximum, and then starts to decrease, and when the initial market share is higher, it tends to decrease over time.

Note that the tendency in the extremes keeps on the same direction as in Fig. 9. The tendency of monopolies to exploit as much as they can their high ground is just reduced by a small value in present time, but the intensity of the warfare in these situations makes the effort to keep a monopoly worthless.

Both Figs. 10 and 12 show the dynamics of $\lambda_i(t)$ with the exception of the value of $\lambda_i(T)$, to allow the reader to better appreciate the behaviour of the variable.

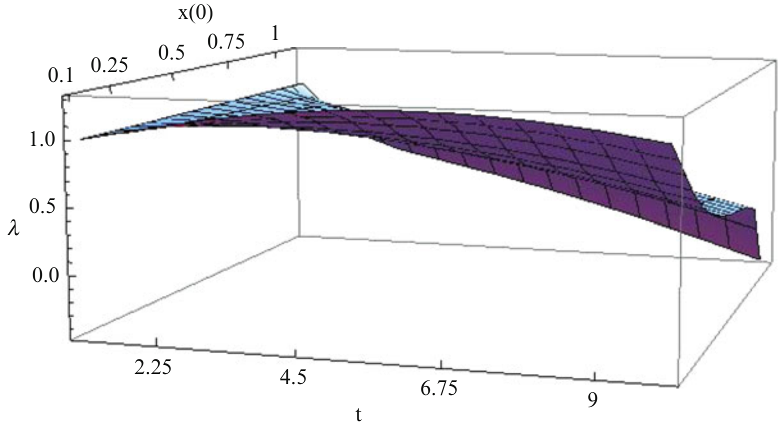

Fig. 12 The dynamics of the shadow price with low discount rates

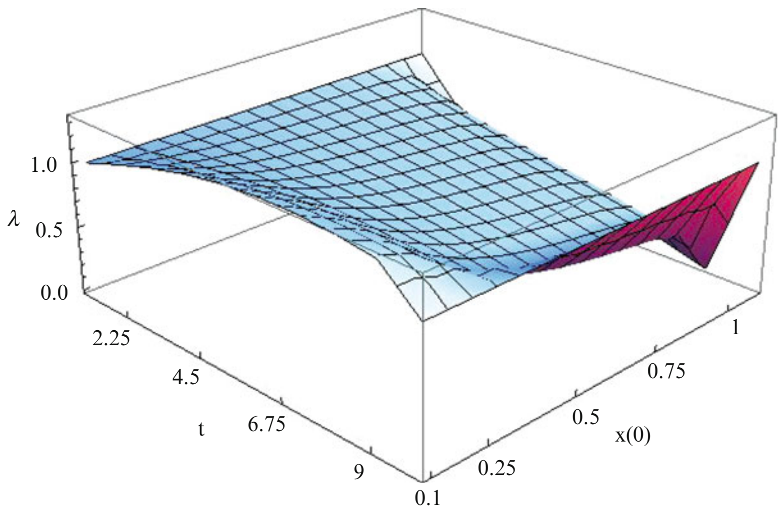

Fig. 13 The dynamics of the shadow price with $\lambda(T) = s_i = 1$

To plot the complete dynamics we simply have to set the final shadow price as in Eq. (11). Figure 13 shows the motion of λ_i with the shadow price in terminal time included, with its value stated by transversality conditions.

Likewise, Figs. 9 and 11 show the market share dynamics with a relatively low terminal value for the shadow price in terminal time. That is, a terminal reward function with a value of 1 like the one shown in Fig. 13 will not be visible in these simulations. Nonetheless, a relatively high level of terminal reward as the one shown in Fig. 14 will be visible in the final market share dynamics.

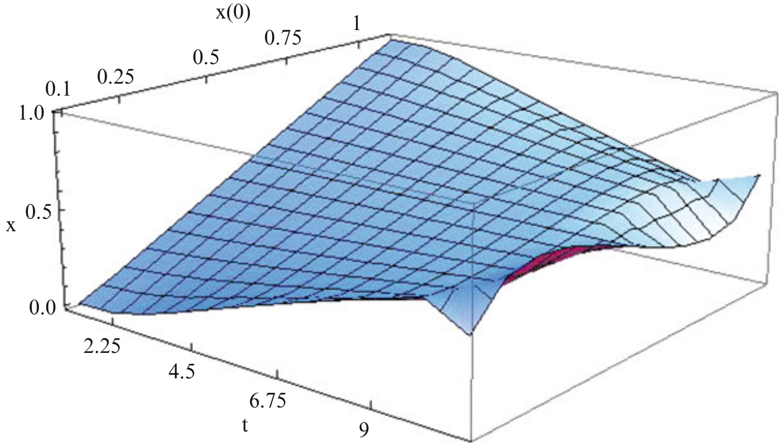

Fig. 14 Market share dynamics with a high value for $\lambda(T) = s_i$

5 Conclusions

We have found that the existence of Network Externalities gives opportunity to a firm to become a monopoly in the long run. Such an opportunity is not present for a market that does not have this kind of structure and is not very common in Lanchester models but is what we would expect on a market with this kind of externalities, such as telecommunications.

By solving the dynamic model, we found that the firms must evaluate how much they value market share and adapt their strategy to this value. This yields an optimal dynamics that will depend on the value of the market share for both firms.

Although we found an approximation of the steady state and performed a geometrical analysis on its existence and stability, we believe this procedure can be improved in further research. However, an interesting finding is that as one player leaves the other advertise for her own brand, his own sales will remain stationary.

Also, by building simulations, we found that the final market share presents a high dependence on the value of the market over time, the value of the market share tends to decrease quickly when a high discount rate is present. When the discount rate is small, this process is slowed down, but eventually, the monopolies tend to give up and the entrant becomes the new monopoly over time.

Further research on the matter includes the use of an analysis with the use of Dynamic Programming to derive Nash equilibria for this game (which might yield a more precise statement of the steady state). It is well known that although simpler to compute, the open-loop strategies present some important drawbacks with respect to the more interesting case of feedback strategies, with come at the cost of a much difficult characterization.

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