Static & Dynamic Game Theory:
Foundations & Applications

Leon A. Petrosyan,
Vladimir V. Mazalov
Editors

Recent Advances in Game Theory and Applications

European Meeting on Game Theory, Saint Petersburg, Russia, 2015, and Networking Games and Management, Petrozavodsk, Russia, 2015

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Preface

The importance of strategic behavior in the human and social world is increasingly recognized in theory and practice. As a result, game theory has emerged as a fundamental instrument in pure and applied research. The discipline of game theory studies decision-making in an interactive environment. It draws on mathematics, statistics, operations research, engineering, biology, economics, political science, and other subjects. In canonical form, a game takes place when an individual pursues an objective in a situation in which other individuals concurrently pursue other (possibly overlapping, possibly conflicting) objectives, and at the same time, these objectives cannot be reached by the individual actions of one decision-maker. The problem then is to determine each object's optimal decisions, how these decisions interact to produce an equilibrium, and the properties of such outcomes. The foundation of game theory was laid more than 70 years ago by John von Neumann and Oskar Morgenstern. Theoretical research and applications are proceeding apace, in areas ranging from aircraft and missile control to inventory management, market development, natural resources extraction, competition policy, negotiation techniques, macroeconomic and environmental planning, capital accumulation, and investment. In all these areas, game theory is perhaps the most sophisticated and fertile paradigm applied mathematics can offer to study and analyze decision-making under real-world conditions.

It is necessary to mention that in 2000, Federico Valenciano organized GAMES 2000, the first meeting of the Game Theory Society in Bilbao. During this conference, Fioravante Patrone took the initiative of setting up a "joint venture" between Italy and Spain, suggesting meetings be held alternately in the said countries. The agreement on this idea led to the meetings in Ischia (2001), Seville (2002), Urbino (2003), and Elche (2004). During the meeting in Urbino, the Netherlands asked to join the Italian-Spanish alternating agreement, and so SING (Spanish-Italian-Netherlands Game Theory Meeting) was set up. The first Dutch edition was organized by Hans Peters in Maastricht from the 24th to 26th of June 2005. It was then agreed that other European countries wishing to enter the team had to participate first as guest organizers and only after a second participation in this role could they then actually join SING. As a result, the following countries acted as guest organizers: Poland in 2008 (Wrocław, organized by Jacek Mercik), France in 2011 (Paris, Michel Grabisch), and Hungary in 2012 (Budapest, László Kóczy). Poland was the guest organizer for the second time in 2014 (Kraków, Izabella Stach) and became an actual member of SING. The 2015 edition took place in St. Petersburg.

Parallel to this activity, every year starting from 2007 at St. Petersburg State University (Russia), an international conference "Game Theory and Management (GTM)" and, at Karelian Research Centre of Russian Academy of Sciences in Petrozavodsk, a satellite international workshop "Networking Games and Management" took place. In the past years, among plenary speakers of the conference were Nobel Prize winners Robert Aumann, John Nash, Reinhard Selten, Roger Myerson, Finn Kydland, and many other world famous game theorists.

In 2014 in Krakow, the agreement was reached to organize the joint SING-GTM conference at St. Petersburg State University, and this meeting was named "European Meeting on Game Theory, SING11-GTM2015."

Papers presented at the "European Meeting on Game Theory, SING11-GTM2015" and the satellite international workshop "Networking Games and Management" certainly reflect both the maturity and the vitality of modern-day game theory and management science in general and of dynamic games in particular. The maturity can be seen from the sophistication of the theorems, proofs, methods, and numerical algorithms contained in most of the papers in this volume. The vitality is manifested by the range of new ideas, new applications, and the growing number of young researchers and wide coverage of research centers and institutes from where this volume originated.

The presented volume demonstrates that "SING11-GTM2015" and the satellite international workshop "Networking Games and Management" offer an interactive program on a wide range of latest developments in game theory. It includes recent advances in topics with high future potential and existing developments in classical fields.

St. Petersburg, Russia Leon Petrosyan
Petrozavodsk, Russia Vladimir Mazalov
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Ranking Journals in Sociology, Education, and Public Administration by Social Choice Theory Methods

Fuad T. Aleskerov, Anna M. Boriskova, Vladimir V. Pislyakov, and Vyacheslav I. Yakuba

Abstract An analysis of journals' rankings based on five commonly used bibliometric indicators (impact factor, article influence score, SNIP, SJR, and H-index) has been conducted. It is shown that despite the high correlation, these single-indicator-based rankings are not identical. Therefore, new approach to ranking academic journals is proposed based on the aggregation of single bibliometric indicators using several ordinal aggregation procedures. In particular, we use the threshold procedure, which allows to reduce opportunities for manipulations.

Keywords Bibliometrics • Journal rankings • Ordinal aggregation procedures • Threshold procedure

1 Introduction

Scientific information is published in academic journals, which are playing an increasingly important role in covering the innovations in academic community. Moreover, the number of journals is growing very fast. Journals' rankings have gained more interest, visibility, and importance recently. The debates over the use and abuse of journal rankings are heated and have recently heightened in their intensity. For the evaluation of journal's scientific significance, various indices are

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used. For these and other reasons, several indicators, such as impact factor, Hirsch index, SNIP, and others, had been proposed to evaluate the various qualities and merits of individual journals. Based on these indicators we obtain different rankings, which do not fully coincide.

Detailed descriptions of these indices can be found in [16, 24, 25]. Furthermore, it was recently understood that the use of single factor to rank scientific journals does not give comprehensive view on the quality of the journals. Therefore, several studies have been performed to construct more complex indices evaluating journals. For example, in [3, 6] several aggregation methods, such as the Copeland rule, the Markov ranking, the uncovered set, and the minimal externally stable set, have been used. Harzing and Mingers [19] investigated relationships between the different rankings, including those between peer rankings and citation behavior and developed a ranking based on four groups. The purpose of that paper was to present a journal ranking for business and management based on a statistical analysis of the Harzing dataset. In [14] a ranking list of journals for the information systems and decision-making is presented. The analysis of journal rankings including several indices had been made.

Indeed, there is no sufficient reason to presume that any simple indicator is somehow inferior to others. Ranking based on only one bibliometric indicator may not fully reflect the quality and significance of an academic journal due to the complexity and multidimensionality of these objects. In addition, single-indicator-based rankings give more opportunities for journal editors to manipulate. For example, according to [13] the impact factor, which is the most popular and commonly used citation indicator, is incredibly easy to manipulate. There are several ways to do it, e.g., self-citation, review articles, increasing non-citable items in the journal, and others.

In this paper, we use such procedures, which reduce opportunities for manipulations. This means that it is impossible to compensate low values of some citation indicators by high values of the others.

The key purpose of our paper is to construct consensus rankings of journals in education, public administration, and sociology based on the social choice procedures, applied to the problem of multi-criteria evaluation, and on the theory of the threshold aggregation developed in [2] and applied, in particular, to authors' evaluation in [4]

- We evaluate the degree of consistency between the bibliometric indicators (impact factor, article influence score, SNIP, SJR, and H-index) for each set of journals separately,
- We construct aggregate rankings using the threshold procedure and other aggregation procedures, such as Hare's procedure, Borda's rule, Black's procedure, Nanson's procedure, Copeland's rules, Simpson's procedure, Threshold procedure, and Markovian method.
- We found that the ranking constructed is more effective tool in evaluation of journal influence than the ranking based on the value of one individual index.

The approach we use evaluates journals according to a set of criteria, which, in our case, consists of impact factor, article influence score, SNIP, SJR, and H-index.

The text is organized as follows. In Sect. 2, we provide the definitions of the used bibliometric indicators. Section 3 contains description of the empirical data and the correlation analysis of single-indicator-based rankings. In Sect. 4, the threshold procedure and other ordinal ranking methods are formally described. Section 5 presents the analysis of the obtained aggregated rankings. The summary of the results is given in the Conclusion. Appendix 1 contains the ranks of journals in single-indicator-based and aggregate rankings for 10 most important journals.

2 Bibliometric Indicators

We will give brief definitions of several measures of journals citedness that are used in this study.

2.1 The Impact Factor

The impact factor (IF), first introduced in [15], is the most popular and commonly used journal citation indicator. It shows the average number of citations to the published paper in a particular journal. In order to calculate IF of a journal, the number of citations received in a given year by journal's papers published within several previous years is divided by the number of these papers. Stated more formally [12, 25], let $PUB(t)$ be the total number of papers published in a journal j during the year t and $CIT(T,t)$ be the total number of citations received in the year T by all papers published in the journal j during the year t. Then the n-year impact factor for the year T can be defined as follows:

$$IF = \frac{\sum_{t=1}^{n} CIT(T, T-t)}{\sum_{t=1}^{n} PUB(T-t)}. \quad (1)$$

The impact factor is published by Thomson Reuters Corporation, in its database Journal Citation Reports (JCR),[1] for $n = 2$ and $n = 5$. However, the optimal "publication window" (parameter n) is still being debated. The 2-year impact factor ($n = 2$) is thought to be the classical case. However, sometimes the 5-year impact factor is more appropriate than 2-year because in certain fields of science it takes a longer time to assimilate new knowledge. Moreover, depending on the area of research and type of the papers, there are differences between how quickly they become obsolete and stop being cited in the literature.

[1]This product is based on another Thomson database, Web of Science (WoS). WoS contains citation data on an individual paper level, while JCR aggregates citation indicators for journals as a whole.

Both the abovementioned publication windows have been analyzed. However, the discrepancies between rankings based on IF with different publication windows were found to be insignificant. Therefore, we use only 2-year impact factor for the further analysis.

2.2 Source Normalized Impact per Paper

The source normalized impact per paper (SNIP) indicator, introduced in [23], measures the citation impact of scientific journals corrected for the differences in citation practice between scientific fields. Another advantage of this indicator is that it does not require a field classification system in which the boundaries of fields are explicitly defined and not flexible. A journal's subject field is defined as the set of papers published in a current year and is citing at least one of the 1–10-year-old papers published in the journal.

The SNIP is defined as the ratio of journal's raw impact per paper (RIP) to the relative database citation potential (RDCP):

$$SNIP = \frac{RIP}{RDCP}. \quad (2)$$

The RIP is similar to the impact factor except that three instead of 2 years of cited publications are used and only citations to publications of the specific document types (article, conference paper, or review) are included.

To calculate the RDCP, a journal's database citation potential (DCP) is divided by the median DCP value for all journals in the database. In its turn, the DCP equals the average number of "active references" in the papers belonging to the journal's subject field. "Active references" are references to papers that appeared within the three preceding years in sources covered by the database (Scopus). All references to documents older than 3 years or not indexed by Scopus do not affect DCP.

Thus, SNIP: (a) corrects for different citation practices in different fields (average number of references); (b) equalizes a field relatively well represented in the database and a field where there are many references to sources outside the database (for instance, a discipline where books are cited more frequently than journal articles); (c) makes equal those fields where most recent literature is cited with those where older documents receive a considerable number of citations.

The SNIP indicator is made available in Elsevier's Scopus database, together with another journal indicator, the SCImago Journal Rank (SJR), which is described below.

Data on SNIP are regularly updated. In our analysis we use data downloaded from the Scopus web site[2] in 2013.

[2]http://www.journalmetrics.com/values.php. As of 2013 "optimized" values of SNIP (the so-called SNIP2) are published. We use older version of SNIP intentionally, since it has already been tested for a while by the academic community. The latest published data are the values for the first half of 2013. The same is to be said about SJR (see below).

2.3 SCImago Journal Rank

The indicator was introduced in [17]. It evaluates journal taking into account not just the number of citations received but also the quality of the source of these citations. For this reason, weights are assigned to all citations based on a "prestige" of the journals where they come from, so that citations received from the more prestigious journals are more valuable than those from less prestigious ones. The prestige is computed recursively, i.e., the prestigious journals are those which receive many citations from other prestigious journals.

At the first stage of the procedure all journals get the equal level of prestige. Then the new level of prestige is computed based on citations received by a journal. On the next stage we re-evaluate the prestige of each journal counting citations it received, each citation is taken with the weight corresponding to the prestige of the citing journal. The algorithm iterates until a steady-state solution is reached, and the final prestige values reflect the journals' scientific importance. Precise mathematical description can be found in [17].

It should be noted that this procedure is equivalent to counting how often a reader would take a certain journal, if she randomly walks from journal to journal following citation links.

Only citations made to papers published within last 3 years are taken into account in SJR. If the number of journal self-citations is large, then it is artificially reduced and is set to 33 % of all citations made to this journal. Finally, journal's SJR is normalized by the number of its articles; therefore the value of this indicator is independent of journal's volume. In this study we use values for 2013.

2.4 Article Influence Score

Another "weighted" indicator, the article influence score, also takes into account the relative importance of citing journals. It is calculated similarly to SJR, the main difference being citation database it is based on. For calculating article influence the Web of Science is used as a source of the data, so the values for this indicator are published in JCR database.

There are several other technical distinctions from SJR methodology, the main are: (a) the publication window for the article influence calculation is 5 years, not 3 years as for SJR; (b) self-citations are totally excluded, whereas for SJR they just have upper limit of 33 % of all citations.

JCR publishes article influence values since 2007; they also may be found with 1-year embargo in open access at http://eigenfactor.org/ (but see [21] on differences in data obtained from two different systems). In this study we use values for 2013.

2.5 Hirsch Index

Hirsch index (H-index) [20] evaluates both the number of papers and their citedness. By definition, the H-index for a set of publication equals h, if exactly h papers from the set have received no less than h citations, while the others have received no more than h citations. This indicator does not involve calculation of the averages, thus the H-index is robust with respect to outliers (e.g., when there is one paper with enormously large number of citations which significantly affect their average number). To have a high value of H-index a journal has to publish many frequently cited papers.

Initially H-index was introduced to assess the output of a scientist, but it can also be applied to journals. For instance, Braun et al. [8] consider the set of articles published in a journal in a certain year and calculate their citedness at present (in their case, 4 years after publication). In this paper we use a more balanced approach adopted in the work on computation of aggregate rankings for economic journals [4]: we take into account papers published in a journal over 5 years (from 2009 to 2013) and citations received over the same period. The values of H-index depend upon a database one uses. We use the Web of Science database to calculate H-index.

It should also be noted that H-index has certain disadvantages. The most evident one is the following: the papers with low citedness (below and, in certain cases, equal to h) are completely ignored. Indeed, suppose there are two journals with 50 papers published in each of them. In the first journal each paper has received 10 citations, while 10 papers in the second one have received 10 citations each, but the other 40 papers have not been cited at all. The journals are clearly unequal by their "influence," but their H-index values are the same—10.

3 Data and the Analysis of Single-Indicator-Based Rankings

Three sets of journals are studied hereafter, representing three academic disciplines: education, public administration, and sociology. We analyze the degree of consistency between the bibliometric indicators (impact factor, article influence score, SNIP, SJR, and H-index) for each set of journals separately. In 2013, the SJR database included 138 journals in sociology, 219 journals in education, and 46 journals in public administration, which were also indexed in the Scopus database. Thus, the values of indicators for the selected journals could be extracted (or calculated in the case of H-index). However, for 8 journals in sociology some of the indicators were missing from JCR. Six more journals did not have their SJR and/or SNIP values. These 14 journals are excluded, leaving 124 journals in sociology for further analysis. For the same reason 46 education and 8 public administration journals are excluded as well. As a result, for 124, 173, and 38 journals in sociology, education, and public administration the values of impact factor (2013), article influence (2013), H-index (2009–2013), SNIP (2013), and SJR (2013) have been extracted. The data sources are summarized in Table 1.

Table 1 Data sources

Indicator	Database	Year(s)
Impact factor (2 year)	JCR/WoS	2013
SNIP	Scopus	2013
SJR	Scopus	2013
Article influence	JCR/WoS	2013
H-index	WoS	2009–2013 (papers and citations)

The values of these bibliometric indicators are used to rank journals. Basically, ranking is a set of positions (called ranks) in which one or more journals can be put. Journals with matching values are given the same position in the ranking, and this corresponds to the same rank. Meanwhile, journals with different values are given different positions, which are ordered by descending values of indicators and are identified by natural numbers, from the "best" value to the "worst" one.

As our ranks are ordinal variables, rank correlation can be estimated by *Spearman's coefficients*. Since percentage of duplicate values in the rankings is relatively low, this coefficient is calculated as follows:

$$\rho = 1 - \frac{6\sum_{i=1}^{n}(x_i - y_i)^2}{n(n^2-1)}, \quad (3)$$

where x_i, y_i are ranks of journal i in two compared rankings X and Y, and n is the total number of journals.

To make it clear, let us suppose that there are two rankings, which rank journals as follows:

	Ranking 1	Ranking 2
Journal A	1	7
Journal B	2	4
Journal C	3	5
Journal D	4	1
Journal E	5	3
Journal F	6	2
Journal G	7	8
Journal H	8	6

In this case, $\rho = 1 - \frac{6((1-7)^2+(2-4)^2+(3-5)^2+(4-1)^2+(5-3)^2+(6-2)^2+(7-8)^2+(8-6)^2)}{8(8^2-1)}$. Hence, the Spearman correlation between the two rankings is approximately 0.07.

However, if ranks of journals are equal, their values are recalculated so that they are given by the arithmetic average of their positions in ranking. Then, the whole procedure is repeated as mentioned above.

Table 2 Spearman's ρ (sociology)

	Impact factor	Article influence score	SNIP	SJR	H-index
Impact factor	1.00	0.85	0.76	0.87	0.86
Article influence score	0.85	1.00	0.78	0.86	0.81
SNIP	0.76	0.78	1.00	0.87	0.70
SJR	0.87	0.86	0.87	1.00	0.84
H-index	0.86	0.81	0.70	0.84	1.00

Table 3 Spearman's ρ (education)

	Impact factor	Article influence score	SNIP	SJR	H-index
Impact factor	1.00	0.87	0.82	0.86	0.83
Article influence score	0.87	1.00	0.80	0.91	0.81
SNIP	0.82	0.80	1.00	0.88	0.73
SJR	0.86	0.91	0.88	1.00	0.82
H-index	0.83	0.81	0.73	0.82	1.00

Table 4 Spearman's ρ (public administration)

	Impact factor	Article influence score	SNIP	SJR	H-index
Impact factor	1.00	0.92	0.85	0.85	0.91
Article influence score	0.92	1.00	0.90	0.90	0.89
SNIP	0.85	0.90	1.00	1.00	0.84
SJR	0.85	0.90	1.00	1.00	0.84
H-index	0.91	0.89	0.84	0.84	1.00

Spearman's ρ, unlike broadly used Pearson's coefficient, is not affected by outliers too much, as it limits them to the values of their ranks. Its value ranges from +1 to -1. $\rho = 1$ means that rankings are the same and $\rho = -1$ that they are completely different. Results for Spearman's ρ measure for all academic disciplines under consideration are given in Tables 2, 3 and 4.

For all academic disciplines, ρ reveals significant correlation between rankings based on each bibliometric indicator. In fact, Spearman's ρ for every pair of rankings is not less than 0.70 for journals in sociology, 0.73 for educational journals, and 0.84 for journals in public administration.

Concerning the highest level of correlation, for social science journals it is between SJR and SNIP rankings (1.00) for public administration, and about 0.85 in other academic disciplines; the second highest correlation is between impact factor and article influence score rankings (0.87) in education and public administration disciplines. Correlation between public administration journals' rankings is high: the ρ coefficient exceeds 0.9. We should note that the correlation coefficients could be biased in the case of public administration science because of the small sample of the available journals. For the other pairs of rankings ρ coefficient is not less than 0.70 for journals in all fields.

Thus, the analysis of correlations presented in this section shows that different indicators generate similar but not identical rankings. We believe that the disparities result mainly from complexity and multidimensionality of the journal quality and significance. Furthermore, the indicators differ largely conceptually. Therefore, rather than trying to choose the best indicator it is worth using ordinal methods developed in the theory of social choice that combine information contained in separate variables. Thus, ranking of journals becomes a multi-criteria evaluation problem.

4 The Description of Threshold Procedure and Other Ordinal Ranking Methods

The obtained values of the rank correlation coefficients show that the use of different indicators leads to a similar, but not coincident rankings of journals. Furthermore, the indicators differ to a great extent conceptually.

A standard solution to a multi-criteria evaluation problem is to calculate a weighted sum of criteria values for each alternative, and then rank alternatives by the value of this sum. However, there is a severe restriction on this approach—the weights should be justified. We have no such justification for the problem under consideration. Therefore, we cannot be sure that a linear convolution of bibliometric indicators is a correct procedure yielding meaningful results.

The alternative solution could be the use of ordinal methods developed in the theory of social choice and, in particular, an application of the threshold procedure [2].

Social Choice Rules

Let us introduce several important notions. The concepts and rules used below can be found in [1–5, 8–10, 26, 27, 29].

Definition 1. Majority relation for a given profile $\vec{P}$ is a binary relation μ which is constructed as follows:

$$x\mu y \Leftrightarrow card\{i \in N\,|xP_iy\} > card\{i \in N\,|yP_ix\},$$

where P_i is a weak order, i.e., irreflexive $x\overline{P}x$ for all ($x \in A$), transitive ($xPyPz \to xPz$), and negatively transitive ($x\overline{P}y\overline{P}z \to x\overline{P}z$).

Definition 2. Condorcet winner $CW(\vec{P})$ in the profile $\vec{P}$ is an element undominated in the majority relation μ (constructed according to the profile), i.e.,

$$CW(\vec{P}) = \left\{a \,\middle|\, \overline{\exists} x \in A,\ x\mu a\right\}.$$

Definition 3. A construction of a profile $\vec{P}$ onto the set $X \subseteq A$, $X \neq \emptyset$ is a profile

$$\vec{P}/X = (P_1/X, \ldots, P_n/X), P_i/X = P_i \cap (X \times X).$$

Definition 4. Upper counter set of an alternative x in the relation P is the set $D(x)$ such that

$$D(x) = \{y \in A \,|yPx\}.$$

Lower counter set of x in the relation P is the set $L(x)$ such that

$$L(x) = \{y \in A \,|xPy\}.$$

The rules under study can be divided into several groups:

(a) Scoring Rules;
(b) Rules, using value function; and
(c) Rules, using tournament matrix.

Scoring Rules

Hare's Procedure. Firstly simple majority rule is used. If such alternative exists, the procedure stops, otherwise, the alternative x with the minimum number of first votes is omitted. Then the procedure again applied to the set $X = A \backslash \{x\}$ and the profile $\vec{P}/X$.

Borda's Rule. Put to each $x \in A$ into correspondence a number $r_i(x, \vec{P})$ which is equal to the cardinality of the lower contour set of x in $P_i \in \vec{P}$, i.e., $r_i(x, \vec{P}) = card\,(L_i(x))$. The sum of that numbers over all i is called Borda's count for alternative x.

Alternative with maximum Borda's count is chosen, i.e.,

$$a \in C(\vec{P}) \Leftrightarrow \left[\forall b \in A, r(a, \vec{P}) \geq r(b, \vec{P})\right], r(a, \vec{P}) = \sum_{i=1}^{n} r_i(a, P_i).$$

Black's Procedure. If Condorcet winner exists, it is to be chosen. Otherwise, Borda's rule is applied.

Inverse Borda's Procedure. For each alternative Borda's count is calculated. Then the alternative a with minimum count is omitted. Borda's count is recalculated for profile $\vec{P}/X, X = A \backslash \{a\}$, and procedure is repeated until choice is found.

Nanson's Procedure. For each alternative Borda's count is calculated. Then the average count is calculated, $\bar{r} = \left(\sum_{a \in A} r(a, \vec{P})\right) / |A|$, and alternatives $c \in A$ are omitted for which $r(c, \vec{P}) < \bar{r}$. Then the set $X = \left\{a \in A \mid r(a, \vec{P}) \geq \bar{r}\right\}$ is considered, and the procedure applied to the profile $\vec{P}/X$. Such procedure is repeated until choice is not empty.

Rule, Using Tournament Matrix

Copeland's rule 1. Construct function $u(x)$, which is equal to the difference of cardinalities of lower and upper contour sets of alternative x in majority relation μ, i.e., $u(x) = card\,(L(x)) - card\,(D(x))$. Then the social choice is defined by maximization of u, that is,

$$x \in C(\overrightarrow{P}) \Leftrightarrow [\forall y \in A, u(x) \geq u(y)]\,.$$

Copeland's rule 2. Function $u(x)$ is defined by cardinality of lower contour set of alternative x in majority relation μ. Social choice is defined by maximization of u.

Copeland's rule 3. Function $u(x)$ is constructed by cardinality of upper contour set of alternative x in majority relation μ. Social choice is defined by minimization of u.

Simpson's Procedure (Maxmin Procedure).
Construct matrix S^+, such that

$$\forall a, b \in X, S^+ = (n(a, b))\,,$$

$$n(a, b) = card\,\{i \in N\,|aP_i b\}\,, n(a, a) = +\infty.$$

Social choice is defined as

$$x \in C(\overrightarrow{P}) \Leftrightarrow x = \arg\max_{a \in A}\min_{b \in A}(n(a, b)).$$

Threshold Procedure

To find a solution to a multi-criteria evaluation problem we propose to apply the threshold procedure [2], which possesses the so-called non-compensatory nature. This means that high values of some citation indicators cannot be traded for low values of the others. Therefore, this procedure reduces opportunities for improving the simulated place of the journal in the ranking by increasing one of the used indices. The "non-compensatory" procedure also reduces the incentive to increase the number of low-quality papers and to attract insignificant citations, as the journals with no many frequently cited publications are not able to take a very high place in the rankings [4].

Before we give a formal definition of the procedure, let us provide some informal explanation of it. Assume that we have only three journals $J1, J2, J3$ evaluated with respect to 3 criteria, such as impact factor, H-index, and SJR. Let the ranks of the journals with respect to the indicators be given in Table 5, the smaller is the number of rank, the better is the journal.

Then, according to the threshold procedure, for $J1$ the value of 1 for SJR index does not compensate the worst values for IF and H-index, so $J1$ in aggregated ranking gets lower rank than $J2$. Even $J3$ since it has worse ranks than *J1* is placed in the final ranking above $J1$. The final ranking looks as $J2 > J3 > J1$.

Table 5 Example

	IF	H-index	SJR
J1	3	3	1
J2	2	2	2
J3	3	2	2

In other words, the procedure punishes low values of indicators stronger than rewards high values. This is exactly the reason why we suggest using it in the construction of aggregated ranking.

Now, let us give a formal definition of the procedure. Let A be a finite set of alternatives, which are evaluated on n criteria. In the present paper different journals are assumed to be alternatives and different bibliometric indicators are considered as criteria.

For each indicator, the sample is split into m grades, where the first grade corresponds to the "best" journals. On the next stage, to each alternative x from A, a vector $(x_1, x_2, \ldots, x_n)$ is assigned, where x_j is the grade of the alternative according to the criterion j, i.e., $x_j \in \{1, \ldots, m\}$.

The threshold procedure ranks the set A based on the vector of grades $(x_1, x_2, \ldots, x_n)$ for each $x \in A$. We assume that the set A consists of all possible vectors of this form.

Let $v_j(x)$ be the number of ranks j in the vector x, i.e., $v_j(x) = |\{1 \leq i \leq n : x_i = j\}|$. It should be noted that $0 \leq v_j(x) \leq n$ for all $j \in \{1, \ldots, m\}$ and $x \in A$, and $v_1(x) + \ldots + v_m(x) = n$ for all $x \in A$.

The alternative $x \in A$ is said to be (strictly) preferred to the other alternative $y \in A$ (x dominates y or, shortly, xPy) if we can find the number k, $1 \leq k \leq n$, such that $v_j(x) = v_j(y)$ for all numbers $k + 1 \leq j \leq m$ and $v_k(x) < v_k(y)$ (if $k = m$, the condition $v_j(x) = v_j(y)$ can be omitted). The relation P is called the threshold relation.

In other words, a vector x is more preferable than a vector y if x has less grades m than y; if both of these vectors have the same number of grades m, then the numbers of grades $m - 1$ are compared, and so on.

After making these comparisons, we obtain a weak order P, the undominated elements of which are the best journals; to these journals the rank 1 is assigned. After excluding these journals, we get the set of the second best alternatives to which we assign the rank 2. Then, we proceed in this way until all the journals are ranked.

The Markovian Method

Finally, we would like to apply a version of a ranking called the Markovian method, since it is based on an analysis of Markov chains that model stochastic moves from vertex to vertex via arcs of a digraph representing a binary relation μ. The earliest versions of this method were proposed by Daniels [11] and Ushakov [28]. References to other papers can be found in [9].

To explain the method let us consider its application in the following situation. Suppose alternatives from A are chess-players. Only two persons can sit at a chess-board, therefore in making judgments about players' relative strength, we are compelled to rely upon results of binary comparisons, i.e., separate games. Our aim is to rank players according to their strength. Since it is not possible with a single game, we organize a tournament.

Before the tournament starts we separate patently stronger players from the weaker ones by assigning each player to a certain league, a subgroup of players who are relatively equal in their strength. To make the assignments, we use the sorting procedure described in the previous subsection. The tournament solution that is used for the selection of the strongest players is the weak top cycle *WTC* [18, 26, 27, 29]. It is defined in the following way. A set *WTC* is called the weak top cycle if (1) any alternative in *WTC* μ-dominates any alternative outside WTC: $\forall x \notin WTC,\ y \in WTC \Rightarrow y\mu x$, and (2) none of its proper subsets satisfies this property.

The relative strength of players assigned to different leagues is determined by a binary relation μ, therefore in order to rank all players all we need to know is how to rank players of the same league. Each league receives a chess-board. Since there is only one chess-board per league, the games of a league form a sequence in time.

Players who participate in a game are chosen in the following way: a player who has been declared a (current) winner in the previous game remains at the board, her rival is randomly chosen from the rest of the players, among whom the loser of the previous game is also present. In a given league, all probabilities of being chosen are equal. If a game ends in a draw, the previous winner, nevertheless, loses her title and it passes to her rival. Therefore, despite ties being allowed, there is a single winner in each game. It is evident that the strength of a player can be measured by counting a relative number of games where he has been declared a winner (i.e., the number of his wins divided by the total number of games in a tournament).

In order to start a tournament, we need to decide who is declared a winner in a fictitious "zero-game." However, the longer the tournament goes (i.e., the greater the number of tournament games there are), the smaller the influence of this decision on the relative number of wins of any player is. In the limit when the number of games tends to infinity, relative numbers of wins are completely independent of who had been given "the crown" before the tournament started.

Instead of calculating the limit of the relative number of wins, one can find the limit of the probability a player will be declared a winner in the last game of the tournament since these values are equal. We can count the probability and its limit using matrices M and T.

For computational purposes a majority relation μ is represented by a majority matrix $M = [m_{xy}]$, defined in the following way:

$$m_{xy} = 1 \Leftrightarrow (x, y) \in \mu, \text{ or } m_{xy} = 0 \Leftrightarrow (x, y) \notin \mu.$$

A matrix $T = [t_{ij}]$ representing a set of ties τ is defined in the same way.

Suppose we somehow know the relative strength of players in each pair of them. Also, suppose this strength is constant over time and is represented by binary relations μ and τ. Therefore, if we know μ and the names of the players who are sitting at the chess-board, we can predict the result of the game: the victory of x (if $x\mu y$), the victory of y (if $y\mu x$), or a draw (if $x\tau y$).

Let $p^{(k)}$ denote a vector, i-th component $p_i^{(k)}$ of which is the probability a player number i is declared the winner of a game number k. Two mutually exclusive situations are possible. The first case—the player number i is declared the winner in both previous games (game number $k-1$) and the current game. She can be declared the winner in the game number k, if and only if her rival (who has been chosen by lot) belongs to the lower section of i. The probability that the i-th player was declared the winner in the game number $k-1$ is $p_i^{(k-1)}$, the probability of her rival being in $L(i)$ equals $\frac{S_2(i)}{m-1}$, where $S_2(i)$ is the Copeland score (the 2nd version), $S_2(x) = |L(x)|$. Thus, the probability of the i-th player being declared the winner in game number k is

$$p_i^{(k-1)} \cdot \frac{S_2(i)}{m-1}.$$

The second case—the player number i is declared the winner in the current game, but not in the previous one. He can be declared the winner in game number k, if and only if (1) he has been chosen by lot as a rival to the winner in the game number $k-1$, the probability of which equals $\frac{1}{m-1}$; and (2) if the $(k-1)$-th winner is in the lower section or in the horizon in τ of the i-th player, a probability of which equals

$$\sum_{i=1}^{m}(m_{ij} + t_{ij})p_j^{(k-1)}.$$

Thus the probability $p_i^{(k)}$ can be determined from the following equation:

$$p_i^{(k)} = p_i^{(k-1)} \cdot \frac{S_2(i)}{m-1} + \frac{1}{m-1} \cdot \sum_{i=1}^{m}(m_{ij} + t_{ij})p_j^{(k-1)}. \quad (4)$$

Formula (4) can be rewritten in a matrix–vector form as

$$p^{(k)} = W \cdot p^{(k-1)} = \frac{1}{m-1} \cdot (M + T + S) \cdot p^{(k-1)}. \quad (5)$$

The matrix $S = [s_{ij}]$ is defined as $s_{ii} = S_2(i)$ and $s_{ij} = 0$ when $i \neq j$. Consequently, passing the title of the current winner from player to player is a Markovian process with the transition matrix W.

We are interested in vector $p = \lim_{k\to\infty} p^{(k)}$. It is not hard to prove that no matter what the initial conditions are (i.e., what the value of $p^{(0)}$ is), the limit vector is an eigenvector of the matrix W corresponding to the eigenvalue $\lambda = 1$ (see, for instance, [22]. Therefore p is determined by solving the system of linear equations $W \cdot p = p$. To rank players in a league, one needs to order them by decreasing values of p_i. Since we have pre-sorted players using WTC, none of the components p_i is equal to zero [22].

5 Aggregated Rankings for Journals

Aggregate journal ratings, based on paired comparisons of journals by five bibliometric indicators using Hare's procedure, Borda's rule, Black's procedure, Nanson's procedure, Copeland's rule, Simpson's procedure, Threshold procedure, and Markovian method, are given for top-10 journals below. Complete list of journals can be seen in [7]. Based on the values of bibliometric indicators the journal ratings are constructed. Rating—is a ranking, which consists of positions (places to which you can put one or several journals). Journals with the same values of the index correspond to the one position in ranking, and with mismatched index values correspond to different positions. Positions are ordered by "deterioration" (in our case—descending order) of indices values and numbered by natural numbers, starting at the position corresponding to the "best" value.

Tables 6, 7 and 8 contain the results of the correlation analysis of the aggregated ratings, constructed using the rules, which were discussed above.

Correlation analysis also shows that aggregate rankings reduce the number of contradictions. Finally, we quantified the degree of consistency between the initial single bibliometric indicators and consensus indices for each set of journals separately. As a result, we could note that there are high values of coherence between individual and aggregate indices. It means that single-indicator-based rankings could be successfully replaced by aggregate rankings, because the latter ones combine information contained in the set of single-indicator-based rankings. Tables 9, 10 and 11 contain the results of the correlation analysis of the aggregated rankings, constructed using the social choice rules and rankings, based on initial indicators.

6 Conclusion

The question of how to assess research outputs published in journals is now a global concern for academics. Numerous journal ratings and rankings exist. However, rankings based on different measures are different, and that poses a problem. Different approaches to the measurement of journal influence stipulate the existence of different indices of influence, each of them has its own theoretical justification. Measuring the level of influence of scientific publications is a task for which there is no single correct solution.

Table 6 Correlation coefficients between the aggregated ratings of journals in sociology

	Borda grades	Hare grades	Copeland 1 grades	Copeland 2 grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Markovian method	Threshold grade
Borda grades	1.00	0.93	1.00	1.00	0.99	0.99	0.99	1.00	0.99	0.98	0.98
Hare grades	0.93	1.00	0.94	0.94	0.94	0.95	0.95	0.94	0.95	0.96	0.93
Copeland 1 grades	1.00	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98	0.98
Copeland 2 grades	1.00	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98	0.98
Copeland 3 grades	0.99	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98	0.98
Nanson grades	0.99	0.95	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98	0.98
Duo-Simpson grades	0.99	0.95	1.00	1.00	1.00	1.00	1.00	0.99	1.00	0.98	0.98
Black grades	1.00	0.94	1.00	1.00	1.00	1.00	0.99	1.00	0.99	0.98	0.98
Inverse Borda grades	0.99	0.95	1.00	1.00	1.00	1.00	1.00	0.99	1.00	0.98	0.98
Markovian method	0.98	0.96	0.98	0.98	0.98	0.98	0.98	0.98	0.98	1.00	0.99
Threshold grade (the more the better)	0.98	0.93	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.99	1.00

Table 7 Correlation coefficients between the aggregated ratings of journals in education

	Borda grades	Hare grades	Copeland 1 grades	Copeland 2 grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Markovian method	Threshold grade
Borda grades	1.00	0.95	1.00	0.99	1.00	0.99	0.99	1.00	0.99	0.97	0.97
Hare grades	0.95	1.00	0.96	0.96	0.96	0.96	0.96	0.95	0.96	0.96	0.94
Copeland 1 grades	1.00	0.96	1.00	1.00	1.00	1.00	0.99	1.00	1.00	0.97	0.97
Copeland 2 grades	0.99	0.96	1.00	1.00	1.00	1.00	0.99	1.00	1.00	0.97	0.97
Copeland 3 grades	1.00	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.97	0.97
Nanson grades	0.99	0.96	1.00	1.00	1.00	1.00	1.00	0.99	1.00	0.97	0.97
Duo-Simpson grades	0.99	0.96	0.99	0.99	1.00	1.00	1.00	0.99	1.00	0.97	0.97
Black grades	1.00	0.95	1.00	1.00	1.00	0.99	0.99	1.00	0.99	0.97	0.97
Inverse Borda grades	0.99	0.96	1.00	1.00	1.00	1.00	1.00	0.99	1.00	0.97	0.97
Markovian method	0.97	0.96	0.97	0.97	0.97	0.97	0.97	0.97	0.97	1.00	0.99
Threshold grade (the more the better)	0.97	0.94	0.97	0.97	0.97	0.97	0.97	0.97	0.97	0.99	1.00

Table 8 Correlation coefficients between the aggregated ratings of journals in public administration

	Borda grades	Hare grades	Copeland 1 grades	Copeland 2 grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Markovian method	Threshold grade
Borda grades	1.00	0.93	0.99	0.99	0.98	0.98	0.97	1.00	0.98	0.96	0.84
Hare grades	0.93	1.00	0.96	0.96	0.96	0.96	0.98	0.93	0.97	0.96	0.86
Copeland 1 grades	0.99	0.96	1.00	1.00	1.00	1.00	0.99	0.99	1.00	0.98	0.86
Copeland 2 grades	0.99	0.96	1.00	1.00	1.00	1.00	0.99	0.99	1.00	0.98	0.86
Copeland 3 grades	0.98	0.96	1.00	1.00	1.00	0.99	0.99	0.99	0.99	0.98	0.86
Nanson grades	0.98	0.96	1.00	1.00	0.99	1.00	0.99	0.99	1.00	0.98	0.86
Duo-Simpson grades	0.97	0.98	0.99	0.99	0.99	0.99	1.00	0.98	0.99	0.98	0.86
Black grades	1.00	0.93	0.99	0.99	0.99	0.99	0.98	1.00	0.99	0.97	0.85
Inverse Borda grades	0.98	0.97	1.00	1.00	0.99	1.00	0.99	0.99	1.00	0.98	0.86
Markovian method	0.96	0.96	0.98	0.98	0.98	0.98	0.98	0.97	0.98	1.00	0.88
Threshold grade (the more the better)	0.84	0.86	0.86	0.86	0.86	0.86	0.86	0.85	0.86	0.88	1.00

Table 9 Correlation coefficients between the aggregated rankings and single-indicator-based rankings of journals in public administration

	Impact factor	Article influence score	SNIP	SJR	H-index	Borda grades	Hare grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Markovian method	Threshold grade
Impact factor	1.00	-	-	-	-	-	-	-	-	-	-	-	-	-
Article influence score	0.92	1.00	-	-	-	-	-	-	-	-	-	-	-	-
SNIP	0.85	0.90	1.00	-	-	-	-	-	-	-	-	-	-	-
SJR	0.85	0.90	1.00	1.00	-	-	-	-	-	-	-	-	-	-
H-index	0.91	0.89	0.84	0.84	1.00	-	-	-	-	-	-	-	-	-
Borda grades	0.95	0.96	0.92	0.92	0.96	1.00	-	-	-	-	-	-	-	-
Hare grades	0.85	0.91	1.00	1.00	0.84	0.93	1.00	-	-	-	-	-	-	-
Copeland 3 grades	0.93	0.97	0.96	0.96	0.92	0.98	0.96	1.00	-	-	-	-	-	-
Nanson grades	0.93	0.96	0.96	0.96	0.92	0.98	0.96	0.99	1.00	-	-	-	-	-
Duo-Simpson grades	0.92	0.95	0.98	0.98	0.90	0.97	0.98	0.99	0.99	1.00	-	-	-	-
Black grades	0.95	0.97	0.93	0.93	0.95	1.00	0.93	0.99	0.99	0.98	1.00	-	-	-
Inverse Borda grades	0.93	0.96	0.97	0.97	0.92	0.98	0.97	0.99	1.00	0.99	0.99	1.00	-	-
Markovian method	0.92	0.96	0.96	0.96	0.88	0.96	0.96	0.98	0.98	0.98	0.97	0.98	1.00	-
Threshold grade	0.74	0.84	0.87	0.87	0.77	0.84	0.86	0.86	0.86	0.86	0.85	0.86	0.88	1.00

Table 10 Correlation coefficients between the aggregated rankings and single-indicator-based rankings of journals in sociology

	Impact factor	Article influence score	SNIP	SJR	H-index	Borda grades	Hare grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Markovian method	Threshold grade
Impact factor	1.00	-	-	-	-	-	-	-	-	-	-	-	-	-
Article influence score	0.85	1.00	-	-	-	-	-	-	-	-	-	-	-	-
SNIP	0.76	0.78	1.00	-	-	-	-	-	-	-	-	-	-	-
SJR	0.87	0.86	0.87	1.00	-	-	-	-	-	-	-	-	-	-
H-index	0.86	0.81	0.70	0.84	1.00	-	-	-	-	-	-	-	-	-
Borda grades	0.93	0.93	0.88	0.96	0.89	1.00	-	-	-	-	-	-	-	-
Hare grades	0.91	0.88	0.84	0.90	0.91	0.93	1.00	-	-	-	-	-	-	-
Copeland 3 grades	0.93	0.93	0.88	0.97	0.89	0.99	0.94	1.00	-	-	-	-	-	-
Nanson grades	0.93	0.92	0.88	0.97	0.90	0.99	0.95	1.00	1.00	-	-	-	-	-
Duo-Simpson grades	0.93	0.93	0.87	0.97	0.90	0.99	0.95	1.00	1.00	1.00	-	-	-	-
Black grades	0.93	0.93	0.88	0.97	0.90	1.00	0.94	1.00	1.00	0.99	1.00	-	-	-
Inverse Borda grades	0.93	0.92	0.88	0.97	0.90	0.99	0.95	1.00	1.00	1.00	0.99	1.00	-	-
Markovian method	0.94	0.91	0.86	0.94	0.91	0.98	0.96	0.98	0.98	0.98	0.98	0.98	1.00	-
Threshold grade	0.92	0.92	0.87	0.95	0.87	0.98	0.93	0.98	0.98	0.98	0.98	0.98	0.99	1.00

Table 11 Correlation coefficients between the aggregated rankings and single-indicator-based rankings of journals in education

	Impact factor	Article influence score	SNIP	SJR	H-index	Borda grades	Hare grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Markovian method	Threshold grade
Impact factor	1.00	-	-	-	-	-	-	-	-	-	-	-	-	-
Article influence score	0.87	1.00	-	-	-	-	-	-	-	-	-	-	-	-
SNIP	0.82	0.80	1.00	-	-	-	-	-	-	-	-	-	-	-
SJR	0.86	0.91	0.88	1.00	-	-	-	-	-	-	-	-	-	-
H-index	0.83	0.81	0.73	0.82	1.00	-	-	-	-	-	-	-	-	-
Borda grades	0.93	0.94	0.90	0.96	0.91	1.00	-	-	-	-	-	-	-	-
Hare grades	0.92	0.89	0.87	0.92	0.90	0.95	1.00	-	-	-	-	-	-	-
Copeland 3 grades	0.93	0.95	0.90	0.97	0.89	1.00	0.96	1.00	-	-	-	-	-	-
Nanson grades	0.94	0.95	0.90	0.97	0.88	0.99	0.96	1.00	1.00	-	-	-	-	-
Duo-Simpson grades	0.93	0.95	0.90	0.97	0.88	0.99	0.96	1.00	1.00	1.00	-	-	-	-
Black grades	0.93	0.94	0.90	0.96	0.91	1.00	0.95	1.00	0.99	0.99	1.00	-	-	-
Inverse Borda grades	0.94	0.95	0.90	0.97	0.89	0.99	0.96	1.00	1.00	1.00	0.99	1.00	-	-
Markovian method	0.94	0.93	0.85	0.93	0.88	0.97	0.96	0.97	0.97	0.97	0.97	0.97	1.00	-
Threshold grade	0.92	0.93	0.87	0.94	0.87	0.97	0.94	0.97	0.97	0.97	0.97	0.97	0.99	1.00

Despite the increasing popularity of journal rankings to evaluate the quality of research contributions, the individual rankings for journals are usually feature only modest agreement. In this paper, five most popular bibliometric indices were used as initial empirical data: the impact factor, SNIP , SJR, article influence score, and Hirsch index. Correlation analysis of rankings for journals in education, sociology, and public administration in general reproduced the results of previous studies [3].

Nevertheless, despite the fact that the ratings, based on various indices, are very similar, there are significant discrepancies between them, and the selection of the rating that should be used for particular solutions is problematic.

Our purpose was to answer the question—whether the aggregated ratings, constructed using ordinal methods and models of social choice theory, the use of which eliminates the issue of homogeneity of different measurements, are more efficient tool for estimation than the individual ratings.

We have calculated ten rankings, using Hare's procedure, Borda's rule, Black's procedure, Nanson's procedure, three Copeland's rules, Simpson's procedure, Threshold procedure, and Markovian method.

Correlation analysis showed that the value of the correlation indices for each of the constructed aggregated rankings exceeds the values obtained by the comparison of the individual bibliometric indices, i.e., the transition from the initial ratings to aggregated ones is reasonable. In other words, the calculated rankings can serve as integral journal ratings. If the individual indices show less coherence, the aggregated values show high correlation with each other, which means that they are more effective.

Not all social choice ranking methods have been employed in this study. The next logical step would be to widen both the arsenal of aggregation techniques and the set of empirical data.

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Appendix

See Tables 12, 13, and 14.

Table 12 Ranks of sociology science journals in single-indicator-based and aggregate rankings (journals are ordered by journal impact factor)

	Journal impact factor	Article influence score	SNIP	H-index	SJR	Borda grades	Hare grades	Copeland 1 grades	Copeland 2 grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Threshold grade	Markovian method
American sociological review	1	2	1	2	2	1	2	2	2	2	2	2	2	2	2	1
American journal of sociology	2	3	3	4	3	3	3	3	3	3	3	3	3	3	3	2
Annual review of sociology	3	1	5	1	1	1	1	1	1	1	1	1	1	1	1	3
Annals of tourism research	4	51	28	22	11	14	5	10	10	10	10	10	13	11	19	4
Sociological theory	5	4	39	3	6	8	4	4	4	4	4	4	4	4	4	5
Population and development review	6	16	28	5	7	6	5	5	5	5	5	5	7	5	8	6
Sociological methods and research	7	7	39	36	20	13	10	13	13	13	13	14	14	14	10	7
Sociology of education	8	6	39	6	4	9	5	5	5	5	6	5	8	6	5	8
Social networks	9	5	14	11	10	5	5	8	8	8	8	5	5	8	6	9
Sociology of health and illness	10	26	21	37	34	18	11	23	22	25	26	23	18	26	31	10

Table 13 Ranks of education science journals in single-indicator-based and aggregate rankings (journals are ordered by journal impact factor)

	Journal impact factor	Article influence score	SNIP	H-index	SJR	Borda grades	Hare grades	Copeland 1 grades	Copeland 2 grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Threshold grade	Markovian method
Review of educational research	1	1	3	2	4	1	1	1	1	1	1	1	1	1	1	1
Educational psychologist	2	4	29	5	7	4	3	4	4	4	2	2	4	2	3	2
Educational research review	3	15	29	1	12	10	3	8	8	8	8	9	9	8	2	2
Learning and instruction	4	10	6	13	11	6	8	7	7	7	7	8	7	7	11	4
Journal of research in science teaching	5	7	2	9	2	3	3	2	2	2	2	2	3	2	4	5
Educational researcher	6	2	12	3	9	2	3	2	2	2	2	2	2	2	6	6
Science education	7	9	12	11	3	7	9	6	6	6	6	7	6	6	5	7
Journal of the learning sciences	8	8	91	12	13	11	11	13	13	12	14	13	11	9	9	8
Journal of engineering education	9	46	1	31	1	21	1	9	8	9	8	2	12	9	13	9
Advances in health sciences education	10	20	29	26	31	13	11	17	17	17	17	15	14	17	15	10

Table 14 Ranks of public administration science journals in single-indicator-based and aggregate rankings (journals are ordered by journal impact factor)

	Journal impact factor	Article influence score	SNIP	H-index	SJR	Borda grades	Hare grades	Copeland 1 grades	Copeland 2 grades	Copeland 3 grades	Nanson grades	Duo-Simpson grades	Black grades	Inverse Borda grades	Threshold grade	Markovian method
Journal of public administration research and theory	1	2	1	5	1	1	1	1	1	1	1	1	1	1	1	1
Policy studies journal	2	6	8	24	9	6	9	6	6	6	6	6	6	6	26	2
Journal of policy analysis and management	3	1	3	5	7	5	7	5	5	5	5	5	5	5	2	2
Public administration	4	7	7	3	4	3	4	4	4	4	4	4	4	4	25	4
Journal of European social policy	5	4	2	11	3	2	3	3	3	3	3	3	3	3	3	4
Climate policy	6	13	27	11	20	12	17	13	12	13	15	14	12	15	27	6
Journal of social policy	7	11	4	2	8	8	8	7	7	7	7	7	8	7	5	7
Governance	8	3	6	9	10	7	10	7	7	7	8	9	7	8	6	8
Policy sciences	9	10	21	3	16	11	16	12	12	11	13	13	11	13	12	9
Public management review	10	14	16	24	19	14	20	18	18	18	18	14	14	18	14	10

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On the Position Value for Special Classes of Networks

Giulia Cesari and Margherita Maria Ferrari

Abstract This paper deals with a particular class of TU-games, whose cooperation is restricted by a network structure. We consider a *communication situation* (or *graph game*) in which a network is produced by subsequent formation of links among players and at each step of the network formation process, the surplus generated by a link is shared between the players involved, according to some rule. As a consequence, we obtain a family of solution concepts that we investigate on particular network structures. This approach provides a different interpretation of the position value, introduced by Borm et al. (SIAM J Discret Math 5(3):305–320, 1992), since it turns out that a specific symmetric rule leads to this solution concept. Moreover, we investigate the problem of computing the position value on particular classes of networks.

Keywords TU-games • Networks • Communication situations • Coalition formation • Allocation protocols • Position value

1 Introduction

A TU-game (a cooperative game with transferable utility) also referred to as *coalitional game* describes a situation in which all players can freely interact with each other, i.e. every coalition of players is able to form and cooperate. However, this is not the case in many real world scenarios. A typical situation is when there exists

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a restriction on the communication possibilities among players, as in the context of social interactions between groups of people, political alliances within parties, economic exchange among firms, research collaborations and so on. In order to represent and study such situations it is necessary to drop the assumption that all coalitions are feasible. Then a natural question arises: *how can we model restrictions of the interaction possibilities between players?*

A typical way to do so is that of considering a *network structure*[1] that describes the interaction possibilities between the players: the nodes of the network are the players of the game and there exists a link between two nodes if the corresponding players are able to interact directly. In this context it is usual to refer to such networks as *communication networks*, since a typical situation they model is a restriction of communication possibilities between players.

This approach leads to the definition of a so-called *communication situation* [13] and to the search for solution concepts that take into account the constraints imposed by the underlying network structure. For the class of graph games, a crucial point is to study how the communication constraints influence the allocation rules. There are at least two ways to measure this impact that correspond to two different main streams in the recent literature.

In a first approach, the communication constraints determine how a coalition is evaluated. There is no actual restriction constraint on the set of feasible coalitions, but if a coalition is not connected through the communication graph, its worth is evaluated on the connected components in the induced graph. This approach is investigated in the seminal paper by Myerson [13], who introduces the Myerson value in order to generalize the Shapley value from TU-games to graph games. Jackson and Wolinsky [11] extend Myerson's model by considering a function assigning values to networks as a basic ingredient. Borm et al. [4] introduce the position value for communication situations. Like the Myerson value, the position value is based on the Shapley value, but it stresses the role of the pairwise connections in generating utility, rather than the role of the players. The value of a pairwise connection is derived as the Shapley value of a game on the set of links of the network and the position value equally divides the value of each link among the pair of players who form it. The position value has been extended in [19] to the setting of network situations introduced in [11] and an axiomatic characterization in this context is given in [21].

In a second approach, the communication constraints determine which coalitions can actually form. The definition of the Shapley value relies on the idea of a one-by-one formation of the grand coalition: its interpretation assumes that the players gather one by one in a room; each player entering the room gets his marginal contribution to the coalition that was already there and all the different orders in which the players enter are equiprobable. To take into account the communication constraints, the orderings of the players that induce disconnected coalitions are

[1] Other models introduced in the literature are discussed in [17], including extensions of the interaction channels to hypergraphs and probabilistic networks, among others.

ruled out: the formation of the grand coalition requires a communication at any stage. In order to satisfy the communication constraints Demange [5] proposes to model the sequential formation of the grand coalition by a rooted spanning tree of the communication graph. Each rooted spanning tree represents a partial order on the players set such that the arrival of a new player forms a connected coalition. Demange [5] introduces the hierarchical outcome in order to extend the concept of marginal contribution from orderings of the players to rooted spanning trees. This second approach is also studied by Herings et al. [9] who introduce the average tree solution for graph games in which the communication graph is a forest (cycle-free graph). This allocation is the average of the hierarchical outcomes associated with all rooted spanning trees of the forest. Hering et al. [10] and Baron et al. [2] show how an extension of the average tree solution to arbitrary graph games can be seen as another generalization of the Shapley value. In [3], the principle of compensation formulated by Eisenman [6] is generalized from orderings of the players to rooted spanning trees and the compensation solution for graph games is introduced.

Based on the idea that the formation of the grand coalition requires a communication at any stage, our approach is different in spirit with respect to the aforementioned models. We assume indeed a different mechanism of coalition formation which results from subsequent connection of links among players. This idea naturally leads to consider a communication situation where a network between the players is produced by a permutation of links and we suppose that, at each step of the network formation process, the surplus generated by a link is shared between the players involved according to a certain protocol. Taking into account this mechanism, we propose a class of solution concepts where each solution corresponds to a different allocation protocol. In particular, at a certain step when a link between two players forms, it is reasonable to equally share the surplus between the players that are responsible for this connection, i.e. the two nodes incident to the link that is formed. It turns out that the solution obtained by this particular allocation protocol is indeed the position value. Our model thus provides a different interpretation for this well-known solution concept and proposes a family of solution that embraces the basic principles of both the approaches described above, providing a bridge between two different ways of modeling the restriction of communication possibilities between players in a coalitional game.

The paper is structured as follows. In Sect. 2 we introduce basic definitions and notations regarding coalitional games and networks. Section 3 describes the concept of communication situation and related solutions in literature. In Sect. 4 we introduce the notion of allocation protocol and the class of solution concepts that derives. Section 5 presents some preliminary results and in Sect. 6 we give formulas for the position value on specific communication situations. Section 7 concludes the paper.

2 Games and Networks

In this section, we introduce some basic concepts and notations on coalitional games and networks.

A *coalitional game* is a pair (N, v), where N denotes the set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$. A group of players $S \subseteq N$ is called *coalition* and $v(S)$ is called the *value* or *worth* of the coalition S. If the set N of players is fixed, we identify a coalitional game (N, v) with the characteristic function v.

We shall denote by $\mathscr{G}$ the class of all coalitional games and by $\mathscr{G}^N$ the class of all coalitional games with players set N. Clearly, $\mathscr{G}^N$ is a vector space of dimension $2^n - 1$, where $n = |N|$. The *canonical basis* for this vector space is given by the family of canonical games $\{e_S, S \subseteq N\}$. The game e_S is defined as:

$$e_S(T) = \begin{cases} 1 \text{ if } S = T \\ 0 \text{ otherwise} \end{cases} \forall S \subseteq N, S \neq \emptyset.$$

It is possible to consider another basis for $\mathscr{G}^N$, the family of the *unanimity games* $\{u_S, S \subseteq N\}$, where u_S is defined as:

$$u_S(T) = \begin{cases} 1 \text{ if } S \subseteq T \\ 0 \text{ otherwise} \end{cases} \forall S \subseteq N, S \neq \emptyset.$$

Every coalitional game v can be written as a linear combination of unanimity games as follows:

$$v = \sum_{S \subseteq N, S \neq \emptyset} c_S(v) u_S, \tag{1}$$

where the constants c_S, referred to as *unanimity coefficients* of v, can be inductively defined in the following way: let $c_{\{i\}}(v) = v(\{i\})$ and, for $S \subseteq N$ of cardinality $s \geq 2$,

$$c_S(v) = v(S) - \sum_{T \subsetneq S, T \neq \emptyset} c_T(v). \tag{2}$$

An equivalent formula for the unanimity coefficients is given by:

$$c_S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|}\, v(T). \tag{3}$$

Given a coalitional game, it is usual in many applications to consider as a solution the Shapley value of the game. The Shapley value was introduced by Shapley [16] in the context of cooperative games with transferable utility. The approach followed by Shapley consists of providing a set of properties that a solution for TU-games should satisfy. A formula to compute the Shapley value is the following:

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \big(v(S \cup \{i\}) - v(S)\big), \ \forall i \in N, \tag{4}$$

where $s = |S|$ is the cardinality of coalition S and $n = |N|$.

The Shapley value belongs to the broader class of *semivalues*:

$$\Psi_i^{\mathbf{p}}(v) = \sum_{S \subseteq N \setminus \{i\}} p_s^n \big(v(S \cup \{i\}) - v(S)\big) \ \ \forall i \in N, \tag{5}$$

where $n = |N|$, $s = |S|$ and p_s^n is such that $p_s^n \geq 0$ for all $s = 0, 1, \ldots, n-1$, $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s^n = 1$ and represents the probability that a coalition of size $s+1$ forms. Furthermore, if $p_s^n > 0$ for all s, then the semivalue is called *regular semivalue*. We shall write p_s instead of p_s^n when there is no ambiguity about the players set. In particular, the Shapley value is a regular semivalue with

$$p_s = \frac{1}{n\binom{n-1}{s}}. \tag{6}$$

and the *Banzhaf index* [1] is defined by (5) with $p_s = \frac{1}{2^{n-1}}$.

Another class of regular semivalues is the one of the *q-binomial semivalues* [14] Ψ^q, where

$$p_s = q^s (1-q)^{n-s-1} \tag{7}$$

with $q \in [0, 1]$ (by convention, we take $0^0 = 1$ if $q = 0$ or $q = 1$). In particular if $q = 0$ we obtain the *dictatorial index* $\Psi_i^0(v) = v(\{i\})$, while if $q = 1$ we get the *marginal index* $\Psi_i^1(v) = v(N) - v(N \setminus \{i\})$. Note that $q = 1/2$ gives $\Psi^{1/2} = \beta$, the Banzhaf value.

An undirected *graph* or *network* Γ is the pair (V, E), where V is a set of nodes and $E \subseteq \{\{i, j\} : i, j \in V, i \neq j\}$ is the set of links between the nodes.

We denote by $deg(i)$ the degree of a node $i \in V$, i.e. the number of links incident to i in Γ. Given a subset $S \subseteq V$ of nodes, we define the induced subgraph $\Gamma_S = (S, E_S)$, where E_S is the set of links $\{i, j\} \in E$ such that $i, j \in S$. Similarly, we denote by Γ_A the graph (V_A, A) induced by a subset $A \subseteq E$ of links, where V_A is the set of nodes incident to at least one link of A.

A *path* between i and j in a graph Γ is a sequence of distinct nodes $(i_0, i_1, \cdots, i_k)$ such that $i_0 = i$, $i_k = j$ and $\{i_s, i_{s+1}\} \in E \ \forall s = 0, \ldots, k-1$. Two nodes i and j are said to be *connected* in Γ if $i = j$ or if there is a path between them in Γ. We call *chain* the set of nodes on a path with different endpoints and we denote by *s-chain* a chain with s nodes. A *connected component* in Γ is a maximal subset of V with the property that any two nodes of V are connected in Γ. We denote by C_Γ the set of connected components in Γ. A graph Γ is said to be *connected* if there exists a path between every two elements of V. A subset of nodes $S \subseteq V$ (respectively, a set of links $A \in E$) is *connected* if the induced graph Γ_S (respectively, Γ_A) is connected.

A *cycle* in Γ is a path $(i_0, i_1, \cdots, i_k)$ such that $i_0 = i_k$. A *forest* is a graph without cycles. A *tree* is a forest with only one connected component.

3 Cooperative Games with Restricted Communication: The Position Value

A coalitional game describes a situation in which every coalition of players is able to form and cooperate. If there exists a restriction on the interaction possibilities among players, not all coalitions are feasible. We can represent this situation by introducing a network structure that models the interactions between players. This leads to the definition of a communication situation.

Given a graph Γ and a coalitional game (N, v) we can define the so-called *communication situation* [13] as the triple (N, v, Γ), where N is the set of players, (N, v) is a coalitional game and Γ is an undirected graph with N as set of vertices. The graph $\Gamma = (N, E)$ describes the communication possibilities between players: an indirect communication between i and j is possible if there is a path that connects them; if $\{i, j\} \in E$, then i and j can communicate directly.

Borm et al. [4] introduce a solution concept for a communication situation based on the approach of Meessen [12]: given $\Gamma = (N, E)$ and $A \subseteq E$, the *link game* v^L is defined by:

$$v^L(A) = \sum_{T \in C_{\Gamma_A}} v(T), \tag{8}$$

where C_{Γ_A} is the set of connected components in Γ_A. We denote by $\mathscr{G}_L$ the vector space of all link games on $\Gamma = (N, E)$, $E \subseteq \{\{i, j\} : i, j \in V, i \neq j\}$, where N is a fixed set of players. Note that the dimension of $\mathscr{G}_L$ is equal to the number of connected subsets of E; i.e. the cardinality of $\{A \subseteq E : \Gamma_A \text{ is connected}\}$.

Every link game v^L can be written as a linear combination of unanimity link games as follows:

$$v^L = \sum_{A \subseteq E} c_A(v^L) u_A, \tag{9}$$

where c_A are the unanimity coefficients of v^L:

$$c_A(v^L) = \sum_{B \subseteq A} (-1)^{|A|-|B|}\, v^L(B); \tag{10}$$

or equivalently $c_{\{l\}}(v^L) = v^L(\{l\})$ and for $A \subseteq E$, $|A| \geq 2$:

$$c_A(v^L) = v(A) - \sum_{B \subsetneq A, B \neq \emptyset} c_B(v^L). \tag{11}$$

Given a communication situation (N, v, Γ), the *position value* $\pi(N, v, \Gamma)$ is defined as:

$$\pi_i(N, v, \Gamma) = \frac{1}{2} \sum_{a \in A_i} \Phi_a(v^L) \ \forall i \in N, \tag{12}$$

where $A_i = \{\{i,j\} \in E, j \in N\}$ is the set of all links for which player i is an endpoint. Note that, since the players in v^L are the elements of E, i.e. the links of Γ, in formula (12) we compute the Shapley value of a link. We shall write $\pi(v)$ when there is no ambiguity about the underlying network.

We point out here a particular property satisfied by the position value that will be useful for our purpose, namely the *superfluous arc property* [21]. Given a communication situation (N, v, Γ), with $\Gamma = (N, E)$, we call *superfluous* a link a such that $v^L(A \cup \{a\}) = v^L(A) \ \forall A \subseteq E$.

The superfluous arc property states that if a is a superfluous arc, then $\pi(N, v, \Gamma) = \pi(N, v, \Gamma')$, where $\Gamma' = (N, E \backslash \{a\})$. The property follows directly by formula (12): the links (or arcs) that provide a marginal contribution equal to zero to every coalition of links (not containing the link itself) do not give contribution to the sum in (12), thus the position value does not change if they are removed from the network.

Note that, like the Shapley value, every semivalue $\Psi^{\mathbf{p}}$ induces a solution concept $\psi^{\mathbf{p}}$ for communication situations given by:

$$\psi_i^{\mathbf{p}}(N, v, \Gamma) = \frac{1}{2} \sum_{a \in A_i} \Psi_a^{\mathbf{p}}(v^L). \tag{13}$$

We write $\psi(v)$ when there is no ambiguity about the underlying network. Note that, by definition of semivalue, the superfluous arc property still holds for every solution ψ corresponding to a given semivalue.

See [18, 19, 21] for an axiomatic characterization of the position value for network situations, which generalize the context of communication situations.

4 Coalition Formation and Allocation Protocols

In Sect. 2 we introduced the Shapley value and gave a formula to compute it. Formula (4) has the following interpretation: suppose that the players gather one by one in a room to create the grand coalition. Each player entering the room gets his marginal contribution to the coalition that was already in the room. Assuming that all the different orders in which they enter are equiprobable, one gets the formula, where $n!$ is the number of permutations on a set of n elements.

Let us consider a different mechanism of coalition formation: let us assume that a coalition forms by subsequent formation of links among players. This naturally leads to consider a communication situation, where a network between the players is produced by a permutation of links and all the different orders in which the links form are considered to be equiprobable.

In this scenario, we can imagine that, when a link between two players forms, the players that are connected to each other receive a certain value according to some rule. Let us suppose that, at each step of the network formation process, when a link between two players i and j forms, the value of the coalition S, where S is the connected component containing i and j, reduced by the values of the connected components formed by the players of S at the previous step, is shared between the players involved according to a certain protocol. Then a natural question rises: *How to share this value?*

Given a communication situation (N, v, Γ), let us consider a possible permutation σ of links. At each step k of the network formation process, when the k-th link $a = \{i, j\}$ in the sequence determined by σ forms, let us consider the *surplus* produced by a:

$$S_k^\sigma = v(S) - v(C^i_{k-1,\sigma}) - v(C^j_{k-1,\sigma}) \tag{14}$$

where S is the connected component in Γ containing i and j at the step k, and $C^i_{k-1,\sigma}$ and $C^j_{k-1,\sigma}$ are the connected components in Γ at the step $k-1$, containing i and j, respectively.

An *allocation protocol* is a rule that specifies how to divide S_k^σ between the players in S. Given an allocation protocol r and a communication situation (N, v, Γ), a solution of v, that we shall denote by $\phi^r(v)$, is given by:

$$\phi_i^r(v) = \frac{1}{|E|!} \sum_{\sigma \in \Sigma_E} \sum_{k=0}^{|E|} f_i^r(S_k^\sigma), \ \forall i \in N, \tag{15}$$

where Σ_E is the set of possible orders on the set of links E in Γ and f_i^r is a function that assigns to each player $i \in N$ a fixed amount of the surplus S_k^σ, depending on the allocation protocol r.

In other words, the solution $\phi^r(v)$ is computed by considering all possible permutations of links, and summing up, for each player i, all the contributions he gets with the allocation procedure r, averaged by the number of permutations over the set of links among the players, with the interpretation discussed at the beginning of this section.

This idea leads to the introduction of a class of solution concepts: different choices of the allocation protocol define different solutions for a communication situation. At a certain step, when a link $a = \{i, j\}$ forms, it is possible to consider the allocation protocol that equally divides the surplus between players i and j only. The solution obtained by this particular allocation protocol is indeed the position value π defined in (12).

Note that other solution concepts can be achieved by sharing the surplus among the players involved in a different way.

5 Preliminary Results

In this section we present some preliminary results that will be useful in the next sections.

Proposition 1. *Let (N, v, Γ) be a communication situation and v^L the corresponding link game. Then $c_A(v^L) = 0$ for any coalition $A \subseteq E$ which is not connected in Γ, where $c_A(v^L)$ are the unanimity coefficients of v^L.*

Proof. We prove the result by induction on $a = |A|$.

Suppose $a = 2$, i.e. $A = \{l_1, l_2\}, l_1, l_2 \in E$, where l_1 and l_2 belong to two different connected components. From this hypothesis and from (8) and (11) we get

$$\begin{aligned} v^L(A) &= v(\{l_1\}) + v(\{l_2\}) \\ &= c_{\{l_1\}}(v^L) + c_{\{l_2\}}(v^L) \end{aligned} \tag{16}$$

and

$$\begin{aligned} v^L(A) &= \sum_{B \subseteq A} c_B(v^L) \\ &= c_{\{l_1\}}(v^L) + c_{\{l_2\}}(v^L) + c_A(v^L). \end{aligned} \tag{17}$$

By comparing (16) and (17), we get that $c_A(v^L) = 0$.

Let us now consider $k \geq 2$ and suppose by inductive hypothesis that $c_B(v^L) = 0, \forall B$ such that $|B| \leq k$ and B is not connected in Γ. We shall prove that $c_A(v^L) = 0\ \forall A$ such that A is not connected and $|A| = k + 1$. Let $B_1 \subset A$ be a connected component in Γ, i.e. $B_1 \in C_{\Gamma_A}$. Then by hypothesis $A \backslash B_1 \neq \emptyset$. It follows that:

$$v^L(A) = v^L(B_1) + v^L(A \backslash B_1). \tag{18}$$

Moreover it holds:

$$
\begin{aligned}
v^L(A) &= \sum_{B\subseteq A} c_B(v^L) \\
&= \sum_{B\subseteq B_1} c_B(v^L) + \sum_{B\subseteq A\setminus B_1} c_B(v^L) + \sum_{B\subseteq A:A\cap B_1\neq\emptyset\wedge B\cap(A\setminus B_1)\neq\emptyset} c_B(v^L) \\
&= v^L(B_1) + v^L(A\setminus B_1) + \sum_{B\subsetneq A:B\cap B_1\neq\emptyset\wedge B\cap(A\setminus B_1)\neq\emptyset} c_B(v^L) + c_A(v^L) \\
&= v^L(B_1) + v^L(A\setminus B_1) + c_A(v^L), \qquad (19)
\end{aligned}
$$

where (19) follows from the inductive hypothesis.

Then, by comparing (18) and (19) we get: $c_A(v^L) = 0$, which ends the proof. □

Note that an equivalent result has been proved by Van den Nouweland et al. [21] for a value function (i.e. a characteristic function over subsets of links).

Corollary 1. *The family of unanimity games $\{u_A, A \subseteq E$, where A is connected in $\Gamma\}$ is a basis for $\mathscr{G}_L$*

Proof. From Proposition 1, we get that $\{u_A, A \subseteq E$, where A is connected in $\Gamma\}$ is a spanning set for the vector space $\mathscr{G}_L$. Moreover the cardinality of this set is equal to the dimension of $\mathscr{G}_L$. □

Equivalent results hold in the context of graph-restricted games and the proofs can be found, for example, in [8]. Moreover, the previous results hold for a generic value function v that satisfies the *component additivity property*, i.e. such that $v(A) = \sum_{T\in C_{\Gamma_A}} v(T)$ for every network Γ over the set of nodes N.

6 The Position Value on Particular Classes of Communication Situations

In general, given a communication situation (N, v, Γ) it is not easy to compute the position value. However, it is so for particular classes of games and graphs. In this section we give formulas to compute the position values on particular classes of communication situations, where the underlying network is described by a tree or a cycle. We assume w.l.o.g throughout our work that $v(\{i\}) = 0\ \forall i \in N$.

6.1 The Position Value on Trees

Let (N, v, Γ) be a communication situation, where $\Gamma = (N, E)$ is a tree and $|N| = n$. Given a node $i \in N$ and a coalition $S \subseteq N$, we define $fringe(S) = \{j \in N \backslash S$ such that $\{i, j\} \in E$ for some $i \in S\}$. Let $f(S) := |fringe(S)|$, $deg_S(i)$ the degree of i in S, i.e. the number of nodes in S that are directly connected to i in Γ and $deg_{fringe(S)}(i)$ the number of nodes in $fringe(S)$ that are directly connected to i in Γ.

We provide a formula for the position value on e_S, with $S \subseteq N$ connected in Γ such that $|S| \geq 2$. If S is not connected, it doesn't make sense to consider the position value of e_S, since the associated link game e_S^L is the null game.

Proposition 2. *Let $S \subseteq N$ connected in Γ, where Γ is a tree and $|S| \geq 2$. Then the position value on the canonical game e_S is given by:*

$$\pi_i(e_S) = \begin{cases} \frac{1}{2}\frac{(s-2)!(f(S)-1)!}{(m-1)!}\,\delta_i(s) & \textit{if } i \in S \\ \\ -\frac{1}{2}\frac{(s-1)!(m-s-1)!}{(m-1)!} & \textit{if } i \in fringe(S) \\ \\ 0 & \textit{otherwise} \end{cases} \tag{20}$$

where $m = s + f(S)$ and $\delta_i(s) = f(S)deg_S(i) - (s-1)deg_{fringe(S)}(i)$.

Proof. We observe that every link not in $E_{S \cup fringe(S)}$ is superfluous. Therefore $\pi_i(e_S) = 0$ for every $i \notin S \cup fringe(S)$ and we can reduce the network to $(S \cup fringe(S), E_{S \cup fringe(S)})$.

Consider $i \in S$. Node i gets a positive contribution (equal to $1/2$) every time a link incident to it is the last one to form inside S and no link outside S already formed. This happens $deg_S(i)$ times. Moreover it gets a negative contribution (equal to $-1/2$) when all the links in S already formed and a link incident to i is the first one to form outside S. This happens $deg_{fringe(S)}(i)$ times. Therefore we get

$$\pi_i(e_S) = \frac{1}{2}\Big[\frac{(s-2)!(m-s)!}{(s-1)!}deg_S(i) - \frac{(s-1)!(m-s-1)!}{(m-1)!}deg_{fringe(S)}(i)\Big],$$

where $m = s + f(S)$ and the expression (20) follows directly.

Consider $i \in fringe(S)$. Node i gets a negative contribution when all the links in S already formed and the only link that connects i to S is the first one to form outside S. Therefore we get

$$\pi_i(e_S) = -\frac{1}{2}\frac{(s-1)!(m-s-1)!}{(m-1)!}.$$

□

Note that the formula holds also for $S = N$, with $fringe(N) = \emptyset$, which implies that $f(N) = 0$. On the other hand, when $S = \{i\}$, the associated link game e_S^L is the null game, as for S not connected.

Let us consider the unanimity games $\{u_S, S \subseteq N\}$. We also provide a formula for the position value on u_S, with $S \subseteq N$ connected in Γ such that $|S| \geq 2$.

Proposition 3. *Let $S \subseteq N$ connected in Γ, where Γ is a tree and $|S| \geq 2$. Then the position value on the unanimity game u_S is given by:*

$$\pi_i(u_S) = \begin{cases} \frac{1}{2} deg_s(i) \frac{1}{s-1} \ \ if\ i \in S \\ \\ 0 \qquad\qquad\qquad otherwise. \end{cases} \tag{21}$$

Proof. We observe that every link not in E_S is superfluous. Therefore $\pi_i(u_S) = 0$ for every $i \notin S$ and we can reduce the network to (S, E_S).

Consider $i \in S$. Node i gets a positive contribution (equal to $1/2$) every time a link incident to it is the last one to form inside S. This happens $deg_S(i)$ times. Therefore we get

$$\pi_i(e_S) = \frac{1}{2} \frac{(s-2)!(m-s)!}{(m-1)!} deg_S(i),$$

where $m = s$ and the result follows directly. □

Moreover, if $S = \{j\}$, easy calculations show that:

$$\pi_i(u_S) = \begin{cases} \frac{1}{2} & \text{if } i = j \\ \frac{1}{2f(S)} & \text{if } i \in fringe(\{j\}) \\ 0 & \text{otherwise.} \end{cases}$$

6.2 The Position Value on Cycles

Let (N, v, Γ) be a communication situation, where $\Gamma = (N, E)$ is a cycle and $|N| = n$.

We provide a formula for the position value on e_S, where $S \subseteq N$ is a s-chain (i.e. S is connected in Γ) with $2 \leq s \leq n-2$. If S is not connected, or $S = \{i\}$, it doesn't make sense to consider the position value of e_S, since the associated link game e_S^L is the null game.

Proposition 4. *Let $S \subseteq N$ be a s-chain in Γ, where Γ is a cycle and $2 \leq s \leq n-2$. Then the position value on the canonical game e_S is given by:*

$$\pi_i(e_S) = \begin{cases} \frac{1}{2}\frac{(s-2)!(m-s-1)!}{(m-1)!}(m-2s+1) & \textit{if } i \in S_e \\ \frac{(s-2)!(m-s)!}{(m-1)!} & \textit{if } i \in S_i \\ -\frac{1}{2}\frac{(s-1)!(m-s-1)!}{(m-1)!} & \textit{if } i \in \textit{fringe}(S) \\ 0 & \textit{otherwise} \end{cases} \quad (22)$$

where $m = s + f(S)$*,* S_e *is the set of the extremal nodes and* $S_i = S \backslash S_e$ *is the set of the internal nodes.*

Proof. We observe that every link not in $E_{S \cup fringe(S)}$ is superfluous. Therefore $\pi_i(e_S) = 0$ for every $i \notin S \cup fringe(S)$ and we can reduce the network to $(S \cup fringe(S), E_{S \cup fringe(S)})$.

Consider $i \in S$. We shall distinguish between the internal and extremal nodes of the chain S. Let $i \in S_e$ the set of endpoints in Γ_S. Node i gets a positive contribution when the link incident to it in the chain is the last one to form inside S and no link outside S already formed. Moreover it gets a negative contribution when all the links in S already formed and the link incident to i in *fringe*(S) is the first one to form outside S. Therefore we get

$$\pi_i(e_S) = \frac{1}{2}\Big[\frac{(s-2)!(m-s)!}{(m-1)!} - \frac{(s-1)!(m-s-1)!}{(m-1)!}\Big],$$

where $m = s + f(S)$ and the expression (22) follows directly.

Let $i \in S_i = S \setminus S_e$. Node i gets a positive contribution when one of the two links incident to it in S is the last one to form inside S. Therefore we get

$$\pi_i(e_S) = 2\Big[\frac{1}{2}\frac{(s-2)!(m-s)!}{(m-1)!}\Big].$$

Consider $i \in fringe(S)$. Node i gets a negative contribution when all the links in S already formed and the only link that connects i to S is the first one to form outside S. Therefore we get

$$\pi_i(e_S) = -\frac{1}{2}\frac{(s-1)!(m-s-1)!}{(m-1)!}.$$

□

On the other hand, if $S = N \backslash \{j\}$, the following proposition holds:

Proposition 5. *Let* $S \subseteq N$ *be a s-chain in* Γ*, where* Γ *is a cycle and* $s = n - 1$*. Then the position value on the canonical game* e_S *is given by:*

$$\pi_i(e_S) = \begin{cases} \frac{4-n}{2n(n-1)(n-2)} & \textit{if } i \in S_e \\ \\ \frac{2}{n(n-1)(n-2)} & \textit{if } i \in S_i \\ \\ -\frac{1}{n(n-1)} & \textit{if } i \in \textit{fringe}(S) \end{cases} \tag{23}$$

where S_e is the set of the extremal nodes and $S_i = S \backslash S_e$ is the set of the internal nodes.

Proof. Using the same argument of the previous proof, formulas for $i \in S$ are derived by noting that there is no superfluous link and $m = n$. Moreover, the only node $i \in \textit{fringe}(S)$ gets twice the contribution he gets in the previous case since it is directly connected to S by its incident links. □

Note that if $S = N$, there is no superfluous link and by symmetry $\pi_i(e_S) = \frac{1}{n}$, for all $i \in N$.

We provide a formula for the position value on u_S, with $S \subseteq N$ an s-chain. If $2 \leq |S| \leq n-1$, the following proposition holds:

Proposition 6. *Let $S \subseteq N$ be a s-chain in Γ, where Γ is a cycle and $2 \leq s \leq n-1$. Then the position value on the unanimity game u_S is given by:*

$$\pi_i(u_S) = \begin{cases} \frac{1}{2}\left[\frac{(n-s+1)}{n(s-1)} + (2s-3)\frac{1}{n(n-1)}\right] & \textit{if } i \in S_e \\ \\ \frac{1}{2}\left[2\frac{(n-s+1)}{n(s-1)} + 2(s-2)\frac{1}{n(n-1)}\right] & \textit{if } i \in S_i \\ \\ (s-1)\frac{1}{n(n-1)} & \textit{if } i \notin S \end{cases}$$

where S_e is the set of the extremal nodes, i.e. the endpoints in Γ_S, and $S_i = S \backslash S_e$ is the set of the internal nodes.

Proof. We observe that there is no superfluous link. Consider $i \in S_e$. Node i gets a positive contribution (equal to $1/2$) every time the link incident to it in the chain is the last one to form inside S (no matter which links already formed outside S).

Moreover it gets a positive contribution when the link incident to it outside the chain is the last one to form in $E \setminus \{a\}$, where a is the link incident to i in the chain S and every time one of the two links incident to i is the last one to form in $E \setminus \{b\}$, where b is one of the links in the chain S not incident to i. Note that the first case happens $\sum_{k=0}^{n-s} \binom{n-s+1}{k}$ times; the second one only occurs once and the last case happens $2(s-2)$ times. This yields the following formula for $i \in S_e$:

$$\pi_i(u_S) = \frac{1}{2}\left[\sum_{k=0}^{n-s} \binom{n-s+1}{k} \frac{(s-2+k)!(n-s-k+1)!}{n!} + (2s-3)\frac{1}{n(n-1)}\right].$$

Consider $i \in S_i$. Node i gets a positive contribution (equal to $1/2$) every time one of the two links incident to it in the chain is the last one to form inside S (no matter which links already formed outside S). Moreover it gets a positive contribution whenever one of the two links incident to it is the last one to form in $E \setminus \{a\}$, where a is other link incident to i in the chain S and every time one of the two links incident to i is the last one to form in $E \setminus \{b\}$, where b is one of the links in the chain S not incident to i. Note that the first case happens $2\sum_{k=0}^{n-s}\binom{n-s+1}{k}$ times; the second one only occurs twice and the last case happens $2(s-3)$ times.

Consider $i \notin S$. Node i gets a positive contribution (equal to $1/2$) every time one of the two links incident to it is the last one to form in $E \setminus \{a\}$, where a is one of the links in the chain S. This happens $2(s-1)$ times. It follows that:

$$\pi_i(u_S) = \begin{cases} \frac{1}{2}\Big[\sum_{k=0}^{n-s}\binom{n-s+1}{k}\frac{(s-2+k)!(n-s-k+1)!}{n!} + (2s-3)\frac{1}{n(n-1)}\Big] & \text{if } i \in S_e \\ \frac{1}{2}\Big[2\sum_{k=0}^{n-s}\binom{n-s+1}{k}\frac{(s-2+k)!(n-s-k+1)!}{n!} + 2(s-2)\frac{1}{n(n-1)}\Big] & \text{if } i \in S_i \\ (s-1)\frac{1}{n(n-1)} & \text{if } i \notin S \end{cases}$$

This formula can be simplified by using Lemma (1) (see Appendix):

$$\begin{aligned} &\sum_{k=0}^{n-s}\binom{n-s+1}{k}\frac{(s-2+k)!(n-s-k+1)!}{n!} \\ &= \frac{1}{n}\sum_{k=0}^{n-s}\binom{n-s}{k}\frac{n-s+1}{n-s-k+1}\frac{(s-2+k)!(n-s-k+1)!}{(n-1)!} \\ &= \frac{n-s+1}{n}\sum_{k=0}^{n-s}\binom{n-s}{k}\frac{(s-2+k)!(n-s-k)!}{(n-1)!} \\ &= \frac{(n-s+1)}{n(s-1)}, \end{aligned} \tag{24}$$

where (24) follows from identity (27). This ends the proof. □

Note that if $S = N$, all players are symmetric and $\pi_i(u_S) = \frac{1}{n}$. On the other hand, if $S = \{j\}$, the position value is very easy to compute. In fact, the links $a = \{i,j\}$ and $b = \{j,k\}$ are symmetric players in the link game, while all the remaining links are superfluous. This implies that $\Phi_a = \Phi_b = \frac{1}{2}$ and $\Phi_c = 0\ \forall c \in E\setminus\{a,b\}$.

$$\pi_i(u_S) = \begin{cases} 1/2 \text{ if } i = j \\ 1/4 \text{ if } i \neq j, \{i,j\} \in E \\ 0 \quad \text{otherwise.} \end{cases}$$

6.3 The Position Value for a Generic Coalitional Game

In the last two sections we provided formulas for the position value on particular classes of games. We shall use those formulas and the results of Sect. 5 to derive an expression that shows the relation between the position value for a generic game and the position value of unanimity games.

Proposition 7. *Let (N, v, Γ) be a communication situation. Then the position value for $i \in N$ is given by*

$$\pi_i(v) = \sum_{A \subseteq E \text{ connected}} c_A(v^L)\pi_i(w), \tag{25}$$

where w is such that $w^L = u_A$.

Proof. By definition of position value and by Corollary 1 we get that:

$$\pi_i(v) = \frac{1}{2}\sum_{a \in A_i} \Phi_a(v^L) = \frac{1}{2}\sum_{a \in A_i}\sum_{A \subseteq E \text{ connected}} c_A(v^L)\Phi_a(u_A)$$
$$= \sum_{A \subseteq E \text{ connected}} c_A(v^L)\pi_i(w)$$

where w is such that $w^L = u_A$. □

This result implies that, in order to compute the position value of a generic game, we have to consider the position value on those games whose corresponding link game is a unanimity game on a connected subset of links.

However, when Γ is a tree, the formula (25) can be simplified and a direct relation between the position value for a generic game and the position value of unanimity games can be obtained.

Corollary 2. *Let (N, v, Γ) be a communication situation, where Γ is a tree. Then the position value for $i \in N$ is given by*

$$\pi_i(v) = \sum_{S \subseteq N \text{ connected}} c_{E_S}(v^L)\pi_i(u_S). \tag{26}$$

Proof. Consider $A \subseteq E$ connected in Γ. Let S be the set of nodes in Γ_A. This definition of S induces a bijection between the set $\{w : w^L = u_A, A \text{ connected in } \Gamma\}$ and the set $\{u_S : S \subseteq N, S \text{ connected in } \Gamma\}$. Therefore the result follows directly. □

However, the computation of the position value for a generic game remains difficult even if the underlying graph is a tree. Indeed, deriving the position value using formula (26) requires listing all subtrees of a tree (the problem has been extensively addresses in the literature, see, for example, [7, 15, 20] without) and computing the corresponding unanimity coefficients.

7 Conclusions

In this paper we proposed a family of solution concepts for communication situations that embraces the principles of the two main approaches existing in the related literature. We also provided a different interpretation of the position value, as the solution concept arising from a particular symmetric allocation protocol, which prescribes how to share the surplus generated by a link among the players involved in the network formation process. Moreover, we provide an expression for the position value of a game when the underlying network is a tree, which relates its computation to the one for unanimity games.

The computation of the position value and its complexity remains an open problem, which has not been studied in the literature and deserves, in our opinion, further investigation. Another interesting direction for future research is to provide a characterization of the family of solution concepts we introduced, based on some reasonable properties that an allocation protocol should satisfy. Moreover, it would be interesting to investigate the relationship between different allocation protocols and known solution concepts, besides the position value.

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Appendix

Lemma 1. *Given $n, s \in \mathbb{N}$ and $2 \le s \le n$, the following combinatorial identity holds:*

$$\sum_{k=0}^{n-s} \binom{n-s}{k} \frac{(s-2+k)!(n-s-k)!}{(n-1)!} = \frac{1}{s-1}. \tag{27}$$

Proof. To derive the position value on trees for the unanimity games u_S, we made use of the superfluous arc property. An equivalent formula can be obtained directly by considering all the possible coalitions to whom a given link provides a positive marginal contribution.

Each $i \notin S$ gets a null contribution from every incident link, because of the superfluous arc property. Consider $i \in S$. Node i gets a positive contribution (equal to $1/2$) every time a link incident to it is the last one to form inside S (no matter which links already formed outside S). This happens $deg_S(i)$ times. It follows that:

$$\pi_i(u_S) = \begin{cases} \frac{1}{2} deg_S(i) \sum_{k=0}^{n-s} \binom{n-s}{k} \frac{(s-2+k)!(n-s-k)!}{(n-1)!} & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

From the equivalence of formulas (21) and (28), the result follows directly. □

We point out that other combinatorial identities arise by considering a generic regular semivalue Ψ and computing the corresponding $\psi(N, u_S, \Gamma)$ as in (13), when Γ is a tree.

As for the position value, the solution $\psi(u_S)$ can be obtained directly or by using the superfluous arc property. From the equivalence of the corresponding formulas, it follows that:

$$\sum_{k=0}^{n-s} \binom{n-s}{k} p_{s+k-2}^{n-1} = p_{s-2}^{s-1} \tag{29}$$

where $\{p_j^m\}_j$ is a probability distribution over the subsets of links in a network with m links. Precisely, p_j^m represents the probability for a link to join a coalition of cardinality j, with $0 \leq j \leq n-1$.

For example, by considering the Banzhaf index, we get the trivial identity

$$\sum_{k=0}^{n-s} \binom{n-s}{k} \frac{1}{2^{n-2}} = \frac{1}{2^{s-2}}. \tag{30}$$

Non-trivial identities can be derived by considering other regular semivalues, such as the p-binomial semivalues:

$$\sum_{k=0}^{n-s} \binom{n-s}{k} q^{s+k-2}(1-q)^{n-s-k} = q^{s-2}, \tag{31}$$

where $q \in (0, 1)$.

Note that the combinatorial identities that we derived can be easily obtained through classical game-theoretical tools by computing the corresponding power indices on the unanimity games.

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A Differential Game of a Duopoly with Network Externalities

Mario Alberto García-Meza and José Daniel López-Barrientos

Abstract In this work, we develop a differential game of a duopoly where two firms compete for market share in an industry with Network Externalities. Here the evolution of the market share is modeled in such a way that the effects of advertising efforts that both firms make are a function of the share itself. This means that the efficacy of marketing efforts are diminished with low market share and enhanced when it is higher. We show that Network Externalities can influence the decision a firm makes about marketing expenditures. Particularly, when a firm is large enough, the creation of a monopoly is easier when this market structure is present. For this, we obtain the optimal strategies for the firms and test them on a simulation, where we compare the market with and without this kind of externalities. We find that the value of the market share in proportion with the cost of obtaining it by advertising efforts is the key to know the long term equilibrium market share.

Keywords Differential games • Advertising competition • Network externalities

1 Introduction

There are many imperfections that can affect the structure of a market. Among these, *Network Externalities* is a class that cannot be ignored, since many important markets, such as telecommunications and software, are under its influence. Network Externalities (NE) emerge when the user's utility from consuming a certain product is a function of the number of people that use the same brand [5, 13].

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When some industry presents this kind of particularities, we must expect that the strategies from incumbents and entrants differ from those in a standard market, in order to be adapted to that particular environment. Network Externalities (also known as *Network Effects* or *demand-side economies of scale*) are found in a large variety of markets from telecommunications to computer software.

The telecommunications market is just one example of *direct* externalities. In [11], the authors study the existence of NE in the wireless telecommunication market in Europe and North America and the way their governments dealt with them. Whereas Europe used mandated standards, North America opted to let the market determine its own structure, yielding a worse result in terms of market concentration.

Computer software, on the other hand, is an example of an *indirect* type of NE. There is plenty of evidence for this kind of externality in a market [10, 12, 14]. We can find another example of indirect NE in DVD's market (formerly VCR's market) in [3, 16, 17], where standards are shown to affect the market. In video games [2, 18, 20], for instance, it is clearer the fact that two-sided markets can create Network Externalities as well.

The existence of NE in the market, in direct or indirect forms, means that the strategies the firms use must be adapted to be more akin with the reality they face. In particular, we might find that these effects act as barriers of entry that give an advantage to the incumbent and make a more expensive entry for new firms in the market. We represent this situation by means of a variation of the well-known Lanchester model (see, for instance, [4, 21]).

The way the structures make such an effect in the market is reviewed in Sect. 2. Here, we lay the ground needed to state that NE act as barriers of entry, protecting incumbents from new firms and making the permanence in the market more expensive for the latter.

In Sect. 3 we dwell on the details of the model (1) and (2) below, a model of a duopoly under NE, whose market dynamics presents barriers of entry created by NE. These barriers of entry are modeled by making the effectiveness of the marketing efforts dependent on the size of the market share. In this section, we find the analytical solution for the game by means of Pontryagin's Maximum Principle (PMP) to get optimal marketing expenditures and the evolution of the market share under these controls. It is important to acknowledge the fact that, although PMP yields just open-loop controls, we have chosen this technique over, for instance, the Principle of Dynamic Programming because the interpretation and simulation of the formulas is straight-forward.

In Sect. 4 we make simulations of the market, in order to get some insights about the effects of NE in the interaction between entrant and incumbent. Here, we present graphically the motion of the market, plotting it for every initial state.

The conclusions of this work can be found in Sect. 5, along with some ideas for future research.

2 Network Externalities

In microeconomics, a good is believed to provide some utility to whomever buys it and consumes it. The form of the utility can vary, but is generally a function of the consumption of the good itself alone. That may not be the case in reality; we may value the goods we buy in terms of subjective values dependent on factors external to the item at hand. An interesting example is when the utility function of the agents takes into account the number of people that buy the same product. This phenomenon is called *Network Externalities* (NE), since the purchase of an individual yields an externality for other buyers by increasing the utility of the purchase and, therefore, distorting its behaviour.

NE can be found in several markets such as telecommunications, software, video games and banking. There are *direct* NE, when the utility of a good is in direct proportion to the number of users. The telecommunications market is a canonical example; the utility of owning a phone line depends directly on how many consumers are there to communicate with. There are also *indirect* NE, when the utility is derived as a byproduct of the size of the network. Two-sided markets such as software or video games typically present this kind of NE: a large size of the network of users of some hardware's brand gives an incentive to developers to make more applications for that particular brand of console or hardware equipment, which in turn yields a higher utility for the users, enabling them to choose between more and better software applications.

A third (and often neglected) kind of NE is the *post-service* Network Externality. It is a more subtle kind, where the utility from the post-service is the one affected by the size of the network. One example is the purchase of a car: if there are enough cars of the same brand available in a given city, the user might find it easier (and cheaper) to hire the services of a mechanic and the parts of the car would be available more easily.

An important factor in the formation of this kind of structure in the market is the compatibility between products. For example, in the telecommunications market, if all competitors share infrastructure, the cost of communicating between brands is lower than it would be if every firm had its own. The latter case would yield a higher utility for the users of the firm where the network is larger, for it would mean a lower price to communicate with their peers, whereas the former would yield the same result regardless of the chosen brand, as if all firms were a single network.

2.1 How Network Externalities Affect the Market

When NE are present, it is not only important to know how many units a certain brand sells, but also the size of the network associated with it. A key characteristic that determines the size of the network is *compatibility*. An example of this can be found in the smartphones market, where a great variety of mobile devices can share a single operative system or carrier.

Some models have been developed to explain this phenomenon. In [13], the authors explore a model with homogeneous goods where the firms have an expectation of the number of consumers of their brand. In this model, the consumers interact with the firms by making the decision of their purchase, according to the utility the brand gives them. Nonetheless, the consumer's utility comes from two sources: the standalone utility from the consumption of the product, which in perfect competition determines the so-called *hedonic price* (see [9, 15]), and the utility derived from the size of the network.

It is worth mentioning that different firms or brands can share the same network. If the network is big enough, the consumers might be more inclined to choose the brands in a shared network, so long as the sum of the standalone utility and the one provided by the network is bigger than other options she can afford.

In the model provided by [13], the consumers create *expectations* of the size of the network and decide their purchase with imperfect information. The benefits of a firm are directly related to the size of the network by the quantities they sell, and by the added value resultant of being in a large network. If we assume that there are only two networks in the market, then the firms in the largest network will have an advantage with respect to the ones in a smaller one.

The value of a firm to the customer can be reflected in the decision process. If we had a dynamic model instead of a static one, then the agents would need to decide in every moment of time the firm they want to buy from. This would give the firms an incentive to make an effort to increase the size of the network.

We might think that there are two ways to increase the size of a network. One is to make the product compatible with the largest network, but one can make an argument that this is an expensive action, an extreme measure that cannot be done in a flexible, continuous adjustment, unless it is done by creating some kind of coalitions which might be illegal in some countries. The other way to get a larger network is by advertising efforts. Section 3 analyses a market where direct Network Externalities are present. The firms will expend in advertising to get a larger market share. Thus, here we will assume that either the size of the network is equal to the quantity sold, or that all firms in a network act in a tacit collusion.

3 The Model

In this section we develop a dynamic model of a duopoly where the firms compete in a market where NE are present. We do this as a duopoly analysis, but it can also be thought of as a situation where two different standards which concentrate any number of firms.

In this model, our main concern is the way NE affect the market share of the firms over time. Consider a duopoly based on the well-known Lanchester model (see [4, 21]) where each firm wants to maximize

$$J_i = \int_0^T \mathrm{e}^{-rt} \left\{ p_i x_i(t) - \frac{c_i}{2} u_i(t)^2 \right\} \mathrm{d}t + \mathrm{e}^{-rT} S_i(x_i(T)), \tag{1}$$

subject to the state dynamics:

$$\dot{x}_i(t) = u_i(t) x_j(t)^{\alpha_{NE}} - u_j(t) x_i(t)^{\beta_{NE}}, \quad x_i(0) \in [0,1]; \tag{2}$$

For $i = 1, 2$ and $i \neq j$. Here, $x_i(t)$ is the market share of brand i at time $t \in [0, T]$, a normalized quantity that will be sold by the i-th firm at the (exogenously given) price p_i. The qualifier "normalized" means that $x_i \in [0, 1]$ and the value of the market share of firm j is given by $x_j = 1 - x_i$, i.e. we care about the captive market only: we assume that there are no agents in the market that can opt for not buying. The value function J_i that the firms want to maximize is a function of the discounted sales minus expenditures. Here, r is the discount rate, which for simplicity can be thought of as the risk-free interest rate of the economy.

We assume that there is no fixed cost nor cost of production other than the advertising expenditures. Note that (1) yields that the marginal cost of obtaining a new unit of market share is equal to c_i, which is a constant for each firm, and u_i is the amount of units of advertising purchased. The terminal surplus given at terminal time $t = T$ is given by the function

$$S(x_i(T)) = s_i x_i(T), \tag{3}$$

where s_i is a given constant, times the market share at the end of the horizon.

3.1 The Network Externalities Modeled on a Lanchester Dynamics

The exponents in restriction (2) stand for the so-called *saturation effects*, i.e., the expenditure made by the i-th firm on advertising is pretended to affect the market share of the j-th firm. In the popular approach adopted by, for example, [4, 6, 8], saturation effects are fixed, and therefore, the dynamics takes the form

$$\dot{x}_i(t) = u_i(t) x_j(t)^{\alpha} - u_j(t) x_i(t)^{\beta}, \quad x_i(0) \in [0,1], \tag{4}$$

where α, β are given constants such that $\alpha + \beta = 1$. If $\alpha = \beta = 1/2$ (see [1, 7]), then both firms' marginal expenditure in advertising would *steal* the same market share from the competition for all moments of time. To see this, note that we can approximate $x_i(t)^{1/2}$ by $x_i(t) + x_i(t)x_j(t)$ for $i = 1, 2$. (This approximation, valid for small values, has also been used in [19, 21] for the analysis of this kind of models.) This way, (4) turns into

$$\dot{x}(t) = u_i(t)x_j(t) - u_j(t)x_i(t) + [u_i(t) - u_j(t)]x_i(t)x_j(t).$$

Here, the first two terms show us the direct effect of the expenditure. The third term shows the interaction of the firms, affected by the difference in expenditures of the i-th and j-th firms. It is easy to see that if the expenditure of the i-th firm is larger than that of j-th firm, the market will favour the i-th firm. Note that this effect depends on the expenditure each firm makes in advertising, because $\alpha = \beta$. Otherwise, if $\alpha > \beta$, the marginal expenditure of the i-th firm would have a greater effect and the j-th firm would be in disadvantage in every moment of time. It is worth to mention that, since we consider only two players, the sum of their markets should always equal one. This means that

$$\dot{x_1}(t) + \dot{x_2}(t) = 0. \tag{5}$$

But, on the aforementioned case, this holds only when $\alpha = \beta = 1/2$. The dynamics used in our model is such that the saturation effects vary with the market share itself. Moreover, our model satisfies (5) as well. We achieve this by considering $\alpha_{NE}(x_i)$ and $\beta_{NE}(x_j)$ as variables that depend on the market share. Thus, we let

$$\alpha_{NE}(x_i) := 1 - x_i = x_j,$$
$$\beta_{NE}(x_j) := 1 - x_j = x_i.$$

Therefore, the restriction in (2) becomes

$$\dot{x}_i(t) = u_i(t)x_j(t)^{x_j(t)} - u_j(t)x_i(t)^{x_i(t)}, \quad x(0) = x_0 \in]0, 1[. \tag{6}$$

This is the main contribution of this work. To the knowledge of the authors, a formulation of Lanchester dynamics whose saturation effects vary in function of the market share in the way stated above has not been analysed before. This assumption complicates the formulation of the dynamics of the market share and, hence, makes more difficult the characterization of the optimal strategies of the differential game at hand. To overcome this problem we will consider open-loop strategies and use the PMP to solve the differential game.

With NE, the effect of advertising gets *stronger* when the firm's market share gets closer to 1 and, conversely, it gets weaker as it approaches 0. This behaviour is our way to model NE in a dynamic game: the efforts made by the i-th firm get an additional effect when it has a large market share. The reason behind this behaviour of the market is supposed to be exogenous, but we consider it to be a side effect of compatibility. This matches a situation where direct Network Externalities are present, since the agents will decide their next purchase by looking at the size of the network and their preferences will be distributed in the same proportion as the market share. In this sense, the saturation effects are a measure of the compatibility of the network.

Figure 1 shows a simulation of the dynamics of x_i with equal marketing efforts for all initial states $x_i(0) \in [0, 1]$. This simulation shows the motion of the market share for a system like (4), without NE. Similarly, Fig. 2 represents a system dynamics

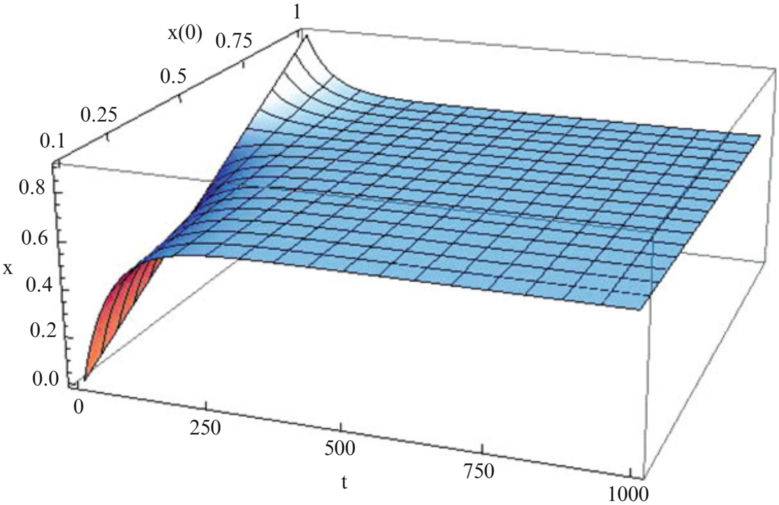

Fig. 1 The market share dynamics for all initial states in [0, 1] when $u_1 = u_2$ without Network Externalities for $T = 1000$

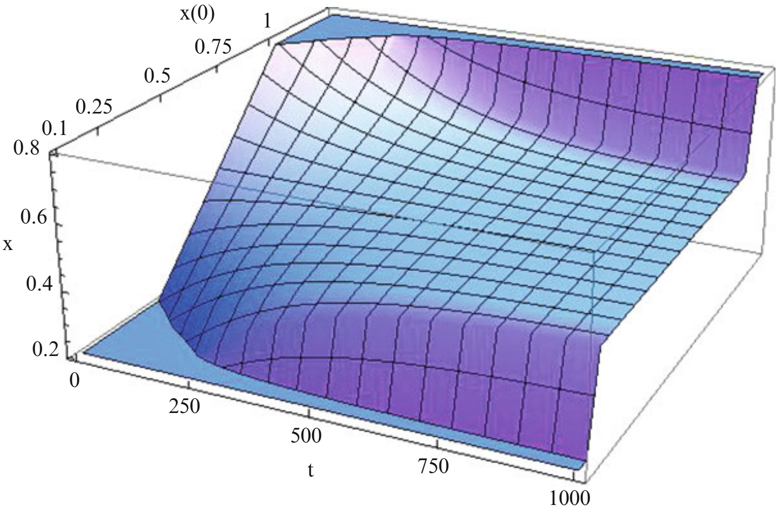

Fig. 2 The market share dynamics for all initial states in [0, 1] when $u_1 = u_2$ with Network Externalities for $T = 1000$

as in (2), with NE present. Note that, for both cases, the only initial state where $\dot{x}_i$ is equal to zero, that is, the market share will remain constant, is in the middle, where $x_i = x_j = 1/2$. In the other initial states, we can see that the market share will move towards this middle point of market shares. An interpretation is that, since consumers are exposed to the same amount of advertising from both firms, they tend in the long run to divide the market. Therefore, any initial state below the halfway point means an increase in the market share for the next moment in time until the market share is divided in half and, analogously, if the initial market share is larger, the value of $\dot{x}$ will be negative until the share is equally divided for both firms.

The difference between a market with Network Externalities [i.e. a market whose dynamics is as in (2)], and one without them [see (4)], is the *speed* of convergence. As we can appreciate from Fig. 2, whereas the market without externalities will always end up in the middle point no matter what the initial value is, if we have that $x_0 = 0$ and $x_0 = 1$, the dynamics for a system with NE will equal zero. This can be interpreted by saying that a monopoly would endure in the economy indefinitely.

Figure 3 presents a system where firm 1 is expending more on advertising than firm 2, i.e., $u_1 > u_2$, in particular, $u_1 = 1.25, u_1 = 1$. Here we can see that the effects of the marketing expenditures favour the firms with larger initial share. If firm 1 kept this proportion of expenditure in advertising for a sufficiently large amount of time, the final market share would be larger if the market was the one with NE. Moreover, if the initial state was larger than the second initial state, firm 1 would be able to become a monopoly. This cannot be done in a market without NE.

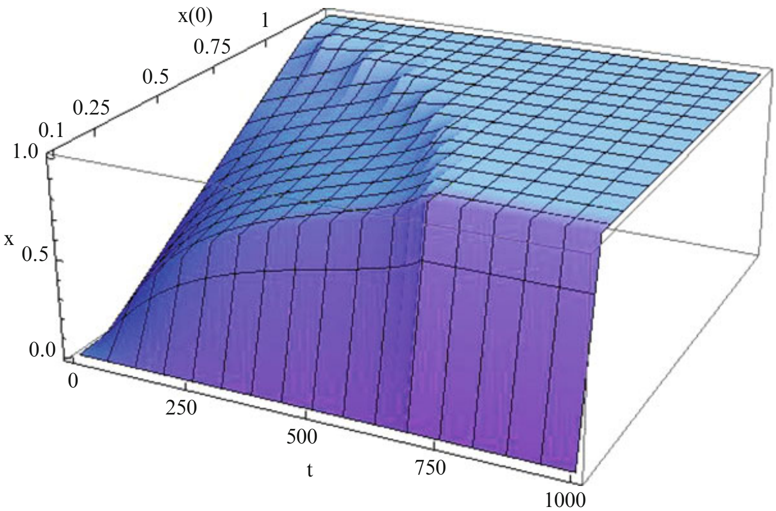

Fig. 3 The market share dynamics for different initial states when $u_1 > u_2$ with and without Network Externalities for $T = 10$

3.2 Optimal Marketing Expenditures

To solve the problems (1)–(6), we will use PMP (see [22]), and find first order conditions to determine the optimal expenditures in advertising for both firms. The Hamiltonian for the *i*-th firm is given by

$$H^i(x_i, u_i, \lambda_i, t) = p_i x_i - \frac{c_i}{2} u_i(t)^2 + \lambda_i[u_i(t) x_j(t)^{x_j(t)} - u_j(t) x_i(t)^{x_i(t)}].$$

By stating the problem in these terms, we turn a problem with three variables (x, u, t) and one restriction into a four variables problem with no restrictions. A firm might be tempted to make the advertising expenditures equal to zero, but if the other firm's expenditure is positive, then that would mean sacrificing market share in the long run. The co-state variable can be seen as the *shadow price* of the market share. That is, the value assigned to the market share, when it is difficult to know or calculate. Its dynamics, according to PMP, is given by

$$\begin{aligned}
\dot{\lambda}_i(t) &= r_i \lambda_i(t) - \frac{\partial}{\partial x_i} H^{i*}_{u^*}(x(t), \lambda^i(t), t) \\
&= r_i \lambda_i(t) - \frac{\partial}{\partial x_i} \left[p_i x_i - \frac{c_i}{2} u_i^*(t)^2 + \lambda_i[u_i^*(t) x_j(t)^{x_j(t)} - u_j^*(t) x_i(t)^{x_i(t)}] \right] \\
&= r_i \lambda_i(t) - p_i + \lambda_i \left(u_i^* x_j^{x_j}(1 + \ln(x_j)) + u_j^* x_i^{x_i}(1 + \ln(x_i)) \right), \qquad (7)
\end{aligned}$$

where u^* stands for the control that optimizes the problem; with transversality condition

$$\lambda_i(T) = S_i'(x_i(T)). \qquad (8)$$

That is, we take the first derivative of the Hamiltonian with respect to the state variable, assuming that $u_i(t)$ is already optimal to know the motion of the market share's value. To find optimal expenditures in advertising, we equal the derivatives of the Hamiltonian (with respect to the controllers) to zero. That is,

$$\frac{\partial}{\partial u_i} H_i(x_i, u_i, \lambda_i, t) = -c_i u_i(t) + \lambda_i x_j(t)^{x_j(t)} = 0.$$

By Theorem 3.2, in [22], this equality gives us as a result that the optimal simultaneous expenditures on advertising of firm $i = 1, 2$ are given by

$$u_i^* = \frac{\lambda_i(t)}{c_i} x_j(t)^{x_j(t)}. \qquad (9)$$

That is, the optimal expenditure will vary in proportion with the market share of the competitor (i.e. the other firm), but its effects will also be affected by it. In addition, the shadow price will also affect the optimal control of the system. Note that (9) displays a division by the marginal cost of advertising c_i. This means that at every moment of time, the firms will evaluate the value of the market share with respect to the amount they have to expend to get an additional unit of it.

If we plug the optimal controllers referred to in (9) into the shadow price motion (7), we have that

$$\dot{\lambda}_i(t) = r_i \lambda_i(t) - p_i + \frac{\lambda_i(t)^2}{c_i} x_j(t)^{2x_j(t)}(1 + \ln(x_j(t))) + \frac{\lambda_i(t)\lambda_j(t)}{c_i} x_i(t)^{2x_i(t)}(1 + \ln(x_i(t))). \tag{10}$$

This expression describes the dynamics of the shadow price. As we can see, it depends almost exclusively on the market share and λ itself. This shadow price influences the optimal behaviour of the firms. To obtain the value of the shadow price in the terminal time, we use the transversality condition (8). For this purpose, we state the terminal payoff function as in (3). This yields a simplified transversality condition:

$$\lambda_i(T) = s_i. \tag{11}$$

Plugging the optimal controls from Eq. (9) in the dynamics described in Eq. (6) we get

$$\dot{x}(t) = \frac{\lambda_i(t)}{c_i} x_j(t)^{2x_j(t)} - \frac{\lambda_j(t)}{c_j} x_i(t)^{2x_i(t)}. \tag{12}$$

This is the behaviour of the market share. Note that the value of $\frac{\lambda_i(t)}{c_i}$ is key to know how the market share would behave. This means that, in an optimal path, the agents make their optimal choices of expenditure on advertising according to how important it is for them to get more market share. Note that if one assumes that the cost remains constant, then the control variables will be highly dependent of $\lambda(t)$.

3.3 Steady State

Finally, we want to find the steady state of our dynamics. That is, the market share of the firms where the dynamic is stable over time. To achieve this, we will state the model in absolute terms, that is, we will turn (4) into

$$\dot{s}_i(t) = u_i(m - s_i)^{\alpha} - u_j s_i^{\beta},$$

where s_i stands for the sales of firm i, and m is the size of the whole market, therefore, $x_i = m/s_i$. This way, for example, when $\alpha = \beta = 1/2$ we have that the steady sales $\bar{s}$ is the level of sales that results from making the dynamics $\dot{s}_i(t)$ equal to zero, that is,

$$\bar{s}_i = \frac{u_i^2 m}{u_i^2 + u_j^2}.$$

In the case of the dynamics described by (6), when we state them in absolute terms they become

$$\dot{s}_i(t) = u_i(t)(m - s_i)^{1-s_i/m} - u_j(t)s_i^{s_i/m}, \quad s_i(0) = 0. \tag{13}$$

Since (13) is an implicit function of s_i, we use Figs. 4, 5, 6, 7 to obtain some insights on the existence of a steady state of the i-th firm.

Figures 4 and 5 show the dynamics of the sales when the sales and one of the controllers vary (while the control of the other player is fixed at 0.5). From these pictures, we may argue the existence of a steady state.

In Figs. 6 and 7, we can look how the state dynamics does reach the plane $\dot{s}_i = 0$ without having to fix any of the controllers. Now, by means of a first order Taylor series around $s = 1$ (with error term $O(s-1)^2$), we approximate an expression for the steady state where sales stabilize ($\dot{s}_i = 0$). Therefore we have

$$\bar{s}_i(t) = \left[m(m-1)^{1/m} \left(u_j(t) - (m-1)^{1-1/m} - \frac{\phi}{m(m-1)^{1/m}} \right) \right] \frac{1}{\phi}. \tag{14}$$

where $\phi = u_i(t)(m-1)(1+\ln(m-1)) + u_j(t)(m-1)^{1/m}$. We now plug (14) into (13) and plot the dynamics $\dot{s}_i$ when the controllers vary. See Fig. 8.

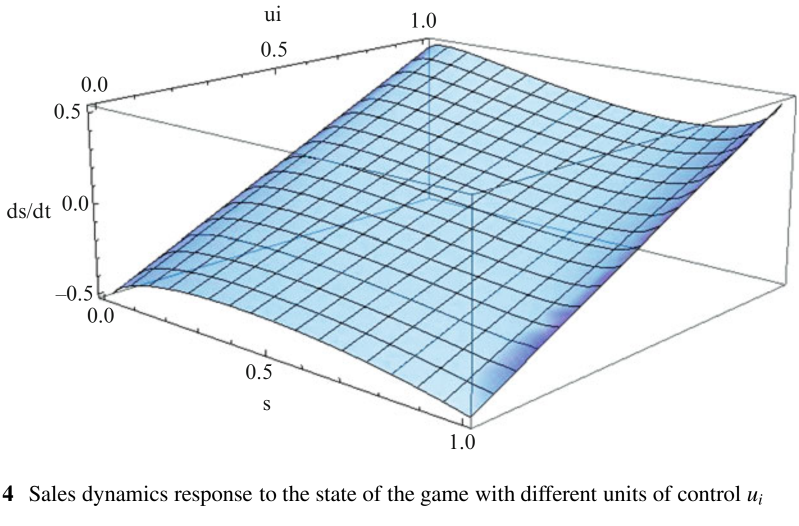

Fig. 4 Sales dynamics response to the state of the game with different units of control u_i

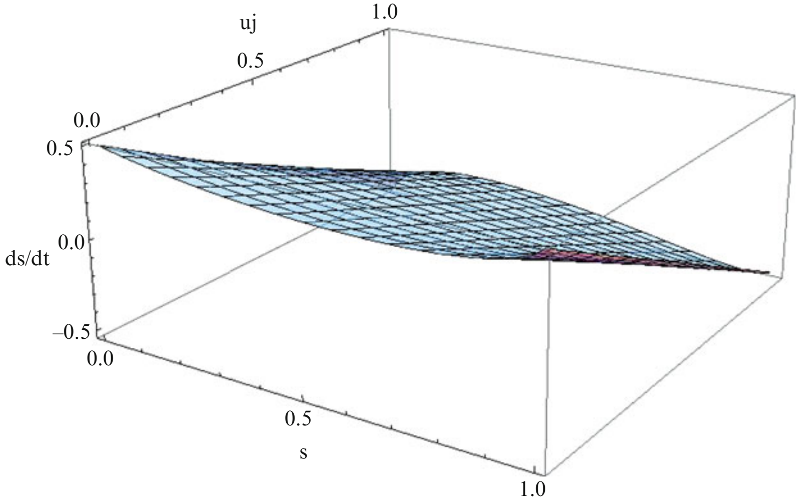

Fig. 5 Sales dynamics response to the state of the game with different units of control u_j

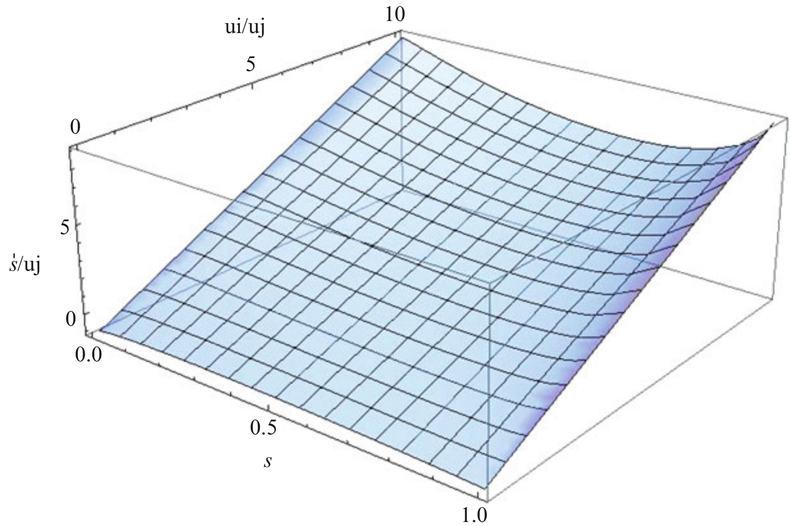

Fig. 6 Sales dynamics response to the state in face of the proportion of advertising efforts u_i/u_j

From Fig. 8 we can infer that, as the controller from the j-th player becomes greater with respect to the controller of the i-th player, the steady state of the latter agent (14) remains stable. This is of particular interest, because it implies that as one player leaves the other act, his own sales will stay stationary. The situation is analogous for the other firm.

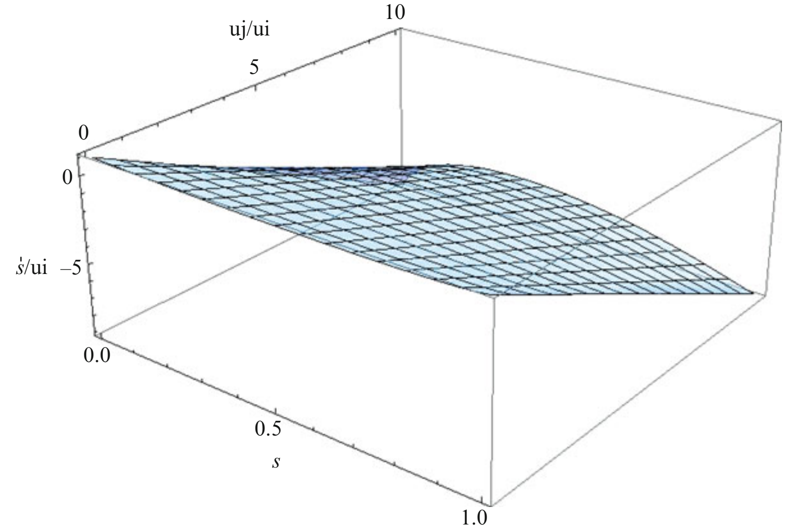

Fig. 7 Sales dynamics response to the state in face of the proportion of advertising efforts u_j/u_i

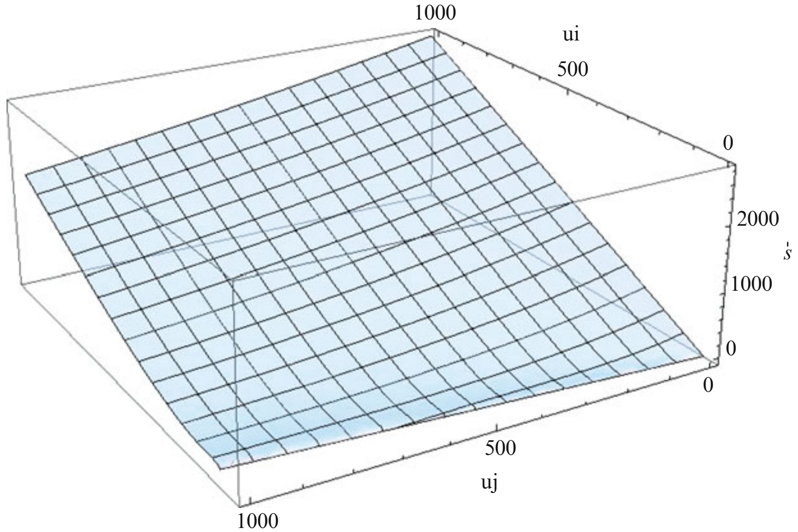

Fig. 8 Sales dynamics response to the state of the game with different units of control u_i

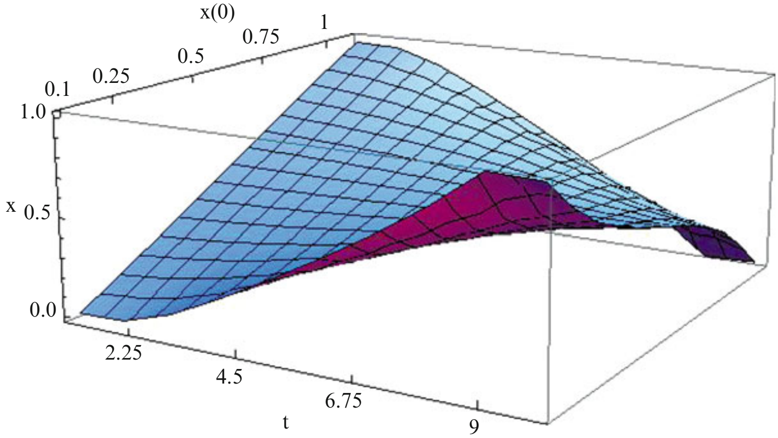

Fig. 9 The market share dynamics for different initial states with Network Externalities and optimal advertising efforts for $T = 10$. High discount rates

4 Simulations

In this section we analyse the dynamics in which the game develops when both players are optimizing, that is, when the game is at equilibrium. For this section, we take the dynamics in (12) and use computer simulations to find the state of the market share of the players in the game over time.

To make a complete analysis, we plot the three-dimensional surface for different initial states $x_0 \in]0, 1[$. Besides, we also plot the surface for the dynamics of the shadow price of market share shown in (10), which is representative of the dynamics of the advertising expenditure.

Figure 9 shows the motion of the market share over time in different initial states. For notational convenience, we assume that x means x_i for $i = 1, 2$. For synoptic purposes, as initial values, we set the same costs for both firms (at a level of 1). We state that the prices are the same for the agents, and started with a discount rate of $r = 1$. Naturally, these variables can be fed with different values more akin to specific situations. Additionally, an initial state of $\lambda_i(0) = 1$ was fed to the system for both firms $i = 1, 2$.

With this settings, we can see that a firm with a very small initial market share can end up owning the market, and analogously, a monopoly might give up on the market over time.

This behaviour can be explained by the choice of a high value for the discount rate. At this rate, a firm with a high market share is interested in exploiting its market power in the present, but is not willing to engage in an attrition war for a long time. The breaking point of the interest that the monopoly has in the market share can be visualized in Fig. 10, that shows the motion of the shadow price.

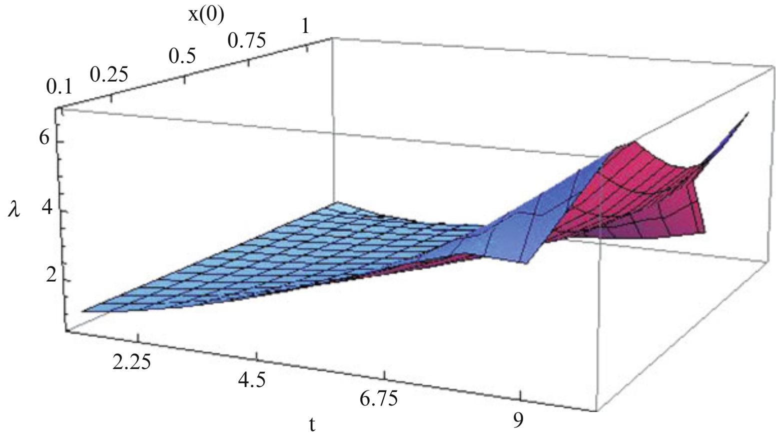

Fig. 10 The dynamics of the shadow price with high discount rates

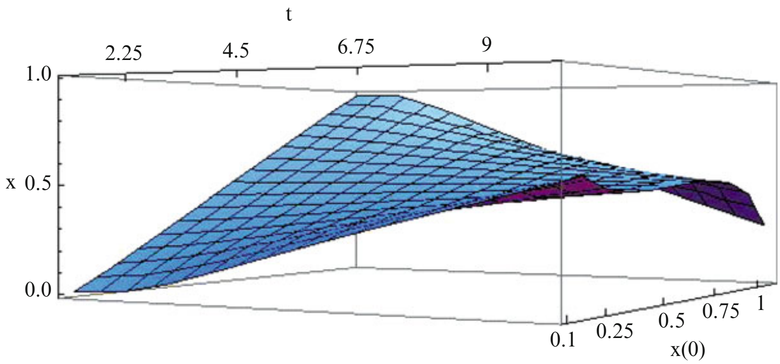

Fig. 11 The dynamics of the market share with low discount rates

In contrast, if we see the same dynamics of Fig. 11, where the discount rate is $r = 0.01$, we can observe that the dynamics, although are not as stable as in Sorger's model (see [21]), have some tendency towards the middle point. More importantly, the shadow price shown in Fig. 12 with a small initial market share reaches a maximum, and then starts to decrease, and when the initial market share is higher, it tends to decrease over time.

Note that the tendency in the extremes keeps on the same direction as in Fig. 9. The tendency of monopolies to exploit as much as they can their high ground is just reduced by a small value in present time, but the intensity of the warfare in these situations makes the effort to keep a monopoly worthless.

Both Figs. 10 and 12 show the dynamics of $\lambda_i(t)$ with the exception of the value of $\lambda_i(T)$, to allow the reader to better appreciate the behaviour of the variable.

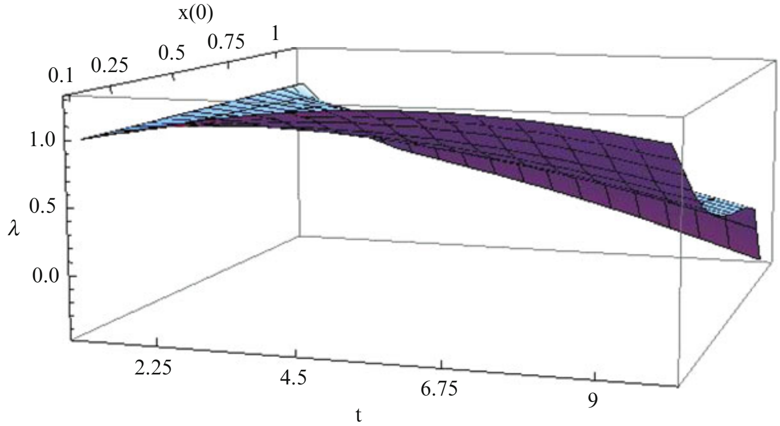

Fig. 12 The dynamics of the shadow price with low discount rates

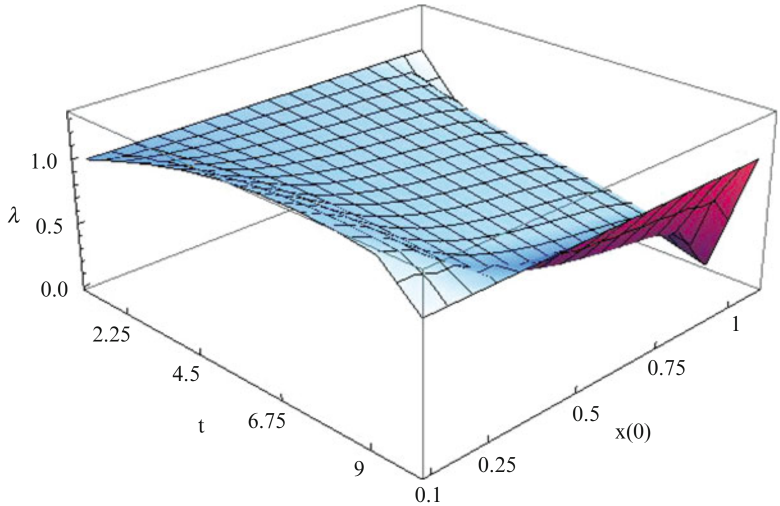

Fig. 13 The dynamics of the shadow price with $\lambda(T) = s_i = 1$

To plot the complete dynamics we simply have to set the final shadow price as in Eq. (11). Figure 13 shows the motion of λ_i with the shadow price in terminal time included, with its value stated by transversality conditions.

Likewise, Figs. 9 and 11 show the market share dynamics with a relatively low terminal value for the shadow price in terminal time. That is, a terminal reward function with a value of 1 like the one shown in Fig. 13 will not be visible in these simulations. Nonetheless, a relatively high level of terminal reward as the one shown in Fig. 14 will be visible in the final market share dynamics.

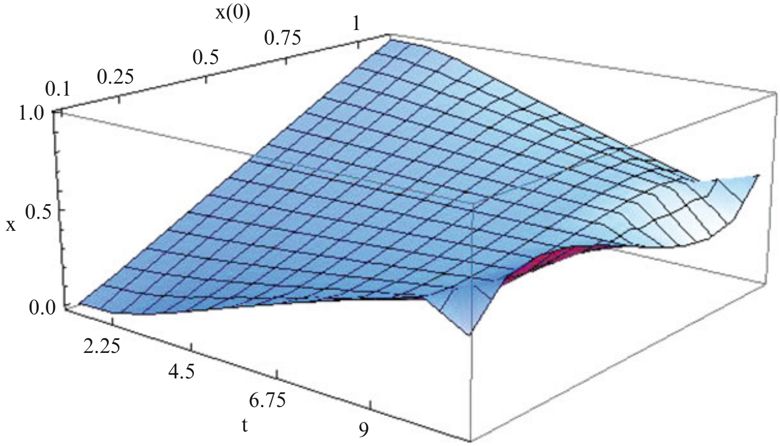

Fig. 14 Market share dynamics with a high value for $\lambda(T) = s_i$

5 Conclusions

We have found that the existence of Network Externalities gives opportunity to a firm to become a monopoly in the long run. Such an opportunity is not present for a market that does not have this kind of structure and is not very common in Lanchester models but is what we would expect on a market with this kind of externalities, such as telecommunications.

By solving the dynamic model, we found that the firms must evaluate how much they value market share and adapt their strategy to this value. This yields an optimal dynamics that will depend on the value of the market share for both firms.

Although we found an approximation of the steady state and performed a geometrical analysis on its existence and stability, we believe this procedure can be improved in further research. However, an interesting finding is that as one player leaves the other advertise for her own brand, his own sales will remain stationary.

Also, by building simulations, we found that the final market share presents a high dependence on the value of the market over time, the value of the market share tends to decrease quickly when a high discount rate is present. When the discount rate is small, this process is slowed down, but eventually, the monopolies tend to give up and the entrant becomes the new monopoly over time.

Further research on the matter includes the use of an analysis with the use of Dynamic Programming to derive Nash equilibria for this game (which might yield a more precise statement of the steady state). It is well known that although simpler to compute, the open-loop strategies present some important drawbacks with respect to the more interesting case of feedback strategies, with come at the cost of a much difficult characterization.

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The Shapley Value as a Sustainable Cooperative Solution in Differential Games of Three Players

Ekaterina Gromova

Abstract The contribution of the paper is twofold: first, it has been shown that the Yeung's conditions can be used to construct a strongly time-consistent core. In this core there is a supporting imputation which has the property that a single deviation from this imputation in favor of any other imputation from the core still leads to the payment from the core. The obtained results were formulated for the Shapley value taken as the supporting imputation. Second, a particular class of differential games was considered. For this class of games the δ-characteristic function turns out to be superadditive and the Yeung's conditions are satisfied without any additional restrictions on the parameters of the model. All results are presented in the analytic form.

Keywords Shapley value • Core • Cooperative game • Differential game • Pollution control • Time-consistency • Strong time-consistency • Irrational behavior proofness

1 Introduction

In the classical cooperative games theory there has been a constant interest in studying the properties of the Shapley value, first formulated by L. Shapley in 1953, [21]. It is well known that this cooperative solution has a number of advantages in the static setting (see, e.g., [6] for an overview of the results in this direction). At the same time, for the dynamic games the properties of cooperative solutions are not yet studied as well as for the static ones. In particular, there are a number of problems occurring when transferring the results obtained for static games to their dynamic counterparts. One of such issues is the problem of time-inconsistency (i.e., a temporal non-realizability) of the cooperative solution upon which the players agree at the beginning of the game. In this work we studied the situation where the Shapley value can be chosen as a sustainable cooperative solution. Note that this

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solution turns out to be not only time-consistent (in the sense of [13, 17, 18]), but also it possesses some other useful properties such as stability w.r.t. an irrational behavior [25], strong time-consistency [5, 14], with respect to the core and others.

The paper is organized as follows. Section 2 describes the problem statement and present results on the characterization of the strongly time-consistent core. Section 3 presents a specific 3-player differential game of pollution control. Finally, Sect. 4 considers the time-consistency property for a specific imputation taken as the supporting imputation. Appendices 1 and 2 contain analytical expressions for the Shapley value and the respective IDP for the game described in Sect. 3.

2 Problem Statement

2.1 Cooperative Game in the Form of Characteristic Function

Consider a 3-player cooperative differential game $\Gamma(x_0, T - t_0)$ with prescribed duration $T - t_0$ [5, 17] which evolves on the interval $t \in [t_0, T]$ from the initial state $x_0 \in R^n$, governed by differential equations

$$\dot{x} = f(x, u_1, u_2, u_3), \qquad x(t_0) = x_0, \tag{1}$$

$x \in R^n$, $u_i \in U_i \subset compR^k$, and with payoff functions

$$K_i(x_0, T - t_0; u_1, u_2, u_3) = \int_{t_0}^{T} h_i(x, u_1, u_2, u_3)dt,$$

where $x(t)$ is the solution of the system (1) under controls (u_1, u_2, u_3), and $i \in N$, where N is the set of players such that $|N| = 3$.

We consider the cooperative form of the game $\Gamma(x_0, T - t_0)$. This means that all players join to maximize their total payoff $\sum_{i=1}^{3} K_i(x_0, T - t_0; u_1, u_2, u_3)$. The trajectory $x^*(t)$, $t \in [t_0, T]$ is said to be the cooperative trajectory if the following holds:

$$\begin{aligned} V(N, x_0, T - t_0) &= \max_{u_1,u_2,u_3} \textstyle\sum_{i=1}^{3} K_i(x_0, T - t_0; u_1, u_2, u_3) = \\ &= \max_{u_1,u_2,u_3} \textstyle\sum_{i=1}^{3} \int_{t_0}^{T} h_i(x, u_1, u_2, u_3)dt = \sum_{i=1}^{3} \int_{t_0}^{T} h_i(x^*(t), u_1^*(t), u_2^*(t), u_3^*(t))dt. \end{aligned} \tag{2}$$

We assume that trajectory $x^*(t)$, $t \in [t_0, T]$ exists and is unique. The controls (u_1^*, u_2^*, u_3^*) are said to be optimal controls. In the cooperative form of the game each player i uses his /her optimal strategy u_i^* which contribute to the maximization of the total joint payoff $V(N, x_0, T - t_0)$ of all players $1, 2, 3$. The class of the strategies in differential games may differ depending on the particular problem. However, most often the open-loop $u_i(t)$ or the close-loop $u_i(x,t)$ classes of controls are used. Within this framework we consider the strategy of the i-th player to be the function $u_i(\cdot)$ which takes values in the set of admissible instantaneous controls U_i.

To define the cooperative game we have to construct the characteristic function (c.f.) $V(S, x_0, T - t_0)$ for every coalition $S \subset N$ in the game $\Gamma(x_0, T - t_0)$. Note that the value of the characteristic function for the grand coalition N equals to $V(N, x_0, T - t_0)$. For the case of a 3-player game the possible coalitions are $S = N = \{1,2,3\}$ along with $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1\}$, $\{2\}$, $\{3\}$, and the empty coalition $S = \emptyset$. There are several main approaches to the construction of the characteristic function which shows the power of the coalition S (see, for example, [4, 19]). It can be constructed, for example, as α-c.f. [10], i.e., as the lower value of the zero-sum game between the coalition S, acting as the first (maximizing) player and the coalition $N \backslash S$, acting as the second (minimizing) player. The payoff of coalition S is the sum of respective payoffs of the players from the coalition S, and the strategy of coalition S is an element of Cartesian product of players' strategy sets from the coalition S. The lower value of the game always exists and is a superadditive function of coalition $S \subset N$ [17], i.e.,

$$\begin{aligned} &V(S_1 \cup S_2, x_0, T - t_0) \geq V(S_1, x_0, T - t_0) + V(S_2, x_0, T - t_0), \\ &\forall S_1, S_2 \subseteq N, \ S_1 \cap S_2 = \emptyset. \end{aligned} \tag{3}$$

It is rather easy to construct the characteristic function $V(S, x_0, T - t_0)$ in the form of δ-c.f. [18]. The characteristic function of coalition S is computed in two stages: first, we have to calculate the Nash equilibrium strategies for all players and next, we fix (freeze) it for players from $N \setminus S$ while players from coalition S seek to maximize their joint payoff $\sum_{i \in S} K_i$. In general, this δ-c.f. is not superadditive function but it is superadditive for a class of games with negative externalities [19].

One of the novel approaches is to use ζ-c.f. [4, 5]. The characteristic function of coalition S is computed in two stages: first, we find optimal controls maximizing the total payoff of the players; next, the cooperative optimal strategies are used by the players from the coalition S while the left-out players from $N \setminus S$ use the strategies minimizing the total payoff of the players from S. The characteristic function defined in this way is superadditive.

Consider the subgames of the main game $\Gamma(x_0, T - t_0)$ with initial conditions $(x^*(t), t)$ belonging to the cooperative trajectory, i.e., $\Gamma(x^*(t), T - t)$. In each subgame $\Gamma(x^*(t), T - t)$ one can define the characteristic function $V(S, x^*(t), T - t; S)$, where $S \subset N$, in the same way as in the overall game $\Gamma(x_0, T - t_0)$.

2.2 The Core

In cooperative game theory [11, 12, 24] the main question is how to divide the total amount $V(N, x_0, T - t_0)$ earned by all the players when using the optimal controls

$\{u_i^*\}$. The rule of the allocation of the total maximum payoff $V(N, x_0, T - t_0)$ among players from N is referred to as the optimality principle [24] or the cooperative solution [8, 11].

Now define the set of imputations $M(x^*(t), T - t)$ in the subgame $\Gamma(x^*(t), T - t)$ as

$$M(x^*(t), T - t) = \{\alpha^t = (\alpha_1^t, \alpha_2^t, \alpha_3^t) : \textstyle\sum_{i=1}^3 \alpha_i^t = V(N, x^*(t), T - t),$$
$$\alpha_i^t \geq V(\{i\}, x^*(t), T - t), i \in N\}.$$

Superadditivity of the characteristic function implies that the set $M(x^*(t), T - t)$ is not empty for all $t \in [t_0, T]$.

Define also the core $C(x^*(t), T - t) \subset M(x^*(t), T - t)$ in the game $\Gamma(x^*(t), T - t)$ and assume that for any $t \in [t_0, T]$, $C(x^*(t), T - t) \neq \emptyset$.

We recall that the core of the game $\Gamma(x^*(t), T - t)$ with three players is the set of imputations $\alpha^t = (\alpha_1^t, \alpha_2^t, \alpha_3^t)$ satisfying the inequalities

$$\sum_{i \in S} \alpha_i^t \geq V(S, x^*(t), T - t), \qquad \forall\ S \subset N,$$

i.e., $\alpha^t = (\alpha_1^t, \alpha_2^t, \alpha_3^t)$ is a vector of allocation of the $V(N, x^*(t), T - t)$ such that

$$\begin{aligned} &\textstyle\sum_{i=1}^3 \alpha_i^t = V(N, x^*(t), T - t),\\ &\alpha_i^t \geq V(\{i\}, x^*(t), T - t) \quad i \in N,\\ &\alpha_1^t + \alpha_2^t \geq V(\{1, 2\}, x^*(t), T - t),\\ &\alpha_1^t + \alpha_3^t \geq V(\{1, 3\}, x^*(t), T - t),\\ &\alpha_2^t + \alpha_3^t \geq V(\{2, 3\}, x^*(t), T - t). \end{aligned} \tag{4}$$

The core is the set of feasible allocations that cannot be improved upon by a subset of players [8, 12]. The Bondareva–Shapley theorem describes a necessary and sufficient condition for the non-emptiness of the core of a cooperative game [1, 22].

2.3 The Shapley Value

In the following we will consider a particular cooperative solution referred to as the Shapley value [21]. In contrast to the core, the Shapley value is defined in a unique way and is particularly suitable for the applications. We note that the Shapley value also has a number of other advantages for static problems, see, e.g., [6].

The Shapley value $Sh = \{Sh_i\}_{i \in N}$ in the game $\Gamma(x_0, T - t_0)$ is a vector, such that:

$$Sh_i(x_0, T - t_0) = \sum_{i \in S \subset N} \frac{(n - s)!(s - 1)!}{n!} \Big(V(S, x_0, T - t_0) - V(S \backslash \{i\}, x_0, T - t_0 \Big). \tag{5}$$

It is well known that (5) is an imputation because $\sum_{i \in N} Sh_i(x_0, T - t_0) = V(N, x_0, T - t_0)$, $Sh_i(x_0, T - t_0) \geq V(\{i\}, x_0, T - t_0)$, i.e., $Sh(x_0, T - t_0) \in M(x_0, T - t_0)$.

For the game with three players the general formula (5) reduces to the following simple form:

$$
\begin{aligned}
&Sh_1(x_0, T - t_0) = \tfrac{1}{3}[V(N, x_0, T - t_0) - V(\{2,3\}, x_0, T - t_0)]+\\
&\tfrac{1}{6}[V(\{1,2\}, x_0, T - t_0) - V(\{2\}, x_0, T - t_0) + V(\{1,3\}, x_0, T - t_0)-\\
&V(\{3\}, x_0, T - t_0)] + \tfrac{1}{3}V(\{1\}, x_0, T - t_0),\\
&Sh_2(x_0, T - t_0) = \tfrac{1}{3}[V(N, x_0, T - t_0) - V(\{1,3\}, x_0, T - t_0)]+\\
&\tfrac{1}{6}[V(\{1,2\}, x_0, T - t_0) - V(\{1\}, x_0, T - t_0) + V(\{2,3\}, x_0, T - t_0)-\\
&V(\{3\}, x_0, T - t_0)] + \tfrac{1}{3}V(\{2\}, x_0, T - t_0),\\
&Sh_3(x_0, T - t_0) = \tfrac{1}{3}[V(N, x_0, T - t_0) - V(\{1,2\}, x_0, T - t_0)]+\\
&\tfrac{1}{6}[V(\{1,3\}, x_0, T - t_0) - V(\{1\}, x_0, T - t_0) + V(\{2,3\}, x_0, T - t_0)-\\
&V(\{2\}, x_0, T - t_0)] + \tfrac{1}{3}V(\{3\}, x_0, T - t_0),
\end{aligned}
\tag{6}
$$

where the values $V(\cdot, x_0, T - t_0)$ are values of the characteristic function $V(S, x_0, T - t_0)$, $S \subseteq N$ constructed by any relevant method.

Note that for the case of 2-player game the set of imputations and the core coincide, i.e., $M(x_0, T - t_0) = C(x_0, T - t_0)$ as well as the sets $M(x^*(t), T - t) = C(x^*(t), T - t)$, $t \in [t_0, T]$. Then from superadditivity of the c.f. it follows that the core (the set of imputations) is a non-empty set and the Shapley value belongs to the core.

In general, however, the Shapley value does not belong to the core. There are a lot of examples in the literature also for 3-player game with non-empty core and the Shapley value outside of the core. But if it happens, the Shapley value has the property of "stability" (see, e.g., [8, 11, 22]). It was proved [12], for instance, that for convex games, i.e., for games with characteristic function such that

$$V(S, x_0, T - t_0) + V(T, x_0, T - t_0) \leq V(S \cup T, x_0, T - t_0) + V(S \cap T, x_0, T - t_0),$$

the core is non-empty [22] and includes the Shapley value.

2.4 *Time-Consistent Shapley Value*

Apart from the stability of the cooperative solution in the sense of the static cooperative game theory, it is very important to have stable cooperation in the dynamic setting of the problem [13, 15, 17, 26]. In dynamic cooperative games players which participate in the cooperative agreement want to establish the dynamically stable (time-consistent) cooperative agreement which is the Shapley value in the framework of this paper. Time-consistency implies that, as the cooperation evolves, cooperating partners are guided by the same optimality principle at each instant of time and hence do not have any incentives to deviate from the previously adopted cooperative behavior. A special imputation distribution procedure (IDP) was proposed in [13] to avoid the problem of time-inconsistency of the solution.

Function $\beta_i(\tau)$, $\tau \in [t_0, T]$, $i \in N$ is said to be the IDP [13] for the imputation $\alpha \in M(x_0, T - t_0)$ if

$$\alpha_i = \int_{t_0}^{T} \beta_i(\tau) d\tau, \qquad i \in N.$$

The IDP is a rule of allocation of the amount of imputation for each player over the time interval of the game $[t_0, T]$.

Consider the IDP for the Shapley value in the game $\Gamma(t_0, x_0, T - t_0)$. Then

$$Sh_i(x_0, T - t_0) = \int_{t_0}^{T} \beta_i(\tau) d\tau, \qquad i \in N. \tag{7}$$

For the case of 3-player game we have $i = 1, 2, 3$.

The Shapley value $Sh(x_0, T - t_0)$ in the game $\Gamma(x_0, T - t_0)$ is said to be time consistent [17] if there exists an IDP $\beta(t) = \{\beta_i(t)\}$ such that the vector, calculated by formula $\{\int_{\vartheta}^{T} \beta_i(\tau) d\tau\}$, $\forall \vartheta$, $\vartheta \in [t_0, T]$ is also the Shapley value in the subgame $\Gamma(x^*(\vartheta), T - \vartheta)$, i.e.,

$$Sh_i(x^*(\vartheta), T - \vartheta) = \int_{\vartheta}^{T} \beta_i(\tau) d\tau, \qquad i \in N, \qquad \forall \vartheta \in [t_0, T]. \tag{8}$$

From (8) we get

$$Sh_i(x_0, T - t_0) = \int\limits_{t_0}^{\vartheta} \beta_i(t) dt + Sh_i(x^*(\vartheta), T - \vartheta), \qquad i \in N, \qquad \forall \vartheta \in [t_0, T]. \tag{9}$$

The form of the Shapley value in (9) shows that at any intermediate time instant ϑ the players stick to the same optimality principle (the Shapley value) for the remaining part of the game.

Taking the first derivative w.r.t. ϑ in (9) gives the analytic formula for the IDP which makes the Shapley value time-consistent cooperative solution [17]:

$$\beta_i(\vartheta) = -\frac{d}{d\vartheta} Sh_i(x^*(\vartheta), T - \vartheta), \quad i \in N, \quad \forall \vartheta \in [t_0, T]. \tag{10}$$

Thus by using the IDP (10) we construct the time-consistent Shapley value in the sense of dynamic stability, i.e., distributing the components of the Shapley value along the whole duration of the game we guarantee the realizability of the cooperative agreement for long-term projects. Following this scheme, players use their optimal controls $u_i^*(t)$, the system evolves along the cooperative trajectory $x^*(t)$ and by the end of the game each player accumulates the payoff which is equal to that calculated according to the cooperative agreement.

2.5 *Irrational Behavior Proofness*

The problem of time-consistency of the cooperative solution is studied under the assumption of rational behavior of the players. At the same time, it is important to investigate the question of time-consistency of a cooperative solution under the assumption that some players may behave irrationally.

As is shown in [25], the condition of protection of a player from the irrational behavior of other players can be formulated as follows:

$$\int_{t_0}^{\vartheta} \beta_i(t)dt + V(\{i\}, x^*(\vartheta), T - \vartheta) \geq V(\{i\}, x_0, T - t_0), \quad \forall \vartheta \in [t_0, T], \qquad i \in N. \tag{11}$$

This means that in the case of irrational rupture of cooperation at time ϑ, the payoff obtained by the i-th player will still be not less than the payoff obtained in the non-cooperative scenario of the game, i.e., the payoff which the respective player would obtain acting individually.

This condition (11) can be rewritten in the following form (by taking the first derivative w.r.t. ϑ) [26]:

$$\beta_i(t) \geq -\frac{d}{dt}V(\{i\}, x^*(t), T - t), \quad \forall t \in [t_0, T] \qquad i \in N. \tag{12}$$

Let us consider the Shapley value $Sh(x_0, T - t_0)$ as an initial cooperative solution (the following is also true for any imputation from the set $M(x_0, T - t_0)$ which players try to make time-consistent). Then from (9) [and (10)] the Yeung's condition (11) [and (12)] can be rewritten in the following useful forms:

$$Sh_i(x_0, T - t_0) - Sh_i(x^*(t), T - t) \geq V(\{i\}, x_0, T - t_0) - V(\{i\}, x^*(t), T - t), \tag{13}$$

and

$$\frac{d}{dt}Sh_i(x^*(t), T - t) \leq \frac{d}{dt}V(\{i\}, x^*(t), T - t), \qquad i \in N. \tag{14}$$

It shows that the Shapley value of the i-th player is decreasing slower than the payoff obtained by the respective player when acting alone. In fact, condition (14) is a "refinement" of the individual rationality condition $Sh_i(x^*(t), T - t) \geq V(\{i\}, x^*(t), T - t)$ for the first order derivative.

It is easy to extend the idea above for the case of partial cooperation preservation. For the case of three players we wish to guarantee that the coalition of two players will still do better when acting together in the case of the deviation of the remaining single player. Let us consider the coalition $\{1, 2\}$. Proceeding in the similar way we obtain conditions for this case:

$$\sum_{i=1}^{2} \int_{t_0}^{\vartheta} \beta_i(t)dt + V(\{1,2\}, x^*(\vartheta), T-\vartheta) \geq V(\{1,2\}, x_0, T-t_0), \quad \forall \vartheta \in [t_0, T]. \tag{15}$$

This means that in the case of the irrational rupture of cooperation at time ϑ by the players 3, the payoff obtained by $\{1, 2\}$ will still be not less than the payoff obtained in the non-cooperative scenario of the game, i.e., the payoff which the respective coalition $\{1, 2\}$ would obtain when acting as a single player in a non-cooperative game with player 3. This result can also be formulated for any possible 2-player coalition $S = \{2, 3\}$ or $S = \{1, 3\}$. We have

$$\begin{aligned} &\sum_{i \in S} \int_{t_0}^{\vartheta} \beta_i(t)dt + V(S, x^*(\vartheta), T-\vartheta) \geq V(S, x_0, T-t_0), \\ &\forall \vartheta \in [t_0, T], \qquad S \in \{\{1,2\}, \{1,3\}, \{2,3\}\}. \end{aligned} \tag{16}$$

From (16) we get

$$\sum_{i \in S} [Sh_i(x_0, T-t_0) - Sh_i(x^*(t), T-t)] \geq [V(S, x_0, T-t_0) - V(S, x^*(t), T-t)], \tag{17}$$

and

$$\frac{d}{dt} \sum_{i \in S} Sh_i(x^*(t), T-t) \leq \frac{d}{dt} V(S, x^*(t), T-t). \tag{18}$$

For an n-player game the inequality (18) can be naturally generalized for any possible coalition $S \subset N$. This results in the condition of partial cooperation preservation as described above.

2.6 Strongly Time-Consistent Core on the Base of the Shapley Value

In the previous section the scenario of deviating from cooperative behavior in favor of non-cooperative behavior for some irrational reasons was considered. It turns out that the same conditions may also protect players from deviating within a fully cooperative scenario. The results below show that by using the same IDP under conditions (14), (18) the deviation—due to whatever reason—from one imputation from the core to another imputation from the core at some intermediate time instant will lead to payments for players which also belong to the core.

Using the IDP (10) players ensure time-consistency of the Shapley value. Strong time-consistency [14] of the core considered as a cooperative solution implies that a single deviation from the chosen imputation from the core in favor of another imputation from the core does not lead to non-realizability of the cooperative agreement (the core) defined for the whole duration of the game.

In the following, we present some definitions from [5] along with a new one.

The core $C(x_0, T - t_0)$ is said to be strongly time-consistent in the game $\Gamma(x_0, T - t_0)$, if

1. $C(x^*(t), T - t) \neq \emptyset$, $t \in [t_0, T]$.
2. There exists an imputation $\bar{\alpha} \in C(x_0, T - t_0)$ and an IDP $\beta(\tau) = (\beta_1(\tau), \ldots, \beta_n(\tau))$, $\tau \in [t_0, T]$, such that

$$\bar{\alpha}_i = \int_{t_0}^{T} \beta_i(\tau) d\tau, \ \forall i = 1, \ldots, n,$$

and

$$C(x_0, T - t_0) \supset \int_{t_0}^{t} \beta(\tau) d\tau \oplus C(x^*(t), T - t), \ t \in [t_0, T].$$

Here symbol $\oplus$ is defined as follows. Let $a \in R^n$, $B \subset R^n$, then $a \oplus B = \{a + b : b \in B\}$.

This definition means that when a game evolves along the cooperative trajectory $x^*(t)$ there exists an imputation $\bar{\alpha}$ inside the core $C(x_0, T - t_0)$ such that a single deviation from this imputation at time t in favor of another imputation from the core $C(x^*(t), T - t)$ [for the subgame starting at time t from the state $x^*(t)$] will lead to the overall payment for players which is also contained in the core $C(x_0, T - t_0)$.

Definition 1. The imputation $\bar{\alpha}$ from the core $C(x_0, T - t_0)$ which guarantees the strong time-consistency of the core is called a supporting imputation.

This definition of the strong time-consistency slightly differs from the previous one published in [14]. Here we state that the core is a strongly time-consistent solution if inside the core it is possible to find the supporting imputation and to redistribute it over time accordingly IDP such that any deviation from this supporting solution in favor of another imputation during the game will lead to the payment for players which is also contained in the core.

It turns out that if the Shapley value belongs to the core and Yeung's conditions (14), (18) are satisfied, then it can be considered as the supporting imputation with IDP calculated by (10). The following theorem is proved:

Theorem 1. *Let the core $C(x^*(\vartheta), T-\vartheta)$ is non-empty set for any subgames $\Gamma(x^*(\vartheta), T-\vartheta)$ occurring along the optimal trajectory $x^*(t)$. Let the Shapley value $Sh(x^*(\vartheta), T-\vartheta)$ (6) be such that*

(A1) It belongs to the core for any $\vartheta \in [t_0, T]$, i.e.,

$$Sh(x^*(\vartheta), T-\vartheta) = \{Sh_1(\cdot), Sh_2(\cdot), Sh_3(\cdot)\} \in C(x^*(\vartheta), T-\vartheta).$$

(A2) The Yeung's conditions (13), (17) are satisfied.

Then the Shapley value $Sh(x_0, T-t_0)$ (6) can be used as a supporting imputation $\bar{\alpha}$ for the core $C(x_0, T-t_0)$ in the game $\Gamma(x_0, T-t_0)$.

Proof. From the condition (A1) and definition of the core in the form (4) we directly get

$$\begin{array}{l} \sum_{i=1}^3 Sh_i(x^*(t), T-t) = V(N, x^*(t), T-t), \\ Sh_i(x^*(t), T-t) \geq V(\{i\}, x^*(t), T-t) \quad i \in N, \\ Sh_1(x^*(t), T-t) + Sh_2(x^*(t), T-t) \geq V(\{1,2\}, x^*(t), T-t), \\ Sh_1(x^*(t), T-t) + Sh_3(x^*(t), T-t) \geq V(\{1,3\}, x^*(t), T-t), \\ Sh_2(x^*(t), T-t) + Sh_3(x^*(t), T-t) \geq V(\{2,3\}, x^*(t), T-t). \end{array} \tag{19}$$

To make the Shapley value $Sh(x_0, T-t_0)$ a time-consistent cooperative solution during the whole period of the game we use the IDP $\{\beta_i(\vartheta)\}_{i\in N}$ defined by (10). Then we get

$$\sum_{i=1}^3 \int_t^T \beta_i(\tau)d\tau = \sum_{i=1}^3 Sh_i(x^*(t), T-t) = V(N, x^*(t), T-t). \tag{20}$$

Let us consider the Shapley value as a supporting solution $\bar{\alpha}$.
We have to show the inclusion

$$\int_{t_0}^t \beta(\tau)d\tau \oplus C(x^*(t), T-t) \subset C(x_0, T-t_0)$$

for all $t \in [t_0, T]$, where $\beta(\tau)$ is the IDP for the Shapley value $Sh(x_0, T-t_0)$.

Indeed, choose an arbitrary imputation from the core: $\alpha^t \in C(x^*(t), T-t)$. Denote

$$\hat{\alpha}_i = \int_{t_0}^t \beta_i(\tau)d\tau + \alpha_i^t,$$

then we have to prove that this new vector $\{\hat{\alpha}_i\}$ belongs to the core $C(x_0, T-t_0)$, i.e., inequalities (4) holds for $\{\hat{\alpha}_i\}$.

Obviously, it follows from (19), (20) that

$$\sum_{i=1}^{3} \int_{t_0}^{t} \beta_i(\tau)d\tau = \sum_{i=1}^{3} \int_{t_0}^{T} \beta_i(\tau)d\tau - \sum_{i=1}^{3} \int_{t}^{T} \beta_i(\tau)d\tau =$$
$$= V(N, x_0, T - t_0) - V(N, x^*(t), T - t).$$

Furthermore, it follows from the property of the imputation that $\sum_{i=1}^{3} \alpha_i^t = V(N, x^*(t), T - t)$.

Then we obtain

$$\sum_{i=1}^{3} \hat{\alpha}_i = \sum_{i=1}^{3} \int_{t_0}^{t} \beta_i(\tau)d\tau + \sum_{i=1}^{3} \alpha_i^t = V(N, x_0, T - t_0).$$

Let us check the property of the individual rationality for $\{\hat{\alpha}_i\}$. From (8) we easily get that

$$\int_{t_0}^{t} \beta_i(\tau)d\tau = Sh_i(x_0, T - t_0) - Sh_i(x^*(t), T - t).$$

Then from (13) we obtain

$$\hat{\alpha}_i = \int_{t_0}^{t} \beta_i(\tau)d\tau + \alpha_i^t = Sh_i(x_0, T - t_0) - Sh_i(x^*(t), T - t) + \alpha_i^t \geq$$
$$= Sh_i(x_0, T - t_0) - Sh_i(x^*(t), T - t) + V(\{i\}, x^*(t), T - t) \geq$$
$$\geq V(\{i\}, x_0, T - t_0) - V(\{i\}, x^*(t), T - t) + V(\{i\}, x^*(t), T - t) = V(\{i\}, x_0, T - t_0).$$

By the similar argument, from (17) we get

$$\begin{aligned} &\sum_{i \in S} \hat{\alpha}_i = \sum_{i \in S} \int_{t_0}^{t} \beta_i(\tau)d\tau + \sum_{i \in S} \alpha_i^t \geq \\ &\geq \sum_{i \in S} \int_{t_0}^{t} \beta_i(\tau)d\tau + V(S, x^*(t), T - t) = \\ &= \sum_{i \in S} Sh_i(x_0, T - t_0) - \sum_{i \in S} Sh_i(x^*(t), T - t) + V(S, x^*(t), T - t) \geq \\ &\geq V(S, x_0, T - t_0), \qquad S \in \{1, 2\}, \{1, 3\}, \{2, 3\}. \end{aligned} \tag{21}$$

Thus, the imputation $\hat{\alpha}$ satisfies conditions (4) and belongs to the core. □

If conditions (A1), (A2) are satisfied, then the Shapley value may be considered as a time-consistent cooperative solution (with IDP (10)) and also it provides strongly time-consistency of the core.

The result of the Theorem 1 can be generalized for the case of n players. In this case the inequality (21) should be shown to hold for all possible coalitions $S \subset N$. This follows directly from (17) for all possible S.

We note that the above result implies that any imputation from the core satisfying Yeung's conditions (13), (17) can be taken as the supporting imputation. In the case of the Shapley value it is easy to formalize this choice.

3 A Cooperative Differential Game of Pollution Control

3.1 Model

Consider a game-theoretic model of pollution control based on models published in [2, 3, 7], see also [23]. There are three players (companies, countries) that participate in the game, $N = \{1, 2, 3\}$. Each player has an industrial production site. It is assumed that the production is proportional to the pollutions u_i. Thus, the strategy of a player is to choose the amount of pollutions emitted to the atmosphere, $u_i \in [0; b_i]$. In this example the solution will be considered in the class of open-loop strategies $u_i(t)$.

The dynamics of the total amount of pollution $x(t)$ is described by

$$\dot{x} = u_1 + u_2 + u_3 - \delta x, \quad x(t_0) = x_0,$$

where δ is the absorption coefficient corresponding to the natural purification of the atmosphere.

In the following we assume that the absorption coefficient δ is equal to zero:

$$\dot{x} = u_1 + u_2 + u_3, \quad x(t_0) = x_0. \tag{22}$$

The instantaneous payoff of i-th player is defined as:

$$R(u_i(t)) = b_i u_i(t) - \frac{1}{2}u_i^2(t), \qquad i \in N.$$

Each player has to bear expenses due to the pollution removal. Thus the instantaneous payoff (utility) of the i-th player is equal to $R(u_i(t)) - d_i x(t), d_i > 0$.

Thus the payoff of the i-th player is defined as

$$K_i(x_0, T - t_0, u) = \int_{t_0}^{T} \left(\left(b_i - \frac{1}{2}u_i \right) u_i - d_i x \right) dt \to max, \quad i = 1, 2, 3. \tag{23}$$

3.2 Cooperative Game

Consider cooperative form of the game. It means that all players join together to maximize their total payoff. We seek optimal profile of strategies $u^* = (u_1^*, u_2^*, u_3^*)$ such that $\sum_{i=1}^{3} K_i \to \max_{u_1,u_2,u_3}$.

The optimization problem is as following:

$$\sum_{i=1}^{3} K_i(t_0, x_0, T, u) = \sum_{i=1}^{3} \int_{t_0}^{T} \left(\left(b_i - \tfrac{1}{2}u_i\right) u_i - d_i x\right) dt \to \max_{u_1,u_2,u_3}, \tag{24}$$
$$s.t.\ x(t) \text{ satisfies } (22).$$

Let us denote the maximum value in (24) as $V(N, x_0, T - t_0)$.

To solve the problem (24) we use classical Pontryagin's maximum principle.

The Hamiltonian is

$$H_i(x, u, \psi) = \sum_{i=1}^{3} \left(b_i - \frac{1}{2}u_i\right) u_i - \sum_{i=1}^{3} d_i x + \psi(u_1 + u_2 + u_3), \tag{25}$$

its first order partial derivatives w.r.t. u_i's are

$$\frac{\partial H}{\partial u_i}(x, u, \psi) = b_i - u_i + \psi, \tag{26}$$

and the Hessian matrix $\frac{\partial^2 H}{\partial u^2}(x, u, \psi)$ is negative definite whence we conclude that the Hamiltonian H is concave w.r.t. u_i.

We get the optimal control

$$u^*(t) = \begin{pmatrix} b_1 - d_s(T - t) \\ b_2 - d_s(T - t) \\ b_3 - d_s(T - t) \end{pmatrix}$$

and from (22) we get the optimal (cooperative) trajectory:

$$x^*(t) = x_0 + (t - t_0)\,(b_s - 3Td_s) + \frac{3d_s}{2}(t^2 - t_0^2), \tag{27}$$

where $d_s = d_1 + d_2 + d_3$, $b_s = b_1 + b_2 + b_3$.

Given the initial conditions (t, x) and the final time T, the value function $V(t, x(t))$ can be represented as

$$V(t, x(t)) = \alpha_1(t, T)x(t) + \alpha_0(t, T),$$

where

$$\alpha_1 = -d_s\,(T-t)$$
$$\alpha_0 = \tfrac{1}{2}\,(T-t)\left(d_s^2(T-t)^2 - b_s d_s(T-t) + \tilde{b}_s\right),$$

where d_s, b_s defined below and $\tilde{b}_s = b_1^2 + b_2^2 + b_3^2$.

For our problem it was possible also to use the dynamic programming method based on solution of the Hamilton–Jacobi–Bellman equation. It is easy to check that for the Bellman function in the form $V(t, x(t)) = A(t)x + B(t)$ we obtain the same result as from Pontryagin's maximum principle.

3.3 Nash Equilibrium

We can also compute the Nash equilibrium. For Nash equilibrium we solve the problem in which every player i, $i = 1, 2, 3$, has to maximize his payoff (23) under restriction (22). By the similar way as above we obtain the optimal controls

$$u^N(t) = \begin{pmatrix} b_1 - d_1(T-t) \\ b_2 - d_2(T-t) \\ b_3 - d_3(T-t) \end{pmatrix}$$

and corresponding trajectory (for Nash equilibrium case)

$$x^N(t) = x_0 + (b_s - Td_s)\,(t - t_0) + \frac{d_s}{2}(t^2 - t_0^2). \tag{28}$$

Comparing the evolutions of the state variable (stock of pollution) in cooperative and non-cooperative case of the game (Fig. 1) one can observe that the level of pollution is bigger in the non-cooperative scenario.

The maximum of the payoff for each player $i = 1, 2, 3$ in the subgame starting from $(t, x(t))$ gets the form:

$$V(\{i\}, x(t), T-t) = -d_i\,(T-t)\,x(t) +$$
$$\frac{(T-t)\left(d_i(2d_s - d_i)(T-t)^2 - 3b_s d_i(T-t) + 3{b_i}^2\right)}{6}.$$

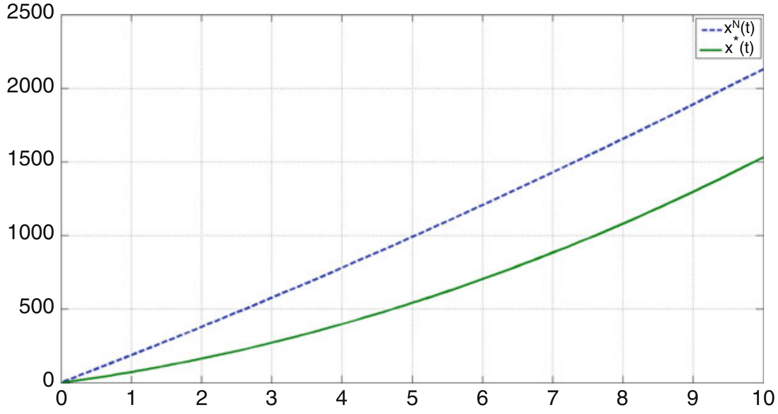

Fig. 1 Stock of pollution for the cooperative and the non-cooperative versions of the game

3.4 Characteristic Function

We can also compute the controls and the respective value functions for different coalitions. However, their form will depend on the way we define the respective optimal control problems.

Let us construct the characteristic function based on the approach described in [18] (δ-c.f., see Sect. 2.1). The characteristic function of coalition S is computed in two stages: the first one is already done (the Nash equilibrium strategies for all players are found in Sect. 3.3); at the second stage we assume that the left-out players implement their Nash optimal strategies $u_i^N(t)$ while the players from coalition S seek to maximize their joint payoff $\sum_{i\in S} K_i$ (see, also, [9]).

Consider the case of $\{1,2\}$-coalition. It seems to instructive to go through the calculations in detail. The respective Hamiltonian is

$$H_{12} = u_1\left(b_1 - \frac{u_1}{2}\right) + u_2\left(b_2 - \frac{u_2}{2}\right) - d_{12}x + \psi\left(u_1 + u_2 + b_3 - d_3(T-t)\right),$$

where $d_{12} = d_1 + d_2$. Note that we substituted u_3 by u_3^N which was found earlier. The optimal strategies of the players are

$$u^0(t) = \begin{pmatrix} b_1 + \psi(t) \\ b_2 + \psi(t) \\ b_3 - d_3(T-t) \end{pmatrix}$$

The differential equation for ψ is $\dot{\psi} = d_{12}$ which is solved to yield $\psi(t) = \psi_0 + d_{12}t - d_{12}t_0$. We choose $\psi_0 = -d_{12}(T - t_0)$ to ensure that $\psi(T) = 0$. Finally we get $\psi(t) = -d_{12}(T - t)$.

We substitute the obtained expression for $\psi(t)$ into $u^0(t)$ to get

$$u^0(t) = \begin{pmatrix} b_1 - d_{12}(T-t) \\ b_2 - d_{12}(T-t) \\ b_3 - d_3(T-t). \end{pmatrix}$$

We see that the 3rd player implements his Nash optimal strategy while the first two players stick to their Pareto optimal strategies.

In the next step we integrate (22) from t_0 to t to get $x(t)$:

$$x(t) = x_0 + b_s(t-t_0) + \left(\frac{d_3}{2} + d_{12}\right)(t^2 - t_0^2) - (d_3T + 2d_{12}T)\,(t-t_0).$$

The respective value of the characteristic function $V(\{1,2\}, x(t), T-t)$ is

$$V(\{1,2\}, x(t), T-t) = -d_{12}\,(T-t)\,x(t) + \\ + \frac{(T-t)\left(3\tilde{b}_{12} + 2d_{12}d_s(T-t)^2 - 3b_sd_{12}(T-t)\right)}{6},$$

where following the same logic as above we denote $\tilde{b}_{12} = b_1^2 + b_2^2$.

In the same way we obtain $V(\{1,3\}, x(t), T-t)$ and $V(\{2,3\}, x(t), T-t)$:

$$V(\{1,3\}, x(t), T-t) = -d_{13}\,(T-t)\,x(t) + \\ + \frac{(T-t)\left(3\tilde{b}_{13} + 2d_{13}d_s(T-t)^2 - 3b_sd_{13}(T-t)\right)}{6},$$

$$V(\{2,3\}, x(t), T-t) = -d_{23}\,(T-t)\,x(t) + \\ + \frac{(T-t)\left(3\tilde{b}_{23} + 2d_{23}d_s(T-t)^2 - 3b_sd_{23}(T-t)\right)}{6},$$

where $\tilde{b}_{13} = b_1^2 + b_2^2,\ \ \tilde{b}_{23} = b_2^2 + b_3^2$.

3.5 Superadditivity

Check the superadditivity condition (3) for constructed characteristic function $V(S,\cdot)$. It turns out that for any $i \neq j \neq k \in \{1,2,3\}$, the following holds:

$$V(N,\cdot) - (V(\{i,j\},\cdot) + V(\{k\},\cdot)) = \frac{1}{6}(T-t)^3(d_1 + d_2 + d_3)^2 > 0.$$

The next condition is $V(\{i,j\},\cdot) \geq V(\{i\},\cdot) + V(\{j\},\cdot)$. Again, we can check that

$$V(\{i,j\},\cdot) - (V(\{i\},\cdot) + V(\{j\},\cdot)) = \frac{1}{6}(T-t)^3(d_i^2 + d_j^2) > 0.$$

Thus, the constructed δ-characteristic function $V(S,\cdot)$ is a superadditive function without any additional conditions on the parameters of the model.

3.6 Shapley Value

Let the players employ the Shapley value as a cooperative solution. Then they allocate the amount $V(N, x_0, T - t_0)$ among players by (6). Using the constructed characteristic function we get

$$\begin{aligned}
&Sh_1(x(t), T-t) = \\
&= \tfrac{(T-t)}{36}(18{b_1}^2 - 36d_1x(t) + 12T^2{d_1}^2 + 3T^2{d_2}^2 + 3T^2{d_3}^2 + 12{d_1}^2t^2 + \\
&+3{d_2}^2t^2 + 3{d_3}^2t^2 - 18Tb_1d_1 - 18Tb_2d_1 - 18Tb_3d_1 + 18b_1d_1t + \\
&+18b_2d_1t + 18b_3d_1t + 16T^2d_1d_2 + 16T^2d_1d_3 + 4T^2d_2d_3 - 24T{d_1}^2t - \\
&-6T{d_2}^2t - 6T{d_3}^2t + 16d_1d_2t^2 + 16d_1d_3t^2 + 4d_2d_3t^2 - 32Td_1d_2t - \\
&-32Td_1d_3t - 8Td_2d_3t).
\end{aligned}$$

All components of the Shapley value in analytic form are presented in Appendix 1. Using the obtained formulas one can readily check that the Shapley value belongs to the core.

The graphic representation of the Shapley value for subgames along the optimal cooperative trajectory $x^*(t)$ is given in Fig. 2.

3.7 Time-Consistency Problem: IDP

Since we obtained the Shapley value in analytic form we can apply the formula (10) for $x(t) = x^*(t)$ to calculate the components of the IDP. In this way we redistribute the Shapley value $Sh(x_0, T - t_0)$ over time $[t_0, T]$ and for any intermediate time instant $\vartheta \in [t_0, T]$ it will have the form (8). Then the cooperative agreement will not be violated and the pollution control will be implemented by all countries cooperatively. This is particularly important for the pollution reduction problem as the cooperative solution leads to a smaller level of pollution compared to the non-cooperative one.

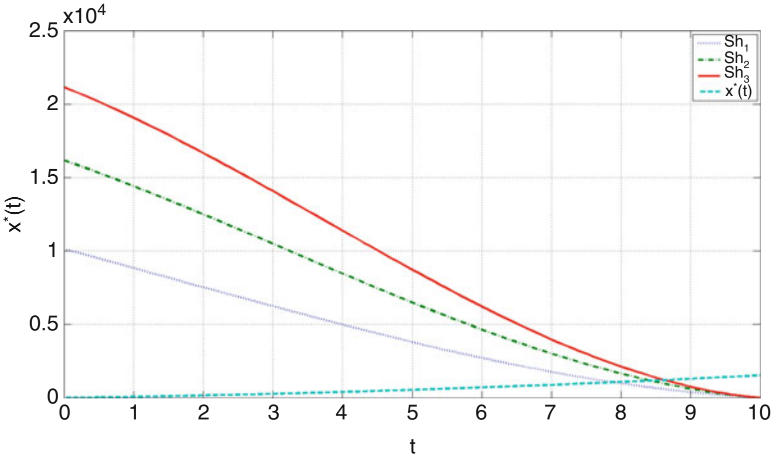

Fig. 2 The Shapley value for three players and the respective cooperative trajectory $x^*(t)$

Let's write the component of IDP $\beta(t) = \{\beta_i(t)\}_{i=1,2,3}$ for the first player:

$$\beta_1(t) = \frac{b_1{}^2}{2} - d_1\left(x_0 + (b_s - 3Td_s)(t - t_0) + \frac{3t^2 d_s}{2} - \frac{3t_0{}^2 d_s}{2}\right) - \frac{(T-t)}{36}(18b_s d_1 -$$
$$-18Td_1{}^2 - 6\tilde{d}_s(T-t) + 18d_1{}^2 t - 36d_1(b_s - 3Td_s + 3td_s) - 32Td_1d_2 - 32Td_1d_3 -$$
$$-8Td_2d_3 + 32d_1d_2t + 32d_1d_3t + 8d_2d_3t) + \frac{T^2d_1{}^2}{3} + \frac{T^2d_2{}^2}{12} + \frac{T^2d_3{}^2}{12} + \frac{d_1{}^2t^2}{3} + \frac{d_2{}^2t^2}{12} +$$
$$+\frac{d_3{}^2t^2}{12} - \frac{b_sd_1(T-t)}{2} + \frac{4T^2d_1d_2}{9} + \frac{4T^2d_1d_3}{9} + \frac{T^2d_2d_3}{9} - \frac{2Td_1{}^2t}{3} - \frac{Td_2{}^2t}{6} - \frac{Td_3{}^2t}{6} + \frac{4d_1d_2t^2}{9} +$$
$$+\frac{4d_1d_3t^2}{9} + \frac{d_2d_3t^2}{9} - \frac{8Td_1d_2t}{9} - \frac{8Td_1d_3t}{9} - \frac{2Td_2d_3t}{9}.$$

All components of the IDP in analytic form are given in Appendix 2. Figure 3 illustrates the IDP for all three players.

3.8 *Yeung's Condition*

To prevent the deviation of players from the cooperative agreement for irrational reasons the condition (12) should be held. Computation yields

$$\beta_i(t) + \frac{d}{dt}V(\{i\}, x^*(t), T-t) = \frac{(T-t)^2\left(3d_i^2 + 2\tilde{d}_s + d_s^2\right)}{12} > 0$$

for any i. Thus this condition holds without any additional restrictions on the parameters of the model. It can also be proved in a similar way that the extension of Yeung's condition (18) holds.

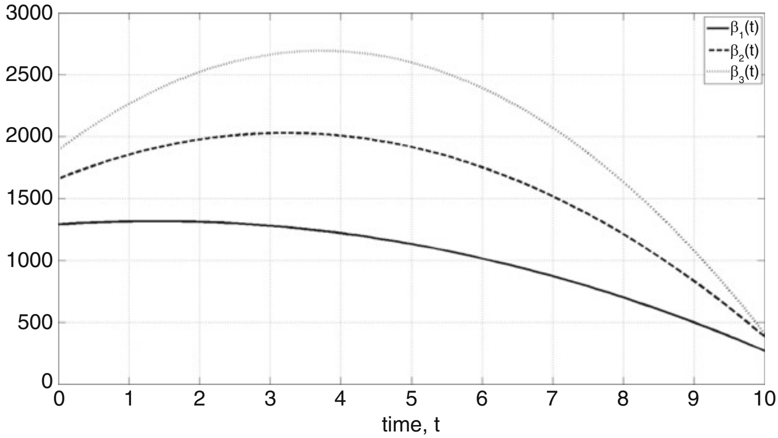

Fig. 3 Components of the IDP for all players

4 Strong Time-Consistency

Based on Theorem 1 and the results for the considered model of the pollution control we can use the Shapley value as a supporting imputation which provides strongly time-consistency of the core. Another approach see also at [20].

Unfortunately, it is difficult to illustrate this situation as there is no clear way to choose another solution from the core. For this reason we considered the Shapley value as it is easily and, not less important, uniquely defined.

To illustrate the converse scenario we consider the case where one switches from a particular imputation $\bar{\alpha}$ from the core as defined in [16] to the Shapley value. We will skip the computations and directly write the final expression for $\bar{\alpha}$.

Note that $\bar{\alpha}$ is taken from a family of imputations. This family forms a subset of possible supporting imputations which, however, does not contain the Shapley value. The reason for this is that the algorithm [16] does not use the values of the characteristic function for one-element coalitions.

Below we obtain the supporting solution by the method of Petrosyan and then show what happens if players decide to switch from this solution to the Shapley value at some intermediate time. The supporting solution is

$$\bar{\alpha}_i = \frac{(T-t_0)}{18}\Big((25d_s^2 - 24d_sd_i)(T-t_0)^2 - 18b_s(d_s-d_i)(T-t_0) + 9(\tilde{b}_s - b_i^2) - \\ -18x_0(d_s-d_i)\Big)$$

and the corresponding IDP

$$\bar{\beta}_i = \frac{1}{2}(\tilde{b}_s - b_i^2) - (d_s - d_i)\big(x_0 + (t-t_0)\,(b_s - 3Td_s) + \\ + \frac{3d_s}{2}(t^2 - t_0^2)\big) + \big(\frac{7}{6}d_s^2 - d_sd_i\big)(T-t)^2 - b_s(d_s-d_i)(T-t), \quad i \in N.$$

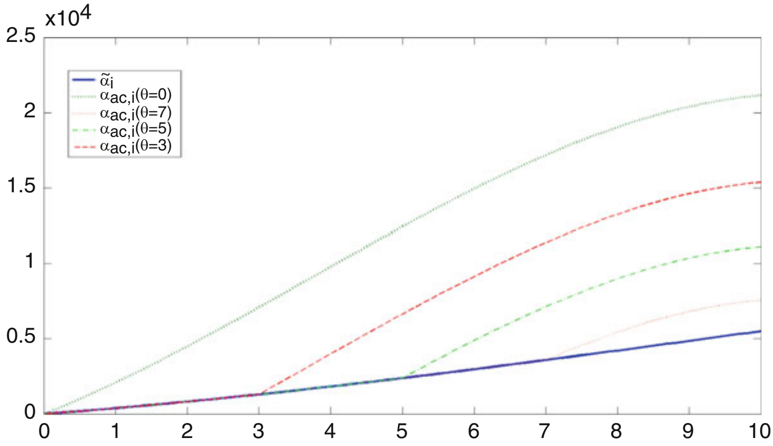

Fig. 4 Composite imputations from the core for different switching times

Figure 4 shows a number of imputations obtained by switching from the supporting imputation $\bar{\alpha}$ to the Shapley value at different time instants.

The resulting imputations always belong to the core, but it can be seen that the Shapley value dominates all the composite imputations. It is the subject of future research to check whether this property holds in general.

5 Conclusion

For a 3-player differential game it was shown that the Yeung's conditions can be used to construct a strongly time-consistent core. Within this core there exists a supporting imputation which has the property that a single deviation from this imputation in favor of any other imputation from the core still leads to the payment from the core. The obtained results were formulated for the Shapley value taken as the supporting imputation.

A particular class of differential games of pollution control was considered. It was shown that the δ-characteristic function computed for this game is superadditive and that the Yeung's conditions are satisfied without any additional restrictions on the parameters of the model. Both the Shapley value and the corresponding IDP were obtained in the analytic form. This result allows for a stable cooperation on the basis of the Shapley value taken as the cooperative solution. This leads to a substantial improvement as the cooperative solution leads to a lower level of pollution.

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The author is grateful to the anonymous reviewers for the valuable comments.

Appendix 1

The expression for the Shapley value calculated for a game of pollution control.

$$\begin{aligned}
&Sh_1(x(t), T-t) = \\
&= \tfrac{(T-t)}{36}(18{b_1}^2 - 36d_1x(t) + 12T^2{d_1}^2 + 3T^2{d_2}^2 + 3T^2{d_3}^2 + 12{d_1}^2t^2 + \\
&+3{d_2}^2t^2 + 3{d_3}^2t^2 - 18Tb_1d_1 - 18Tb_2d_1 - 18Tb_3d_1 + 18b_1d_1t + \\
&+18b_2d_1t + 18b_3d_1t + 16T^2d_1d_2 + 16T^2d_1d_3 + 4T^2d_2d_3 - 24T{d_1}^2t - \\
&-6T{d_2}^2t - 6T{d_3}^2t + 16d_1d_2t^2 + 16d_1d_3t^2 + 4d_2d_3t^2 - 32Td_1d_2t - \\
&-32Td_1d_3t - 8Td_2d_3t); \\
&Sh_2(x(t), T-t) = \\
&= \tfrac{(T-t)}{36}(18{b_2}^2 - 36d_2x(t) + 3T^2{d_1}^2 + 12T^2{d_2}^2 + 3T^2{d_3}^2 + 3{d_1}^2t^2 + \\
&+12{d_2}^2t^2 + 3{d_3}^2t^2 - 18Tb_1d_2 - 18Tb_2d_2 - 18Tb_3d_2 + 18b_1d_2t + \\
&+18b_2d_2t + 18b_3d_2t + 16T^2d_1d_2 + 4T^2d_1d_3 + 16T^2d_2d_3 - 6T{d_1}^2t - \\
&-24T{d_2}^2t - 6T{d_3}^2t + 16d_1d_2t^2 + 4d_1d_3t^2 + 16d_2d_3t^2 - 32Td_1d_2t - \\
&-8Td_1d_3t - 32Td_2d_3t); \\
&Sh_3(x(t), T-t) = \\
&= \tfrac{(T-t)}{36}(18{b_3}^2 - 36d_3x(t) + 3T^2{d_1}^2 + 3T^2{d_2}^2 + 12T^2{d_3}^2 + 3{d_1}^2t^2 + \\
&+3{d_2}^2t^2 + 12{d_3}^2t^2 - 18Tb_1d_3 - 18Tb_2d_3 - 18Tb_3d_3 + 18b_1d_3t + \\
&+18b_2d_3t + 18b_3d_3t + 4T^2d_1d_2 + 16T^2d_1d_3 + 16T^2d_2d_3 - \\
&-6T{d_1}^2t - 6T{d_2}^2t - 24T{d_3}^2t + 4d_1d_2t^2 + 16d_1d_3t^2 + 16d_2d_3t^2 \\
&-8Td_1d_2t - 32Td_1d_3t - 32Td_2d_3t).
\end{aligned}$$

Appendix 2

The expression for the IDP calculated for a game of pollution control.

$$\begin{aligned}
&\beta_1(t) = \tfrac{{b_1}^2}{2} - d_1\left(x_0 + (b_s - 3Td_s)(t - t_0) + \tfrac{3t^2d_s}{2} - \tfrac{3{t_0}^2d_s}{2}\right) - \tfrac{(T-t)}{36}(18b_sd_1 - \\
&-18T{d_1}^2 - 6\tilde{d}_s(T-t) + 18{d_1}^2t - 36d_1(b_s - 3Td_s + 3td_s) - 32Td_1d_2 - 32Td_1d_3 - \\
&-8Td_2d_3 + 32d_1d_2t + 32d_1d_3t + 8d_2d_3t) + \tfrac{T^2{d_1}^2}{3} + \tfrac{T^2{d_2}^2}{12} + \tfrac{T^2{d_3}^2}{12} + \tfrac{{d_1}^2t^2}{3} + \tfrac{{d_2}^2t^2}{12} + \\
&+\tfrac{{d_3}^2t^2}{12} - \tfrac{b_sd_1(T-t)}{2} + \tfrac{4T^2d_1d_2}{9} + \tfrac{4T^2d_1d_3}{9} + \tfrac{T^2d_2d_3}{9} - \tfrac{2T{d_1}^2t}{3} - \tfrac{T{d_2}^2t}{6} - \tfrac{T{d_3}^2t}{6} + \tfrac{4d_1d_2t^2}{9} + \\
&+\tfrac{4d_1d_3t^2}{9} + \tfrac{d_2d_3t^2}{9} - \tfrac{8Td_1d_2t}{9} - \tfrac{8Td_1d_3t}{9} - \tfrac{2Td_2d_3t}{9};
\end{aligned}$$

$$\beta_2(t) = \frac{b_2{}^2}{2} - d_2\left(x_0 + (b_s - 3Td_s)(t - t_0) + \frac{3t^2 d_s}{2} - \frac{3t_0{}^2 d_s}{2}\right) - \frac{(T-t)}{36}(18b_s d_2 - -18Td_2{}^2 - 6\tilde{d}_s(T-t) + 18d_2{}^2 t - 36d_2(b_s - 3Td_s + 3td_s) - 32Td_1d_2 - 32Td_2d_3 - -8Td_1d_3 + 32d_1d_2t + 8d_1d_3t + 32d_2d_3t) + \frac{T^2d_1{}^2}{12} + \frac{T^2d_2{}^2}{3} + \frac{T^2d_3{}^2}{12} + \frac{d_1{}^2t^2}{12} + \frac{d_2{}^2t^2}{3} + + \frac{d_3{}^2t^2}{12} - \frac{Tb_1d_2}{2} - \frac{Tb_2d_2}{2} - \frac{Tb_3d_2}{2} + \frac{b_1d_2t}{2} + \frac{b_2d_2t}{2} + \frac{b_3d_2t}{2} + \frac{4T^2d_1d_2}{9} + \frac{T^2d_1d_3}{9} + \frac{4T^2d_2d_3}{9} - -\frac{Td_1{}^2t}{6} - \frac{2Td_2{}^2t}{3} - \frac{Td_3{}^2t}{6} + \frac{4d_1d_2t^2}{9} + \frac{d_1d_3t^2}{9} + \frac{4d_2d_3t^2}{9} - \frac{8Td_1d_2t}{9} - \frac{2Td_1d_3t}{9} - \frac{8Td_2d_3t}{9};$$

$$\beta_3(t) = \frac{b_3{}^2}{2} - d_3\left(x_0 + (b_s - 3Td_s)(t - t_0) + \frac{3t^2 d_s}{2} - \frac{3t_0{}^2 d_s}{2}\right) - \frac{(T-t)}{36}(18b_s d_3 - -18Td_3{}^2 - 6\tilde{d}_s(T-t) + 18d_3{}^2 t - 36d_3(b_s - 3Td_s + 3td_s) - 8Td_1d_2 - 32Td_1d_3 - -32Td_2d_3 + 8d_1d_2t + 32d_1d_3t + 32d_2d_3t) + \frac{T^2d_1{}^2}{12} + \frac{T^2d_2{}^2}{12} + \frac{T^2d_3{}^2}{3} + \frac{d_1{}^2t^2}{12} + \frac{d_2{}^2t^2}{12} + + \frac{d_3{}^2t^2}{3} - \frac{Tb_1d_3}{2} - \frac{Tb_2d_3}{2} - \frac{Tb_3d_3}{2} + \frac{b_1d_3t}{2} + \frac{b_2d_3t}{2} + \frac{b_3d_3t}{2} + \frac{T^2d_1d_2}{9} + \frac{4T^2d_1d_3}{9} + \frac{4T^2d_2d_3}{9} - -\frac{Td_1{}^2t}{6} - \frac{Td_2{}^2t}{6} - \frac{2Td_3{}^2t}{3} + \frac{d_1d_2t^2}{9} + \frac{4d_1d_3t^2}{9} + \frac{4d_2d_3t^2}{9} - \frac{2Td_1d_2t}{9} - \frac{8Td_1d_3t}{9} - \frac{8Td_2d_3t}{9}.$$

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Impact of Propagation Information in the Model of Tax Audit

Elena Gubar, Suriya Kumacheva, Ekaterina Zhitkova, and Olga Porokhnyavaya

Abstract An effective tax system is an important part of economic and social interactions in human society. The key element of the tax system is tax control which provides the main functions of taxation and allows for increasing tax revenue and fees to the state budget. However, total tax audits of a population of taxpayers is economically unreasonable, and even selective tax audits are not always profitable. In this case the propagation of information can be viewed as an "infection of the mind," and its spread shows an interesting resemblance to that of epidemics. We thus use a modification of the classical Susceptible-Infected-Recovery model to describe the process. We assume that information propagates through the population by pairwise contacts between spreaders and others in the population and Informed agents disseminate information through their network of contacts or social networks. We study a model of spreading information in a large population of taxpayers and describe the dynamics of this process in complex social networks. We formulate an optimal control problem of tax auditing and analyze the behavior of agents in different subgroups depending on received information.

Keywords Tax control • Information spreading • SIR model • Epidemic process • Optimal control • Social networks

1 Introduction

One of the most important aspects of modeling taxation is tax control and so it has been the subject of continuous interest in much recent research. The standard model of taxation used to describe the behavioral relationship between taxpayers and the tax authority is static. For example, [1, 17–19] formulate mathematical models of tax evasion and auditing. Two of the most famous works [17] and [18] applied

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a game-theoretical approach to describe the taxation problem for the first time: they have presented the interaction between tax authority and taxpayers based on a hierarchical game "principal-to-agent." As well as in [1] and [19], optimal strategies are defined as optimal scheme or optimal contract, including tax and penalty rates and the probabilities of auditing.

However as tax collection is a periodic event, this problem can be considered as a dynamic process which occurs, for example, on a yearly basis. Moreover, it has been proved that a total tax audit is quite expensive and so the tax authority must define a method which helps to collect taxes while minimizing the costs of inspections. In particular, spreading information about the positive social benefits of tax collection or possible penalties for tax evasion through social networks and media provides a tool of control of large group of taxpayers.

A major difference between the models which have been studied in the past and our current study is that we combine an approach based on game theory with one accounting for the process of spreading information. Recent studies have been shown that the process of spreading information resembles an epidemic process and hence it is possible to use a modification of the Susceptible-Infected-Recovered (SIR) model to characterize the propagation of information.

As in classical SIR model we consider a large but finite population of taxpayers which is divided into several subgroups subject to the relevance of information. The entire population may be sorted out to Uninformed, Informed, Indifferent, and Resistant. The subgroup Uninformed consists of agents, who do not have information about a future tax auditing campaign. Informed agents received information and disseminate it if it is important to them. Indifferent taxpayers get information, but is not interesting to them and they may not transfer it.

We suppose that information is propagated through the population by pairwise contacts between spreaders and others in the population of taxpayers. The important fact is that if someone has adopted information and it is important, or she believes in it, then the agent is capable of spreading it to others.

In real life, social networks offer a good platform for interactions among agents in a population and many taxpayers have extensive social contacts and can disseminate information through their contacts network. It is a fact that in a social network information spreads rapidly through different channels without many restrictions and it is possible to consider the Internet and social networks as an effective tool for the propagation of information. However people are more likely to believe news from their friends and relatives while network information must be verified and from time to time as it has a lack of credibility. Hence we have to take into account a group of agents who ignore the received news. The scale-free structure of the Internet implies that each agent with an access to the social network has a statistically significant probability of having a very large number of contacts, which can be estimated by the average connectivity of the network.

During the past decades different models for the propagation viruses and information in networks have been developed. One of the first papers, applying epidemic processes to the spreading of the rumors, ideas, and information, is [3]. In [14] the spreading of computer virus over the network is considered as the propagation of epidemic process.

In this paper, we establish a complex control-theoretic model to design tax authority control strategies through the propagation of information and advertising the bonuses of participating in tax collection campaign to mitigate the impact of nonpayment on society. Information transmission can be represented by dynamics on a graph where vertices denote individuals and an edge connecting a pair of vertices indicates interaction between individuals. Due to a large population of people involved in the process of spreading rumors and information, random graph models such as scale-free networks in [7, 13] are convenient to capture the heterogeneous patterns in the large scale complex network.

This paper is organized as follows. In Sect. 2, we review a static game-theoretical model of tax control. In Sect. 3, we formulate and analyze the complex model of propagation information through the social network in a population of taxpayers; we formulate an optimal control problem and present a structure of an optimal program for spreading information by the tax authority, employing Pontryagin's maximum principle. In Sect. 4 we show a modification of algorithm which forms a scale-free network. Finally, we present simulations and conclusions about the model and discuss the impact of parameters to the system so that we can come up with suggestions for possible preventative or control methods.

2 Static Model

Based on a game-theoretical model presented in [1] in this section we present a static model of tax control. In the mentioned model the tax authority (high level of the hierarchy) and N taxpayers (low level of the hierarchy) are players. Each taxpayer has income level equal to i_j, where $j = \overline{1, N}$. At the end of every tax period the jth taxpayer can declare her income as r_j which can be less or equal to her true income i_j ($r_j \leq i_j$ for each $j = \overline{1, N}$).

After collecting the tax returns the tax authority audits taxpayers with the probability $\overline{p}$. The tax auditing supposed to be absolutely effective, that is, it reveals the existing evasion.

Let ξ be the tax rate, π be the penalty rate (these rates are assumed to be constants). If the evasion is revealed as a result of a tax audit, then the evaded taxpayer should pay unpaid tax and the penalty, which depends on the evasion level: $(\xi + \pi)(i_j - r_j)$.

Then the jth taxpayer's expected payoff is defined from the equation

$$\beta_j = i_j - \xi r_j - \overline{p}\,(\xi + \pi)(i_j - r_j), \quad (1)$$

where the first summand is always paid by the taxpayer (pre-audit payment), and the second—as the result of the tax auditing—made with probability $\overline{p}$ (post-audit payment).

Then the tax authority's net income can be defined as

$$J = \sum_{j=1}^{N} \left(\xi r_j + \overline{p}(\xi + \pi)(i_j - r_j) - \overline{p} \cdot \overline{c} \right), \tag{2}$$

where $\overline{c}$ is the unit cost of one audit. Let's define it as

$$\overline{c} = \frac{B}{\nu}, \tag{3}$$

where B is the tax authority's budget, ν is the share of the audited taxpayers:

$$\overline{p} = \frac{\nu}{N}. \tag{4}$$

Naturally, every players' aim is to maximize their expected payoffs.

For obtaining the further results we should use the next proposition, which was formulated for the model, described above, and proved in [1].

Proposition 1. *Let the inequality*

$$(\xi + \pi)i_j \geq \overline{c}, \tag{5}$$

be fulfilled for the subset $\{1, N_0\}$ *(*$N_0 < N$*) of* N *taxpayers. The optimal tax authority's strategy is* $p^* = \frac{\xi}{\xi+\pi}$ *for every* $j = \overline{1, N_0}$*. The jth taxpayer's optimal strategy is*

$$r_j^*(\overline{p}) = \begin{cases} 0, \text{ if } \overline{p} < p^*, \\ i_j, \text{ if } \overline{p} \geq p^*. \end{cases}$$

Let the inequality (5) be not fulfilled for every $j = \overline{N_0 + 1, N}$*. The optimal tax authority's strategy is* $\overline{p} = 0$*. The jth taxpayer's optimal strategy is* $r_j^*(0) = 0$*.*

The first case of the Proposition 1 is about the optimal strategy of tax authority in terms of what the tax audit is profitable for it (inequality (5) is satisfied). In response, the optimal strategy for lower level players is to decide to pay taxes or not, depending on the probability of being audited, chosen by the top player. This result is similar to the "threshold rule," obtained in [19] for another mathematical model of tax control.

The second case is a pessimistic situation. In this case, the tax authority does not have sufficient funds to carry out the necessary tax audits. Taxpayers are rational, and in these conditions they can afford not to pay anything.

3 Dynamic Model of Spreading Information

As we indicated above, taxation is a regular process and can be formulated as a dynamic model that takes into account the dissemination of information as a factor to stimulate tax compliance.

Let's assume that at the first moment of the process the audit probability $\overline{p} = 0$. The total population of rational taxpayers evades of taxation in accordance with the second case of the Proposition 1.

In practice there is no information about the relation of parameters in (5) for every $j = \overline{1, N}$, therefore, the tax authority does not know whether auditing is profitable or not. Moreover, as the tax authority's budget B is strongly limited, auditing with probability p^* is practically impossible.

Therefore, the tax authority has to stimulate unaudited taxpayers to pay tax that corresponds to their true income level. The means of such stimulation is spreading information about future audits to the taxpayers (which can be irrelevant in general). This information makes rational taxpayers think that the audit probability is high enough that paying taxes is less costly than evading them and risking having to pay back-taxes along with the penalties. Within the framework of this model, this information is given by inequality

$$\overline{p} \geq p^*, \tag{6}$$

where $p^* = \frac{\xi}{\xi + \pi}$ (due to Proposition 1).

The tax authority spreads this information with the intensity $u(t)$ (the share of the Informed taxpayers per unit time), $u \in [0; \overline{u}]$, where $\overline{u}$ is the possible maximum value of control function u.

3.1 Scheme of Spreading Information

Here we consider the process of spreading information over the network of contacts modeled as a scale-free network. Each node of such network represents a taxpayer, who gets the information and propagates it over her social contacts, internet, social networks, etc.

The entire population of taxpayers is divided into four subgroups according to their relation to spreading information (see [6, 8, 13]):

- Uninformed taxpayers S (we denote the number of agents in this group as n_S). They do not have any information about future auditing and, therefore, do not pay taxes ($r_j = 0$ due to Proposition 1).

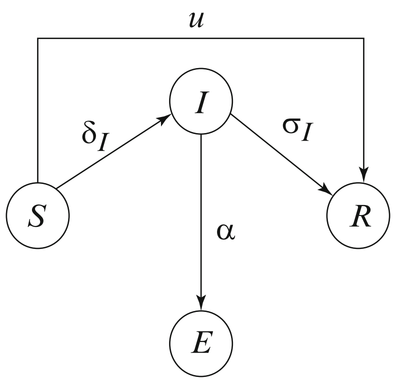

Fig. 1 Scheme of information spreading in a taxpayers population

- Informed taxpayers I (group n_I). These taxpayers receive information and propagate it: at first, they pay the taxes corresponding to their true income levels (ξi_j); second, they spread the information over Uninformed taxpayers.
- Indifferent taxpayers E (group n_E). They get information, but do not propagate it: they do not spread the information over Uninformed taxpayers and pay the taxes and penalties ($(\xi + \pi)i_j$) if and only if they were audited;
- Resistant taxpayers R (group n_R). The taxpayers from this subgroup are those who lost their interest in the information, because they paid taxes and propagated the information or, vice versa, did not propagate it and, so, paid penalties. In any case, the information becomes irrelevant for them.

Let's denote shares of Uninformed, Informed, Indifferent, and Resistant as

$$S^k = \frac{n_S}{N}, I^k = \frac{n_I}{N}, E^k = \frac{n_E}{N}, R^k = \frac{n_R}{N},$$

where $S^k + E^k + I^k + R^k = 1$ and k is the degree of each taxpayer's connections at time t. Initial states are $I^k(t_0) = I_0^k > 0$, $E^k(t_0) = E_0^k > 0$, $R^k(t_0) = R_0^k > 0$, $S^k(t_0) = 1 - I_0^k - E_0^k - R_0^k$.

The scheme of information spreading in a population of taxpayers is presented in the following diagram. See Fig. 1.

3.2 Constructing the Aggregated System Profit

In the current study, the aggregated system profit consists of two different parts: first, profit which was received by the tax authority from the propagation of information and, second, tax auditing. This step gives us the following conclusions.

The first conclusion is that in the model examined there is a two-component budget

$$B = \int_0^T (b_1(\overline{p}(t)) + b_2(u(t)))\, dt, \tag{7}$$

where $b_1(\overline{p}(t))$ is the cost of auditing with probability $\overline{p}$ and $b_2(u(t))$ is the cost for activating the spread of information: it is twice differentiable and increasing function in $u(t)$, such that $b_2(0) = 0$, $b_2(u) > 0$, when $u(t) > 0$.

The second conclusion is that the aggregated system profit also consists of two components:

$$J = J_{aud} + J_{inf}, \tag{8}$$

where J_{aud} is the tax authority's net income, obtained as a result of auditing, and J_{inf} is the profit, obtained from spreading information.

The first summand is the post-audit payments of Indifferent taxpayers (from the subgroup E) without total audit cost:

$$J_{aud} = g_E(E^k(T)) - b_1(\overline{p}(T)), \tag{9}$$

where $b_1(\overline{p}(T))$ is defined from

$$b_1(\overline{p}(T)) = n_E\overline{p}(T)\overline{c}, \tag{10}$$

and the post-audit payments of the taxpayers from E^k are

$$g_E(E^k(T)) = (\xi + \pi)\overline{p}\sum_{j=1}^{n_E} i_j. \tag{11}$$

To simplify the following reasoning, we will substitute $g_E(E^k(t))$ in the next continuous estimation:

$$\widehat{g_E}(E^k(T)) = (\xi + \pi)\overline{p}NE^k(T)\hat{i}, \tag{12}$$

where $\hat{i}$ is the average taxpayers' income.

The second summand of the aggregated system profit is the profit from the propagation of information:

$$J_{inf} = \int_0^T \big(f_R(R^k(t)) - f_E(E^k(t)) - b_2(u(t))\big)\, dt, \tag{13}$$

where the integrand is a sum of the taxes $f_R(R^k(t))$ collected from the Resistant taxpayers R^k, without the taxes unpaid by the Indifferent taxpayers from E^k and the cost of activating information spreading $b_2(u(t))$.

The first summand under the integral in (13) is

$$f_R(R^k(t)) = \xi \left(\sum_{j=1}^{n_R} i_j \right). \tag{14}$$

The second summand with minus is the unpaid taxes

$$f_E(E^k(t)) = \xi \left(\sum_{j=1}^{n_E} i_j \right). \tag{15}$$

We should use the continuous estimations for $f_R(R^k(t))$ and $f_E(E^k(t))$, as it was done for $g_E(E^k(t))$ in (12):

$$\widehat{f_R}(R^k(t)) = \xi N R^k(t)\hat{i}, \tag{16}$$

$$\widehat{f_E}(E^k(t)) = \xi N E^k(t)\hat{i}, \tag{17}$$

where $\hat{i}$ is the average taxpayers' income.

$f_R(R^k(t))$ and $f_E(E^k(t))$ (from (16) and (17) correspondingly) are non-decreasing and differentiable functions, such as $f_R(0) = 0$, $f_E(0) = 0$, $f_R(R^k(t)) > 0$, $f_E(E^k(t)) > 0$ for $R^k(t) > 0$, $E^k(t) > 0$.

The cost for activating information spreading $b_2(u(t))$ can be defined as

$$b_2(u(t)) = Nu(t)\tilde{c}, \tag{18}$$

where $\tilde{c}$ is the unit cost of information spreading.

Thus, the aggregated system profit (the tax authority's net income) is

$$J = \int_0^T \left[f_R(R^k(t)) - f_E(E^k(t)) - b_2(u(t)) \right] dt + g_E(E^k(T)) - b_1(\overline{p}(T)). \tag{19}$$

3.3 Constructing the System of Equations

We define a process of spreading information as a system of nonlinear differential equations corresponding to the scheme (Fig. 1). In our study, we use a modification of a classical epidemic model (see [10, 14]):

$$\begin{aligned}
\dot{S}^k &= -\delta_I I^k S^k \Theta_I - uS^k;\\
\dot{I}^k &= \delta_I I^k S^k \Theta_I - (\sigma_I + \alpha) I^k;\\
\dot{E}^k &= \alpha I^k;\\
\dot{R}^k &= \sigma_I I^k + uS^k;
\end{aligned} \tag{20}$$

where control

$$0 \leq u(t) \leq \overline{u} \leq 1, \text{ for all } t \in [0, T], \tag{21}$$

where $\overline{u}$ is a boundary value of control; δ_I is the rate of spreading of information in subgroup I; σ_I is the rates of forgetting of information in subgroup I; and α is a probability that received information is not important for an agent.

$\Theta_I(t)$ represents a probability that any given link points to an Informed or Indifferent taxpayer (see [7, 14]), as

$$\Theta_I(t) = \sum_{k'} \frac{\tau(k')P(k'|k)I^I_{k'}}{k'}, \tag{22}$$

where $\tau(k)$ denotes the infectivity of a node with degree k [7, 14]:

1. $\tau(k) \leq k$;
2. $\tau(k)$ is monotonically increasing;
3. $\lim\limits_{k\to\infty} \tau(k) = M > 0$;

$P(k'|k)$ shows the probability of a node with degree k pointing to a node with degree k': $P(k'|k) = \frac{k'P(k')}{\langle k \rangle}$, where mean value $\langle k \rangle = \sum\limits_k kP(k)$.

Within the framework of a model statement (20)–(21), we solve the optimal control problem. We find the optimal intensity of information spreading $u(t)$, which gives maximum to the functional (13)

$$J_{inf} = \int_0^T \left[f_R(R^k(t)) - f_E(E^k(t)) - b_2(u(t)) \right] dt \to \max. \tag{23}$$

3.4 Optimal Control Problem of Propagation Information

We find the optimal propagation strategy u to the problem described above applying Pontryagin's maximum principle [5, 15]. We define the associated Hamiltonian H and adjoint functions λ_S, λ_I, λ_E, λ_R as follows:

$$\begin{aligned} H = -b_2(u) - f_E(E^k) + f_R(R^k) + (\lambda_I - \lambda_S)\delta_I S^k I^k \Theta_I + (\lambda_R - \lambda_S)uS^k + \\ (\lambda_E - \lambda_I)\alpha I^k + (\lambda_R - \lambda_I)\sigma_I I^k. \end{aligned} \tag{24}$$

Adjoint system is

$$\begin{aligned} \dot{\lambda}_S(t) &= -\tfrac{\partial H}{\partial S^k} = (\lambda_S - \lambda_I)\delta_I I^k \Theta_I + (\lambda_S - \lambda_R)u; \\ \dot{\lambda}_I(t) &= -\tfrac{\partial H}{\partial I^k} = (\lambda_S - \lambda_I)\delta_I S^k \Theta_I + (\lambda_I - \lambda_E)\alpha + (\lambda_I - \lambda_R)\sigma_I; \\ \dot{\lambda}_E(t) &= -\tfrac{\partial H}{\partial E^k} = f'_E(E^k); \\ \dot{\lambda}_R(t) &= -\tfrac{\partial H}{\partial R^k} = -f'_R(R^k); \end{aligned} \tag{25}$$

with the transversality conditions given by

$$\lambda_S(T) = 0,\ \lambda_I(T) = 0,\ \lambda_E(T) = 0,\ \lambda_R(T) = 0. \tag{26}$$

According to the Pontryagin's maximum principle [15], there exist continuous and piecewise continuously differentiable co-state functions $\overline{\lambda} = (\lambda_S, \lambda_I, \lambda_E, \lambda_R)$ that at every point $t \in [0, T]$, where u is continuous, satisfy (25) and (26). In addition, we have

$$u \in \arg \max_{\underline{u} \in [0, \overline{u}]} H(\overline{\lambda}, (S^k, I^k, E^k, R^k), \underline{u}). \tag{27}$$

The derivative of Hamiltonian by u is

$$\frac{\partial H}{\partial u} = -b_2'(u) + (\lambda_R - \lambda_S)S^k \geq 0. \tag{28}$$

It is easy to see that Hamiltonian H reaches its maximum if condition (28) is satisfied.

According to the standard approach our main results are formulated in the following proposition and auxiliary lemmas:

Lemma 1. *Function ϕ is decreasing over the time interval $[0, T)$.*

Lemma 2. *For all t, $0 < t < T$ the following condition holds $(\lambda_R - \lambda_S) < 0$.*

Proofs of Lemmas 1 and 2 follow the same technique as in [4, 11].

Proposition 2. *In the problem statement (20), (21) (23) optimal control $u(t)$ has the following structure:*

- *When $b_2(\cdot)$ is concave function for (23), then there exist the time moments $\overline{t}$, $\overline{t} \in [0, T]$ such as*

$$u(t) = \begin{cases} \overline{u}, \text{ if } \phi > b_2(\overline{u})/\overline{u}, \text{ for } 0 < t < \overline{t}; \\ 0, \text{ if } \phi < b_2(\overline{u})/\overline{u}, \text{ for } \overline{t} < t < T. \end{cases} \tag{29}$$

- *When $b_2(\cdot)$ is strictly convex function, then there exist the time moments t_0, $\overline{t} \in [0, T]$, $0 \leq t_0 \leq \overline{t} \leq T$ such as:*

$$u(t) = \begin{cases} \overline{u}\ , \text{ on } 0 < t \leq t_0; \\ \text{is continually decreasing function}\ , \text{ on } t_0 < t \leq \overline{t}; \\ 0\ , \text{ on } \overline{t} \leq t \leq T; \end{cases} \tag{30}$$

where $\phi = (\lambda_R - \lambda_S)S^k$ is switching function for control problem (20), (24), (25), $\overline{u}$ is defined in (21).

Rewrite Hamiltonian in the following form:

$$H = -f_E(E^k) + f_R(R^k) + (\lambda_I - \lambda_S)\delta_I S^k I^k \Theta_I + (\phi u - b_2(u)) + \\ (\lambda_E - \lambda_I)\alpha I^k + (\lambda_R - \lambda_I)\sigma_I I^k. \quad (31)$$

To prove the main statement of the Proposition 2 we consider two cases:

(1) Consider a case when $b_2(\cdot)$ is concave.

Since function b_2 is concave ($b_2'' \le 0$), then $(u\phi - b_2(u))$ is convex function of u in (31) and for any $t \in [0, T]$ it reaches its maximum either at $u(t) = \overline{u}$ or $u(t) = 0$. From (31) we have that optimal $u(t)$ satisfies $u\phi - b_2(u) \ge \underline{u}\phi - b_2(\overline{u})$, where $\underline{u}$ is any admissible control, $\underline{u} \in [0, \overline{u}]$. If $u = \overline{u}$, then switching function is satisfied $\phi \ge b_2(\overline{u})/\overline{u}$ and if $u = 0$, then $\phi \le b_2(\overline{u})/\overline{u}$.

Lemma 1 suggests that ϕ is decreasing function, then there can be at most one moment $t \in [0, T]$ at which $\phi(t) = b_2'(\overline{u})$, moreover if such moment exists, for example, $\bar{t}$, then $\phi(t) > b_2(\overline{u})/\overline{u}$ on $0 \le t < \bar{t}$ and $\phi(t) < b_2(\overline{u})/\overline{u}$ on $\bar{t} < t \le T$. Then, (29) is satisfied.

(2) Let cost function $b_2(\cdot)$ be strictly convex.

If function b_2 is strictly convex ($b_2'' > 0$), then minimizer of $(u\phi - b_2(u))$ is unique. Expression (28) implies that if $\frac{dH}{du} = -b_2'(u) + \phi = 0$ at optimal u else $u \in [0, \overline{u}]$.

Thus, from continuity of functions ϕ and b_2' follows that u is continuous at all $t \in [0, T]$. As far as b_2 is strictly convex, then $b_2'(\overline{u}) > b_2'(0)$, $\overline{u} > 0$. Lemma 1 requires that there exist time moments t_0, $\bar{t}$, such as $0 < t_0 < \bar{t} < T$, which are defined from the following conditions:

$$u(t) = \begin{cases} 0\,, & \text{if } \frac{db_2(0)}{du} \le \phi; \\ \frac{db_2^{-1}(\phi)}{du}\,, & \text{if } \frac{db_2(\overline{u})}{du} \le \phi < \frac{db_2(0)}{du}; \\ \overline{u}\,, & \text{if } \phi < \frac{db_2(\overline{u})}{du}. \end{cases} \quad (32)$$

4 Scale-Free Network

Having considered scale-free network as a tool to structure the population of taxpayers and an engine for effective information spreading we estimated the number of contacts as the average connectivity of the network $\langle k \rangle$ and suppose that each node has approximately the same number of connections. Usually scale-free network is defined as a random graph whose degree distribution follows a power law and the main characteristic of the network does not depend on its size [7, 14]. The probability that a node of these networks has k connections follows a scale-free distribution $P(k) \sim k^{-\gamma}$ with an exponent γ that ranges between 2 and 3.

We studied the SEIR model of spreading information over a scale-free network (SF network), taking into account the impact of scale-free connectivity into the process of propagations. Based on the algorithm which had been studied in [14] in the present paper we introduce a modification of the algorithm of constructing a SF network. We develop a software product that allows for the observation of the process of dissemination of information and tracking changes in the networks settings. Below we show the key point of the algorithm:

- Initially (in the moment t_0), the number of unrelated nodes m_0 is small.
- At any time $t_i = t_{i-1} + 1$ we add a new node with m links that point to an existing node i with k_i links according to the probability

$$P(i) = \frac{k_i}{\sum_j k_j}. \tag{33}$$

Here m_0 and m are the parameters, defined by the user, which have some restrictions. The parameter m characterizes the average number of connections of a single individual $\langle k \rangle = 2m$. Suppose that the parameter m_0 must be not more than m due to the following considerations: if $m_0 > m$ and the next node is added then, according to (33), the nodes with a zero probabilities of further connection remain in this network. Then the constructed graph will be disconnected and contains nodes with no neighbors. These nodes represent individuals who do not have any contacts in the population and, therefore, are not involved in the epidemic process. These individuals can then be eliminated. After iterating this process we obtain a network with N nodes with connectivity distribution $P(k) \sim k^{-\gamma}$.

The detailed process of forming the SF network can be divided into two main stages:

Step 1. We build m_0 disconnected nodes. Then, while the number of available nodes is not more than m, we add node by node, which immediately get communication with others. This approach helps to avoid the unacceptable situation when two nodes are connected by two or more links.

Step 2. When there are more than m nodes in the network we can use Eq. (33). Before the size of the network reaches N, the nodes are added one by one. New node gets the link with one of old nodes according to the calculated probability. At each iteration the denominator in Eq. (33) changes. Nodes that have already connected to the added one are not involved in the process anymore. When the size of the network reaches N, the algorithm is stopped.

Using a scale-free network to assign the connections between agents in the populations of taxpayers we consider a process of propagating information which resembles an epidemic, then we must define the parameters of the proposed SEIR model: δ_I, σ_I are the transition coefficients; the initial distribution of the states of nodes; $\overline{p}$ is the audit probability and $\overline{c}$ is the audit cost; ξ and π are the tax and penalty rates correspondingly; distribution of the population by income level.

As has been shown in the previous research, an epidemic process can take place in different ways on a network, even when given the same parameters. This occurs because the initial distribution of the states of nodes introduces an element of chance—the user defines the number of nodes in a particular state, but does not choose exactly which nodes belong to each group. This also changes the initial value of Θ_I, since in one case we can obtain a hub as an Informed, which will increase Θ_I, and in the other case we can obtain a node with a small number of connections.

After all the parameters of the epidemic process are defined, the user can run it step by step. At each step, the following actions occur:

- For each Uninformed S^k node which has a connection with an Informed "neighbor" I^k, we check if he receives and adopts information in accordance with the specified transition coefficients δ_I. If the transmission of information is successful, the Uninformed node changes its status from S^k to I^k.
- If a node changed its status from $S^k \rightarrow I^k$, then it is necessary to determine if the received information is important to the agent. It means that information is important for agent with probability $(1 - \alpha)$ and it is indifferent with the probability α. If the node becomes indifferent, then it belongs to E^k.
- Each Informed I^k node loses interest in information in accordance with the transition coefficient σ_I and becomes resistant to information (R^k).
- after all transitions we recalculate values Θ_I and draw the new network.

Below we depict an example of process of propagation of information on small population of taxpayers. In Figs. 2–4 blue dots correspond to Uninformed taxpayers, red dots—Informed, orange dots—Indifferent, and green dots—Resistant.

5 Numerical Simulations

In this section, we present numerical simulations which are used to corroborate the results of the main propositions. We study the model of spreading information with the following parameters: tax and penalty rates are $\xi = 0.13$ and $\pi = 0.13$ correspondingly; the value of optimal probability is $p^* = 0.5$ (according to the fixed values of ξ and π); the value of the actual auditing probability is $\overline{p} = 0.2$.

We use the distribution of income among the population of Russian Federation in April of 2014 (see [2]) and calculate the average income level as the expected value of the uniform and Pareto distributions [9] (as it was previously done in [12]) to illustrate the simulation results.

We estimate an average monthly income of taxpayers as $\hat{i} = 30{,}000$ (rub) (see Table 1). According to the statistical data, costs of audit and information announcements approximately are equal to $\overline{c} = 7455$ (rub) and $\tilde{c} = 200$ (rub), respectively. We assume that the duration of time period which is valued to propagate information is $T = 0.5$ (130 days). In our paper we consider population of size $N = 1000$ and initial fractions of Uninformed, Informed, Indifferent, and Resistant are $S^k(0) = 0.9$, $E^k(0) = 0$, $I^k(0) = 0,1$, $R^k(0) = 0$. We use as a model parameters $\sigma = \frac{1}{\overline{T}} = 0.0083$, where $\overline{T} = 120$ is a period of obsolescence of information, and $\alpha = 0.1$. We construct a scale-free network (for $N = 1000$), according to the algorithms, which are presented in [14, 16], using the

Table 1 The distribution of income among the taxpayers

No	Average income level (rub)	Share of taxpayers
1	1750	0.038
2	4250	0.056
3	6000	0.094
4	8500	0.146
5	12,500	0.202
6	20,000	0.235
7	30,000	0.108
8	70,000 more	0.121

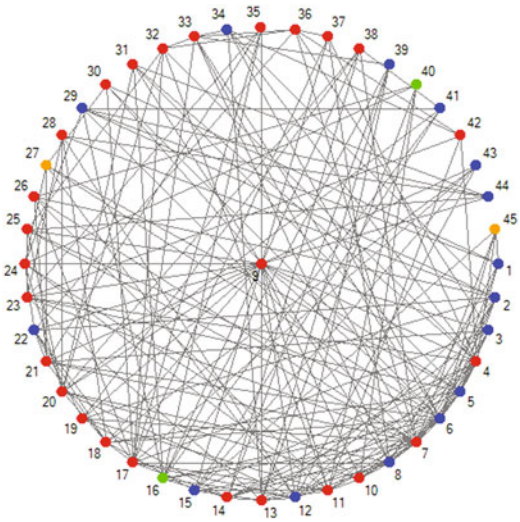

Fig. 2 An example of the network at different moments t. $t = 8$ seconds, $N = 45$, $S^k(t) = 15$, $E^k(t) = 2$, $I^k(t) = 25$, $R^k(t) = 2$, $\delta_I = 0.4$, $\alpha = 0.1$, $\sigma_I = 0.05$, $\overline{p} = 0.2$

next parameters: $\langle k \rangle = 6$, $P(k) = \frac{2m^2}{k^{-3}}$, $m = 5$, $m_0 = 4$, $\Theta_I \sim 0.33$. Examples of the networks are presented in Figs. 2, 3, and 4.

In Figs. 5, 6, and 7 we estimate an impact of transitions rate α on SEIR system and compare fractions of I^k in five different cases. A higher value of α suggests that the fraction I^k is less and application of control decreases a number of I^k for the same values of α. This fact shows that spreading information guarantees the increasing of taxpayers in group R^k who will pay taxes.

Figures 8 and 9 show the aggregated system profit and demonstrate the influence of parameter δ to collected taxes. We observe that total system costs which consist of J_{inf} and J_{aud} persistently grow depending on increasing of α at large value of δ (Figs. 10 and 11).

By studying the impact of various parameters to the population of taxpayers where tax authority propagates information about future tax audit we can draw a conclusion that the amount of collected taxes increases if a number of Informed

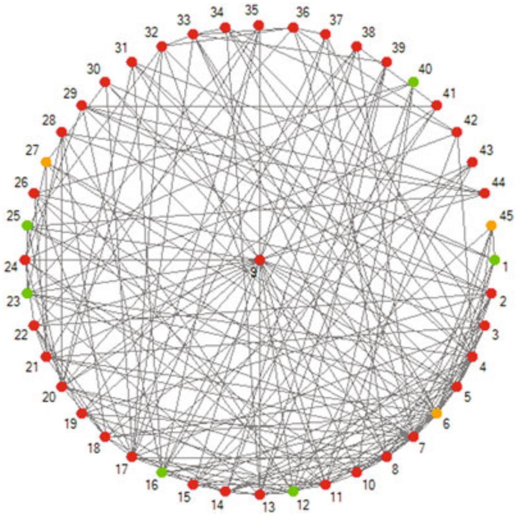

Fig. 3 An example of the network at different moments t. $t = 13$ seconds, $N = 45$, $S^k(t) = 0$, $E^k(t) = 3$, $I^k(t) = 33$, $R^k(t) = 9$, $\delta_I = 0.4$, $\alpha = 0.1$, $\sigma_I = 0.05$, $\overline{p} = 0.2$

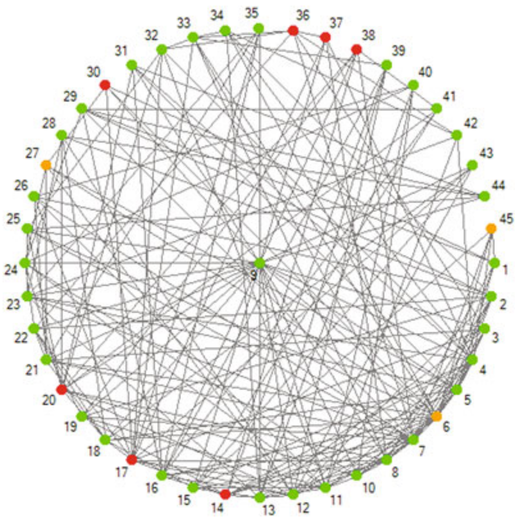

Fig. 4 An example of the network at different moments t. $T = 42$ seconds, $N = 45$, $S^k(T) = 0$, $E^k(T) = 3$, $I^k(T) = 7$, $R^k(T) = 35$, $\delta_I = 0.4$, $\alpha = 0.1$, $\sigma_I = 0.05$, $\overline{p} = 0.2$. Results: $S^k(T) = 0$, $I^k(T) = 0$, $E^k(T) = 3$, $R^k(T) = 42$, aggregated system profit is $J = 109{,}980$ monetary units

taxpayers and spreaders grow. At the same time even if the probability α is high, the spreading information provokes the augmentation of tax collection with minimum costs. Therefore we are able to say that this approach can be considered as effective and reasonable method to improve taxation system.

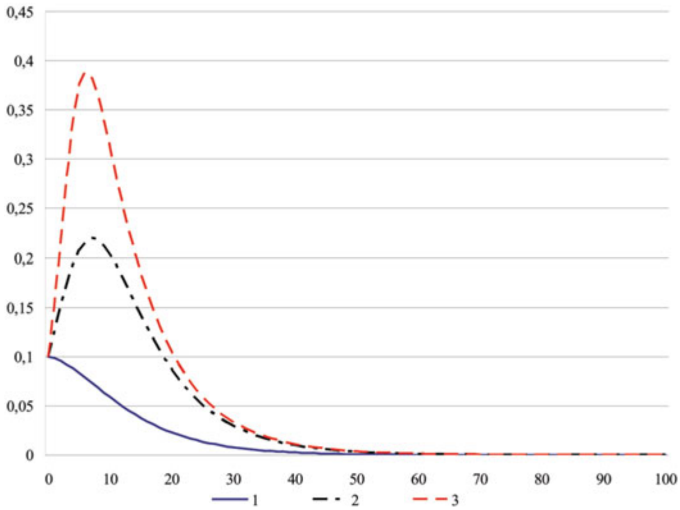

Fig. 5 The fraction of Informed taxpayers. Controlled case. Initial states: $S^k(0) = 0.9$, $E^k(0) = 0$, $I^k(0) = 0, 1$, $R^k(0) = 0$, $\sigma_I = 0.0083$, $\alpha = 0.1$. (*1*) $\delta_I = 0.1$, $I^k_{max} = 0.1$, $t_{max} = 0$. (*2*) $\delta_I = 0.4$, $I^k_{max} = 0.21985$, $t_{max} = 7$. (*3*) $\delta_I = 0.7$, $I^k_{max} = 0.38927$, $t_{max} = 6$

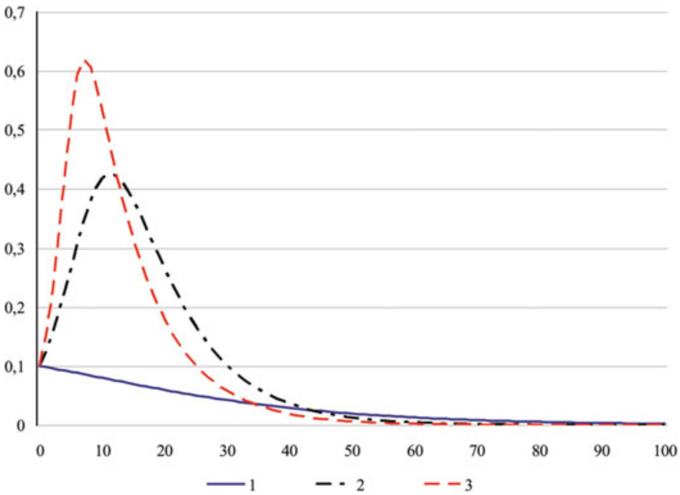

Fig. 6 The fraction of Informed taxpayers. Uncontrolled case. Initial states: $S^k(0) = 0.9$, $E^k(0) = 0$, $I^k(0) = 0, 1$, $R^k(0) = 0$, $\sigma_I = 0.0083$, $\alpha = 0.1$. (*1*) $\delta_I = 0.1$, $I^k_{max} = 0.1$, $t_{max} = 0$. (*2*) $\delta_I = 0.4$, $I^k_{max} = 0.425751$, $t_{max} = 11$. (*3*) $\delta_I = 0.7$, $I_{max} = 0.619338$, $t_{max} = 7$

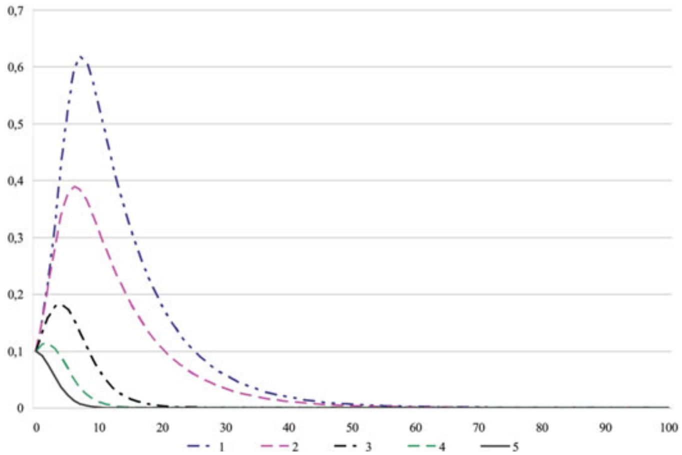

Fig. 7 Fractions of I^k in SEIR model, controlled and uncontrolled case. $S^k(0) = 0.9$, $E^k(0) = 0$, $I^k(0) = 0, 1$, $R^k(0) = 0$, $\sigma_I = 0.0083$, $\delta_I = 0.7$. (*1*) Uncontrolled system. $\alpha = 0.1$, $I^k_{max} = 0.6193$, $t_{max} = 7$, (*2*) Controlled system. $\alpha = 0.1$, $I^k_{max} = 0.3892$, $t_{max} = 6$, (*3*) Controlled system. $\alpha = 0.3$, $I^k_{max} = 0.1831$, $t_{max} = 4$, (*4*) Controlled system. $\alpha = 0.5$, $I^k_{max} = 0.1138$, $t_{max} = 2$, (*5*) Controlled system. $\alpha = 0.3$, $I^k_{max} = 0.1$, $t_{max} = 0$

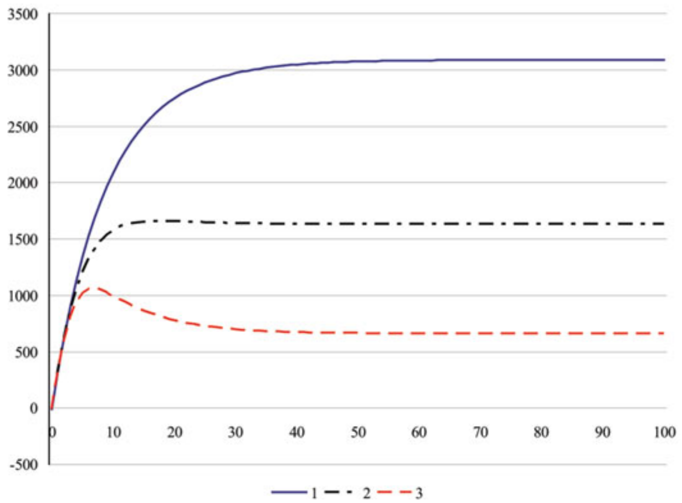

Fig. 8 Aggregated system profit tax authority throws information into the taxpayers population. $S^k(0) = 0.9$, $E^k(0) = 0$, $I^k(0) = 0.1$, $R^k(0) = 0$, $\sigma_I = 0.0083$, $\alpha = 0.7$. (*1*) $\delta_I = 0.1$, $J_{inf} = 282{,}717$, $J = 282{,}945$ monetary units. (*2*) $\delta_I = 0.4$, $J_{inf} = 158{,}526$, $J = 159{,}171$ monetary units. (*3*) $\delta_I = 0.7$, $J_{inf} = 71{,}178$, $J = 72{,}101$ monetary units

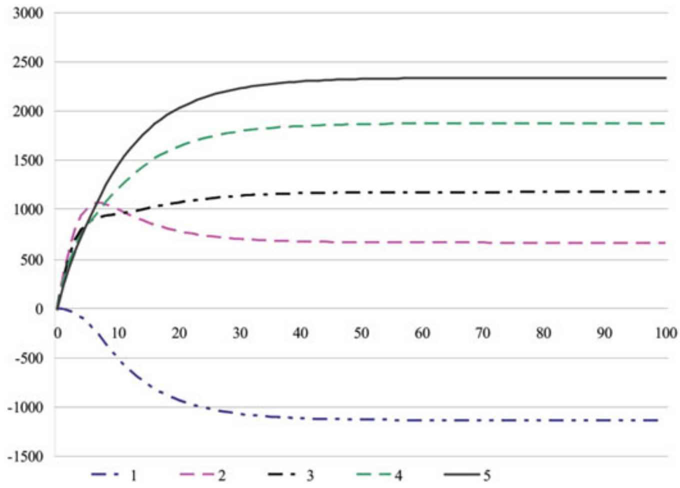

Fig. 9 Aggregated system profit tax authority throws information into the taxpayers population. $S^k(0) = 0.9$, $E^k(0) = 0$, $I^k(0) = 0, 1$, $R^k(0) = 0$, $\sigma_I = 0.0083$, $\delta_I = 0.7$. (*1*) Uncontrolled case. $\alpha = 0.1$, $J = -98{,}018$ monetary units. (*2*) Controlled case. $\alpha = 0.1$, $J = 701{,}013$. (*3*) Controlled case. $\alpha = 0.3$, $J = 111{,}655$ monetary units. (*4*) Controlled case. $\alpha = 0.5$ $J = 171{,}343$ monetary units. (*5*) Controlled case. $\alpha = 0.7$, $J = 211{,}925$ monetary units

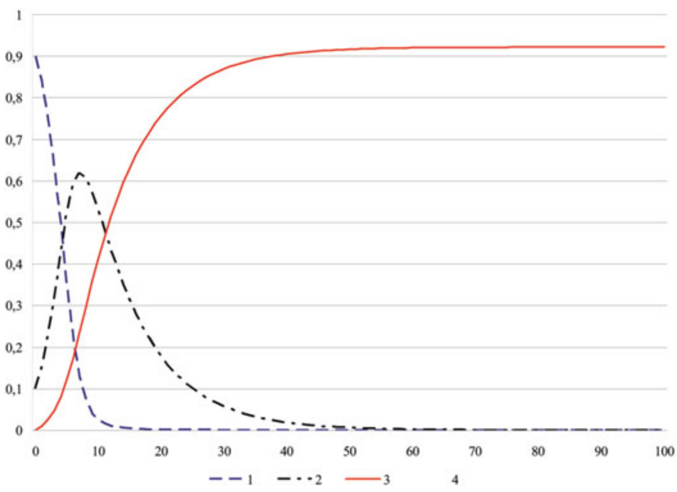

Fig. 10 SEIR model without application of control. $S^k(0) = 0.9$, $E^k(0) = 0$, $I^k(0) = 0.1$, $R^k(0) = 0$, $\sigma_I = 0.0083$, $\delta_I = 0.7$. $I^k_{max} = 0.6193$, $t_{max} = 7$

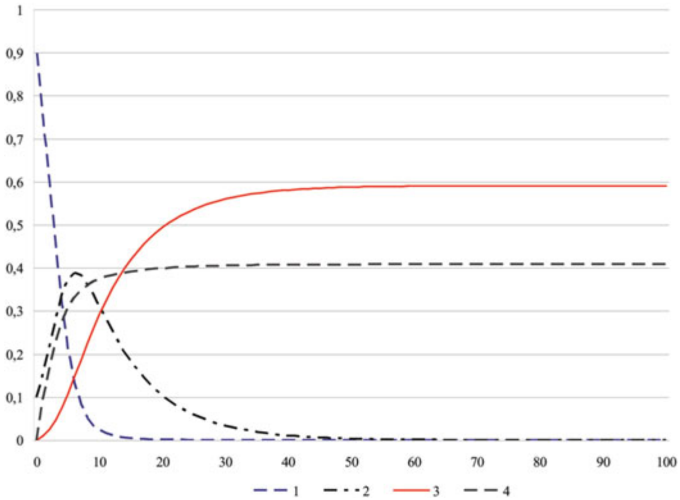

Fig. 11 SEIR model without application of control. $S^k(0) = 0.9$, $E^k(0) = 0$, $I^k(0) = 0.1$, $R^k(0) = 0$, $\sigma_I = 0.0083$, $\delta_I = 0.7$. $I^k_{max} = 0.3892$, $t_{max} = 4$

6 Conclusion

In the present paper we have investigated a complex model in which we combine a game-theoretical approach of tax control with a dynamic model of information propagation over a structured population of taxpayers. We formulated an optimal control problem for a tax auditing policy and analyzed the behavior of agents depending on social contacts and specific cost functions. All theoretical results are supported by numerical simulations with the real statistical data. Connections between taxpayers are modeled as a scale-free network constructed by a specially developed algorithm. As a result of our research we attempted to provide a new method of tax collection which can be more efficient and cost-effective.

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An Infinite Horizon Differential Game of Optimal CLV-Based Strategies with Non-atomic Firms

Gerasimos Lianos and Igor Sloev

Abstract We study the structure of optimal customer acquisition and customer retention strategies as a differential game over an infinite horizon in an industry with a large number of non-atomic firms. The optimal retention effort is constant over time and the optimal acquisition effort is proportional to the size of potential customer base. Greater customer profitability leads to higher per- capita acquisition and retention efforts, larger size of firms, and lower churn rate. A greater discount rate leads to lower per-capita acquisition and retention efforts, smaller firm size, and a greater churn rate. Tougher competition lowers the firms' acquisition and retention expenditures and it does not affect per-capita values. Both the churn rate and the share of acquisition expenditures in the total marketing budget decrease as firms grow over time. We revisit the concepts of the customer lifetime value (CLV) and the value of the firm in the dynamic equilibrium of an industry with a large number of players and demonstrate the equivalence between maximization of the value of the firm and maximization of a firm's individual CLV.

Keywords Differential games • Non-atomic games • Dynamic competition • Customer acquisition • Customer retention

Subject Classification: 90B60, 90B50, 91A80, 90A13

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1 Introduction

Customer lifetime value (CLV), the present value of the stream of profits accruing to a firm over the whole period of its relationship with customer, is a central concept in marketing science and business practice. Closely related to the idea of CLV are the ideas of customer acquisition (attracting new customers to a firm's service) and customer retention (preventing existing customers from leaving for other firms offering competitive suppliers of the service).

The literature on CLV has taken several directions. A stream of research builds probabilistic models [22, 23], econometric models [26, 27, 29], and diffusion/growth models [7, 15, 24]. A comprehensive review of different types of models based on CLV concept is given in [16]. A different stream of research examines the implications of CLV for managerial decisions. Blattberg and Deighton [6] propose a model of maximization of CLV giving the optimal allocation of expenditures between customer acquisition and customer retention. Berger and Nasr [5] propose models for the determination of CLV in the context of the contractual relationship between a customer and firm. Berger and Bechwati [4] examine the maximization of customer equity and the optimal allocation of promotional budget with applications to different market conditions.

A few studies have analyzed the explicit effects of competition and the marketplace on a firm's optimal CA and CR strategies (see [30] and [20] for a discussion). Syam and Hess [25] have built an analytical model to investigate the optimal CR or CA of an incumbent firm when faced with the threat of entry by a new firm. Fruchter and Zhang [14] have analyzed the strategic use of targeted promotions for CA and CR in dynamic duopoly game with firms of different size. They show that a firm with a larger market share should focus on CR whereas a firm with a smaller market share should focus on CA. Martin-Herran et al. [21] have investigated the optimal spending allocation between CA and CR in a dynamic market when two firms compete for market share and they have showed that a firm's CR expenditures can either increase or decrease with its own market share depending on the parameters of the model.

We study the equilibrium structure of optimal customer acquisition and customer retention strategies as a differential game over an infinite horizon in a market with a large number of non-atomic firms. The infinite-horizon optimal control approach to the problem of CLV is a distinguishing feature of our approach. CLV is an inherently dynamic concept most naturally defined over an infinite planning horizon. Customer acquisition entails incurring an acquisition cost in the present with the anticipation of a benefit from a new cohort of customers in the future. Similarly, customer retention entails incurring a cost in the present with the anticipation of a larger customer base in the future. The nature of the decision-making process points naturally to a dynamic control problem. The static approach of maximizing current profits under the assumption of a fixed exogenous future would ignore the effects of present decisions on future profits and the dynamic interplay between the time paths of decisions. A steady-state, or long-term equilibrium, of the dynamic model would be similar to the solution of the static model, but the dynamics of reaching the long-term equilibrium may be of importance. One might be interested in knowing, rather than assuming, whether an economic environment of relevance and fully spelled out dynamics can give rise to such a long-term stable equilibrium as it is implicitly assumed in static models. A finite-horizon model, moreover, presents a number of difficulties regarding the customer acquisition and customer retention decisions, which would have to be set optimally at zero in the last period of the horizon. This would mean, however, that there is no long-term equilibrium in the model. Therefore an infinite horizon seems to be the natural point of departure for a dynamic model. A simple optimal control problem of customer acquisition and customer retention would correspond to one of a monopolistic firm. Although understanding the optimal dynamic structure of decisions for a monopolistic firm would have an undoubted merit, our purpose is to understand the equilibrium structure of decisions in a competitive industry. In this paper we consider an industry with a large number of firms as the natural benchmark—the one closest to classical perfect competition in which customer acquisition and customer retention are meaningfully defined. The framework is reminiscent of the classical economic perfect competition in that there is a large number of firms, each of negligible size, and that firms do not engage in price competition. The deviation from perfect competition is the presence of some implicit friction that makes the product of one firm distinguishable from the product of the other, so firms engage in non-price competition. The framework is similar to some of the recent models of monopolistic competition—only in our case competition takes place over variables other than prices. We think of such environments as almost competitive markets in which personal relations matter. We do not try to justify the nature of discrimination between sellers—we just assume non-price competition in a market with a large number of sellers.

The industry consists of an infinite number of firms and an infinite number of customers. There is a continuum of firms of measure M, and each firm is represented by a point on the interval $[0, M]$. Each firm is of measure zero, it is non-atomic. Considering a continuum of non-atomic firms of measure M instead of discrete number of firms M offers simplicity in calculations. The first application of this approach is [1] in modeling perfectly competitive markets. The approach was further developed in [2] and it became known as non-atomic games. (For more details on this type of game we refer the reader to [18]). From an applied economic perspective, the approach corresponds to monopolistic competition: a market structure in which: (a) an individual firm has a degree of market power over its product, so it takes actions to maximize profit, however (b) individual firm's actions have no impact on whole market because the size of each single firm is negligible comparing to the market. Although actions taken by individual firms are of negligible influence on the market, when many firms (precisely, a subset of positive measure) change their actions there will be an influence on markets aggregates and so on an individual firm's optimal strategy. In such situations, the measure of firm M has impact on an individual firm's optimal strategy. (We refer the reader to [9] for detailed exposition of the topic). There is also a continuum of customers of measure N. The continuum of customers allows us to ignore the problem that each firm has to serve an integer number of customers.

Our model extends the formal Blattberg–Daighton's (see [6]) logic of balancing acquisition and retention expenditures by (1) considering a dynamic optimization model with infinite horizon of planning (instead of a single period model), (2) allowing for competition between non-atomic firms for customers through an acquisition contest (instead of considering a case of a single firm). We analyze how changes in the customer profit margin and the concentration of firms reshape the optimal marketing strategy (CA and CR expenditures and their shares in marketing budget) and the market outcome (size of firms, set of untapped customers, and churn rate) in the long run. Finally, we demonstrate an equivalence of maximization of the firm's value and maximization of individual CLV (defined as net present value of a flow of net per-period profits generated by a customer). For economy of expression, in the rest of the paper we will be writing: "number of customers" meaning: "measure of set of customers."

2 The Model

We model a situation when there are many firms in the industry and each is negligibly small compared to the industry. We assume that there is a continuum of firms of measure M in the industry. We let $I = [0, M]$ be the set of firms, and $i \in I$ be an individual firm in the industry. Each firm lives infinitely long and is run by a fully rational manager whose objective is to maximize the present value of the firm. We assume that there is a continuum of customers of measure N, $[0, N]$. At each period, each customer may purchase or not purchase the goods provided by some firm. There is a turnover of customers: in each period, a fraction of a firm's customers stops buying from the firm and they become potential customers for the whole industry. Firms may engage in customer acquisition effort to attract new customers from the pool of potential customers as well as customer retention effort to reduce the fraction of leaving customers.

Our approach to finding the market equilibrium is as follows: first we describe the economic environment in which firms operate, restrictions on individual behavior, and the nature of competition. Second, we set up the problem of an individual firm in the industry and characterize its solution. The individual firm maximizes the present value of profits taking the behavior of all other firms in the industry as given. In solving this problem the individual firm recognizes that its individual behavior will not affect market aggregates due to its negligible size. Finally, we derive the symmetric equilibrium of the industry.

2.1 The Individual Firm

Firm i has a constant exogenous gross-of-marketing-cost per-capita profit m and $l^i(t)$ customers at time t. The number of potential customers in the market is

$$N^p(t) = N - \int_0^M l^j(t)\,dj. \quad (1)$$

Following [12] we assume that customer acquisition works through attracting untapped customers. If firm i at time t makes acquisition expenditure $K^i(t)$, it acquires the share of potential customers:

$$s^i(t) = \left(K^i(t)\right)^\theta / \int_0^M \left(K^j(t)\right)^\theta dj, \quad (2)$$

where $0 < \theta < 1$ reflects the degree of decreasing returns in the acquisition expenditure.

(As we shown below, in equilibrium $K^i(t) > 0$ for all t and all i. Thus the denominator in (2) is always positive).

The share of new customers acquired by a firm increases at a decreasing rate in the acquisition expenditure of the firm and decreases in the competitors' acquisition expenditure. If the acquisition expenditure chosen by all firms is the same, the firms get the same share of potential customers, which decreases in the measure of firms. As an example of such an acquisition effort one may think of combative advertising in markets where customers are well informed about the existence of a product/service [3, 10].

The number of new customers for firm i in the interval from t to $t + dt$ is

$$N_0^i(t) = s^i(t)\,N^p(t). \quad (3)$$

In the interval between times t and $t + dt$ a fraction $\beta(1 - r^i(t))$ of the time-t customers is lost, where $\beta \in (0, 1)$ is the baseline attrition rate and is the customer retention effort exerted by firm i at time t. The per-customer cost of retention effort is: $r^i(t)$ is $f(r^i(t))$ and is payable at time t. We assume that the function $f(\cdot)$ is continuously differentiable, increasing, convex, and satisfies the conditions: $f(0) = 0, f'(0) = 0, f(1) > m$. Examples of such retention actions are loyalty programs offering personalized bonuses [8, 11, 19].

Within the interval between the times t and $t + dt$ there is the following sequence of events: first, each of the l^i customers of firm i pays for the service; firm i decides on the level of retention effort, r^i, and spends $f(r^i)$ per customer. Second, firm i makes acquisition expenditures K^i, which determines the distribution of date-t potential clients among firms at the time $t + dt$. Third, a fraction $1 - \beta(1 - r^i(t))$ of customers leaves firm i joining the set of $t + dt$-period potential clients.

The profit of firm i at time t is

$$\pi^i(t) = ml^i(t) - [f\left(r^i(t)\, l^i(t) + K^i(t)\right].$$

The first term in the expression for profits is the operations profit, gross of all marketing costs. The second term and third terms are the marketing costs: total customer retention cost and total customer acquisition cost.

Given the marketing decisions of the firm at time t the number of its customers at time $t + dt$ is determined by the number of customers at time t, reduced by customer attrition and increased by customer acquisition:

$$l^i(t + dt) = l^i(t) - \beta\left[1 - r^i(t)\right] l^i(t)\, dt + N^p(t)\, s^i(t)\, dt.$$

In the following we suppress the notation for the dependence of the variables on time.

2.2 The Problem of a Firm

The manager of the firm i at the initial time ($t = 0$) chooses the complete time paths for acquisition expenditures K^i, and retention effort r^i and number of customers l^i to maximize the present value of the firm's profits subject to the law of motion of the firm's customers and the initial number of customers, taking as given the number of customers of the other firms, $(l^j)_{j\in I\setminus\{i\}}$, and the choices of customer acquisition expenditures by the other firms, $(K^j)_{j\in I\setminus\{i\}}$. The maximization problem of firm i is given by:

$$\max_{(r,k,l)_{t=0}^{\infty}} \left(\int_0^\infty \exp\{-\rho t\}\, \pi^i dt \right), \tag{4}$$

$$\pi^i = (m - f\left(r^i\right))l^i - K^i,$$

$$s.t.\ \frac{dl^i}{dt} = -\beta\left(1 - r^i\right) l^i + N^p s^i,$$

where $\rho \in (0, 1)$ is the discount rate, N^p and s^i are defined by (1) and (2), and l_0^i is given. The problem is a standard problem of infinite-horizon Optimal Control (as, for instance, in [17]). The firm maximizes the sum of present values of the function: $f(t, l(t), r(t), K(t)) = e^{-\rho t}\pi(l(t), r(t), K(t))$, which depends on time t, the state variable $l(t)$, and the control variables $r(t)$ and $K(t)$. The state variable (after the substitutions) follows the law of motion: $l'(t) = -\beta(1 - r(t))\, l(t) +$

$[N - \int_0^M l^j(t)\,dj] \frac{K(t)^\theta}{\int_0^M (K^j(t))^\theta dj}$ which is just a function of t, the $l(t)$, and the controls $r(t)$ and $K(t)$ (the integrals $\int_0^M l^j(t)\,dj = \Phi(l(t))$ and $\int_0^M (K^j(t))^\theta dj = \Psi(K(t))$ are just functions of the state and controls), so that the law of motion of the state variable is of the form: $l'(t) = g(t, l(t), r(t), K(t))$. The solution to this problem can be characterized by the Pontryagin's Maximum Principle, obtained from the system of Hamiltonian dynamics derived below.

As each firm is of measure zero, the decisions of a single firm do not affect market aggregates, i.e.,

$$\frac{dN^p}{dl^i} = 0, \quad \frac{d}{dK^i} \int_0^M (K^j)^\theta dj = 0.$$

The Hamiltonian function for firm i's problem is given by:

$$H^i\left(r^i, K^i, l^i, \lambda^i, t\right) = \exp\{-\rho t\} \left(\left[m - f\left(r^i\right)\right] l^i - K^i\right) + \lambda^i \left\{-\beta\left(1 - r^i\right) l^i + N^p s^i\right\},$$

where $\lambda^i(t)$ is a co-state variable expressing the contribution of an additional customer at time t to the present value of firm i's profit. The first-order condition of the problem with respect to the retention effort is: $dH^i/dr^i = 0$. This is equivalent to the condition:

$$\lambda^i \beta = \exp\{-\rho t\} f'\left(r^i\right), \tag{5}$$

which equates the marginal per-capita gain and the marginal per-capita cost of increasing the retention rate.

With respect to customer acquisition the first-order condition is: $dH^i/dK^i = 0$. This is equivalent to the condition:

$$\exp\{-\rho t\} = \lambda^i N^p \frac{ds^i}{dK^i}. \tag{6}$$

The marginal cost from increasing the acquisition effort equals the marginal benefit from increasing it, the latter being the product of the increase in the number of the new customers with the marginal value of an additional customer for the firm's present value of profit.

With respect to the co-state variable the first-order condition is given by: $dl^i/dt = \partial H^i/\partial \lambda^i$. This is equivalent to the law of motion of firm's number of customers:

$$\frac{dl^i}{dt} = -\beta\left(1 - r^i\right) l^i + N^p s^i. \tag{7}$$

With respect to the number of customers the first-order condition: $d\lambda^i/dt = -dH^i/dl^i$ is equivalent to the following condition:

$$\frac{d\lambda^i}{dt} = -\exp\{-\rho t\}\,\left[m - f\left(r^i\right)\right] + \lambda^i\beta\left(1 - r^i\right). \tag{8}$$

Along the optimal path the marginal rate of decline of the shadow value of the number of customers on the firm's profit, $-d\lambda^i/dt$, equals the effect of a new customer on the value of the Hamiltonian.

The solution paths must also satisfy the transversality condition:

$$\lim_{t\to\infty} \lambda^i = 0.$$

The present value of adding one more customer for the firm should be zero at the limit of the time horizon.

It can be shown that the function $H^i\left(r^i, K^i, l^i, \lambda^i, t\right)$ is quasi-concave jointly in (r^i, K^i, l^i), so that a path that satisfies the first- and second-order necessary conditions for an optimum gives the optimal solution for the problem of the firm, for reasons along the lines of the Mangasarian Sufficiency Theorem.

Equations (5) and (6) combined give the following condition:

$$f'\left(r^i\right) N^p \frac{ds^i}{dK^i} = \beta. \tag{9}$$

Differentiating (5) with respect to t, we obtain the expression:

$$\frac{d\lambda^i}{dt} = -\rho\exp\{-\rho t\}\,\beta^{-1}f'\left(r^i\right) + \exp\{-\rho t\}\,\beta^{-1}f''\left(r^i\right)\frac{dr^i}{dt}.$$

Then substituting into (8) we have

$$\frac{dr^i}{dt} = \left\{-\beta\left[m - f\left(r^i\right)\right] + \left[\beta\left(1 - r^i\right) + \rho\right]f'\left(r^i\right)\right\}/f''(r^i). \tag{10}$$

Along an optimal path the marginal benefit for the firm from retaining one more customer equals the marginal cost of retaining her, while the rate of decrease of the marginal benefit equals to the marginal benefit of having one more customer.

3 The Symmetric Industry Equilibrium

In the symmetric industry equilibrium all firms make the same choices of decision variables: $r^i = r$, $K^i = K$, $l^i = l$. We have $s^i = 1/M$ and $ds^i/dK^i = \theta/(MK)$. Numbers of potential and newly acquired customers are the following, respectively:

$$N^p = N - Ml, \; N_0 = (N - Ml)/M. \tag{11}$$

The number of newly acquired customers is proportional to the number of potential customers, with both linearly decreasing in the number of customers served in the same period. Substituting for the above expressions into (7) and (10), the dynamic behavior of the industry equilibrium can be represented as the following system:

$$\frac{dl}{dt} = -\beta\,(1-r)\,l + \frac{N - Ml}{M}, \tag{12}$$

$$\frac{dr}{dt} = \left\{-\beta\,[m - f\,(r)] + [\beta\,(1-r) + \rho] f'\,(r)\right\}/f''(r). \tag{13}$$

Equation (12) is the law of motion of the number of customers for the representative firm in the industry. It shows how the number of customers depends on the exogenous rate of customer attrition, the firm's choice of the retention effort, and the share of newly acquired customers, which is $1/M$ in the symmetric equilibrium. Equation (13) is the rate of change of the firm's optimal choice of retention rate over time.

In a similar way (9) gives the optimal choice of customer acquisition:

$$K = \theta\beta^{-1}\frac{f'\,(r)\,(N - Ml)}{M}. \tag{14}$$

The firms' acquisition expenditure is proportional to the number of potential customers. The transversality condition is also written: $\lim_{t\to\infty} \exp\{-\rho t\} f'\,(r) = 0$.

The dynamic behavior of the monopolistically competitive industry is represented by the system of (12)–(14), and the transversality condition, given the initial condition for l.

3.1 The Industry Dynamics

Equation (13) does not include decision variables other than the retention effort. This allows us to characterize the equilibrium path of the retention effort without solving the whole system of equations.

Lemma 1. *There exists a unique equilibrium path that satisfies Eq. (13) and maximizes the firm's value. Moreover, it is stationary:* $r(t) = r^*$.

Proof. Considering the optimal path of $r(t)$ determined by (13), we define the function:

$$G\,(r) = -\beta\,[m - f\,(r)] + [\beta\,(1-r) + \rho] f'\,(r)\,.$$

Differentiating $G(r)$ in respect to r we have

$$\frac{\partial G(r)}{\partial r} = [\beta (1 - r) + \rho] f''(r) > 0.$$

Thus, $G(r)$ increases in r. By assumption we have that $f(1) > m$ and $f'(0) = 0$, and thus $G(1) = -\beta [m - f(1)] + \rho f'(1) > 0$ and $G(0) = -\beta m < 0$. Therefore, there exists the unique level of retention effort, $r = r^*$, such that $G(r^*) = 0$. Thus, $r(t) = r^*$ is the only stationary solution of $dr/dt = 0$. For paths that start at $r > r^*$, we have $dr/dt > 0$, which implies that they lead to $r = 1$. These paths cannot be equilibrium, because $f(1) > m$ implies that firms get negative per-period profit in the long run. For all paths started at $r < r^*$ we have $dr/dt < 0$, thus they lead to $r = 0$. They cannot be equilibrium paths for the following reason. Undertaking a small effort δ for cost $f'(\delta)\delta$ each firm would save the fraction $\beta\delta$ of customers, which would provide the net profit $\beta\delta m$ in the consequent period. Assumption $f'(0) = 0$ provides that $f'(\delta) < \beta m$, and thus paths do not maximize the firm's profit. □

Given that the optimal retention effort is constant over time, we may describe explicitly dynamics of the firms' size, l, over time. It may be checked directly that (12) has the following solution:

$$l(t) = \frac{N}{M} \frac{1 - \alpha \exp\{-(1 + \beta (1 - r^*)) t\}}{1 + \beta (1 - r^*) + 1}, \quad (15)$$

where α is determined by the initial condition:

$$\alpha = 1 - (M/N)(1 + \beta \left(1 - r^*\right)) l_0.$$

In particular, if firms start with zero size ($l_0 = 0$), then $\alpha = 1$, whereas if at $t = 0$ all customers were served ($l_0 = N/M$), then $\alpha = -\beta(1 - r^*)$. Having $\alpha > 0$ ($\alpha < 0$) implies that the firms' size increases (decreases) over time. In both cases the firm size converges over time to the steady state value:

$$l^* = \frac{N}{M(1 + \beta (1 - r^*))}. \quad (16)$$

Let F be the *firm's level of retention expenditure*, i.e., $F = f(r^*)l$. Thus, F grows proportionally to the size of the firm, l. In contrast, by (14), we have that the firm's acquisition expenditure declines with l. This implies that the ratio K/F decreases in l.

Let k be the *per-new-customer* acquisition expenditure, $k = K/N_0$. Equations (14) and (11) give the following:

$$k = \frac{K}{N_0} = \beta^{-1} \theta f' \left(r^*\right). \quad (17)$$

Both the optimal per-customer retention effort and the *per-new-customer* acquisition effort do not depend on the firm's size. Let h be the customer churn rate as: $h = N_0/l$. We summarize the results on the dynamics in the following proposition:

Proposition 1. *As the firm's size increases (decreases) over time, (1) the per-capita expenditures remain constant, (2) the firm's level acquisition expenditure decreases (increases), while the firm's level of retention expenditure increases (decreases); (3) both the share of acquisition expenditures in the total marketing expenditures,* $K/(K+F)$*, and the customer churn rate, h, decrease (increase).*

3.2 Effects of Changes in the Economic Environment

As we have shown above the optimal retention effort is determined by the following equation:

$$-\beta\left[m - f\left(r^*\right)\right] + \left[\beta\left(1 - r^*\right) + \rho\right] f'\left(r^*\right) = 0. \tag{18}$$

Applying the implicit function theorem to (18) we obtain the following result:

Proposition 2. *The steady state level of the retention effort,* r^**: (1) increases in the gross per-client profit, m; (2) decreases in the discount rate,* ρ*, and increases the baseline attrition rate,* β*; (3) does not depend on the market size, N, and the number of firms in the industry, M.*

When customers become more profitable to firms, firms are willing to spend more to retain them. A higher baseline retention rate, $1 - \beta$, leads to a lower incentive to incur the cost of the retention effort. Thus an increase in β brings higher investment in retention. A higher discount rate means that firms put lower values to future profits, so they have lower incentive to invest in retention. The independence of r from M and N reflects the fact that the value of each customer for a firm is independent from its competitors' strategies.

It follows immediately from (16) and Proposition 1 that l^* satisfies the properties given in the following proposition:

Proposition 3. *The number of customers,* l^**, served by each firm: (1) increases in the market size, N, and the gross per-customer profit, m; (2) decreases in both the number of firms, M, and the discount rate,* ρ*; (3) the impact of the change in the baseline retention rate,* β*, on* l^* *is ambiguous.*

A larger market size and a higher firm concentration (lower M) imply a larger firm size. An increase in the customer profitability increases the firm's incentives to invest in retention and acquisition resulting in a larger firm size. A higher discount rate implies a lower incentive to invest, leading to a lower firm size. An increase in the baseline attrition rate, β, has a negative direct impact on the effective retention rate and a positive indirect impact through an increase in the optimal retention effort. Which effect dominates depends on curvature of the retention cost function.

The number of potential customers, N^p, in (11), and of new clients, $N_0 = N^p/M$, in the steady state can be written as:

$$N^{p^*} = N\frac{\beta\,(1-r^*)}{1+\beta\,(1-r^*)},\quad N_0^* = \frac{N}{M}\frac{\beta\,(1-r^*)}{1+\beta\,(1-r^*)}. \tag{19}$$

Using (16) and (14) we obtain the firm's acquisition expenditure in the steady state:

$$K^* = \theta N\frac{\beta\,(1-r^*)}{1+\beta\,(1-r^*)}f'\left(r^*\right). \tag{20}$$

The firm's level retention expenditure in the steady state is

$$F^* = f\left(r^*\right)l^* = N\frac{f\,(r^*)}{1+\beta\,(1-r^*)}. \tag{21}$$

Using (20) and (21) we obtain the ratio of the firm's acquisition expenditure K^* to retention expenditure F^* in the steady state:

$$\frac{K^*}{F^*} = \frac{\theta\,(1-r^*)f'\,(r^*)}{f'\,(r^*)}. \tag{22}$$

Equation (17) gives the per-new-customer acquisition expenditure in the steady state:

$$k^* = \beta^{-1}\theta f'\left(r^*\right). \tag{23}$$

Dividing k^* by $f(r^*)$ we obtain the ratio of the per-new-customer acquisition expenditure to per-customer retention expenditure. This is given by:

$$\frac{k^*}{f\,(r^*)} = \beta^{-1}\frac{\theta f'\,(r^*)}{f\,(r^*)}. \tag{24}$$

Now we can characterize the firm's optimal response to different changes in the market environment. The following results on the *per-customer* values follow directly from (24) and Proposition 1.

Proposition 4. *The per-new-customer acquisition expenditure, k^*, and the per-customer retention expenditure, $f(r^*)$ (1) increase in per-client gross profit m, (2) decrease in the discount rate, ρ, (3) do not depend on the number of firms, M, and market size, N.*

An increase in the gross profit increases both the firms' incentives to acquire new customers and retain the existing ones. A higher discount rate makes all expenditures less profitable. As both the total acquisition cost, K^*, and the number of potential customers, N^{p*} are proportional to the ratio N/M, the per-new-customer acquisition expenditure does not depend on the market size, N, or the number of competitors, M.

The following proposition, related to the properties of the firm's total levels of expenditures, follows from (20) and (21) and Proposition 1.

Proposition 5. *(1) A firm's total acquisition expenditure, K^*, and total retention expenditure, F^*, increase in the market size, N, and decrease in the number of firms in the industry, M. (2) The share of total acquisition expenditure in the total marketing expenditure, $K^*/(K^* + F^*)$, does not depend on the market size, N, and the number of firms, M. (3) The total retention expenditure, F^*, increases in the gross customer's profit, m, and decreases in the discount rate, ρ.*

In contrast to the per-capita values, a firm's expenditures on acquisition and retention do not depend on market size and the number of competitor. However, as both K^* and F^* are proportional to N/M, neither the market size nor the number of firms affects the distribution of the total marketing expenditures between acquisition and retention.

4 CLV and Firm's Value

Suppose that in $t = 0$ the industry is at the steady state, with $K^j = K^*$ and $r^j = r$, for all $j \in (0, M]$. Consider the profit of firm i generated by a cohort of customers acquired at time $t = 0$, assuming that firm i makes the customer acquisition expenditure K_0^i at $t = 0$ and keeps the retention effort r^i for all $t > 0$.

Making expenditures K_0^i firm i acquires a cohort of N_0^i customers. By (2) and (3) we have

$$N_0^i = \frac{N^{p^*} \left(K_0^i\right)^\theta}{\left(M \left(K_0^i\right)^\theta\right)}. \tag{25}$$

During any period customers in this cohort decrease at the rate $\beta(1 - r^i)$. The dynamics of the remaining customers are given by:

$$dl_0^i(t)/dt = -\beta\left(1 - r^i\right) l_0^i(t), \tag{26}$$

given $l_0^i(0) = N_0^i$,

where $l_0^i(t)$ is the number of customers in the cohort acquired at $t = 0$ and still remaining with firm i at time t. It is readily seen that the solution of (26) is

$$l_0^i(t) = \exp\left\{-\beta\left(1 - r^i\right)t\right\} N_0^i, \tag{27}$$

and $l_0^i(t)$ smoothly converges to zero over time.

Let π^0 denote the net profit generated by the cohort over its lifetime:

$$\pi^0 = \int_0^\infty \exp\{-\rho t\} \left[m - f\left(r^i\right) l_0^i\right] dt - K_0^i.$$

Using (27) we can rewrite the above as:

$$\pi^0 = \int_0^\infty \exp\left\{[-\rho + \beta(1 - r^i)]t\right\} \left[m - f\left(r^i\right)\right] N_0^i dt - K_0^i. \tag{28}$$

As $N_0^i\left[m - f\left(r^i\right)\right]$ is constant over time, we have:

$$\pi^0 = N^{p*} \frac{N_0^i}{N^{p*}} \left(\frac{m - f\left(r^i\right)}{\rho + \beta\left(1 - r^i\right)} - \frac{K_0^i}{N^{p^*}} \right). \tag{29}$$

Let A be the *per-prospect* acquisition expenditure: $A = K_0^i / N^{p^*}$, and a be *the probability* of acquiring a customer given the per-prospect acquisition expenditures A: $a = N_0^i / N^{p^*}$. Using (25) and $K_0^i = AN^{p^*}$ we obtain

$$a = \left(AN^{p^*}\right)^\theta / [M\left(K^{p^*}\right)^\theta]. \tag{30}$$

Finally, let R denote the effective retention rate, $R = 1 - \beta(1 - r)$ and $\phi(R) = f([1 - \beta - R]/\beta)$ be the per-customer retention expenditure as a function of R. Then, (29) can be written as:

$$\pi^0 = N^{p^*} \left(a \frac{m - \phi\left(R\right)}{1 + \rho - R} - A \right). \tag{31}$$

The term in brackets, $(m - \phi(R))/(1 + \rho - R) - A$, is just the standard formula for the lifetime value of a customer who generates the net per-period profit $m - \phi(R)$ and is retained by the firm at the constant rate R.

We will show that the solution of the maximization of the profit, π^0, given by (29), subject to (25), solves also the problem of firm's value maximization, which is given by (4). The first-order condition with respect to r^i gives

$$-f'\left(r^i\right)\left[\beta\left(1-r^i\right)+\rho\right]+\beta\left[m-f\left(r^i\right)\right]=0,$$

which is just (18).

The first-order condition with respect to acquisition cost K_0^i gives us:

$$\frac{\theta\left(K_0^i\right)^{\theta-1}}{M(K^*)^\theta}\frac{m-f(r^i)}{\rho+\beta(1-r^i)}=\frac{1}{N^{p*}}.$$

Equation (18) implies that in the optimum the following holds:

$$\frac{f'\left(r^i\right)}{\beta}=\frac{m-f\left(r^i\right)}{\beta\left(1-r^i\right)+\rho},$$

Thus, we have

$$\frac{\theta N^{p^*}}{\beta M}\frac{(K_0^i)^{\theta}}{(K^*)^{\theta}}f'(r^i)=1,$$

which is equivalent to (20) in the symmetric equilibrium (i.e., when $K_0^i=K^*$). This proves the following proposition:

Proposition 6. *The problems of maximization of a firm's net present value and maximization of firm's individual CLV result in the same steady-state equilibrium solution* (r^*, K^*).

5 Discussion

5.1 *Non-atomic Firms Approach as the Limit Case of Atomic Firms*

One assumption we employed in our model is that each firm is non-atomic (or equivalently, each firm is of negligible size with respect to the industry). Here we demonstrate that the same result may be obtained in the case of the large number of atomic firms.

We assume that there are M firms. Equations (1) and (2) take the following form:

$$N^p(t)=N-\sum_{j=1}^{M}l^j(t),\ s^i(t)=\left(K^i(t)\right)^\theta/\sum_{j=1}^{M}\left(K^j(t)\right)^\theta.$$

The firm's maximization problem is still given by (4) and the Hamiltonian function is the same as in Sect. 2.2. Now, actions of each atomic firm have an impact on markets aggregates. We have

$$\frac{dN^p}{dl^i} = -1, \quad \frac{d}{dK^i}\sum_{j=1}^{M}\left(K^j(t)\right)^{\theta} = \theta\left(K^i(t)\right)^{\theta-1}.$$

The optimality conditions: $dH^i/dr^i = 0$ and $dl^i/dt = \partial H^i/\partial\lambda^i$ give (5) and (7), respectively. The conditions: $dH^i/dK^i = 0$ and $d\lambda^i/dt = -dH^i/dl^i$ in the symmetric equilibrium give

$$\exp\{-\rho t\} = \lambda^i\left(N - Ml(t)\right)\left(\frac{\theta}{MK(t)} - \frac{\theta}{M^2K(t)}\right).$$

The second term in brackets vanishes as M increases, and this gives (6). Optimality condition $d\lambda^i/dt = -dH^i/dl^i$ in symmetric equilibrium implies

$$\frac{d\lambda^i}{dt} = -\exp\{-\rho t\}\left[m - f\left(r^i\right)\right] + \lambda^i\left(\beta\left(1 - r^i\right) - \frac{1}{M}\right).$$

Again, the term $1/M$ vanishes when M increases and we obtain (8). As the number of firms in the industry grows the impact of each firm's actions on market aggregates becomes smaller and vanishes as M goes to infinity. Optimal firm behavior in an industry with non-atomic can be thought of as an approximation of optimal behavior in an industry with atomic firms when the number of firms is large.

5.2 Managerial Implications and Further Research

The model we developed may be helpful to explain a number of observed phenomena in marketing strategy. For instance, Tretyak and Sloev [28] give an example of a multinational company operating in both the Russian and European markets whose Russian branch had double the number of newly acquired customers per period, but due to low retention rate it lost a large share of them resulting in a churn rate in the Russian market much higher than in the European one. Our present model suggests that such differences may be attributed to lower customer profitability, large market size, and higher risks and uncertainty, which might imply a higher discount rate, in Russia comparing to Europe.

The formal modeling of the managerial process has helped us obtain a number of testable implications about marketing decisions; however, we should exercise care in interpreting them. On the one hand, our model is a normative one, in which a fully rational manager chooses optimal time paths of marketing decisions, under perfect knowledge of the economic environment, with the objective of maximizing shareholders' value, and in the absence of liquidity constraints or adjustment costs (to mention but a few conditions). For this manager under these conditions the model prescribes at least the rough lines of best decisions. This may be of value to managers who, in the absence of a model of normative behavior, are obliged to find the best stream of actions through rules-of-thumb behavior or short-term strategies. A key point of the paper is that there is a long-term marketing strategy, moreover this can, relatively adequately, be described. The manager need not be sure about the precise values of the environmental parameters to decide whether to increase or decrease retention and acquisition effort. On the other hand, any theoretical model should not be devoid of testable implications. Our model provides a rich set of them, summarized in Propositions 1–5, and these can be tested using data at the firm or the industry level. Given the normative character of the model and the number of conditioning assumptions, the maintained hypothesis of the model at this level should be a very composite one, with the alternative hypothesis the violation of at least of one the constituent hypotheses: full rationality, value-maximization, perfect capital markets, absence of adjustment costs, and so on. Evidence against the maintained hypothesis of the model might mean some conditioning assumptions are not satisfied in practice.

Some of our results come from the assumption that the baseline retention rate is not affected by competitors' acquisition effort. This means that a firm's retention effort should independent from competitors' acquisition expenditures, hence from the number of competitors in the industry. This may not hold when firms compete also to attract customers currently served by competitors. When the sensitivity of customers currently served to competitors acquisition effort is small, as, for example, when firm-customer relations take time to build, as, for instance, when monitoring, trust, or gradual revelation of information are important elements (banking, legal, insurance, and medical services might be some good examples) the assumption would not be overly restrictive. Similarly assuming that firms are of negligible size compared to markets clearly does not hold in markets with a small number of large firms. When, however, despite the large firm size, strategic effects are small, the impact of size on strategies might be expected to be negligible.

Several directions for future research open up.

Adjustment Costs. We have assumed that customer acquisition and customer retention can freely adjust according to the firm's management will. In reality these decisions should have a degree of time persistence. There may be learning by doing that makes the productivity of customer acquisition effort time persistent, or legal obligations that do not allow drastic changes in customer retention. The effects of the decisions may be the dynamic adjustment of the firm to the long-term equilibrium position may be different, with a higher dependence on previous decisions. The restricted optimal time path of retention may not be constant over time, and independent of firm size. The presence of technological restrictions of this sort should be an obvious next step in research.

Customers' Heterogeneity. We have assumed that retained customers are identical to newly acquired ones in their baseline retention rate. However, a cohort level retention rate typically increases over time (see [15] or [8]). Individual customer's characteristics may change over time, as, for instance, when customer's loyalty increases as the customer gains more experience with the firm. Similarly, when customers in a cohort have a different baseline retention rate, then the average per-cohort retention rate increases over time [13]. These models may lead to different optimal managerial decisions in dynamic settings. Our framework might be useful in this direction.

The Role of Marketing Budget. Firms often operate under binding restrictions on the marketing budget. Even if increasing expenditures on retention and/or acquisition would be profitable strategy, the firm may not have enough financial resources to do it. This may be particularly true for firms under liquidity constraints, small firms without enough free cash flow, reputational capital, and collateral assets. In such cases the optimal allocation of resources between acquisition and retention may depends on the amount of liquid assets, the maturity and reputation of the firm that gives it access to capital markets, the size and composition of the firm's balance sheet, and even the macroeconomic conditions. The discount rate implied in CLV maximization, and the problem we solve in this paper, should be closely related to the cost of capital for the firm. The relation between marketing and corporate finance decisions may be one of the most promising directions for future research.

6 Concluding Remarks

We have presented a tractable model that allows us to study the impact of competition and market environment on a firm's optimal marketing strategy based on the idea of CLV. We find that while along the equilibrium path the number of customers served by a firm and the number of new customers acquired converges smoothly to a steady state equilibrium, with the optimal path characterized by a constant per-customer retention expenditure. We find that stronger competition leads to lower firm size and lower acquisition and retention expenditure; however, it does not change acquisition and retention expenditures at the per-capita level. In contrast, an increase in the size of the market leaves the per-capita retention and acquisition expenditures unchanged, whereas it increases the firm's total acquisition and retention expenditures. As the firm size grows, the churn rate and the share of acquisition expenditures in the total marketing budget decrease. A number of extensions and generalizations, as, for instance, adjustment costs, customer heterogeneity, and presence of liquidity restrictions shall be the object of future research.

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A Dynamic Model of a Decision Making Body Where the Power of Veto Can Be Invoked

Jacek Mercik and David M. Ramsey

Abstract Classical analysis of the power of individuals or groups in decision making bodies tends to consider processes of coalition formation in isolation from each other. However, the results of attempts to form a winning coalition will affect the dynamics of coalition formation in the future, particularly when certain players are endowed with the power of veto. When a voter invokes their power of veto to block a generally popular motion, this is likely to provoke retaliation from the supporters of the motion. For this reason, even if a player is not in favor of a bill and can block it, it may be favorable for that party to abstain rather than veto in return for support regarding issues to be considered in the future. Hence, players should only use their power of veto if they are very strongly against a bill. In this paper, we present a model of voting in which the results of previous votes can affect the process of coalition formation. We present a model of such a dynamic voting game and present the form of an equilibrium in such a game. This theory is illustrated using an example based on the voting procedures used in the United Nations Security Council.

Keywords Coalition formation • Power of veto • Dynamic voting game • Equilibrium

1 Introduction

Decision making based on various majority voting schemes has been a subject of analysis for many years using the concept of a cooperative game. Using such a voting procedure, for a motion to be accepted, a specified majority of voters must

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vote for it and no player with power of veto can vote against it. Note that some forms of veto can be overturned if there is a sufficient majority supporting the bill (a so-called conditional veto). However, in this article we consider only unconditional vetoes, which cannot be overturned (see Mercik [5]). The existence of conditional vetoes complicates any analysis of such a game (see Mercik and Ramsey [6]).

The classical approach to such votes (see, e.g., Brams [2], von Neumann and Morgenstern [10]) assumes that decisions are made in isolation from each other. However, this may not be true. For example, when a player with the power of veto blocks a generally popular motion, other players may retaliate. Hence, one may consider dynamic models of decision making in which players vote on a sequence of motions and can condition their behavior on the voting patterns of other players in the past. It is assumed that the vote on a motion is carried out in two stages. In the first stage, players can either vote for the motion or abstain. If the required majority is achieved, a motion passes on to the final round, in which players can either vote for, vote against or abstain. If a player with power of veto votes against the motion at this stage, the motion is unconditionally blocked. If no veto is applied, then the same majority as in the first round is required to pass the motion. Note that when a player does not have power of veto, then voting against a motion is equivalent to abstaining.

This paper constitutes a sequel to Ramsey and Mercik [9]. The authors gave the form of an equilibrium in such a game. The present article describes an iterative procedure for deriving such an equilibrium when retaliation occurs. Numerical results are given for an example based on the United Nations Security Council.

2 Model

We consider a class of cooperative games which correspond to weighted voting games with certain players having the right of unconditional veto. Let $N = \{1, 2, \ldots, n\}$ be a finite set of players, w_i be the voting weight of player i, $i \in N$, where the weights are assumed to be integers, $\sum_{i=1}^{n} w_i = W$ and q be the required majority, where $W/2 < q$. For a motion to be passed, it is necessary that the sum of the weights of the voters supporting a motion is at least q and no-one vetoes the motion. It is assumed that the quota q is sufficiently small so that this voting procedure does not require unanimity for a motion to be passed. Any coalition can be described by the subset containing the indexes of the players who are its members.

A cooperative game involving the set of players N can be described by a map $v : 2^N \to R$ with $v(\phi) = 0$ (see Başar and Olsder [1]). We denote the space of all such games by G. The domain $SG \subset G$ of simple games involving the set of players N corresponds to the set of maps $v \in G$, such that

- $v(S) \in \{0, 1\}$, for all $S \in 2^N$;
- $v(N) = 1$;
- v is monotonic, i.e., when $S \subset T$, then $v(S) \leq v(T)$.

A simple game is denoted by (N, v) and interpreted as voting game where no player has right of veto. In such a game, if $v(S) = 1$, then the coalition S is said to be winning. All other coalitions are said to be losing. Passing a motion is equivalent to forming a winning coalition. A simple game (N, v) is called proper, if and only if the following condition is satisfied: for all $S \subset N$, if $v(S) = 1$, then $v(N\backslash S) = 0$. Hence, a specified majority (according to voting weight) must support a motion in order for the motion to be passed. Such games are denoted $(N, q, \mathbf{w})$, where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, since it is defined by the set of players, their weights, and the quota.

When certain players have power of veto, then whether a coalition is winning or not depends on whether the players with power of veto who are not in the coalition veto or abstain. Let N_p be the set of players with power of veto. The remaining players are denoted $N_{\sim p}$. The classical approach to such voting games assumes that players who are not in a coalition vote against the motion. In the more general case where players with power of veto can abstain, we should consider the subgames $(N\backslash A, v_A)$, where $A \subseteq N_p$ is the set of veto players who abstain. In each subgame, $v_A(S)$ is equal to one if and only if the sum of the weights of the players in S is at least q and all of the players in $N_p\backslash A$ belong to S, i.e., the coalition has the required weight to pass the motion and no player wishes to veto the motion. Such a subgame may be denoted by $(N\backslash A, q, \mathbf{w}_A)$, where $\mathbf{w}_A$ is the vector of the voting weights of the players in $N\backslash A$. Let $\#(N_p) = n_p \geq 1$ be the number of players with power of veto.

We consider an infinite sequence of motions which are assumed to be addressed to situations which appear in sequence, e.g., as in the UN Security Council. The value of the jth motion to the ith player is $X_{i,j}$, where the $X_{i,j}$ are independent and identically distributed. Note that this value is measured with respect to the status quo, i.e., a player i is in favor of motion j when $X_{i,j} > 0$ and opposed to the motion when $X_{i,j} < 0$. It is assumed that players maximize their average payoff per vote.

Example. The UN Security Council has 15 members. Only the five permanent members have the right of veto. The five permanent members will be labelled from 1 to 5. A bill is introduced by the player who most strongly supports it and a player cannot cheat by finding an ally to introduce a bill. There are two rounds of voting. In order for the motion to be passed, in both rounds it must obtain at least nine votes. The power of veto can only be used in round two. Since the UN Security Council reacts to crisis situations where action is required, most often a majority of players will support a motion. However, it is possible that a player strongly opposes a motion. Hence, we consider distributions of the value of the motion which have heavy tails compared to the normal distribution (see Clauset et al. [3]), and the probability of a positive realization (i.e., of a member supporting the motion) is relatively large. The value of motion j to player i, $X_{i,j}$, has the same distribution as X, which is assumed to come from the non-central Laplace distribution with density function $f(x) = 0.5e^{-|x-1|}$. It follows that $E(X) = 1$ and $P(X < 0) = 0.5e^{-1} \approx 0.1839$. Hence, on average around 12 of the 15 members will support a motion.

Assume that there are two rounds of voting on each motion and each player has the same voting weight, Players $1, 2, \ldots n_p$ possess power of veto, while Players $n_p + 1, n_p + 2, \ldots, n$ do not. Without loss of generality, we may assume that $w_i = 1, i = 1, 2, \ldots, n$. The same quota q is used in both rounds. Power of veto is not applicable in the first round, thus for a motion to pass onto the second round it suffices that the sum of the weights of the players voting for the motion is at least q. For convenience, it is assumed that the votes in the first round are honest signals, i.e., player i votes for motion j in the initial round if and only if $X_{i,j} > 0$. This initial vote gives information about the set of veto players who might be considering applying their veto in the second round. Whether a veto player applies a veto or not in the second round depends on the likelihood of retaliation from the players supporting the bill and the degree to which the veto player is opposed to a bill. It is assumed that the beliefs of the players about how other players will vote in the second round are based on these initial signals. In this paper we do not consider the possibility of dishonest signaling. However, we hope to address this issue in a future paper.

Assume that retaliation is only practical against individual players and takes the following form: when a motion that would otherwise have been passed is blocked by a single vetoer, the players who supported that bill retaliate against the vetoer by abstaining when the vetoer introduces his next bill, which will thus not be passed. As motions are generally favored to the status quo, in the short term, retaliation is costly to all members (including those who neither retaliate nor are punished). In practice, other forms of retaliation may be used, e.g., trade sanctions, which are easier in practice to observe and interpret. However, in order to internalize the costs of retaliation in the game, we assume that retaliation is carried out within the confines of the voting procedure. For example, after the USSR vetoed a generally popular security council motion regarding its deployment of troops in Afghanistan in 1979 (see Nossal [7]), many economic and cultural sanctions were placed upon them, including a boycott of the 1980 Moscow Olympics. On the other hand, China initially vetoed a motion regarding sending UN observers to Guatemala in 1997, but after negotiations the motion was implemented in return for easing the sanctions placed on China after the Tiananmen Square demonstrations (see Xinbo [11]). Hence, vetoes and sanctions can be used as tools for negotiation. However, such sanctions are much more difficult to implement when more than one permanent member wishes to veto a generally popular motion. For example, the UN security council could not implement any serious measures with regard to the Vietnam War due to the conflicting interests of the USA, the USSR, and France (see Kennedy [4]).

The frequency with which a member introduces motions could be either lower than the frequency with which a member is the lone vetoer of a popular motion or at least as great. In the first case, it is assumed that every motion introduced by that player will be blocked by a coalition of abstainers. In the second case, the first motion that the vetoer introduces after such a veto will be blocked by abstentions. For such a convention to be stable, the following condition must be satisfied: the average reward obtained by veto players by behaving in this way must be greater than the average reward obtained when such a player automatically vetoes a motion that he is opposed to, i.e., veto player i vetoes motion j if and only if $X_{i,j} < 0$.

3 Form of the Equilibrium

Under the assumptions made above, when the players are voting on motion j, the game can be in one of the following two states:

- State 0: A player who is not the subject of retaliation introduced the motion;
- State 1: A player who is the subject of retaliation introduced the motion.

Our initial analysis is based on the following assumptions:

- At least one player supports a motion. It should be noted that otherwise such a motion would not even be presented for an initial vote.
- Retaliation against a lone vetoer is costly in the short term, i.e., members on average prefer a motion which is blocked to the status quo.
- Each veto member applies a lone veto less often than it introduces bills.
- No veto member preferring a motion to the status quo will veto that motion. Note that it is possible that if a threat to veto a motion is always applied, then it may pay a veto member supporting the bill to veto the motion, in order to avoid retaliation.

It follows that in State 0 the voting only affects the outcome of the present vote and of the next vote introduced by a lone vetoer (if such a vetoer emerges). Hence, players should maximize the sum of their payoffs from these two votes. The following results follow from these assumptions (see also Ramsey and Mercik [9]).

Theorem 1. *If $X_{i,j} < 0$, then Player i prefers abstaining to voting for motion j.*

Proof. Suppose that player i can only choose between abstaining and voting for. Given the actions of the other members, Player i may be either pivotal (i.e., when Player i abstains, the motion is blocked and when Player i votes for, the motion is passed) or non-pivotal. In the first case, there will be no retaliation and thus Player i strictly prefers the motion to be blocked rather than being passed. In the second case, Player i is indifferent between voting for and abstaining. It follows that for $X_{i,j} < 0$, Player i's action "abstain" dominates his action of voting for. □

Theorem 2. *Given that no veto member abstains in the first round, in the second round Player i should vote for a motion when $X_{i,j} > 0$ and abstain when $X_{i,j} < 0$. It should be noted that in this case all the veto players will vote for a motion.*

Proof. Given that no veto player abstains in the first round, no-one believes that there will be a veto by any player. Given this belief, each veto player strictly prefers the motion being passed to the motion being blocked and thus voting for the motion strictly dominates both abstaining and vetoing (particularly since it is believed that vetoing will lead to retaliation). Similarly, any non-veto player strictly prefers voting for the motion to abstaining when $X_{i,j} > 0$. Theorem 2 then follows from Theorem 1. It should be noted that given the beliefs of the players and the fact that voting proceeded to the second round a non-veto player who is against the bill is indifferent between voting for and abstaining, since he believes that the bill will be passed anyway. However, if we assume that voters make errors with some small probability, a non-veto player always prefers abstaining to voting for when $X_{i,j} < 0$. □

Theorem 3. *Given that at least two veto players abstain in the first round and the motion passes through to the second round, such veto players should apply their veto in the second round.*

Proof. Suppose two veto players initially abstain. The situation where more than two veto players initially abstain is analogous. The subgame played in the second round between the veto players who abstained in round one can be interpreted as a two-player matrix subgame. From Theorem 1, these two players should either veto or abstain. The payoffs in this matrix are expected payoffs from the current vote and the future vote in which retaliation might occur based on the votes of the other 13 members in the first round. These payoffs are given relative to the case in which the present vote is not passed and no retaliation occurs, which occurs when both of these veto players apply their veto. Suppose one of the players vetoes the motion, but the other does not. The motion is not passed, but the expected payoff of the vetoer in the next vote he initiates is lowered by $u_1 p_r$, where p_r is the probability that retaliation occurs and u_1 is the expected reward to a veto member from introducing a motion when in state 0 (no retaliation). Similarly, the expected reward of the non-vetoing player is lowered by $u_2 p_r$, where u_2 is the expected reward from a motion to a veto member who is not the introducer. From the voting pattern in round 1, $p_r > 0$. When neither veto player applies their veto, there will be no future retaliation. However, the expected rewards of these two veto players from the present game are negative, since they abstained in round one and there is sufficient support for the bill in the initial round to assume that the probability of it being passed is positive. In this case, let the expected payoff of these two players from the present vote be v_1 and v_2, where $v_1, v_2 < 0$. Hence, the payoff matrix for this subgame is given by

	Veto	Abstain
Veto	(0,0)	$(-u_1 p_r, -u_2 p_r)$
Abstain	$(-u_2 p_r, -u_1 p_r)$	(v_1, v_2)

It should be noted that both players vetoing always constitutes a Nash equilibrium. Both players abstaining constitutes a Nash equilibrium when $v_1 > -u_1 p_r$ and $v_2 > -u_1 p_r$. However, this Nash equilibrium is always Pareto dominated by the equilibrium at which both these veto players apply their veto. Hence, given the information from the initial vote, both these veto players should apply their veto. □

Note. In practice, if negotiations in the UN Security Council indicate that two veto players are opposed to a bill, such a bill is normally either withdrawn or appropriately modified before a vote occurs (see also Ramsey and Mercik [8]).

Hence, it suffices to consider votes where in the initial round only one veto player indicates opposition to the motion. Assume that the beliefs of the players regarding the voting patterns of other players come from historical data. We look for an equilibrium where all the veto players follow one strategy and all the non-veto players follow another. In particular, assume that k non-veto players and one veto player abstain in the first round. We may assume that $0 \leq k < N - q$, otherwise the motion would not proceed to round two. We derive an equilibrium where in round two:

- Veto player i votes for motion j when $X_{i,j} > t_2(k)$, abstains when $t_1(k) < X_{i,j} < t_2(k)$ and vetoes when $X_{i,j} < t_1(k)$.
- Each non-veto player votes for motion j when $X_{i,j} > t_3(k)$.

The form of this equilibrium is intuitive. When there is a threat that a motion will be vetoed, a player will only vote for that motion when he is fairly strongly in favor. In this case, the possible gain from passing the motion is strong enough to overcome the threat of a veto. Similarly, a veto member only vetoes when he is strongly opposed to a motion.

Definition 1. Let $E[V_{P,1}], E[V_{P,2}]$, and $E[V_{N,2}]$ be the expected value from voting when a veto player introduces a motion of: (a) the veto player introducing the motion, (b) a veto player who is not introducing the motion, and (c) a non-veto player, respectively. When there is no ambiguity, the expectation operator will be omitted.

Theorem 4. *Given that only one veto member, denoted i, abstains in the initial vote on motion j, he should veto it if and only if* $X_{i,j} < -V_{P,1}$.

Proof. Player i should maximize the expected sum of the payoffs obtained in the votes on the present motion and on the next motion that he introduces. Let p_k be Player i's estimate of the probability that in round two the motion gains at least q votes when k non-veto players initially abstained. By vetoing, Player i obtains a payoff of 0 in the present vote. With probability p_k, retaliation occurs when Player i introduces his next bill and then his payoff from that vote is 0. Otherwise, his expected reward will be $V_{P,1}$. Assuming that Player i abstains in the present vote, conditioning on whether the motion is passed or not, his expected payoff in the present vote is $p_k X_{i,j}$. Since there will be no retaliation, his expected reward from introducing his next motion is $V_{P,1}$. Comparing the sum of the expected payoffs in these two votes, Player i should veto motion j when

$$(1 - p_k)V_{P,1} > p_k X_{i,j} + V_{p,1} \Rightarrow X_{i,j} < -V_{P,1}.$$

□

Note. Hence, at equilibrium the probability of a veto player vetoing a motion is independent of the number of non-veto players initially abstaining. It might seem likely that a veto member is less likely to veto a motion when no-one else initially abstains, since retaliation is almost guaranteed. However, this is balanced by the fact that the expected gain from vetoing in this case is greater, as it is probable that such a motion can only be blocked by using the power of veto.

Theorem 5. *Non-veto player i should vote for motion j if and only if* $X_{i,j} > \frac{sV_{N,2}}{1-s}$, *where s is Player i's prior probability that the veto player who initially abstained will veto in the second round.*

Proof. Assume the cost of punishing a lone vetoer is positive, i.e., $V_{N,2} > 0$. Hence, any non-veto player who abstained in the first round will do the same in the second round. Consider a non-veto player who initially voted for the motion. Denote by q_k his prior probability that the motion obtains at least q votes given that k non-veto players initially abstained and he votes for the motion in round 2. Denote by $\tilde{q}_k$ his prior probability that the motion obtains at least q votes given that k non-veto players initially abstained and he votes against the motion in round 2. Note that $q_k > \tilde{q}_k$.

For the motion to be passed, in round 2 there must be at least q votes for and the veto player who initially abstained also abstains in round 2, rather than vetoing. Hence, if Player i votes for the motion in round 2, his expected reward is $X_{i,j}(1-s)q_k$. For retaliation to be implemented, it is necessary for both the motion to obtain at least q votes and be vetoed. Hence, player i's expected reward from the next motion to be introduced by the potential vetoer is $(1 - sq_k)V_{N,2}$.

When Player i abstains in round 2, arguing as above his expected reward from the present motion is $X_{i,j}(1-s)\tilde{q}_k$ and his expected reward from the next motion to be introduced by the potential vetoer is $(1-s\tilde{q}_k)V_{N,2}$. It follows that Player i prefers voting for motion j in round 2 rather than abstaining if and only if

$$X_{i,j}(1-s)q_k + (1-sq_k)V_{N,2} > X_{i,j}(1-s)\tilde{q}_k + (1-s\tilde{q}_k)V_{N,2} \Rightarrow X_{i,j} > \frac{sV_{N,2}}{1-s}.$$

□

Theorem 6. *Veto player i should vote for motion j if and only if* $X_{i,j} > \frac{sV_{P,2}}{1-s}$, *where s is Player i's prior probability that the veto player who initially abstained will veto in the second round.*

The proof of Theorem 6 is analogous to the proof of Theorem 5 and is thus omitted. The following theorem follows from Theorems 5–7 and, together with Theorems 1–4, defines the form of the equilibrium in the voting game.

Theorem 7. *At equilibrium, given that one veto player abstained in the first round, in the second round of voting on motion j.*

(1) A veto player should vote for the motion if $X_{i,j} > t_2$, *veto if* $X_{i,j} < t_1$ *and otherwise abstain.*
(2) A non-veto player should vote for the motion if $X_{i,j} > t_3$ *and otherwise abstain.*
(3) The thresholds t_1, t_2, *and* t_3 *satisfy the following conditions:*

$$t_1 = -V_{P,1}; \qquad t_2 = \frac{P(X < t_1)V_{P,2}}{P(X > t_1)}; \qquad t_3 = \frac{P(X < t_1)V_{N,2}}{P(X > t_1)}.$$

Hence, the form of this equilibrium is relatively intuitive. When just one veto player threatens to veto a motion, this threat should only be carried out when that player is strongly against a motion. In addition, the other players can also vote tactically in such a situation. This is due to the fact that if a player is only weakly in favor of a motion, then it may be worth abstaining in the second round when there is a threat of a veto, in order to avoid the effects of future retaliation.

4 Derivation of the Equilibrium

The main goal of this paper is to present a policy iteration algorithm to approximate such an equilibrium of a dynamic voting game based on the procedures of the UN Security Council. The iteration procedure can be described as follows:

(1) Set $k = 0$ and define initial thresholds $t_{1,0}, t_{2,0}, t_{3,0}$, defining the strategies of the members in the situation where one permanent member abstains in round one.
(2) Calculate $V_{P,1}$, $V_{P,2}$, and $V_{N,2}$ corresponding to the current strategy profile.
(3) The set of thresholds is updated using

$$t_{1,k+1} = -V_{P,1}; \qquad t_{2,k+1} = \frac{P(X < t_{1,k+1})V_{P,2}}{P(X > t_{1,k+1})}; \qquad t_{3,k+1} = \frac{P(X < t_{1,k+1})V_{N,2}}{P(X > t_{1,k+1})}.$$

(4) Unless a convergence criterion is met, increase k by 1 and return to step 2.

We now present an algorithm that calculates $V_{P,1}, V_{P,2}$, and $V_{N,2}$ for a given strategy profile. From the form of the equilibrium profile described above, any motion which is eventually passed will have gained at least q votes in the initial round of voting. Thus obtaining at least q votes with no veto is a necessary and sufficient condition for the motion to be passed. Hence, a motion will be passed if and only if:

(1) All veto players initially vote for the motion and at least $q - n_p$ non-veto players vote for the motion. In this case, the pattern of voting in the second round is exactly the same as in the first round, i.e., all the veto players again for the motion and all the non-veto players cast the same vote as before.
(2) Exactly one veto player abstains in round one. In round two, this veto player abstains again and at least q players support the motion.

Consider the game based on the UN Security Council, in which $n = 15, q = 9, n_p = 5$ and the value of a motion to a player is described by the non-central Laplace distribution function with density function $f(x) = 0.5e^{-|x-1|}$. According to this distribution we expect that the majority of members will favor such a motion, but some members may be strongly against it. This is likely to be the case for the type of motions considered by the UN Security Council. Of course, we may consider other distributions satisfying these conditions to see how robust the conclusions are. The events considered in Point 1 above can be split into seven mutually exclusive events $B_4, B_5, \ldots, B_{10}$, where the event B_k corresponds to all the permanent members favoring the bill and k non-permanent members favoring the bill. In these cases, as described above, the voting patterns are identical in both rounds. Let D_1 denote the union of these seven events. The events considered in Point 2 above can be split into 20 mutually exclusive events $C_{j,k}$, where j and k are the numbers of permanent and non-permanent members, respectively, voting for the motion in the second round, $0 \leq j \leq 4$, $9-j \leq k \leq 10$. Let D_2 denote the union of these 20 events.

Let A be the event that a motion is introduced by a veto player. Note that $P(A) = 1/3$. Given the voting patterns in the first round, the probability that a motion was introduced by a veto player is the proportion of veto players among those initially voting for a motion. Hence, $P(A|B_k) = \frac{5}{5+k}$.

For convenience, it will be assumed that whenever $X_{i,j} > 1$, then Player i will always vote for motion j, i.e., $\max\{t_2, t_3\} \leq 1$. Intuitively, since a permanent member has a stronger voting position than a non-permanent member, we will assume that $V_{P,2} > V_{N,2}$, i.e., $t_2 > t_3$. It follows that the event $C_{j,k}$ corresponds to the following:

(1) One permanent member abstains in both rounds. This corresponds to the value of the motion to this member being between t_1 and 0.
(2) Four permanent members vote for the motion in round one and j of them vote for the motion in round two. Hence, the value of the motion to j members is at least t_2 and the value to $4-j$ members is between 0 and t_2.
(3) The number of non-permanent members voting for the bill in the second round is k. The value of the motion to these members is at least t_3. The value of the motion to the remaining non-permanent members is less than t_3.

Let D be the event that the motion is passed, i.e.,

$$D = \left[\bigcup_{4 \leq k \leq 10} B_k \right] \cup \left[\bigcup_{0 \leq j \leq 4\,;\, 9-j \leq k \leq 10} C_{j,k} \right] = D_1 \cup D_2. \tag{1}$$

The expected reward of a permanent member initiating a motion can be calculated as $E[X_{max} \mathbb{1}_D | A]$, where $\mathbb{1}_D$ is the indicator function for the set D. We have

$$E[X_{max} \mathbb{1}_{D_1} | A] = \sum_{k=4}^{10} E[X_{max} | A \cap B_k] P(B_k | A). \tag{2}$$

Note that from Bayes' law,

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A)} = \frac{15P(B_k)}{5+k}. \quad (3)$$

The probability that the votes correspond to event B_k, i.e., all the permanent members and k of ten non-permanent members vote for the motion, is given by

$$P(B_k) = \binom{10}{k} P[X > 0]^{5+k} P[X < 0]^{10-k} \approx \binom{10}{k} 0.8161^{5+k} 0.1839^{10-k}. \quad (4)$$

When the voting pattern is given by B_k, $E[X_{max}]$ is conditionally independent of the event A, since in round two both permanent and non-permanent members vote for the motion whenever they prefer it to the status quo. We approximate $E[X_{max}|B_k]$ by considering the number of players, M, who ascribe a value of greater than 1 to a motion. Since $5+k$ players initially support the motion $M \sim \text{Bin}(5+k, p_0)$, where

$$p_0 = P[X > 1|X > 0] = \frac{0.5}{0.5(1 + \int_0^1 e^{-x}dx)} \approx 0.6127.$$

From the properties of the exponential distribution, when $m \geq 1$, the expected value of the maximum of these values is given by

$$E[X_{max}|M = m] = 1 + \sum_{j=1}^{m} \frac{1}{j}.$$

Assume that when a motion is passed, then $E[X_{max}|M = 0] \approx 1$. Using the properties of the minimum of a set of independent observations from an exponential distribution, it can be shown that in this case $E[X_{max}|M = 0] \geq \frac{8}{9}$ and $P(M = 0) \leq (1 - p_0)^9 \approx 0.0002$. Hence, the following approximation is reasonably accurate:

$$E(X_{max}|B_k) \approx 1 + \sum_{m=1}^{5+k} \left(\sum_{j=1}^{m} \frac{1}{j} \right) \binom{5+k}{m} p_0^m (1-p_0)^{5+k-m}. \quad (5)$$

The expected value of the motion to a member who voted for it in the initial round is given by

$$E[X|X > 0] = \frac{\int_0^\infty x f(x)dx}{\int_0^\infty f(x)dx} = \frac{2 + e^{-1}}{2 - e^{-1}}.$$

Let $X_{sup,P}$ and $X_{sup,N}$ be the value of a motion to a permanent and non-permanent, respectively, member who is a supporter of the bill in round two, but does not introduce the bill. When the voting pattern corresponds to B_k, the distributions of these two variables are identical. Considering the expected value of the motion to the $5 + k$ members who vote for the motion in such cases, of whom one introduced the bill

$$(4 + k)E[X_{sup,P}|B_k] + E[X_{max}|B_k] = (5 + k)E[X|X > 0]. \quad (6)$$

Having estimated $E[X_{max}|B_k]$ from Eq. (5), we now estimate $E[X_{sup,P}|B_k]$ and $E[X_{sup,N}|B_k]$ from Eq. (6). From the properties of the exponential distribution, the expected value of the motion to a non-permanent member who does not support the bill, $E[X_{opp,N}|B_k]$, is equal to -1. Given $B_k \cap A$, a permanent member introduces the bill, the other four permanent members support the bill, k non-permanent members support the bill, and the remaining non-permanent members abstain. Hence,

$$E[V_{P,2}|B_k \cap A] = E[X_{sup,P}|B_k] \quad (7)$$

$$E[V_{N,2}|B_k \cap A] = \frac{k(1 + E[X_{sup,P}|B_k]) - 10}{10}. \quad (8)$$

Now we consider cases where one of the permanent members abstains in the initial vote and the voting pattern in the second round of voting is given by $C_{j,k}$. From the description of such an event given above, it follows that

$$P(C_{j,k}) = 5\binom{4}{j}\binom{10}{k} p_1 p_2^j p_3^{4-j} p_4^k (1 - p_4)^{10-k}, \quad (9)$$

where $p_1 = 0.5\exp(-1)[1 - \exp(t_1)]$ is the probability that a permanent member abstains in both rounds of voting given the other permanent members initially voted for the motion, $p_2 = 1 - 0.5\exp(t_2 - 1)$ is the probability in this case that a permanent member votes for the motion in the second round, $p_3 = 0.5\exp(-1)[\exp(t_2) - 1]$ is the probability in this case that a permanent member votes for the motion in round 1 and abstains in round 2, and $p_4 = 1 - 0.5\exp(t_3 - 1)$ is the probability in this case that a non-permanent member votes for the motion in round 2.

Assume that if none of the permanent members voted for a motion in the second round and it was passed, then the motion was introduced by a non-permanent member, i.e., $P(A|C_{0,k}) \approx 0$ and from Bayes' law $P(C_{0,k}|A) \approx 0$. Since permanent and non-permanent members use different thresholds to define when they should vote for a motion in the second round, given $C_{j,k}$ the distribution of the value of the motion to the member introducing it will not be independent of the type of player (permanent or non-permanent). For this reason, conditioning on the number of non-permanent members who assign a value of a least t_2 to the motion, L, we obtain

$$E[X_{max}\mathbb{1}_{D_2}|A] = \sum_{j=1}^{4}\sum_{k=9-j}^{10}\sum_{l=0}^{k} E[X_{max}|A \cap C_{j,k} \cap L = l]P(C_{j,k} \cap L = l|A), \quad (10)$$

where

$$P(C_{j,k} \cap L = l|A) = \frac{P(A|C_{j,k} \cap L = l)P(C_{j,k} \cap L = l)}{P(A)}$$
$$= \frac{3j}{j+l} P(L = l|C_{j,k})P(C_{j,k})$$
$$= \frac{3j}{j+l} \binom{k}{l} p_5^l (1-p_5)^{k-l} P(C_{j,k}) \qquad (11)$$

and

$$p_5 = P(X > t_2|X > t_3) = \frac{P(X > t_2)}{P(X > t_3)} = \frac{2 - \exp(t_2 - 1)}{2 - \exp(t_3 - 1)}.$$

Equation (11) results from the fact that the probability of the motion being introduced by a permanent member given $C_{j,k}$ is given by the proportion of members who ascribe a value of at least t_2 to the motion.

Arguing as above, given $C_{j,k} \cap L = l$, the value of the motion to the member introducing it is conditionally independent of the member introducing it. Analogously to the derivation of Eq. (5), we obtain

$$E[X_{max}|C_{j,k} \cap L = l) \approx 1 + \sum_{m=1}^{j+l} \left(\sum_{i=1}^{m} \frac{1}{i} \right) \binom{j+l}{m} p_6^m (1-p_6)^{j+l-m}, \qquad (12)$$

where

$$p_6 = P(X > 1|X > t_2] = \frac{1}{2 - \exp(t_2 - 1)}.$$

Now we consider the expected value of the permanent members who do not introduce the motion and of the non-permanent members given $C_{j,k} \cap A \cap L = l$. First, we consider the expected value of the motion to the $j + l - 1$ supporters who did not introduce the bill, $j - 1$ permanent and l non-permanent, who ascribed the motion a value of at least t_2 to the motion. We obtain

$$(j+l-1)E[X_{sup,P}|C_{j,k} \cap A \cap L = l] + E[X_{max}|C_{j,k} \cap L = l] = (j+l)E[X|X > t_2], \qquad (13)$$

where

$$E[X|X > t_2] = \frac{2 + \exp(t_2 - 1)[1 - t_2]}{2 - \exp(t_2 - 1)}.$$

The expected value of the motion to the $(4-j)$ permanent members who support the motion in round one and abstain in round two is $E[X_{abs1,P}|C_{j,k} \cap A \cap L = l]$, where

$E[X_{abs1,P}|C_{j,k} \cap A \cap L = l] = E[X|0 < X < t_2]$. The expected value of the motion to the permanent member who abstains in both rounds is $E[X_{abs2,P}|C_{j,k} \cap A \cap L = l]$, where $E[X_{abs2,P}|C_{j,k} \cap A \cap L = l] = E[X|t_1 < X < 0]$. Hence,

$$E[V_{P,2}|C_{j,k} \cap A \cap L = l] = 0.25\left\{(j-1)E[X_{sup,P}|C_{j,k} \cap A \cap L = l]+ \right.$$
$$\left. +(4-j)E[X|0 < X < t_2] + E[X|t_1 < X < 0]\right\}, \quad (14)$$

where $E[X_{sup,P}|C_{j,k} \cap A \cap L = l]$ is derived from Eqs. (12) and (13). In addition,

$$E[X|t_1 < X < 0] = -1 - \frac{t_1 \exp(t_1)}{1 - \exp(t_1)}; \quad E[X|0 < X < t_2] = \frac{t_2 \exp(t_2)}{\exp(t_2) - 1} - 1.$$

Analogously,

$$E[V_{N,2}|C_{j,k} \cap A \cap L = l]{=}0.1\left\{(lE[X_{sup,P}|C_{j,k} \cap A \cap L = l]+ \right.$$
$$\left. +(k-l)E[X|t_3 < X < t_2]+(10-k)E[X|<X<t_3]\right\}, \quad (15)$$

where

$$E[X|t_3 < X < t_2] = \frac{t_3 \exp(t_3) - t_2 \exp(t_2)}{\exp(t_3) - \exp(t_2)} - 1; \quad E[X|X < t_3] = t_3 - 1.$$

The expected reward of a permanent member introducing a bill can be calculated from the law of total probability using Eqs. (1), (2), and (10), where Eq. (2) is based on Eqs. (3)–(5) and Eq. (10) is based on Eqs. (9), (11), and (12). Similarly, the expected rewards of the other members, according to the type of membership, can be calculated using Eqs. (6)–(8) and (13)–(15).

Based on these calculations, we can derive an iterative approach to approximating the equilibrium. It is assumed that members use a strategy according to their type. By calculating the expected rewards of the various players when a permanent member introduces a motion, we can derive the optimal response to this initial profile. This approach is repeated until convergence.

4.1 Numerical Results

A program was written to estimate this equilibrium using the procedure described above. Starting from the strategy profile described by $t_1 = -2, t_2 = 0.2, t_3 = 0.1$, after ten iterations the algorithm converged to $t_1 = -2.9160, t_2 = 0.008105, t_3 = 0.006584$. A number of initial strategy profiles were used where $t_1 < 0$ and $0 < t_3 < t_2$ and the procedure always converged to the same result. The probability of a motion being of low enough value to initiate such a veto is $0.5\exp(t_1 - 1) \approx$

0.009960. Hence, as assumed, a member introduces motions more regularly than it is the subject of retaliation. Also, the threat of retaliation seems to be critical in avoiding vetoes being applied to motions that are generally positive.

Note that the threat of a veto does not have a big influence on the members who prefer the motion to the status quo, i.e., such members show little tendency to vote tactically, in order to avoid a situation where retaliation is called for. This is associated with the fact that when there is only one permanent member who initially abstained, the probability that this member vetoes the motion in round two is low (it is equal to $\exp(t_1) \approx 0.0054$). In the context of the UN security council, it is very possible that a permanent member who is against a particular motion can estimate how much support the other members have for a bill. Hence, it is possible that such a member might gain by only abstaining in round one when it feels that accepting the motion is much worse than the status quo. Such a strategy would make the threat of a veto more believable. However, such a strategy could make it more difficult to block a motion without retaliation, as when there are two permanent members against a motion, one might not realize the other is also against it.

5 Conclusions

This paper has considered a model of a voting game in which the players observe a sequence of motions and some players have power of veto. The United Nations Security Council seems to be an obvious example of such a game. Empirical observations suggest that the permanent members of the council only tend to use their power of veto when they feel that the motion being considered is significantly worse than the status quo from their point of view. It might be surprising that the voting behavior of those who are in favor of a motion is unlikely to be affected by the threat of veto and subsequent retaliation, while the behavior of a possible vetoer is strongly affected by the threat of retaliation. However, our model assumes that retaliation is only realistic in cases where one veto member blocks a motion that is supported by a clear majority of members. In such situations, it seems reasonable to assume that a large coalition of members has more power to influence a lone member that the other way round. In other situations, where a possible vetoer sees that he is not isolated, then the threat to veto is very real (in the framework of the UN Security Council, such motions are removed or significantly altered).

According to the model, each motion has a value to each of the members, which is measured relative to the present state. It is assumed that players can react to the behavior of others in previous votes. We derived an equilibrium at which members retaliate when a lone permanent member applies their power of veto to a motion that would have otherwise been passed. The threat of such retaliation is sufficient to lower the frequency with which vetoes are applied. It should be noted that it is assumed that in the initial round of voting players give honest signals of whether they are in favor of a bill or not. It is hoped that future research into such games will relax this assumption. Also, it is assumed that the threat to retaliate is always carried out. Intuitively, after observing what the consequences of retaliation are (e.g., costly sanctions or costs associated with blocking a generally popular motion), a member may decide not to retaliate. For an agreed strategy for retaliation to be stable, it is necessary that the mean reward of players when this strategy is adopted is greater than the mean reward of the players when there is no procedure for retaliation. We intend to look at this problem in a future article.

Future research should consider various distributions describing the value of a motion to analyze how robust our conclusions are. Also, it was assumed that the values the members ascribe to a motion are independent. In practice, some members may have common interests, i.e., the values of the motions to individual countries are correlated. This could also be an area for future research.

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The Selten–Szidarovszky Technique: The Transformation Part

Pierre von Mouche

Abstract A technique due to Selten and Szidarovszky for the analysis of Nash equilibria of games with an aggregative structure is reconsidered. Among other things it is shown that the transformation part of this technique can be extended to abstract games with co-strategy mappings and allows for a purely algebraic setting.

Keywords Aggregative games • Fixed points • Nash equilibria • Selten-Szidarovszky technique

1 Introduction

In this article I reconsider a powerful technique for analysing Nash equilibria of games in strategic form with an aggregative structure. Aggregative games form the most important class of such games. In an aggregative game the strategy sets are subsets of $\mathbb{R}$ and the payoff of a player i depends on two variables: his strategy x_i and the aggregate $\sum_l x_l$ of all strategies.

As the origin of this technique is in the article [11] of Selten dealing with aggregative games and in [12] of Szidarovzsky dealing with homogeneous Cournot oligopolies,[1] I like to refer to this technique as the "Selten–Szidarovszky technique". It is not only used for showing equilibrium existence, but also for equilibrium semi-uniqueness[2] and equilibrium uniqueness.[3]

The power of the Selten–Szidarovszky technique is that it just exploits the aggregative structure and does not rely on assumptions like downward- or upward-sloping best replies or continuous payoff functions. For the (relative difficult)

[1]However, it may be good to mention here also [9].

[2]That is, that there exists at most one equilibrium.

[3]Some recent contributions concerning this technique are [1, 3, 7, 8, 16, 17].

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case of the homogeneous Cournot oligopoly with concave industry revenue this is illustrated by Szidarovzsky and Okuguchi in [13], by Deneckere and Kovenock in [4], by Hirai and Szidarovzsky in [5] and by Cornes and Sato in [3].

In [2] Cornes and Hartley present a more systematic treatment of the Selten–Szidarovszky technique for aggregative games.[4] Nevertheless, in my opinion, this technique still is not well enough understood and there is space for generalisations and improvements, for example, for more general aggregates, for higher dimensional strategy sets and for general results that can be derived by this technique.

A better understanding of the technique can be obtained by dividing it into two parts: the "transformation part" and the "analysis part".[5] The aim of this article is to formulate and analyse the transformation part in a very general setting. The setting concerns what I call "abstract games with co-strategy mappings".[6] In doing so I show that the transformation part is a purely algebraic event.

An n-player abstract game with co-strategy mappings is given by two objects. First, a correspondence $\mathbf{R} = (R_1, \ldots, R_n) : \mathbf{X} \multimap \mathbf{X}$, where $\mathbf{X} = \mathsf{X}_{l=1}^{n} X_l$ is a Cartesian product of non-empty sets, referred to as the "joint best reply correspondence". Second, by a mapping $\boldsymbol{\Phi} = (\varphi_1, \ldots, \varphi_n) : \mathbf{X} \to \mathbf{Y}$, referred to as the "joint co-strategy mapping", where $\mathbf{Y} = \mathsf{X}_{l=1}^{n} Y_l$ with every $Y_i := \varphi_i(\mathbf{X})$ a subset of an Abelian group G_i. The mapping $\boldsymbol{\Phi}$ is not necessarily related to $\mathbf{R}$. It is good to note that a game in strategic form leads in a natural way to an abstract game, but that it does not lead in such a way to co-strategy mappings. However, for aggregative games it does: take $\varphi_i(\mathbf{x}) = \sum_l x_l$. Out of $\mathbf{R}$ and $\boldsymbol{\Phi}$ a correspondence $\mathbf{B} = (B_1, \ldots, B_n) : \mathbf{Y} \multimap \mathbf{X}$ is constructed, referred to as the "joint backward reply correspondence". Here the "backward reply correspondence" $B_i : Y_i \multimap X_i$ of player i provides the best replies of that player that are "compatible with his co-strategy". The correspondence $\boldsymbol{\Phi} \circ \mathbf{B} : \mathbf{Y} \multimap \mathbf{Y}$ is referred to as the "extended backward reply correspondence". (See Sect. 3 for precise definitions.)

The transformation part concerns the study of backward solvability of the game and of dimensional reduction of $\boldsymbol{\Phi} \circ \mathbf{B}$. Backward solvability is defined as a situation where $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) = \mathrm{fix}(\mathbf{R})$. Dimensional reduction concerns the situation where the fixed point problem for $\boldsymbol{\Phi} \circ \mathbf{B}$, which takes place in $\mathbf{Y}$, boils down to a fixed point problem for a correspondence with a "simpler" domain. In the case of dimensional reduction, the analysis of the fixed point problem for the extended backward reply correspondence in principle simplifies. For example, in the case of an n player aggregative game, dimensional reduction together with backward solvability usually implies that the fixed point problem for $\mathbf{R}$ which takes place in an n-dimensional real interval can be handled by studying a fixed point problem in a real interval. So instead of deep theorems like the Brouwers' fixed point theorem, just the intermediate value property may be sufficient.

[4] Also see [3] and [18].

[5] It should be said that in most articles dealing with the Selten–Szidarovszky technique there is no clear distinction between the two parts and that each article uses the technique in its own way. Also there is no standard terminology for the objects in this technique.

[6] The notion of "abstract game" belongs to Vives [15].

The analysis part concerns the study of fix($\mathbf{R}$) by analysing $\mathbf{B}$ and fix($\boldsymbol{\Phi} \circ \mathbf{B}$). In order to perform this analysis, further assumptions, like an aggregative game structure and suitable smoothness and concavity conditions, are very welcome. Using such conditions, in the article [14] of Szidarovzsky and Yakowitz a proof of 12 lines was presented for Cournot equilibrium uniqueness.[7] Important general results for the analysis part are obtained in, for instance, [1, 6, 7, 10].[8] The contribution of [10] by Novshek concerns the handling of discontinuous non-quasiconcave profit functions. The key idea there was later generalised in [6, 7] by Kukushkin to abstract games.[9] Reference [1] by Acemoglu and Jensen deals with comparative statics.

In the present article I further only concentrate on the transformation part. I will deal with the transformation part without imposing further (topological, convexity, monotonicity, smoothness, etc.) properties on the involved objects. As far as I know this has never been done before.[10] The organisation and a short overview of the results of the present article are as follows. Section 2 starts with a motivating example. Section 3 presents the setting, the fundamental objects and notions; in particular the notions of "internal backward solvability", "external backward solvability" and "backward solvability". Section 4 presents general results for abstract games with co-strategy mappings. Theorem 1 there states that fix ($\boldsymbol{\Phi} \circ \mathbf{B}) \supseteq \boldsymbol{\Phi}(\text{fix}(\mathbf{R}))$. Section 5 deals with abstract games with related co-strategy mappings. Such co-strategy mappings satisfy a certain separability property, i.e., $\varphi_i(\mathbf{x}) = \rho_i(\lambda_i(x_i), \mu_i(\mathbf{x}_{\hat{\imath}}))$ with ρ_i injective in its second variable, and have a specific relation to $\mathbf{R}$, i.e., $R_i(\mathbf{x}_{\hat{\imath}}) = \tilde{R}_i(\mu_i(\mathbf{x}_{\hat{\imath}}))$; the $\tilde{R}_i$ are called "reduced best reply correspondences". Theorem 2 states that fix($\mathbf{B} \circ \boldsymbol{\Phi}$) $=$ fix($\mathbf{R}$) and Theorem 3 that the game is internal backward solvable. Theorem 4 states that fix($\boldsymbol{\Phi} \circ \mathbf{B}$) $= \boldsymbol{\Phi}(\text{fix}(\mathbf{R}))$ and that if $\mathbf{B}$ is at most single-valued, then the game is backward solvable and $\boldsymbol{\Phi}$ is injective on fix($\mathbf{R}$). Section 6 deals with abstract games with proportional co-strategy mappings: $\varphi_i = \alpha_i \varphi_1$. Theorem 5 there shows that the fixed point problem for $\boldsymbol{\Phi} \circ \mathbf{B}$ admits a dimensional reduction in the sense that the fixed point problem for the extended backward reply correspondence boils down to the fixed point problem for a correspondence $U : Y_1 \multimap Y_1$; I refer to U as the "aggregate backward reply correspondence". However, such a game may not be backward solvable. Section 7 considers related proportional separable co-strategy mappings of the form $\varphi_i(\mathbf{x}) = \alpha_i \sum_l \sigma_l(x_l)$. Theorem 6 there presents sufficient conditions for backward solvability in terms of the reduced best reply correspondences.

[7] I like to call this proof a "Proof from the Book" when these 12 lines were easier to understand.

[8] Also see [16] where sufficient conditions are provided for an aggregative game to have a unique Nash equilibrium.

[9] The way of obtaining the results in [7] supports my observation on the algebraic nature of the transformation part.

[10] However, see [17].

2 A Motivating Example

Consider the following game in strategic form with n players: each player i has $X_i = \mathbb{R}_+$ as strategy set and his payoff function is given by

$$f_i(\mathbf{x}) = e^{-(x_1+\cdots+x_n)}x_i - \frac{1}{2}x_i^2.$$

This game can be interpreted as a homogeneous Cournot oligopoly with price function $p(y) = e^{-y}$ and cost functions $c_i(x_i) = \frac{1}{2}x_i^2$. I now prove with the Selten–Szidarovszky technique that this game has a unique Nash equilibrium.

Well, as the game is aggregative, each conditional payoff function of player i (i.e. his payoff function as a function of the own strategy, given the strategies of the other players) only depends on the aggregate of the strategies of the other players. Having observed this, I define for $z \in \mathbb{R}_+$ the function $\tilde{f}_i^{(z)} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\tilde{f}_i^{(z)}(x_i) = e^{-x_i-z}x_i - \frac{1}{2}x_i^2.$$

One easily verifies that $D\tilde{f}_i^{(z)}(x_i) = 0 \;\Rightarrow\; D^2\tilde{f}_i^{(z)}(x_i) < 0$. This implies that $\tilde{f}_i^{(z)}$ is strictly pseudo-concave. Define $\tilde{U} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{U}(y) = n\frac{e^{-y}}{1+e^{-y}}.$$

As the continuous function $\tilde{U} - \mathrm{Id}$ is strictly decreasing, has value $n/2$ at $y = 0$ and assumes negative values, the intermediate value property implies that $\tilde{U} - \mathrm{Id}$ has a unique zero. Thus $\tilde{U}$ has a unique fixed point, say $y_\star$.

Now suppose $\mathbf{x}$ is a Nash equilibrium. With $a_i = \sum_{l \neq i} x_l$, this is equivalent with that, for every player i, x_i is a maximiser of $\tilde{f}_i^{(a_i)}$. As $D\tilde{f}_i^{(a_i)}(0) = e^{-a_i} > 0$, I have $x_i \neq 0$. So x_i is an interior maximiser. By Fermat's theorem $D\tilde{f}_i^{(a_i)}(x_i) = 0$, so $e^{-x_i-a_i}(1-x_i) = x_i$. With $y = \sum_l x_l$, I have the identity $e^{-y}(1-x_i) = x_i$. This implies $x_i = \frac{e^{-y}}{1+e^{-y}}$. Summing over i gives $y = \tilde{U}(y)$. So y is a fixed point of $\tilde{U}$. It follows that $y = y_\star$ and $x_i = \frac{e^{-y_\star}}{1+e^{-y_\star}}$. Thus there exists at most one Nash equilibrium.

Finally, I prove the existence of a Nash equilibrium. With $x_i = \frac{e^{-y_\star}}{1+e^{-y_\star}}$, I have $D\tilde{f}_i^{(\sum_{l \neq i} x_l)}(x_i) = 0$. As $\tilde{f}_i^{(\sum_{l \neq i} x_l)}$ is pseudo-concave, I have, as desired, that x_i is a maximiser of $\tilde{f}_i^{(\sum_{l \neq i} x_l)}$.

So I proved equilibrium uniqueness without (for existence) relying on a deep fixed point theorem like that of Brouwer. Just the intermediate value property was needed. The interesting question now is: what makes that such a proof is possible? Well, for a full understanding, in my opinion, one needs to understand the transformation part and the analysis part of the Selten–Szidarovszky technique. The present article provides a detailed understanding of the transformation part. Theory for the analysis part is planned for a follow-up.

3 Fundamentals

In the rest of this article, n is a positive integer, !

$$N := \{1, 2, \dots, n\}$$

and

$$X_1, \dots, X_n$$

are non-empty sets. Although these objects do not concern a game structure, I refer to the elements of N as **players** and to X_i as **strategy set** of player i. I put

$$\mathbf{X} := \times_{l=1}^{n} = X_1 \times \cdots \times X_n$$

and for $i \in N$ put

$$\mathbf{X}_{\hat{\imath}} := X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n.$$

I refer to the elements of $\mathbf{X}$ as **strategy profiles**. Given $i \in N$, I identify $\mathbf{X}$ with $X_i \times \mathbf{X}_{\hat{\imath}}$ and then write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x_i; \mathbf{x}_{\hat{\imath}})$.

3.1 Co-strategy Mappings

Definition 1. A **co-strategy mapping** of player i is a mapping

$$\varphi_i : \mathbf{X} \to G_i$$

where G_i is an Abelian group.

The set

$$Y_i := \varphi_i(\mathbf{X})$$

is called the **co-strategy set** of player i and its elements are called **co-strategies**. If for each player i a co-strategy mapping φ_i is given, then

$$\mathbf{Y} := \times_{l=1}^{n} Y_l$$

and $\boldsymbol{\Phi} : \mathbf{X} \to \mathbf{Y}$ is defined by

$$\boldsymbol{\Phi}(\mathbf{x}) := (\varphi_1(\mathbf{x}), \dots, \varphi_n(\mathbf{x}))$$

and referred to as the $\boldsymbol{\Phi}$ as the **joint co-strategy mapping**. $\diamond$

In the following definition I define separable co-strategy mappings:

Definition 2. A co-strategy mapping $\varphi_i : \mathbf{X} \to G_i$ of player i is called **separable** if there exist mappings

$$\lambda_i : X_i \to G_i \text{ and } \mu_i : \mathbf{X}_{\hat{\imath}} \to G_i$$

and with $L_i := \lambda_i(X_i)$ and $M_i := \mu_i(\mathbf{X}_{\hat{\imath}})$, a mapping

$$\rho_i : L_i \times M_i \to G_i$$

which is injective in its second variable such that for every $\mathbf{x} \in \mathbf{X}$

$$\varphi_i(\mathbf{x}) = \rho_i(\lambda_i(x_i), \mu_i(\mathbf{x}_{\hat{\imath}})).$$

In this case $(\rho_i, \lambda_i, \mu_i)$ is called a **decomposition** of φ_i. ⋄

Up to now the group structure of the sets G_i did not play a role.

Definition 3. Let G be a commutative ring with identity. Co-strategy mappings $\varphi_i : \mathbf{X} \to G$ $(i \in N)$ are called **proportional** if there exist $\alpha_i \in G$ $(i \in N)$ with $\alpha_1 = 1$ such that, with $\varphi := \varphi_1$,

$$\varphi_i = \alpha_i \varphi \ (i \in N). \ \diamond$$

For proportional co-strategy mappings I have with $Y := Y_1$ that

$$Y_i = \alpha_i Y_1 = \alpha_i Y.$$

Now consider proportional co-strategy mappings of the form

$$\varphi_i(\mathbf{x}) = \alpha_i \sum_{l \in N} \sigma_l(x_l) \tag{1}$$

with $\sigma_i : X_i \to G$. Such co-strategy mappings are separable as with

$$\rho_i(l_i, m_i) = l_i + m_i, \ \lambda_i(x_i) = \alpha_i \sigma_i(x_i), \ \mu_i(\mathbf{z}) = \alpha_i \sum_{l \in N \setminus \{i\}} \sigma_l(z_l),$$

$(\rho_i, \lambda_i, \mu_i)$ is a decomposition of φ_i.

3.2 *Abstract Games*

The next definition concerns the notion of “abstract game”.

Definition 4. An (n-player) **abstract game** is an n-tuple $(R_1, \dots, R_n)$ where, with $X_1, \dots, X_n$ non-empty sets, every R_i is a correspondence

$$R_i \colon \mathbf{X}_{\hat{\imath}} \multimap X_i. \ \diamond$$

I refer to R_i as the **best reply correspondence** of player i and to the elements of $R_i(\mathbf{X}_{\hat{\imath}})$ as **best replies** of player i. And defining the correspondence $\mathbf{R} : \mathbf{X} \multimap \mathbf{X}$ by

$$\mathbf{R}(\mathbf{x}) \ := \ R_1(\mathbf{x}_{\hat{1}}) \times \cdots \times R_n(\mathbf{x}_{\hat{n}}),$$

I refer to $\mathbf{R}$ as the **joint best reply correspondence**.

Definition 5. An (n-player) **abstract game with co-strategy mappings** is a $2n$-tuple $(R_1, \dots, R_n; \varphi_1, \dots, \varphi_n)$ such $(R_1, \dots, R_n)$ is an abstract game and every $\varphi_i : \mathbf{X} \to G_i$ is a co-strategy mapping. $\diamond$

Given an abstract game with co-strategy mappings $(R_1, \dots, R_n; \varphi_1, \dots, \varphi_n)$, I fix for every $i \in N$ a subset Y_i' of G_i with $Y_i' \subseteq Y_i$ such that

$$\varphi_i(\mathrm{fix}(\mathbf{R})) \subseteq Y_i'.$$

(Thus, for instance, $Y_i' = Y_i$ is always possible.) Let

$$\mathbf{Y}' \ := \ \mathsf{X}_{l=1}^{n} Y_l',$$
$$X' \ := \ \{\mathbf{x} \in \mathbf{X} \mid \boldsymbol{\Phi}(\mathbf{x}) \in \mathbf{Y}'\},$$
$$\boldsymbol{\Phi}' \ := \ \boldsymbol{\Phi} \upharpoonright X',$$

i.e., $\boldsymbol{\Phi}'$ is the restriction of $\boldsymbol{\Phi}$ to X'. Note that X' may be empty, and that

$$\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R})) \subseteq \mathbf{Y}', \quad \mathrm{fix}(\mathbf{R}) \subseteq X'.$$

In the case of proportional co-strategy mappings I (can and shall) take Y_i' such that $Y_i' = \alpha_i Y_1'$ and then I write

$$Y' \ := \ Y_1'.$$

Working with the Y_i' instead of with the Y_i may have advantages.[11]

[11]For example, in the case of homogeneous Cournot oligopolies where the price function has a market satiation point v, under weak conditions one can take $Y = [0, v]$ (see, for instance, [16]).

3.3 Backward Reply Correspondence

In the following definition I introduce the most important objects: the correspondences B_i $(i \in N)$:

Definition 6. Given an abstract game with co-strategy mappings $(R_1, \dots, R_n; \varphi_1, \dots, \varphi_n)$, for $i \in N$, the **backward reply correspondence** B_i: $Y_i' \multimap X_i$ is defined by

$$B_i(y_i) \ := \ \{x_i \in X_i \mid \exists\, \mathbf{z} \in \mathbf{X}_{\hat{\imath}} \text{ such that } x_i \in R_i(\mathbf{z}) \text{ and } y_i = \varphi_i(x_i; \mathbf{z})\}.$$

The **joint backward reply correspondence** $\mathbf{B} : \mathbf{Y}' \multimap \mathbf{X}$ is defined by

$$\mathbf{B}(\mathbf{y}) \ := \ B_1(y_1) \times \cdots \times B_n(y_n). \ \diamond$$

Thus the backward reply correspondence of a player provides the best replies of that player that are "compatible with his co-strategy".

In the case R_i is single-valued,[12] I have

$$B_i(y_i) = \{R_i(\mathbf{z}) \mid \mathbf{z} \in \mathbf{X}_{\hat{\imath}} \text{ with } y_i = \varphi_i(R_i(\mathbf{z}); \mathbf{z})\}.$$

If R_i is single-valued, then B_i may not be single-valued.[13]

I refer to the correspondence $\boldsymbol{\Phi} \circ \mathbf{B} : \mathbf{Y}' \multimap \mathbf{Y}$ as the **extended backward reply correspondence**. Note that

$$(\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y}) = \cup_{\mathbf{x} \in \mathbf{B}(\mathbf{y})} \{(\varphi_1(\mathbf{x}), \dots, \varphi_n(\mathbf{x}))\}. \tag{2}$$

Finally, I consider the case of an abstract game with proportional co-strategy mappings. For such a game I see with (2) that[14]

$$(\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y}) = \varphi(\mathbf{B}(\mathbf{y}))(\alpha_1, \dots, \alpha_n). \tag{3}$$

For such a game I define the correspondence $U : Y' \multimap Y$ by

$$U(y) \ := \ \varphi(\mathbf{B}(\alpha_1 y, \dots, \alpha_n y)) = \varphi(B_1(\alpha_1 y) \times \cdots \times B_n(\alpha_n y)).$$

[12] I use the following convention for a correspondence $C : A \multimap B$. If $a \in A$ and $C(a) = \{b\}$, then I also write $C(a) = b$. And when C is single-valued, I also consider it as a mapping $C : A \to B$.

[13] For example, the following result holds in the case where X_i are proper intervals of $\mathbb{R}$ and $\varphi_i = \gamma$: if R_i is interior (that is, $R_i(\mathbf{z}) \subseteq X_i^{\circ}$ for each $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$), then $B_i(y_i) = \emptyset$ for all $y_i \in Y_i \setminus Y_i^{\circ}$.

Indeed: Suppose $B_i(y_i) \neq \emptyset$ and $x_i \in B_i(y_i)$. Let $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ such that $x_i = R_i(\mathbf{z})$ and $y_i = x_i + \sum_l z_l$. As $x_i \in X_i^{\circ}$, it follows that $y_i \in Y_i^{\circ}$.

[14] G^n has in a natural way the structure of a left G module: for $A \subseteq G$ and $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ the set $A\mathbf{g}$ is defined as $\{(ag_1, \dots, ag_n) \mid a \in A\}$.

I refer to U as the **aggregate backward reply correspondence**. Theorem 5 below will make the usefulness of U clear.

In the case of co-strategy mappings of the form (1), formula (3) implies

$$U(y) = \sum_{l \in N} \sigma_l(B_l(\alpha_l y)).$$

And in case here G even is a commutative field, I define for $i \in N$ the correspondence $s_i : Y' \setminus \{0\} \multimap G$ by

$$s_i(y) \ := \ y^{-1}\sigma_i(B_i(\alpha_i y))$$

and

$$s \ := \ \sum_{i \in N} s_i.$$

Following the terminology in [2], I refer to s_i as the **share correspondence** of player i and s the **aggregate share correspondence**. As $s(y) = U(y)/y$ $(y \in Y' \setminus \{0\})$, I obtain that for all $y \in Y' \setminus \{0\}$ it holds that

$$y \in \mathrm{fix}(U) \ \Leftrightarrow \ 1 \ \in s(y).$$

3.4 *Various Mappings*

Before proceeding it is good to resume that for an abstract game with co-strategy mappings

$$\boldsymbol{\Phi} : \mathbf{X} \to \mathbf{Y}, \ \ \boldsymbol{\Phi}' : X' \to \mathbf{Y}',$$

$$\mathbf{B} : \mathbf{Y}' \multimap \mathbf{X}, \ \ \mathbf{R} : \mathbf{X} \multimap \mathbf{X},$$

$$\boldsymbol{\Phi} \circ \mathbf{B} : \mathbf{Y}' \multimap \mathbf{Y}, \ \ \mathbf{B} \circ \boldsymbol{\Phi}' : X' \multimap \mathbf{X}.$$

(In case $Y'_i = Y_i$ $(i \in N)$) the fixed point problems for $\mathbf{R}$ and $\mathbf{B} \circ \boldsymbol{\Phi}'$ are problems in $\mathbf{X}$, but that for $\boldsymbol{\Phi} \circ \mathbf{B}$ is one in $\mathbf{Y}$.

In the following I always consider an abstract game with co-strategy mappings $(R_1, \ldots, R_n; \varphi_1, \ldots, \varphi_n)$.

3.5 Backward Solvability

Definition 7. An abstract game with co-strategy mappings is said to be

- **external backward solvable** if $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) \subseteq \mathrm{fix}(\mathbf{R})$;
- **internal backward solvable** if $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) \supseteq \mathrm{fix}(\mathbf{R})$;
- **backward solvable** if $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) = \mathrm{fix}(\mathbf{R})$. $\diamond$

As we shall see, finding sufficient conditions for external backward solvability is more difficult than such conditions for internal solvability: in Sect. 5 we shall see that each abstract game with related co-strategy mappings is internal backward solvable.

As an example I consider the 2-player game in strategic form given by $X_1 = X_2 = \mathbb{R}$ and

$$f_1(x_1, x_2) = f_2(x_1, x_2) = -(x_1 + x_2)^2.$$

The best reply correspondences of this game are singleton-valued and are given by $R_1(x_2) = -x_2$ and $R_2(x_1) = -x_1$. I obtain $\mathrm{fix}(\mathbf{R}) = \{(x, -x) \mid x \in \mathbb{R}\}$.

With the co-strategy mappings $\varphi_1, \varphi_2 : \mathbf{X} \to \mathbb{R}$ defined by $\varphi_1(x_1, x_2) = \varphi_2(x_1, x_2) = x_1 + x_2$, I have an abstract game with co-strategy mappings $(R_1, R_2; \varphi_1, \varphi_2)$. Note that $Y_1 = Y_2 = \mathbb{R}$. I obtain for the backward reply correspondences

$$B_i(0) = \mathbb{R},\ B_i(y_i) = \emptyset \text{ elsewhise.}$$

This leads for the extended backward reaction correspondence $\boldsymbol{\Phi} \circ \mathbf{B}$ to

$$(\boldsymbol{\Phi} \circ \mathbf{B})(0, 0) = \{(x, x) \mid x \in \mathbb{R}\},\ \ (\boldsymbol{\Phi} \circ \mathbf{B})(y_1, y_2) = \emptyset \text{ elsewhise.}$$

So $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \{(0, 0)\}$. As $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) = \mathbf{B}(0, 0) = \mathbb{R} \times \mathbb{R} \supset \mathrm{fix}(\mathbf{R})$, the game is internal backward solvable but not external backward solvable.

4 General Results

In this section I always consider an abstract game with co-strategy mappings $(R_1, \ldots, R_n; \varphi_1, \ldots, \varphi_n)$.

Lemma 1. *1.* $\mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}') \supseteq \mathrm{fix}(\mathbf{R})$.
2. $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}'))$.
3. $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) \subseteq \mathrm{effdom}(\mathbf{B})$.[15] $\diamond$

[15]effdom($\mathbf{B}$) denotes the effective domain of B, i.e., the set of points in the domain of the correspondence $\mathbf{B}$ at which $\mathbf{B}$ is not the empty set.

Proof. 1. Suppose $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$; so $\mathbf{x} \in X'$. Let $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{x}) = \boldsymbol{\Phi}'(\mathbf{x})$. For $i \in N$ one has $x_i \in R_i(\mathbf{x}_{\hat{i}})$ and $y_i = \varphi_i(x_i; \mathbf{x}_{\hat{i}})$. So $x_i \in B_i(y_i)$. Thus $\mathbf{x} \in \mathbf{B}(\mathbf{y}) = (\mathbf{B} \circ \boldsymbol{\Phi}')(\mathbf{x})$.
2. "$\supseteq$": suppose $\mathbf{x} \in \mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}')$, i.e., $\mathbf{x} \in (\mathbf{B} \circ \boldsymbol{\Phi}')(\mathbf{x})$. Then $\boldsymbol{\Phi}(\mathbf{x}) \in \boldsymbol{\Phi}((\mathbf{B} \circ \boldsymbol{\Phi}')(\mathbf{x})) = (\boldsymbol{\Phi} \circ \mathbf{B})(\boldsymbol{\Phi}(\mathbf{x}))$.
"$\subseteq$": suppose $\mathbf{y} \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$; so $\mathbf{y} \in \mathbf{Y}'$. Then $\mathbf{y} \in \boldsymbol{\Phi}(\mathbf{B}(\mathbf{y}))$. Let $\mathbf{x} \in \mathbf{B}(\mathbf{y})$ such that $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{x})$; so $\mathbf{x} \in X'$ and $\mathbf{y} = \boldsymbol{\Phi}'(\mathbf{x})$. It follows that $\mathbf{x} \in \mathbf{B}(\boldsymbol{\Phi}'(\mathbf{x}))$. So $\mathbf{x} \in \mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}')$. Thus $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{x}) \in \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}'))$.
3. Suppose $\mathbf{y} \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$, so $\mathbf{y} \in \boldsymbol{\Phi}(\mathbf{B}(\mathbf{y}))$. Thus, as desired, $\mathbf{B}(\mathbf{y}) \neq \emptyset$. □

Lemma 1(1) implies that in the case $\mathbf{R}$ and $\mathbf{B}$ are single-valued, the identity $\mathbf{B} \circ \boldsymbol{\Phi}' = \mathbf{R}$ holds on $\mathrm{fix}(\mathbf{R})$.

Proposition 1. *If* $\mathbf{B} \circ \boldsymbol{\Phi}'$ *is at most single-valued on the set of its fixed points, then* $\boldsymbol{\Phi}$ *is injective on* $\mathrm{fix}(\mathbf{R})$. ⋄

Proof. Suppose $\mathbf{a}, \mathbf{b} \in \mathrm{fix}(\mathbf{R})$ and $\boldsymbol{\Phi}(\mathbf{a}) = \boldsymbol{\Phi}(\mathbf{b}) =: \mathbf{y}$. By Lemma 1(1), $\mathbf{a} \in (\mathbf{B} \circ \boldsymbol{\Phi})(\mathbf{a}) = \mathbf{B}(\mathbf{y})$ and $\mathbf{b} \in (\mathbf{B} \circ \boldsymbol{\Phi})(\mathbf{b}) = \mathbf{B}(\mathbf{y})$. As $\mathbf{B} \circ \boldsymbol{\Phi}'$ is at most single-valued on the set of its fixed points, I have $\mathbf{a} = \mathbf{B}(\mathbf{y})$ and $\mathbf{b} = \mathbf{B}(\mathbf{y})$. Thus $\mathbf{a} = \mathbf{b}$. □

Theorem 1. $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) \supseteq \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$. ⋄

Proof. By Lemma 1. □

Thus sufficient for $\boldsymbol{\Phi}$ to be constant on $\mathrm{fix}(\mathbf{R})$ is that the extended backward reply correspondence has at most one fixed point.

Part 1 of the next proposition implies for a external backward solvable game that sufficient for $\mathbf{R}$ to have a fixed point is that the extended backward reaction correspondence $\boldsymbol{\Phi} \circ \mathbf{B}$ has a fixed point.

Proposition 2. *1. If the game is external backward solvable, then* $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$.
2. $\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R})) \subseteq \mathrm{effdom}(\boldsymbol{B})$. ⋄

Proof. 1. "$\supseteq$" holds by Theorem 1. To prove "$\subseteq$" let $\mathbf{y} \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$. That is $\mathbf{y} \in \boldsymbol{\Phi}(\mathbf{B}(\mathbf{y}))$. Choose $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x} \in \mathbf{B}(\mathbf{y})$ and $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{x})$. Then, as the game is external backward solvable, $\mathbf{x} \in \mathbf{B}(\mathbf{y}) \subseteq \mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) \subseteq \mathrm{fix}(\mathbf{R})$. Thus $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{x}) \in \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$.

2. Suppose $\mathbf{y} \in \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$. Fix $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$ such that $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{x})$. By Lemma 1(1), $\mathbf{x} \in B(\boldsymbol{\Phi}(\mathbf{x})) = \boldsymbol{B}(\mathbf{y})$. Thus $\boldsymbol{B}(\mathbf{y}) \neq \emptyset$. □

Also Theorem 4(1) will give a sufficient condition for $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$.

The next two results give sufficient conditions for an abstract game with co-strategy mappings to be internal backward solvable; also see Theorem 3 below.

Lemma 2. *If* $\boldsymbol{\Phi}$ *is injective on* $\mathrm{fix}(\mathbf{R}) \cup (\mathbf{B} \circ \boldsymbol{\Phi})(\mathrm{fix}(\mathbf{R}))$*, then*

1. $\mathbf{B}(\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))) \supseteq \mathrm{fix}(\mathbf{R})$*;*
2. the game is internal backward solvable. ⋄

Proof. 1. Suppose $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$. Then, by Theorem 1, with $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{x})$, I have $\mathbf{y} \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$, i.e., $\mathbf{y} \in \boldsymbol{\Phi}(\mathbf{B}(\mathbf{y}))$. Let $\mathbf{z} \in \mathbf{B}(\mathbf{y})$ such that $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{z})$. Now $\boldsymbol{\Phi}(\mathbf{x}) =$

$\boldsymbol{\Phi}(\mathbf{z})$, $\mathbf{x}$ belongs to $\mathrm{fix}(\mathbf{R})$ and $\mathbf{z}$ to $(\mathbf{B} \circ \boldsymbol{\Phi})(\mathrm{fix}(\mathbf{R}))$. Using the injectivity property of $\boldsymbol{\Phi}$, it follows $\mathbf{x} = \mathbf{z}$. Thus $\mathbf{x} = \mathbf{z} \in \mathbf{B}(\mathbf{y}) \subseteq \mathbf{B}(\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R})))$.
2. This follows by part 1 and Theorem 1. □

Proposition 3. *If* $\mathbf{B}$ *is at most single-valued on* $\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$*, then*

1. $\mathrm{fix}(\mathbf{R}) = \mathbf{B}(\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R})))$;
2. *the game is internal backward solvable;*
3. *the game is backward solvable* $\Leftrightarrow$ $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$. ◇

Proof. 1. "$\supseteq$": suppose $\mathbf{x} \in \mathbf{B}(\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$. Let $\mathbf{z} \in \mathrm{fix}(\mathbf{R})$ be such that $\mathbf{x} \in \mathbf{B}(\boldsymbol{\Phi}(\mathbf{z}))$. By Lemma 1(1), $\mathbf{z} \in \mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}')$, i.e., $\mathbf{z} \in (\mathbf{B} \circ \boldsymbol{\Phi})(\mathbf{z}) = \mathbf{B}(\boldsymbol{\Phi}(\mathbf{z}))$. As $\mathbf{B}$ is on $\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$ at most single-valued, it follows that $\mathbf{B}(\boldsymbol{\Phi}(\mathbf{z}))$ is a singleton and from this that $\mathbf{x} = \mathbf{z} \in \mathrm{fix}(\mathbf{R})$.
"$\subseteq$": suppose $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$. By Lemma 1(1), $\mathbf{x} \in \mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}')$, i.e., $\mathbf{x} \in \mathbf{B}(\boldsymbol{\Phi}(\mathbf{x}))$. As $\mathbf{B}$ is on $\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$ at most single-valued, it follows that $\mathbf{B}(\boldsymbol{\Phi}(\mathbf{x}))$ is a singleton and from this that $\mathbf{x} = \mathbf{B}(\boldsymbol{\Phi}(\mathbf{x})) \in \mathbf{B}(\boldsymbol{\Phi}(\mathrm{fix}(R)))$.
2. With part 1 and Theorem 1 I obtain $\mathrm{fix}(\mathbf{R}) = \mathbf{B}(\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))) \subseteq \mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \boldsymbol{B}))$.
3. "$\Rightarrow$": by Proposition 2(1).
"$\Leftarrow$": with part 1 I obtain $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) = \mathbf{B}(\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R})) = \mathrm{fix}(\mathbf{R})$. □

Proposition 4. *1.* $[\mathbf{a}, \mathbf{b} \in \mathrm{fix}(\mathbf{R}),\ \mathbf{a} \neq \mathbf{b},\ \boldsymbol{\Phi}(\mathbf{a}) = \boldsymbol{\Phi}(\mathbf{b})] \ \Rightarrow\ \#\mathbf{B}(\boldsymbol{\Phi}(\mathbf{a})) \geq 2$.
2. If $\mathbf{B}$ *is at most single-valued on* $\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$*, then* $\boldsymbol{\Phi}$ *is injective on* $\mathrm{fix}(\mathbf{R})$. ◇

Proof. 1. By Lemma 1(1), $\mathbf{a} \in (\mathbf{B} \circ \boldsymbol{\Phi})(\mathbf{a})$ and $\mathbf{b} \in (\mathbf{B} \circ \boldsymbol{\Phi})(\mathbf{b})$. So, writing $\mathbf{y} = \boldsymbol{\Phi}(\mathbf{a}) = \boldsymbol{\Phi}(\mathbf{b})$, I have $\mathbf{a} \in \mathbf{B}(\mathbf{y})$ and $\mathbf{b} \in \mathbf{B}(\mathbf{y})$. Thus $\#\mathbf{B}(\boldsymbol{\Phi}(\mathbf{a})) \geq 2$.
2. By part 1. □

Proposition 5. *Sufficient for a external backward solvable game to be backward solvable is that the joint backward reply correspondence* $\mathbf{B}$ *is at most single-valued on* $\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$. ◇

Proof. Theorem 1 and the external backward solvability give $(\mathbf{B} \circ \boldsymbol{\Phi})(\mathrm{fix}(\mathbf{R})) = \mathbf{B}(\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))) \subseteq \mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) \subseteq \mathrm{fix}(\mathbf{R})$. As by Proposition 4(2), $\boldsymbol{\Phi}$ is injective on $\mathrm{fix}(\mathbf{R})$, $\boldsymbol{\Phi}$ also is on $\mathrm{fix}(\mathbf{R}) \cup (\mathbf{B} \circ \boldsymbol{\Phi})(\mathrm{fix}(\mathbf{R}))$ injective. Lemma 3(2) guarantees that the game is external backward solvable. □

5 Abstract Games with Related Co-strategy Mappings

The next definition deals with separable co-strategy mappings which have a special relation to the best reply correspondences.

Definition 8. The co-strategy mappings $\varphi_1, \ldots, \varphi_n$ of an abstract game with co-strategy mappings $(R_1, \ldots, R_n; \varphi_1, \ldots, \varphi_n)$ are **related** if each co-strategy mapping φ_i admits a decomposition $(\rho_i, \lambda_i, \mu_i)$ and for each player i there exists a correspondence

$$\tilde{R}_i : M_i \multimap X_i,$$

such that

$$R_i(\mathbf{z}) = \tilde{R}_i(\mu_i(\mathbf{z})). \diamond$$

Remark. R_i is (at most) single-valued $\Leftrightarrow$ $\tilde{R}_i$ is (at most) single-valued.

For an abstract game with related co-strategy mappings I have $B_i(y_i) = \{x_i \in X_i \mid \exists\, \mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ such that $x_i \in \tilde{R}_i(\mu_i(\mathbf{z}))$ and $y_i = \rho_i(\lambda_i(x_i), \mu_i(\mathbf{z}))\}$. Therefore also

$$B_i(y_i) = \{x_i \in X_i \mid \exists\, m_i \in M_i \text{ such that } x_i \in \tilde{R}_i(m_i) \text{ and } y_i = \rho_i(\lambda_i(x_i), m_i)\}. \quad (4)$$

Now let us reconsider the general results in Sect. 4 for abstract games with related co-strategy mappings.

Lemma 3. *Consider an abstract game with related co-strategy mappings. Let* $\mathbf{y} \in \mathbf{Y}'$ *and suppose* $\mathbf{x} \in \mathbf{B}(\mathbf{y})$.

1. *If* $\boldsymbol{\Phi}(\mathbf{x}) = \mathbf{y}$, *then* $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$.
2. *If* $(\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y}) = \{\mathbf{y}\}$, *then* $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$. $\diamond$

Proof. 1. Fix $i \in N$. I have $y_i = \varphi_i(\mathbf{x}) = \varphi_i(x_i; \mathbf{x}_{\hat{\imath}})$ and $x_i \in B_i(y_i)$. Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ such that $x_i \in R_i(\mathbf{z})$ and $y_i = \varphi_i(x_i; \mathbf{z})$. So $\rho_i(\lambda_i(x_i), \mu_i(\mathbf{x}_{\hat{\imath}})) = \rho_i(\lambda_i(x_i), \mu_i(\mathbf{z}))$. As ρ_i is injective in its second variable, $\mu_i(\mathbf{z}) = \mu_i(\mathbf{x}_{\hat{\imath}})$ follows. Now $x_i \in R_i(\mathbf{z}) = \tilde{R}_i(\mu_i(\mathbf{z})) = \tilde{R}_i(\mu_i(\mathbf{x}_{\hat{\imath}})) = R_i(\mathbf{x}_{\hat{\imath}})$, as desired.

2. $\boldsymbol{\Phi}(\mathbf{x}) \in \boldsymbol{\Phi}(\mathbf{B}(\mathbf{y})) = (\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y}) = \{\mathbf{y}\}$. So $\boldsymbol{\Phi}(\mathbf{x}) = \mathbf{y}$. Apply part 1. $\square$

Theorem 2. *Consider an abstract game with related co-strategy mappings.*

1. $\mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}') = \mathrm{fix}(\mathbf{R})$.
2. *For every* $\mathbf{x} \in X'$: $\mathbf{x} \in \mathrm{fix}(\mathbf{R}) \;\Leftrightarrow\; x_i \in B_i(\boldsymbol{\Phi}(\mathbf{x}))$ $(i \in N)$. $\diamond$

Proof. 1. "$\supseteq$": by Lemma 1(1).
"$\subseteq$": suppose $\mathbf{x} \in (\mathbf{B} \circ \boldsymbol{\Phi}')(\mathbf{x})$. Let $\mathbf{y} = \boldsymbol{\Phi}'(\mathbf{x}) = \boldsymbol{\Phi}(\mathbf{x})$. So $\mathbf{x} \in \mathbf{B}(\mathbf{y})$. By Lemma 3(1), $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$.

2. By part 1. $\square$

Theorem 3. *Consider an abstract game with related co-strategy mappings.*

1. *The game is internal backward solvable.*
2. *If* $\boldsymbol{\Phi} \circ \mathbf{B}$ *is at most single-valued on* $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$, *then the game is backward solvable.* $\diamond$

Proof. 1. With Theorem 2(1) and Lemma 1(2) I obtain $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) = (\mathbf{B} \circ \boldsymbol{\Phi}')(\mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}')) \supseteq \mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}') = \mathrm{fix}(\mathbf{R})$.

2. Having part 1, I still have to prove that $\mathrm{fix}(\mathbf{R}) \supseteq \mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}))$. So suppose $\mathbf{x} \in \mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}))$. Let $\mathbf{y} \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$ be such that $\mathbf{x} \in \mathbf{B}(\mathbf{y})$. As $\mathbf{y} \in (\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y})$ and $\boldsymbol{\Phi} \circ \mathbf{B}$ is at most single-valued on $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$, I have $(\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y}) = \{\mathbf{y}\}$. So by Lemma 3(2), $\mathbf{x} \in \mathrm{fix}(\mathbf{R})$. $\square$

Theorem 4. *Consider an abstract game with related co-strategy mappings.*

1. $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R}))$.
2. *If* $\mathbf{B}$ *is at most single-valued on* $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$, *then*

   a. *The game is backward solvable.*
   b. $\boldsymbol{\Phi}$ *is injective on* $\mathrm{fix}(\mathbf{R})$. $\diamond$

Proof. 1. By Theorem 2(1) and Lemma 1(2).
2a. By Theorem 3(2).
2b. I prove that $\mathbf{B} \circ \boldsymbol{\Phi}'$ is single-valued on $\mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}')$ and then the proof is complete by Proposition 1. Well, suppose $\mathbf{x} \in \mathbf{B}(\boldsymbol{\Phi}(\mathbf{x}))$. This implies $\boldsymbol{\Phi}(\mathbf{x}) \in (\boldsymbol{\Phi} \circ \mathbf{B})(\boldsymbol{\Phi}(\mathbf{x}))$. So $\boldsymbol{\Phi}(\mathbf{x}) \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$. As $\mathbf{B}$ is single-valued on $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$, I have, as desired, $\#(\mathbf{B} \circ \boldsymbol{\Phi})(\mathbf{x})) = 1$. $\square$

So for an abstract game with related co-strategy mappings: if $\boldsymbol{\Phi}$ is injective on $\mathrm{fix}(\mathbf{R})$, then $\#\mathrm{fix}(\mathbf{R}) = \#\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$.

Theorem 2 and Theorem 4(1) imply that for an abstract game with related co-strategy mappings

$$\mathrm{fix}(\mathbf{R}) \neq \emptyset \;\Leftrightarrow\; \mathrm{fix}(\mathbf{B} \circ \boldsymbol{\Phi}') \neq \emptyset \;\Leftrightarrow\; \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) \neq \emptyset. \tag{5}$$

6 Abstract Games with Proportional Co-strategy Mappings

In this section I consider an abstract game with proportional co-strategy mappings.

Theorem 5. *Suppose the co-strategy mappings are proportional.*

1. *The extended backward reply correspondence admits a dimensional reduction in the following sense:* $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \mathrm{fix}(U) \cdot (\alpha_1, \dots, \alpha_n)$.
2. $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) = \bigcup_{y \in \mathrm{fix}(U)} \mathbf{B}(\alpha_1 y, \dots, \alpha_n y)$.
3. $\#\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \#\mathrm{fix}(U)$. $\diamond$

Proof. 1. "$\subseteq$": suppose $\mathbf{y} \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$, so $\mathbf{y} \in (\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y})$. The formula (3) implies that there exists $\mathbf{x} \in \mathbf{B}(\mathbf{y})$ with $\mathbf{y} = \varphi(\mathbf{x})(\alpha_1, \dots, \alpha_n)$, i.e., such that $y_i = \alpha_i \varphi(\mathbf{x}))$ $(i \in N)$. So with $y = \varphi(\mathbf{x})$, I have $\mathbf{y} = (\alpha_1 y, \dots, \alpha_n y)$. The proof is complete if $y \in \mathrm{fix}(U)$, i.e., if $y \in U(y)$. Well, $y = \varphi(\mathbf{x}) \in \varphi(\mathbf{B}(\mathbf{y})) = \varphi(\mathbf{B}(\alpha_1 y, \dots, \alpha_n y)) = U(y)$.
"$\supseteq$": suppose $y \in \mathrm{fix}(U)$, so $y \in \varphi(\mathbf{B}(\alpha_1 y, \dots, \alpha_n y))$. Take $\mathbf{x} \in \mathbf{B}(\alpha_1 y, \dots, \alpha_n y)$ such that $y = \varphi(\mathbf{x})$. Now, as desired, $(\alpha_1 y, \dots, \alpha_n y) = (\varphi_1(\mathbf{x}), \dots, \varphi_n(\mathbf{x})) = \boldsymbol{\Phi}(\mathbf{x}) \in (\boldsymbol{\Phi} \circ \mathbf{B})(\alpha_1 y, \dots, \alpha_n y)$.
2. With part, $\mathbf{B}(\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})) = \bigcup_{\mathbf{y} \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})} \mathbf{B}(\mathbf{y}) = \bigcup_{y \in \mathrm{fix}(U)} \mathbf{B}(\alpha_1 y, \dots, \alpha_n y)$.
3. By part 1, $\#\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B}) = \#\mathrm{fix}(U) \cdot (\alpha_1, \dots, \alpha_n) = \#\mathrm{fix}(U)$. Here the last identity holds as $\alpha_1 = 1$. $\square$

Lemma 4. *U is at most single-valued on* $\mathrm{fix}(U) \Rightarrow \boldsymbol{\Phi} \circ \mathbf{B}$ *is at most single-valued on* $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$. $\diamond$

Proof. In order to prove that $\boldsymbol{\Phi} \circ \mathbf{B}$ is at most single-valued on $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$, I suppose that $\mathbf{y} \in \mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$ and shall prove that $\#(\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y}) \leq 1$. Well, by Theorem 5(1), there exists $y \in \mathrm{fix}(U)$ such that $\mathbf{y} = (y\alpha_1, \ldots, y\alpha_n)$. With (3) I obtain

$$(\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y}) \subseteq \varphi(\mathbf{B}(\mathbf{y}))\alpha_1 \times \cdots \times \varphi(\mathbf{B}(\mathbf{y}))\alpha_n$$

$$= \varphi(\mathbf{B}(\alpha_1 y, \ldots, \alpha_n y))\alpha_1 \times \cdots \times \varphi(\mathbf{B}(\alpha_1 y, \ldots, \alpha_n y))\alpha_n = U(y)\alpha_1 \times \cdots \times U(y)\alpha_n.$$

As $U(y) = \{y\}$, I obtain $\#(\boldsymbol{\Phi} \circ \mathbf{B})(\mathbf{y}) \leq 1$. □

Proposition 6. *Consider an abstract game with proportional related co-strategy mappings.*

1. $\boldsymbol{\Phi}(\mathrm{fix}(\mathbf{R})) = \mathrm{fix}(U)(\alpha_1, \ldots, \alpha_n)$.
2. $\mathrm{fix}(\mathbf{R}) \subseteq \bigcup_{y \in \mathrm{fix}(U)} \mathbf{B}(\alpha_1 y, \ldots, \alpha_n y)$.
3. *If U is at most single-valued on* $\mathrm{fix}(U)$*, then the game is backward solvable,* $\mathrm{fix}(\mathbf{R}) = \bigcup_{y \in \mathrm{fix}(U)} \mathbf{B}(\alpha_1 y, \ldots, \alpha_n y)$ *and* $\#\mathrm{fix}(\mathbf{R}) = \#\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$. ◇

Proof. 1. By Theorems 5(1) and 4(1).
2. By Theorems 3(1) and 5(2).
3. By Lemma 4, $\boldsymbol{\Phi} \circ \mathbf{B}$ is at most single-valued on $\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$. Apply Theorem 4(2). □

7 Abstract Games with Related Proportional Co-strategy Mappings of the Form (1)

In this section I always consider an abstract game with related proportional co-strategy mappings of the form (1).

7.1 *Effective Domain of B_i*

The formula (4) becomes

$$B_i(y_i) = \{x_i \in X_i \mid y_i - \alpha_i \sigma_i(x_i) \in M_i \text{ and } x_i \in \tilde{R}_i(y_i - \alpha_i \sigma_i(x_i))\}. \quad (6)$$

It is interesting to note that $B_i(y_i)$ in (6) concerns a fixed point problem (depending on a parameter y_i), while $\tilde{R}_i(m_i)$ concerns a maximisation problem (depending on a parameter m_i).

Before presenting the following lemma, I note that for every $y_i \in Y_i'$

$$y_i - \alpha_i \sigma_i(B_i(y_i)) \subseteq M_i.$$

And also that for every $m_i \in M_i$

$$\alpha_i(\sigma_i \circ \tilde{R}_i)(m_i) + m_i \subseteq Y_i.$$

Lemma 5. *1.* $B_i(y_i) \subseteq \tilde{R}_i(y_i - \alpha_i\sigma_i(B_i(y_i)))$ $(y_i \in Y_i')$.
2. If $Y_i' = Y_i$*, then* $\tilde{R}_i(m_i) \subseteq B_i(\alpha_i(\sigma_i \circ \tilde{R}_i)(m_i) + m_i)$ $(m_i \in M_i)$.
3. For every $y_i \in Y_i'$ *and* $x_i \in B_i(y_i)$*:* $[y_i \in (\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id})(y_i - \alpha_i\sigma_i(x_i))$. ⋄

Proof. 1. Suppose $x_i \in B_i(y_i)$. Then $x_i \in \tilde{R}_i(y_i - \alpha_i\sigma_i(x_i)) \subseteq \tilde{R}_i(y_i - \alpha_i\sigma_i(B_i(y_i)))$.

2. Suppose $x_i \in \tilde{R}_i(m_i)$. Let $y_i = \alpha\sigma_i(x_i) + m_i$. So $y_i \in Y_i = Y_i'$. Now $x_i \in \tilde{R}_i(y_i - \alpha_i\sigma_i(x_i))$. So I have $x_i \in B_i(y_i)$. It follows that

$$x_i \in B_i(y_i) = B_i(\alpha_i\sigma_i(x_i) + (y_i - \alpha_i\sigma_i(x_i))) \in B_i(\alpha_i(\sigma_i \circ \tilde{R}_i)(m_i) + m_i).$$

3. As $x_i \in B_i(y_i)$, I have $x_i \in \tilde{R}_i(y_i - \alpha_i\sigma_i(x_i))$. It follows that $\alpha_i\sigma_i(x_i) \in \alpha_i(\sigma_i \circ \tilde{R}_i)(y_i - \alpha_i\sigma_i(x_i))$ and thus $y_i = \alpha_i\sigma_i(x_i) + y_i - \alpha_i\sigma_i(x_i) \in (\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id})(y_i - \alpha_i\sigma_i(x_i))$. □

Proposition 7. *If* $\tilde{R}_i$ *is single-valued and* $Y_i' = Y_i$*, then* $\mathrm{effdom}(B_i) = (\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id})(M_i)$. ⋄

Proof. "⊆": suppose $x_i \in \tilde{R}_i(m_i)$. With $y_i = \alpha_i\sigma_i(x_i) + m_i$, I have $y_i \in Y_i = Y_i'$. Also $x_i \in \tilde{R}_i(m_i) = \tilde{R}_i(y_i - \alpha_i\sigma_i(x_i))$. So I have $x_i \in B_i(y_i)$. Thus $x_i \in B_i(y_i) = B_i(\alpha_i\sigma_i(x_i) + m_i) \subseteq B_i(\alpha_i(\sigma_i \circ \tilde{R}_i)(m_i) + m_i)$.
"⊇": suppose $y_i \in (\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id})(m_i)$. As $\tilde{R}_i$ is singleton-valued, $y_i = \alpha_i(\sigma_i \circ \tilde{R}_i)(m_i) + m_i$. So $\tilde{R}_i(m_i) = \tilde{R}_i(y_i - \alpha_i(\sigma_i \circ \tilde{R}_i)(m_i))$. Thus $\tilde{R}_i(m_i) \in B_i(y_i)$. □

7.2 At Most Single-Valuedness of B_i

Remark. If G even is an integral domain and $\sigma_i : X_i \to G$ is injective, then for all $y_i \in Y_i'$: $\#(\alpha_i\sigma_i(B_i(y_i))) = \#B_i(y_i)$.[16] So B_i is (at most) single-valued if and only if $\alpha_i(\sigma_i \circ B_i)$ is (at most) single-valued.

Proposition 8. *Suppose* G *is an integral domain,* $\sigma_i : X_i \to G$ *is injective,* $\tilde{R}_i$ *is single-valued and* $\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id}$ *is injective.*

1. B_i *is at most single-valued.*
2. If $Y_i' = Y_i$*, then* B_i *is single-valued on* $(\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id})(M_i)$. ⋄

Proof. 1. Suppose $y_i \in Y_i'$. I have to prove that $\#B_i(y_i) \leq 1$. So, having the above remark, I have to prove that $\#(\alpha_i\sigma_i(B_i(y_i))) \leq 1$. So suppose $y, y' \in \alpha_i\sigma_i(B_i(y_i))$.

[16] Indeed: as G is an integral domain and $\alpha_i \neq 0$, the set $\alpha_i\sigma_i(B_i(y_i))$ has the same number of elements as the set $\sigma_i(B_i(y_i))$. And, as σ_i is injective, the set $\sigma_i(B_i(y_i))$ has the same number of elements as the set $B_i(y_i)$.

Let $x_i, x_i' \in B_i(y_i)$ be such that $y = \alpha_i \sigma_i(x_i)$ and $y' = \alpha_i \sigma_i(x_i')$. As $\tilde{R}_i$ is single-valued, Lemma 5(3) guarantees

$$y_i = (\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id})(y_i - \alpha_i \sigma_i(x_i)) = (\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id})(y_i - \alpha_i \sigma_i(x_i')).$$

As $\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id}$ is injective, it follows that $y_i - \alpha_i \sigma_i(x_i) = y_i - \alpha_i \sigma_i(x_i'))$. So $\alpha_i \sigma_i(x_i) = \alpha_i \sigma_i(x_i')$. Thus, as desired, $y = y'$.

2. By part 1, B_i is at most single-valued. So B_i is single-valued on effdom(B_i). Finally, apply Proposition 7. □

7.3 *Backward Solvability*

In the next theorem the game is not only backward solvable, but also the extended backward reply correspondence admits a dimensional reduction.

Theorem 6. *Consider an abstract game with related proportional co-strategy mappings of the form (1): $\varphi_i(\mathbf{x}) = \alpha_i \sum_{l \in N} \sigma_l(x_l)$. Suppose that G is an integral domain, that every σ_i injective, that every $\tilde{R}_i$ is single-valued and that every $\alpha_i(\sigma_i \circ \tilde{R}_i) + \mathrm{Id}$ is injective. Then the game is backward solvable, $\boldsymbol{\Phi}$ is injective on* $\mathrm{fix}(\mathbf{R})$ *and* $\#\mathrm{fix}(\mathbf{R}) = \#\mathrm{fix}(U)$. ⋄

Proof. As the co-strategy mappings are proportional, the first statement holds by Theorem 5(1). Proposition 8(1) implies that $\mathbf{B}$ is at most single-valued. So Theorem 4(2) implies that the game is backward solvable, $\boldsymbol{\Phi}$ is injective on $\mathrm{fix}(\mathbf{R})$ and $\#\mathrm{fix}(\mathbf{R}) = \#\mathrm{fix}(\boldsymbol{\Phi} \circ \mathbf{B})$. Finally, apply Theorem 5(3). □

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Generalized Nucleoli and Generalized Bargaining Sets for Games with Restricted Cooperation

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Abstract A generalization of the theory of the bargaining set, the kernel, and the nucleolus for cooperative TU-games, where objections and counter-objections are permitted only between the members of a collection of coalitions $\mathscr{A}$ and can use only the members of a collection of coalitions $\mathscr{B} \supset \mathscr{A}$, is considered. Four versions of generalized bargaining set are possible. Three versions of generalized kernel and two versions of generalized nucleolus are defined. One generalized kernel, one generalized nucleolus, and the corresponding generalized bargaining sets were examined in Naumova (Contributions to Game Theory and Management, vol. 5, pp. 230–242. Graduate School of Management, St. Petersburg University, St. Petersburg, 2012; Contributions to Game Theory and Management GTM2014, vol. 8, pp. 231–242. St. Petersburg State University, St. Petersburg, 2015). Conditions on $\mathscr{A}$ and $\mathscr{B}$ that ensure existence of the second generalized kernel are obtained. Weakly mixed collections of coalitions are defined. For such collections of coalitions, the second generalized nucleolus is contained in the second generalized kernel and in two generalized bargaining sets. If $\mathscr{A}$ does not contain singletons such inclusion is valid for all games only if $\mathscr{A}$ is a weakly mixed collection of coalitions. For weakly mixed collection of coalitions $\mathscr{A}$ an iterative procedure that converges to a point in the second generalized kernel is described.

Keywords Cooperative games • Nucleolus • Kernel • Bargaining set

1 Introduction

The theory of the bargaining set, the kernel, and the nucleolus for cooperative TU-games was born in papers of Aumann, Maschler, Davis [1, 2] and Schmeidler [14]. The first variant of the bargaining set was introduced by Aumann and Maschler [1]. Davis and Maschler [2] defined the kernel and proved that it is a nonempty

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subset of the bargaining set. Different proofs of existence of some other variants of these sets are given by Davis and Maschler [3], Maschler and Peleg [4], and Peleg [11]. All these proofs used fixed point theorems. Stearns [15] proposed an iterative procedure that converges to a point in the kernel. Schmeidler [14] defined the nucleolus which always exists and belongs to these sets.

For each imputation x of TU-cooperative game, an objection of a player i against a player j at x and a counter-objection to this objection were defined. An imputation x^0 belongs to the bargaining set $\mathscr{M}_1^i$ if for each players i, j for each objection of i against j at x^0 there exists a counter-objection. At the same time some objections and counter-objections between coalitions were defined and it was shown that the existence theorem is not fulfilled if objections and counter-objections are permitted between all pairs of disjoint coalitions.

It is natural to consider the case when some inactive players can assert their rights only with the help of some other players, i.e., only the members of a collection of coalitions $\mathscr{A}$ can actively defend their interests. A player can belong to several members of $\mathscr{A}$. Some "active" coalitions can unite and can use the help of some inactive players in the bargaining procedure. The results of these unions form a collection of coalitions $\mathscr{B} \supset \mathscr{A}$ and can be used for objections and counter-objections.

Two versions of objection and two versions of counter-objection generalize the definitions of objection and counter-objection between singletons. Several versions of generalized kernel are possible. The problem is not to generalize the existing notions but to get existing results for different generalizations. If the collection of active coalitions $\mathscr{A}$ is not a partition of the set of players, then generalized bargaining sets and generalized kernels can be empty sets. Generalized nucleoli are always nonempty sets, but sometimes they don't intersect generalized bargaining sets.

The goal of the paper is to describe the properties of $\mathscr{A}$ that ensure existence results for different generalizations. These properties are different for different generalizations.

Naumova [6] considered the case when objections and counter-objections are permitted only between the members of a collection of coalitions $\mathscr{A}$ but all coalitions can be used for objections and counter-objections. She used one version of objection and one version of counter-objection that provide the largest generalized bargaining set, which is called $\mathscr{M}^{sw}_{\mathscr{A} 2^N}$ in this paper. She proved that if $\mathscr{A}$ is a minimal covering of the set of players then for each cooperative TU-game there exists an imputation that belongs to one of the versions of generalized kernel, which is called $\mathscr{K}_{\mathscr{A},2^N}$ in this paper and to the largest generalized bargaining set $\mathscr{M}^{sw}_{\mathscr{A} 2^N}$.

Peleg [12] proposed a model, where each imputation generates a binary relation on the set of singletons. An equilibrium imputation for this system of relations is such that each singleton is maximal with respect to the binary relation that is generated by this imputation. He proved that an equilibrium imputation exists for each system of open acyclic relations on the set of singletons. Naumova [7] generalized his result and considered the case, where each imputation generates a binary relation on a collection of coalitions $\mathscr{A}$. Necessary and sufficient condition on $\mathscr{A}$ for existence an equilibrium imputation for each system of open acyclic relations on $\mathscr{A}$ was obtained. This condition is a sufficient condition on $\mathscr{A}$ for existence the generalized kernel $\mathscr{K}_{\mathscr{A},2^N}$. In [8] it was proved that this condition is also the necessary one.

A sufficient condition on $\mathscr{A}$ for existence of equilibrium imputations without acyclicity assumption was obtained by Naumova [8]. This result is a generalization of Knaster–Kuratowski–Mazurkiewicz fixed point theorem. It permitted to give sufficient conditions on $\mathscr{A}$ for existence of two versions of generalized bargaining sets $\mathscr{M}^{sw}_{\mathscr{A}2^N}$ and $\mathscr{M}^{ss}_{\mathscr{A}2^N}$. It was proved that these conditions are also necessary if the number of players is no more than 5.

The case when objections and counter-objections can use only the members of a collection of coalitions $\mathscr{B} \supset \mathscr{A}$ was examined in [10]. Two versions of objection and two versions of counter-objection provide four generalized bargaining sets. A class of n-person TU-games with nonnegative values of characteristic functions was considered. For each of four versions of generalized bargaining set necessary and sufficient conditions on $\mathscr{A}$ and $\mathscr{B}$ for existence of these bargaining sets at each game in the considered class were obtained.

Two main generalized kernels $\mathscr{K}_{\mathscr{A}\mathscr{B}}$ and $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}$ are considered. It will be shown in this paper that each of these kernels does not include the other. The first version $\mathscr{K}_{\mathscr{A}\mathscr{B}}$ was examined in [6–8, 10]. It is contained in the largest bargaining set $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}$ and its narrowing is contained in two generalized bargaining sets $\mathscr{M}^{ww}_{\mathscr{A}\mathscr{B}}$ and $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}$.

The second generalized kernel $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}$ is contained in the third generalized bargaining set $\mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}$ and in the largest bargaining set $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}$. Sufficient conditions and necessary conditions on $\mathscr{A}$ and $\mathscr{B}$ for existence of $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$ at each v are obtained in this paper.

Generalized nucleoli use in their definitions only elements of $\mathscr{B}$ that are suitable for objections or for counter-objections. Two versions of subsets of $\mathscr{B}$ generate two generalized nucleoli. The first generalized nucleolus $Nucl_{\mathscr{B}^0}(N, v)$ was examined in [10]. It is not single-point and its intersection with nonempty generalized kernel $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v)$ may be the empty set. This generalized nucleolus always intersects the generalized kernel $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v)$ only in trivial case (when "essential" elements of $\mathscr{A}$ are contained in a partition of the set of players).

A special class of collections of coalitions (weakly mixed collections of coalitions) were defined. It was proved that the "weakly mixed property" of $\mathscr{A}$ is sufficient and under addition weak condition is also necessary for inclusion of the first generalized nucleolus in the largest generalized bargaining set $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$ at each v.

In this paper we consider the second generalized nucleolus $Nucl_{\mathscr{C}(\mathscr{A})}(N, v)$. For weakly mixed collection of coalitions $\mathscr{A}$ this nucleolus is contained in the second generalized kernel $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$ at each v and it is contained in two versions of bargaining sets $\mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N, v)$ and $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$. If $\mathscr{A}$ does not contain singletons, then this nucleolus intersects the largest bargaining set $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$ at each v only if $\mathscr{A}$ is a weakly mixed collection of coalitions.

Thus, for weakly mixed collections of coalitions two generalized nucleoli are contained in the largest bargaining set.

For weakly mixed collection of coalitions $\mathscr{A}$ we describe an iterative procedure that converges to a point in the second generalized kernel $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}$. This procedure is a generalization of the procedure proposed by Stearns [15] and the proof of its convergence generalizes the proof of Maschler and Peleg [5].

The paper is organized as follows. Section 2 contains the main definitions concerning generalized bargaining sets and generalized kernels. Section 3 describes conditions on $\mathscr{A}$ and $\mathscr{B}$ for existence of generalized kernels. In Sect. 4 we obtain conditions that permit to use points in the second generalized nucleolus as selectors of the second kernel and of some generalized bargaining sets. In Sect. 5 for weakly mixed collections of coalitions we describe an iterative procedure that converges to a point in the second generalized kernel $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}$.

2 Definitions

For simplicity, this paper considers not all coalition structures but only the case generated by the grand coalition.

Let Γ^0 be the set of cooperative TU-games (N, v) such that $v(\{i\}) = 0$ for all $i \in N$ and $v(S) \geq 0$ for all $S \subset N$. (Such games are 0-normalizations of games (N, v) with $\sum_{i\in S} v(\{i\}) \leq v(S)$ for all $S \subset N$.) Let V_N^0 be the set of v such that $(N, v) \in \Gamma^0$.

Denote $x(S) = \sum_{i\in S} x_i$.

For $(N, v) \in \Gamma^0$, an *imputation* is a vector $x = \{x_i\}_{i\in N}$ such that $x(N) = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$.

Let $\mathscr{A}$ be a collection of subsets of N. A *union closure of* $\mathscr{A}$ is the collection of all unions of elements of $\mathscr{A}$.

Example 1. Let $N = \{1, 2, 3, 4\}$, $\mathscr{A} = \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$. If $\mathscr{B}$ is the union closure of $\mathscr{A}$, then $\mathscr{B} \supset \mathscr{A}$ and $\mathscr{B} \setminus \mathscr{A}$ consists of all three-person coalitions and N.

Let $(N, v) \in \Gamma^0$, $K, L \subset N$, x be an imputation of (N, v), $\mathscr{B}$ be a collection of nonempty subsets of N.

Consider two versions of generalized objections.

A *strong* $\mathscr{B}$*-objection of* K *against* L *at* x is a pair (C, y_C), such that $C \in \mathscr{B}$, $K \subset C$, $L \cap C = \emptyset$, $y_C = \{y_i\}_{i\in C}$, $y(C) = v(C)$, $y_i > x_i$ for all $i \in K$, and $y_i \geq x_i$ for all $i \in C$.

A coalition $C \in \mathscr{B}$ is *suitable for strong* $\mathscr{B}$*-objection of* K *against* L if $K \subset C$, $L \cap C = \emptyset$.

Let $K \cap L = \emptyset$ and $x(L) > 0$. A *weak* $\mathscr{B}$*-objection of* K *against* L *at* x is a pair (C, y_C), such that $C \in \mathscr{B}$, $K \subset C$, $L \not\subset C$, $y_C = \{y_i\}_{i\in C}$, $y(C) = v(C)$, $y_i > x_i$ for all $i \in K$, and $y_i \geq x_i$ for all $i \in C$.

A coalition $C \in \mathscr{B}$ is *suitable for weak* $\mathscr{B}$*-objection of* K *against* L if $K \subset C$, $L \not\subset C$.

Consider two versions of generalized counter-objections.

A *weak* $\mathscr{B}$*-counter-objection to strong or weak* $\mathscr{B}$*-objection* (C, y_C) *of* K *against* L *at* x is a pair (D, z_D) such that $D \in \mathscr{B}$, $L \subset D$, $K \not\subset D$, $z(D) = v(D)$, $z_i \geq x_i$ for all $i \in D$, $z_i \geq y_i$ for all $i \in C \cap D$.

A coalition $D \in \mathscr{B}$ is *suitable for weak* $\mathscr{B}$*-counter-objection to objection of* K *against* L if $L \subset D$, $K \not\subset D$.

A *strong* $\mathscr{B}$*-counter-objection to* $\mathscr{B}$*-objection* (C, y_C) of K against L at x is a pair (D, z_D) such that $D \in \mathscr{B}$, $L \subset D$, $K \cap D = \emptyset$, $z(D) = v(D)$, $z_i \geq x_i$ for all $i \in D$, $z_i \geq y_i$ for all $i \in C \cap D$.

A coalition $D \in \mathscr{B}$ is *suitable for strong* $\mathscr{B}$*-counter-objection to objection of* K *against* L if $L \subset D$, $K \cap D = \emptyset$.

Now we define four types of generalized bargaining sets.

Let $\mathscr{A}$, $\mathscr{B}$ be collections of subsets of N, $\mathscr{A} \subset \mathscr{B}$, and the case $\mathscr{A} = \mathscr{B}$ be possible.

An imputation x of (N, v) belongs to the *strong–weak bargaining set* $\mathscr{M}^{sw}_{\mathscr{A},\mathscr{B}}(N, v)$ if for all $K, L \in \mathscr{A}$, for each strong $\mathscr{B}$-objection of K against L at x there exists a weak $\mathscr{B}$-counter-objection.

In [8] the strong–weak bargaining set $\mathscr{M}^{sw}_{\mathscr{A},2^N}(N, v)$ is called the bargaining set $\mathscr{M}^i_{\mathscr{A}}(N, v)$.

An imputation x of (N, v) belongs to the *weak–weak bargaining set* $\mathscr{M}^{ww}_{\mathscr{A},\mathscr{B}}(N, v)$ if for all $K, L \in \mathscr{A}$ for each weak $\mathscr{B}$-objection of K against L at x there exists a weak $\mathscr{B}$-counter-objection.

An imputation x of (N, v) belongs to the *strong–strong bargaining set* $\mathscr{M}^{ss}_{\mathscr{A},\mathscr{B}}(N, v)$ if for all $K, L \in \mathscr{A}$, for each strong $\mathscr{B}$-objection of K against L at x there exists a strong $\mathscr{B}$-counter-objection.

In [8] the strong–strong bargaining set $\mathscr{M}^{ss}_{\mathscr{A},2^N}(N, v)$ is called the strong bargaining set $\bar{\mathscr{M}}^i_{\mathscr{A}}(N, v)$.

An imputation x of (N, v) belongs to the *weak–strong bargaining set* $\mathscr{M}^{ws}_{\mathscr{A},\mathscr{B}}(N, v)$ if for all $K, L \in \mathscr{A}$, for each weak $\mathscr{B}$-objection of K against L at x there exists a strong $\mathscr{B}$-counter-objection.

For each of these four bargaining sets, a permitted objection is *justified* if it has no permitted counter-objection.

Note that

$$\mathscr{M}^{ws}_{\mathscr{A}\mathscr{B}}(N, v) \subset \mathscr{M}^{ww}_{\mathscr{A}\mathscr{B}}(N, v) \subset \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$$

and

$$\mathscr{M}^{ws}_{\mathscr{A}\mathscr{B}}(N, v) \subset \mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N, v) \subset \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v).$$

If $\mathscr{A}$ is the set of all singletons and $\mathscr{B}$ is the set of all subsets of N, then

$$\mathscr{M}^{ws}_{\mathscr{A}\mathscr{B}}(N, v) = \mathscr{M}^{ww}_{\mathscr{A}\mathscr{B}}(N, v) = \mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N, v) = \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v) = \mathscr{M}^i_1(N, v).$$

If $\mathscr{A}$ is a partition of the set of players and $\mathscr{B}$ is the union closure of $\mathscr{A}$, then

$$\mathscr{M}^{ws}_{\mathscr{A}\mathscr{B}}(N,v) = \mathscr{M}^{ww}_{\mathscr{A}\mathscr{B}}(N,v) = \mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N,v) = \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N,v).$$

For families of coalitions $\mathscr{A}$, $\mathscr{B}$ with $\mathscr{A} \subset \mathscr{B}$, consider the following generalizations of the kernel.

Let $K, L \subset N$ and x be an imputation of (N, v). *K $\mathscr{B}$-outweighs L at x* if $K \cap L = \emptyset$, $x(L) > v(L)$, and $s^{\mathscr{B}}_{K,L}(x) > s^{\mathscr{B}}_{L,K}(x)$, where

$$s^{\mathscr{B}}_{P,Q}(x) = \max\{v(S) - x(S) : S \in \mathscr{B}, P \subset S, Q \not\subset S\}.$$

The set $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N,v)$ is the set of all imputations x of (N,v) such that no $K \in \mathscr{A}$ can $\mathscr{B}$-outweigh any $L \in \mathscr{A}$ at x.

In [8] the set $\mathscr{K}_{\mathscr{A}2^N}(N,v)$ is denoted by $\mathscr{K}_{\mathscr{A}}(N,v)$.

Another generalizations of the kernel are possible.

Let $K, L \subset N$ and x be an imputation of (N, v). *K $\mathscr{B}$-prevails L at x* if $K \cap L = \emptyset$, $x(L) > v(L)$, and $t^{\mathscr{B}}_{K,L}(x) > t^{\mathscr{B}}_{L,K}(x)$, where

$$t^{\mathscr{B}}_{P,Q}(x) = \max\{v(S) - x(S) : S \in \mathscr{B}, P \subset S, Q \cap S = \emptyset\}.$$

The set $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N,v)$ is the set of all imputations x of (N,v) such that no $K \in \mathscr{A}$ can $\mathscr{B}$-prevail any $L \in \mathscr{A}$ at x.

The next version of the generalized kernel will be used only in Sect. 5 for the proof of convergence of an iterative procedure.

Let $K, L \subset N$ and x be an imputation of (N, v). *K $\mathscr{B}$-weakly prevails L at x* if $K \cap L = \emptyset$, $x(L) > 0$, and $t^{\mathscr{B}}_{K,L}(x) > t^{\mathscr{B}}_{L,K}(x)$.

The set $\bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}(N,v)$ is the set of all imputations x of (N,v) such that no $K \in \mathscr{A}$ can $\mathscr{B}$-weakly prevail any $L \in \mathscr{A}$ at x.

$\bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}(N,v) \subset \bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N,v)$ because $v(L) \geq 0$ for all $L \in \mathscr{A}$.

If $\mathscr{A}$ is the set of all singletons and $\mathscr{B}$ is the set of all subsets of N, then $\mathscr{K}_{\mathscr{A}\mathscr{B}}$, $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}$, and $\bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}$ coincide with the kernel.

The following theorem is Theorem 2 in [10].

Theorem 1. *Let $\mathscr{A} \subset \mathscr{B}$, $(N,v) \in \Gamma^0$. Then*

$\mathscr{K}_{\mathscr{A}\mathscr{B}}(N,v) \subset \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N,v)$,
$\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N,v) \subset \mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N,v)$.

3 Existence Conditions for Generalized Kernels

First, we describe the result of the author in [8] that will be used later in this section.

Let $N = \{1, \ldots, n\}$, $X \subset R^n$, $\mathscr{A}$ be a collection of subsets of N, $\{\succ_x\}_{x \in X}$ be a collection of binary relations. Then $x^0 \in X$ is an *equilibrium vector on* $\mathscr{A}$ if $K \not\succ_{x^0} L$ for all $K, L \in \mathscr{A}$.

For $b > 0$, $K \in \mathscr{A}$ denote

$$X(b) = \{x \in R^n : x_i \geq 0, \ \ x(N) = b\},$$

$$F^K(b) = \{x \in X(b) : L \not\succ_x K \quad \text{for all} \quad L \in \mathscr{A}\}.$$

Then x is an equilibrium vector on $\mathscr{A}$ iff $x \in \bigcap_{K \in \mathscr{A}} F^K(b)$.

The following theorem is Theorem 1 in [8].

Theorem 2. *Let a family of binary relations* $\{\succ_x\}_{x \in X(b)}$ *on* $\mathscr{A}$ *satisfy the following conditions:*

1) for all $K \in \mathscr{A}$*, the set* $F^K(b)$ *is closed;*
2) if $x_i = 0$ *for all* $i \in K$*, then* $x \in F^K(b)$*;*
3) for each $x \in X(b)$*, the set of coalitions* $\{L \in \mathscr{A} : K \succ_x L \quad \text{for some} \quad K \in \mathscr{A}\}$ *does not cover* N*.*

Then there exists an equilibrium vector $x^0 \in X(b)$ *on* $\mathscr{A}$*.*

Now we describe a property of $\mathscr{A}$ that ensure existence of $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v)$.

A collection of coalitions $\mathscr{A}$ generates the undirected graph $G = G(\mathscr{A})$, where $\mathscr{A}$ is the set of nodes and $K, L \in \mathscr{A}$ are adjacent iff $K \cap L = \emptyset$.

Consider the following property of $\mathscr{A}$.

C0) *If a single node is taken out from each component of* $G(\mathscr{A})$*, then the union of the remaining elements of* $\mathscr{A}$ *does not contain* N*.*

Example 2. Trivial sufficient condition for C0.

Let $\mathscr{A}^0$ be obtained from $\mathscr{A}$ by deletion all isolated vertices of $G(\mathscr{A})$. Then either $\mathscr{A}^0$ does not cover N or $\mathscr{A}^0$ is a minimal covering of N.

Example 3. Let $n = 4$, $\mathscr{A}$ consist of no more than five two-person coalitions, then condition C0 is fulfilled.

The following theorem from [10] will be used later.

Theorem 3. *Let* $\mathscr{A}$ *be a set of subsets of* N*,* $\mathscr{B} \supset \mathscr{A}$*. If* $\mathscr{A}$ *satisfies Condition C0, then* $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ *for all* $v \in V_N^0$*.*

If $\{i\}, S \in \mathscr{A}$ *implies* $\{i\} \cup S \in \mathscr{B}$ *and* $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ *for all* $v \in V_N^0$*, then* $\mathscr{A}$ *satisfies Condition C0.*

Example 4. Let $N = \{1, 2, 3\}$, $\mathscr{A} = \{\{1\}, \{2\}, \{1, 3\}\}$, and $\mathscr{B}$ be a union closure of $\mathscr{A}$. Then $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ for all $v \in V_N^0$ and $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v) = \emptyset$ for some $v \in V_N^0$.

Proof. $\mathscr{A}$ does not satisfy Condition C0. Indeed, $\mathscr{A}$ is a unique component of the graph $G(\mathscr{A})$ and the sets from $\mathscr{A} \setminus \{1\}$ cover N. Hence, by Theorem 3, $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v) = \emptyset$ for some $v \in V_N^0$.

For each $v \in V_N^0$, we construct imputations in $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$. Consider two cases.

Case 1. $v(\{1,3\}) \geq v(N)$. Take $x = (0, 0, v(N))$. If $x \notin \bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$, then $\{2\}$ prevails $\{1, 3\}$ at x. But

$$t^{\mathscr{B}}_{\{1,3\},\{2\}}(x) = v(\{1,3\}) - v(N) \geq 0,$$
$$t^{\mathscr{B}}_{\{2\},\{1,3\}}(x) = 0 - x_2 = 0.$$

Case 2. $v(\{1,3\}) < v(N)$. Take $y \in R^3$ such that $y_1 = y_2 = (v(N) - v(\{1,3\}))/2$, $y_3 = v(\{1,3\})$. Then

$$t^{\mathscr{B}}_{\{1,3\},\{2\}}(y) = v(\{1,3\}) - y_1 - y_3 = -y_1,$$
$$t^{\mathscr{B}}_{\{2\},\{1,3\}}(y) = 0 - y_2 = -y_2 = -y_1,$$
$$t^{\mathscr{B}}_{\{2\},\{1\}}(y) = 0 - y_2,$$
$$t^{\mathscr{B}}_{\{1\},\{2\}}(y) = \max\{-y_1, v(\{1,3\}) - y_1 - y_3\} = -y_1 = -y_2.$$

Thus, $y \in \bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$. □

Now we describe conditions on $\mathscr{A}$ that ensure the nonemptyness of $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}$ for all games $(N, v) \in \Gamma^0$.

Definition 1. Let $\mathscr{A}$ be a collection of subsets of N, $\mathscr{B} \supset \mathscr{A}$. A directed graph Gr is called *$\mathscr{A}\mathscr{B}$-admissible* if $\mathscr{A}$ is the set of its vertices and there exists a map g defined on the set of the edges of Gr that takes each oriented edge (K, L) to a pair $g(K, L) = (Q, r)$ ($Q \in \mathscr{B}$, $r \in R^1$) and satisfies the following two conditions:

C1) If $g(K, L) = (Q, r)$, then $K \subset Q$, $Q \cap L = \emptyset$.
C2) If $g(K, L) = (Q, r)$, $g(R, P) = (S, t)$, $L \subset S$, $K \cap S = \emptyset$, then $r > t$.

Definition 2. Let $\mathscr{A}$ be a collection of subsets of N, $\mathscr{B} \supset \mathscr{A}$. A directed graph Gr is called *strongly $\mathscr{A}\mathscr{B}$-admissible* if $\mathscr{A}$ is the set of its vertices and there exists a map g defined on the set of the edges of Gr that takes each oriented edge (K, L) to a pair $g(K, L) = (Q, r)$ ($Q \in \mathscr{B}$, $r \in R^1$) and satisfies the following two conditions:

C3) If $g(K, L) = (Q, r)$, then $K \subset Q$, $Q \cap L = \emptyset$, $Q \notin \mathscr{A}$.
C2) If $g(K, L) = (Q, r)$ $g(R, P) = (S, t)$, $L \subset S$, $K \cap S = \emptyset$, then $r > t$.

Theorem 4. *Let $\mathscr{A}$ be a set of subsets of N, $\mathscr{B} \supset \mathscr{A}$.*

If for each $\mathscr{A}\mathscr{B}$-admissible graph Gr the set of the ends of its edges does not cover N, then $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma^0$.

If for some strongly $\mathscr{A}\mathscr{B}$-admissible graph Gr the set of the ends of its edges covers N, then $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v) = \emptyset$ for some $(N, v) \in \Gamma^0$.

Proof. Let for each $\mathscr{A}\mathscr{B}$-admissible graph Gr the set of the ends of its edges does not cover N. For each imputation x define the following binary relation $\succ_x$ on $\mathscr{A}$. $K \succ_x L$ iff K $\mathscr{B}$-prevails L at x. We check that this relation satisfies all conditions of Theorem 2. Condition (1) is realized because if L is not prevailed by any K, then either $x(L) \leq v(L)$ or $t^{\mathscr{B}}_{L,K}(N, v) \geq t^{\mathscr{B}}_{K,L}(N, v)$ for each $K \in \mathscr{A}$. Condition (2) follows from $v \in V^0_N$.

Let us check condition (3). For x, define $\mathscr{A}\mathscr{B}$-admissible graph as follows. (K, L) is the edge iff $K \succ_x L$ and $f(K, L) = (Q, t^{\mathscr{B}}_{K,L}(x))$, where $t^{\mathscr{B}}_{K,L}(x) = v(Q) - x(Q)$. Then the map f satisfies conditions C1, C2 from the definition of $\mathscr{A}\mathscr{B}$-admissible graph. By Theorem 2, $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$.

Now let there exist a strongly $\mathscr{A}\mathscr{B}$-admissible graph Gr such that the set of the ends of its edges covers N. We construct $v \in V^0_N$ such that $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v) = \emptyset$. Take $v(N) = 1$, $v(T) = 0$ if there is no edge (K, L) such that $f(K, L) = (T, r(T))$. For Q with minimal $r(Q)$, take $v(Q) = 1$. Suppose that $v(Q)$ is defined for all Q with $r(Q) < \bar{r}$ and define $v(T)$ for T with $r(T) = \bar{r}$ as follows. If $v(Q) \leq \alpha$ for all Q with $r(Q) < \bar{r}$, then take $v(T) = \alpha + 1$. By this way, all $v(T)$ are defined inductively.

Suppose that $x \in \bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$. We prove that for each edge (K, L) of Gr, $x(L) = 0$. Let $f(K, L) = (Q, r(Q))$, then, by C3, $v(L) = 0$. Suppose that $x(L) > 0$.

If $r(Q)$ is minimal, then, by C2, $v(P) = 0$ for all P such that $L \subset P$ and $K \cap P = \emptyset$, hence $t^{\mathscr{B}}_{L,K}(N, v) < 0$. Moreover, $t^{\mathscr{B}}_{K,L}(N, v) \geq v(Q) - x(Q) > 0$, hence $t^{\mathscr{B}}_{K,L}(N, v) > t^{\mathscr{B}}_{L,K}(N, v)$ and K prevails L at x.

Let $r(Q)$ be not minimal. If $P \supset L$, $P \cap K = \emptyset$, then $v(Q) \geq v(P) + 1$, hence

$$v(Q) - x(Q) \geq v(Q) - 1 \geq v(P) > v(P) - x(L) \geq v(P) - x(P),$$

and K prevails L at x.

Thus, the case, when (K, L) is an edge of Gr and $x(L) > 0$, is impossible. Since the ends of the edges of Gr cover N, we get $x(N) = 0$ and this contradicts $x(N) = 1$. □

Example 5. Let $\mathscr{A}_1 = \{K, L, M\}$, where $K \subset L$, $K \neq L$, $M \cap L = \emptyset$, $M \cup L = N$. Let $\mathscr{B}$ be the union closure of $\mathscr{A}$.

Let Gr_1 be a digraph, where $\mathscr{A}_1$ is the set of vertices and $\{(K, M), (M, L)\}$ is the set of edges. Then Gr_1 is $\mathscr{A}_1\mathscr{B}$-admissible. Indeed, we can take a map g, where $g(M, L) = (M, 1)$, $g(K, M) = (K, 2)$.

Theorem 4 does not work in this case. Indeed, the ends of the edges of Gr_1 cover the set of players N, hence the theorem does not give existence result.

If the ends of the edges of some digraph Gr_2 cover N, then Gr_2 contains the edge (M, L) and $g(M, L) = (M, r)$, but this contradicts C3, hence Gr_2 is not a strongly $\mathscr{A}_1\mathscr{B}$-admissible graph and the theorem does not give negative result.

Example 6. Let $N = \{a, b, c, d\}$, $\mathscr{A} = \{\{a\}, \{b\}, \{c\}, \{c, d\}\}$, $\mathscr{B}$ contain all unions of elements of $\mathscr{A}$. The following digraph Gr_3 is strongly $\mathscr{A}\mathscr{B}$-admissible. $\mathscr{A}$ is the set of vertices, $\{(\{c\}, \{b\}), (\{b\}, \{c, d\}), (\{c, d\}, \{a\})\}$ is the set of edges. Indeed, take

$f(\{c\}, \{b\}) = (\{a, c\}, 2)$,
$f(\{b\}, \{c, d\}) = (\{a, b\}, 1)$,
$f(\{c, d\}, \{a\}) = (\{b, c, d\}, 3)$. By Theorem 4, $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v) = \emptyset$ for some $(N, v) \in \Gamma^0$.

Now we demonstrate that $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}} \not\subset \mathscr{K}_{\mathscr{A}\mathscr{B}}$ and $\mathscr{K}_{\mathscr{A}\mathscr{B}} \not\subset \bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}$.

The game (N, v), where $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ and $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v) = \emptyset$ is possible, was presented in Example 3.

Example 7. The case, where $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N,v) \neq \emptyset$ and $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N,v) = \emptyset$, is possible when $\mathscr{B}$ is the union closure of $\mathscr{A}$.

Let $N = \{1,\ldots,14\}$, $\mathscr{A} = \{S_1,\ldots,S_8\}$, where

$$S_1 = \{1,2,3\}, S_2 = \{4,5,6\}, S_3\{1,7,8\}, S_4 = \{4,9,10\},$$
$$S_5 = \{9,11\}, S_6 = \{2,12\}, S_7 = \{5,13\}, S_8 = \{7,14\}.$$

Take the following digraph Gr, where $\mathscr{A}$ is the set of vertices, $\{(S_1,S_2),(S_2,S_3),(S_3,S_4),(S_4,S_1),(S_1,S_5),(S_2,S_6),(S_3,S_7),(S_4,S_8)\}$ is the set of edges, for an edge (K,L), $g(K,L) = (Q(K,L),1)$, where :

$$Q(S_1,S_2) = S_1 \cup S_5, Q(S_2,S_3) = S_2 \cup S_6,$$
$$Q(S_3,S_4) = S_3 \cup S_7, Q(S_4,S_1) = S_4 \cup S_8,$$
$$Q(S_1,S_5) = S_1 \cup S_2, Q(S_2,S_6) = S_2 \cup S_3,$$
$$Q(S_3,S_7) = S_3 \cup S_4, Q(S_4,S_8) = S_4 \cup S_1.$$

Note that $\mathscr{A}$ is a minimal covering of N, hence, by Theorem 3, $\mathscr{K}_{\mathscr{A}\mathscr{B}}(N,v) \neq \emptyset$ for each (N,v).

The ends of the edges of Gr cover N and Gr is a strongly $\mathscr{A}\mathscr{B}$-admissible graph. Indeed, for each edge (K,L), $Q(K,L) \cap L = \emptyset$, $K \subset Q(K,L)$, and there is no edge (P,T) such that $Q(P,T) \supset L$, $Q(P,T) \cap K = \emptyset$. By symmetry reasons, it is sufficient to check it for $(K,L) = (S_1,S_2)$ and for $(K,L) = (S_1,S_5)$.

Let $(K,L) = (S_1,S_2)$. If $R = Q(P,T) \supset S_2$, then $R \in \{S_1 \cup S_2, S_2 \cup S_3, S_2 \cup S_6\}$. $1 \in S_1 \cap S_3$, $2 \in S_1 \cap S_6$. Let $(K,L) = (S_1,S_5)$. If $R = Q(P,T) \supset S_5$, then $R = S_1 \cup S_5$. Thus, by Theorem 4, $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N,v) = \emptyset$ for some (N,v). □

4 Generalized Nucleoli and Generalized Bargaining Sets

Consider a selector problem for generalized kernels and bargaining sets. For the case of singletons, the nucleolus is a selector of the kernel and of the bargaining sets, therefore it is natural to search for selectors of generalized kernels and generalized bargaining sets in the class of generalized nucleoli.

Definition 3. Let $\mathscr{D}$ be a collection of subsets of N, $(N,v) \in \Gamma^0$. The $Nucl_{\mathscr{D}}(N,v)$ is the set of all imputations x of (N,v) such that

$$\theta^{\mathscr{D}}(N,v,x) \not>_{lex} \theta^{\mathscr{D}}(N,v,y)$$

for all imputations y of (N,v), where $\theta^{\mathscr{D}}(N,v,y) = \{v(S) - y(S)\}_{S\in\mathscr{D}}$ with decreasing coordinates.

If $\mathscr{D} = 2^N$, then $Nucl_{\mathscr{D}}(N,v)$ is the nucleolus (for the grand coalition) defined in [14]. In terms of Definition 5.1.4 in [13], $Nucl_{\mathscr{D}}(N,v)$ is the nucleolus of H with respect to the set of imputations of (N,v), where $H = \{v(S) - y(S)\}_{S\in\mathscr{D}}$.

For different generalized bargaining sets we consider $Nucl_{\mathscr{D}}(N,v)$ with different collection of coalitions $\mathscr{D}$.

The following fact will be used in the proofs.

For $z \in R^m$ denote $\theta(z) = \{z_i\}_{i=1}^m$ with decreasing coordinates.

Proposition 1. *Let $a, b \in R^m$, $a_i \geq b_i$ for all i, $a_{i_0} > b_{i_0}$ for some i_0, then $\theta(a) >_{lex} \theta(b)$.*

Proof. Let $\theta(b) = (b_{i_1}, b_{i_2}, \ldots, b_{i_m})$, then

$$\theta(a) \geq_{lex} (a_{i_1}, a_{i_2}, \ldots, a_{i_m}) >_{lex} \theta(b).$$

Since the binary relation $\geq_{lex}$ is a complete and transitive relation and $>_{lex}$ is its strict part, $\theta(a) >_{lex} \theta(b)$. □

The first generalized nucleolus was introduced in [10] as follows:

Definition 4. $S \in \mathscr{A}$ is *unessential for* $\mathscr{A}$ if $S \cap T \neq \emptyset$ for all $T \in \mathscr{A}$ and $S \supset T_1 \cup T_2$ for all $T_1, T_2 \in \mathscr{A}$ with $T_1 \cap T_2 = \emptyset$.

If S is unessential for $\mathscr{A}$, then there are no objections neither of S against any $L \in \mathscr{A}$ nor against S and unions with S are not suitable for objections and counter-objections.

Generalized nucleolus must use in its definition only elements of $\mathscr{B}$ that are suitable for objections or for counter-objections, therefore it is necessary to reject unessential coalitions.

Definition 5. Denote

$$\mathscr{A}^0 = \{S \in \mathscr{A} : S \quad \text{is not unessential for} \quad \mathscr{A}\}.$$

Let $\mathscr{B}^0$ be the union closure of $\mathscr{A}^0$.

Example 8. Let $N = \{1, 2, 3, 4\}$, $\mathscr{A} = \{\{1, 2\}, \{3, 4\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3\}\}$. Then $\mathscr{A}^0 = \mathscr{A}$.

Example 9. Let $N = \{1, 2, 3, 4, 5\}$, $\mathscr{A} = \{\{1, 2\}, \{3, 4\}, \{2, 3, 5\}, \{1, 2, 3, 4\}\}$. Then $\mathscr{A}^0 = \mathscr{A} \setminus \{1, 2, 3, 4\}$.

The generalized nucleolus $Nucl_{\mathscr{B}^0}(N, v)$ was examined in [10].

It was proved in [10, Theorems 8 and 9] that

$$Nucl_{\mathscr{B}^0}(N, v) \cap \mathscr{M}^{ww}_{\mathscr{A}^0 \mathscr{B}^0}(N, v) \neq \emptyset \quad \text{for all} \quad v \in V_N^0$$

iff $\mathscr{A}^0$ is contained in a partition of N and if $\mathscr{B}$ is a closure union of $\mathscr{A}$, then

$$Nucl_{\mathscr{B}^0}(N, v) \cap \mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset \quad \text{for all} \quad v \in V_N^0$$

iff $\mathscr{A}^0$ is contained in a partition of N. In this case $Nucl_{\mathscr{B}^0}(N, v) \subset \mathscr{K}_{\mathscr{A}\mathscr{B}}(N, v)$.

Thus, $Nucl_{\mathscr{B}^0}$ contains selectors of $\mathscr{K}_{\mathscr{A}\mathscr{B}}$ and of $\mathscr{M}^{ww}_{\mathscr{A}^0 \mathscr{B}^0}$ only in the trivial case.

Later we describe conditions on $\mathscr{A}$ that ensure intersection of $Nucl_{\mathscr{B}^0}(N, v)$ with the bargaining set $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$.

The second version of the generalized nucleolus will be introduced later.

Now we define a property of collections of coalitions, which will figure in further theorems.

For $i \in N$, denote $\mathscr{A}_i = \{T \in \mathscr{A} : i \in T\}$.

Definition 6. A collection of coalitions $\mathscr{A}$ is *weakly mixed at N* if for each $i \in N$, $Q \in \mathscr{A}_i$, $S \in \mathscr{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathscr{A}_j \supset \mathscr{A}_i \cup \{S\} \setminus \{Q\}$.

Weakly mixed collections of coalitions were introduced in [9] for another problem.

Lemma 1. *If $\mathscr{A}$ is a weakly mixed at N collection of coalitions, then $\mathscr{A} = \cup_{i=1}^{k} \mathscr{B}^i$, where*

C4) each $\mathscr{B}^i$ is contained in a partition of N;
C5) $Q \in \mathscr{B}^i$, $S \in \mathscr{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$.

Proof. Let $\mathscr{A}$ be a weakly mixed at N collection of coalitions. Let $\mathscr{B}^i$ be components of the undirected graph $G = G(\mathscr{A})$, where $\mathscr{A}$ is the set of nodes and $K, L \in \mathscr{A}$ are adjacent iff $K \cap L = \emptyset$. Then Condition C5 is fulfilled.

Let us check C4. First we prove that the case, when there exist $P, S, Q \in \mathscr{A}$ such that $P \cap S = \emptyset$, $S \cap Q = \emptyset$, $P \cap Q \neq \emptyset$, is impossible. Take $i_0 \in P \cap Q$ then there exists j such that $\mathscr{A}_j \supset \mathscr{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Then $j \in P \cap S$, but $P \cap S = \emptyset$.

Assume that C4 is not fulfilled, i.e., there exist $\mathscr{B}^i$ and $K, L \in \mathscr{B}^i$ such that $K \cap L \neq \emptyset$. Then there exist $K_0, K_1, \ldots, K_m$ such that $m \geq 2$, $K_0 = K$, $K_m = L$, $K_j \cap K_{j+1} = \emptyset$ for $j = 0, \ldots, m-1$. Let $j_0 = \min\{j : K_j \cap L = \emptyset\}$. Then $0 < j_0 < m$, but for $P = K_{j_0-1}$, $S = K_{j_0}$, $Q = L$, so we get the impossible case as was proved above. □

Remark 1. If $\mathscr{A}$ satisfies C4 and C5 for $k \leq 2$, then $\mathscr{A}$ is weakly mixed at N. Indeed, if $|\mathscr{A}_i| = 1$, then $\mathscr{A}_j \supset \mathscr{A}_i \cup \{S\} \setminus \{Q\}$ for each $j \in S$, and if $\mathscr{A}_i = \{Q, P\}$, then, by condition C5, $S \cap P \neq \emptyset$, hence $\mathscr{A}_j \supset \mathscr{A}_i \cup \{S\} \setminus \{Q\}$ for each $j \in S \cap P$.

Example 10. Let $N = \{1, 2, \ldots, 5\}$, $\mathscr{C} = \mathscr{B}^1 \cup \mathscr{B}^2$, where

$\mathscr{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathscr{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,

then $\mathscr{C}$ is weakly mixed at N.

It was proved in [9] that each weakly mixed collection of coalitions satisfies Condition C0. The next example demonstrates that the weakly mixed property does not follow from Condition C0.

Example 11. Let $N = \{1, 2, \ldots, 6\}$, $\mathscr{C} = \mathscr{B}^1 \cup \mathscr{B}^2 \cup \mathscr{B}^3$, where

$\mathscr{B}^1 = \{\{1, 2\}, \{3, 4\}\}$,
$\mathscr{B}^2 = \{\{1, 3\}, \{2, 4\}\}$,
$\mathscr{B}^3 = \{\{1, 4, 5\}, \{2, 3, 6\}\}$,

then $\mathscr{C}$ satisfies C0, C4, and C5, but is not weakly mixed. (Take $i = 1$ and $Q = \{1, 2\}$).

If $\mathscr{A}$ is weakly mixed at N, then $Nucl_{\mathscr{B}^0}$ is a selector of the largest bargaining set $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}$. It was proved in [10, Theorems 10 and 11] that if $\mathscr{A}$ is a weakly mixed at N collection of coalitions, $\mathscr{B}$ is the union closure of $\mathscr{A}$, then $Nucl_{\mathscr{B}^0}(N, v) \subset \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$ for all $v \in V^0_N$.

Moreover, if $\mathscr{A}$ does not contain singletons, does not contain unessential coalitions, and $Nucl_{\mathscr{B}^0}(N, v) \cap \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ for all $v \in V^0_N$, then $\mathscr{A}$ is a weakly mixed at N collection of coalitions.

Now we define the second generalized nucleolus. We suppose that $\mathscr{B}$ is the union closure of $\mathscr{A}$.

Definition 7. Denote by $\mathscr{C}(\mathscr{A})$ the collection of $P \in \mathscr{B}$ such that for some $S, T \in \mathscr{A}$, $S \subset P$ and $P \cap T = \emptyset$.

Thus, $\mathscr{C}(\mathscr{A})$ is the collection of coalitions that can be used for strong objections and strong counter-objections.

Note that $\mathscr{C}(\mathscr{A}) \neq \mathscr{B}^0$ for some $\mathscr{A}$.

Example 12. Let $\mathscr{A} = \{1,2\}, \{3,4\}, \{1,3\}, \{2,4\}\}$. Then $\mathscr{B}^0$ is the union closure of $\mathscr{A}$ and $\mathscr{C}(\mathscr{A}) = \mathscr{A}$.

We consider the following problems. When imputations in $Nucl_{\mathscr{C}(\mathscr{A})}(N, v)$ can be used as selectors of $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$, of $\mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N, v)$, and of $\mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$?

The following theorem demonstrates that if $\mathscr{A}$ is a weakly mixed at N collection of coalitions, then each imputation in the generalized nucleolus $Nucl_{\mathscr{C}(\mathscr{A})}$ is a selector of the generalized kernel $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}$.

Theorem 5. *Let $\mathscr{A}$ be a weakly mixed at N collection of coalitions, $\mathscr{B}$ be the union closure of $\mathscr{A}$. Then $Nucl_{\mathscr{C}(\mathscr{A})}(N, v) \subset \bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$ for all $v \in V^0_N$.*

Proof. Let $\mathscr{C} = \mathscr{C}(\mathscr{A})$. Suppose that for some v there exists $x \in Nucl_{\mathscr{C}}(N, v)$ such that $x \notin \bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$. Then there exist $S, Q \in \mathscr{A}$ such that $S \cap Q = \emptyset$ and S prevails Q at x. Let $t^{\mathscr{C}}_{S,Q}(x) = v(T) - x(T)$. Then $T \cap Q = \emptyset$ and $x(Q) > v(Q)$. Take $i_0 \in Q$ such that $x_{i_0} > 0$. Since $\mathscr{A}$ is weakly mixed, there exists $j_0 \in N$ such that $\mathscr{A}_{j_0} \supset \mathscr{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Let $0 < \delta \leq x_{i_0}$, then take $y^\delta \in R^n$, where

$$y^\delta_i = \begin{cases} x_i - \delta & \text{for} \quad i = i_0, \\ x_i + \delta & \text{for} \quad i = j_0, \\ x_i & \text{otherwise.} \end{cases}$$

Consider $P \in \mathscr{C}$ such that $y(P) < x(P)$. Then $i_0 \in P$ and $j_0 \notin P$. Since $j_0 \in S$, $P \not\supset S$.

There exists $P^0 \in \mathscr{A}$ such that $P^0 \subset P$, $i_0 \in P^0$, and $j_0 \notin P^0$. By the definition of j_0, only the case $P^0 = Q$ is possible, hence $P \supset Q$. Since S prevails Q at x and $t^{\mathscr{C}}_{K,L}(x) = t^{\mathscr{B}}_{K,L}(x)$, we have

$$v(T) - x(T) = t^{\mathscr{C}}_{S,Q}(x) = t^{\mathscr{B}}_{S,Q}(x) > t^{\mathscr{B}}_{Q,S}(x) = t^{\mathscr{C}}_{Q,S}(x) \geq v(P) - x(P),$$

hence $v(T) - x(T) > v(P) - x(P)$.

If δ is small enough, $v(T) - y(T) > v(P) - y(P)$ for all $P \in \mathscr{C}$ such that $v(P) - y(P) > v(P) - x(P)$. Since $v(T) - y(T) < v(T) - x(T)$, it follows from Proposition 1 that $\theta^{\mathscr{C}}(N, v, x) >_{lex} \theta^{\mathscr{C}}(N, v, y)$ and $x \notin Nucl_{\mathscr{C}}(N, v)$. This contradiction completes the proof. □

Corollary 1. *Let $\mathscr{A}$ be a weakly mixed at N collection of coalitions, $\mathscr{B}$ consist of all unions of elements of $\mathscr{A}$. Then $Nucl_{\mathscr{C}}(N, v) \subset \mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N, v)$ for all $v \in V^0_N$.*

Corollary 2. *Let $\mathscr{A}$ be a weakly mixed at N collection of coalitions, $\mathscr{B}$ consist of all unions of elements of $\mathscr{A}$. Then $Nucl_{\mathscr{C}}(N, v) \subset \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$ for all $v \in V^0_N$.*

Theorem 6. *Let $\mathscr{A}$ do not contain singletons and $\mathscr{B}$ be the union closure of $\mathscr{A}$. If $Nucl_{\mathscr{C}}(N, v) \cap \mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ for all $v \in V^0_N$, then $\mathscr{A}$ is a weakly mixed at N collection of coalitions.*

Let, moreover, $\mathscr{A}$ do not contain K such that $K \cap T \neq \emptyset$ for all $T \in \mathscr{A}$. If $Nucl_{\mathscr{C}}(N, v) \cap \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ for all $v \in V^0_N$, then $\mathscr{A}$ is a weakly mixed at N collection of coalitions.

Proof. Step 1. Let $\mathscr{A}$ do not contain singletons and $Nucl_{\mathscr{C}}(N, v) \cap \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v) \neq \emptyset$ for all $v \in V^0_N$.

We prove that if there exist $S, P, Q \in \mathscr{A}$ such that $P \neq Q$, $P \cap Q \neq \emptyset$, $P \cap S = \emptyset$, $Q \cap S = \emptyset$ then there exists $v \in V^0_N$ with $Nucl_{\mathscr{C}}(N, v) \cap \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v) = \emptyset$. We can suppose that $P \not\supset Q$.

Since $\mathscr{A}$ does not contain singletons, we can take the following $v \in V^0_N$:

$$v(T) = \begin{cases} 1 & \text{for} \quad T = N, S, P, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in Nucl_{\mathscr{C}}(N, v) \cap \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, v)$. Assume that $x(Q) > 0$, then $x(S) < 1$ and there exists a strong objection of S against Q at x. Since $P \not\supset Q$, such strong objection is a justified strong objection. Thus $x(Q) = 0$.

As $x \in Nucl_{\mathscr{C}}(N, v)$, $x(P) = x(S) = 1/2$. Hence, as $x(Q) = 0$, there exists $i_0 \in P \setminus Q$ such that $x_{i_0} > 0$. Fix $j_0 \in P \cap Q$. Let $0 < \delta < x_(i_0)/2$. Take the following $y \in R^n$.

$$y_i = \begin{cases} x_{i_0} - \delta & \text{for} \quad i = i_0, \\ x_i + \delta & \text{for} \quad i = j_0, \\ x_i & \text{otherwise.} \end{cases}$$

Consider 2 cases.

Case 1.1. $i_0 \notin T$ for all $T \in \mathscr{A} \setminus \{P\}$. Then $y(T) \geq x(T)$ for all $T \in \mathscr{B}$ and $y(Q) > x(Q)$, hence by Proposition 1, $x \notin Nucl_{\mathscr{C}}(N, v)$.

Case 1.2. $i_0 \in T$ for some $T \in \mathscr{A} \setminus \{P\}$. Then $T \neq S$, $v(T) = 0$, and

$$v(T) - y(T) \leq -x_{i_0} + \delta.$$

Similarly, if $R \in \mathscr{B}$ and $y(R) < x(R)$, then $R \neq S, P$, hence $v(R) = 0$ and

$$v(R) - x(R) \leq -x_{i_0} + \delta.$$

Since $x(Q) = 0$ and $v(Q) = 0$, $v(Q) - y(Q) = -\delta$. For such R, the condition $\delta < x_(i_0)/2$ implies $v(Q) - y(Q) > v(R) - y(R)$. Therefore, by Proposition 1,

$$\theta^{\mathscr{C}}(N, v, x) >_{lex} \theta^{\mathscr{C}}(N, v, y)$$

and $x \notin Nucl_{\mathscr{C}}(N, v)$.

Step 2. Now we prove that $\mathscr{A}$ is a weakly mixed at N collection of coalitions. Suppose that $\mathscr{A}$ does not satisfy this condition, i.e., there exist $i_0 \in \mathscr{A}$, $Q \in \mathscr{A}_{i_0}$, $S \in \mathscr{A}$ such that $S \cap Q = \emptyset$ and for each $j \in N$, $\mathscr{A}_j \not\supset \mathscr{A}_{i_0} \setminus \{Q\} \cup \{S\}$. Take the following $w \in V_N^0$:

$$w(T) = \begin{cases} 1 & \text{for} \quad T = N, S, \\ 2 & \text{for} \quad T \in \mathscr{A}_{i_0} \setminus \{Q\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in Nucl_{\mathscr{C}}(N, w) \cap \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, w)$. Suppose that $x(Q) > 0$, then $x(S) < 1$ and there exists a strong objection (S, y_S) of S against Q at x.

If $x \in \mathscr{M}^{ss}_{\mathscr{A}\mathscr{B}}(N, w)$, then there exists a strong counter-objection (D, z_D) to this objection, where $D \cap S = \emptyset$, $D \supset Q$, but it was proved at Step 1 that the existence of such D is impossible.

Let $x \in \mathscr{M}^{sw}_{\mathscr{A}\mathscr{B}}(N, w)$ and $\mathscr{A}$ do not contain K such that $K \cap T \neq \emptyset$ for all $T \in \mathscr{A}$. There exists a weak counter-objection (D, z_D) to this objection. Then $D \neq S$, $D \supset Q$, $w(D) \geq x(D) \geq x(Q) > 0$ hence $D \neq Q$ and $D \in \mathscr{A}_{i_0} \setminus \{Q\}$. There exists $D' \in \mathscr{A}$ such that $D \cap D' = \emptyset$. As $D \cap Q \neq \emptyset$, it follows by the proved at Step 1, that $D' \cap Q \neq \emptyset$, thus $D \not\supset Q$ and D is not suitable for weak counter-objection. This contradiction proves that $x(Q) = 0$.

There exists $j_0 \in N$ such that $x_{j_0} \geq 1/n$. Then $j_0 \notin Q$ and $j_0 \neq i_0$. Let $0 < \delta < 1/(2n)$. Take the following $x^\delta \in R^n$:

$$x_i^\delta = \begin{cases} x_{i_0} - \delta & \text{for} \quad i = j_0, \\ x_i + \delta & \text{for} \quad i = i_0, \\ x_i & \text{otherwise.} \end{cases}$$

Consider 2 cases.

Case 2.1. $j_0 \notin S$. Let $x^\delta(T) < x(T)$, then $j_0 \in T$, $i_0 \notin T$, hence $T \neq S$ and $v(T) = 0$ in this case. Then $v(T) - x(T) \leq -x_{j_0}$, $v(T) - x^\delta(T) \leq -x_{j_0} + \delta$, $v(Q) - x(Q) = 0$, $v(Q) - x^\delta(Q) = -\delta$. Since $\delta < 1/(2n)$, we have $v(Q) - x^\delta(Q) > v(T) - x^\delta(T)$ and, by Proposition 1, $x \notin Nucl_{\mathscr{C}}(N, v)$.

Case 2.2. $j_0 \in S$. Then due to our supposition, there exists $P \in \mathscr{A}_{i_0} \setminus \mathscr{A}_{j_0} \setminus \{Q\}$, hence $v(P) = 2$ and

$$v(P) - x(P) > v(P) - x^{\delta}(P) \geq 1.$$

If $v(T) - x(T) < v(T) - x^{\delta}(T)$, then $j_0 \in T$, $i_0 \notin T$ and either $T = S$ or $v(T) = 0$. For $T = S$, $v(T) - x^{\delta}(T) \leq 1 - x_{j_0} + \delta < 1$. For $v(T) = 0$, $v(T) - x^{\delta}(T) < 0$. Thus, $v(P) - x^{\delta}(P) > v(T) - x^{\delta}(T)$ and, by Proposition 1, $x \notin Nucl_{\mathscr{C}}(N, v)$. This contradiction completes the proof. □

5 Dynamic System for $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$

For weakly mixed collection of coalitions we describe a dynamic system that converges to a point in the generalized kernel $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$. This dynamic system is a generalization of the dynamic system that was introduced by Stearns [15]. Definitions and notations are due to Peleg and Sudholter [13].

Let X be a metric space and $d : X \times X \to R$ be a metric for X. A *dynamic system on* X is a set-valued function $\varphi : X \to 2^X$.

A *φ-sequence from* $x_0 \in X$ is a sequence $\{x_t\}_{t=0}^{\infty}$ such that $x_{t+1} \in \varphi(x_t)$ for all $t = 0, 1, \ldots$. A point $x \in X$ is called an *endpoint of* φ if $\varphi(x) = \{x\}$.

A set-valued function $\varphi : X \to X$ is *lower hemicontinuous at* $x \in X$ if for every open set $U \subset X$ such that $\varphi(x) \cap U \neq \emptyset$, there exists an open set $V \subset X$ such that $x \in V$ and $\varphi(z) \cap U \neq \emptyset$ for every $z \in V$.

φ is *lower hemicontinuous*, if it is lower hemicontinuous at each $x \in X$.

A *valuation for* φ is a continuous function $\Psi : X \to R$ such that

$$y \in \varphi(x) \Longrightarrow \Psi(x) - \Psi(y) \geq d(x, y) \text{ for all } x, y \in X.$$

Define the function $\rho_{\varphi} : X \to R \cup \{\infty\}$ by

$$\rho_{\varphi}(x) = \sup\{d(x, y) : y \in \varphi(x)\}.$$

A φ-sequence $\{x_t\}_{t=0}^{\infty}$ is *maximal* if there exists $\alpha > 0$ and a subsequence $\{x_{t_j}\}_{j=0}^{\infty}$ such that

$$d(x_{t_j}, x_{t_{j+1}}) \geq \alpha \rho(x_{t_j}) \text{ for all } j.$$

The proof of convergence of φ-sequence that will be defined in this paper is based on the following result:

Corollary 10.1.9 in [13]. *Let X be a compact metric space and φ be a lower hemicontinuous set-valued function. If φ has a valuation, then every maximal φ-sequence converges to an endpoint of φ.*

Let $\mathscr{A}$ be a weakly mixed family of coalitions. We define a dynamic system for the generalized kernel $\bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}(N,v) \subset \bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N,v)$ by the following rule.

Let x be an imputation of (N,v). Then $y \in \varphi_{\bar{\mathscr{K}}}(x)$ if there exist $i,j \in N$, $\beta \geq 0$ such that

$y_i = x_i - \beta$, $y_j = x_j + \beta$, $y_k = x_k$ for all
$k \in N \setminus \{i,j\}$,
$i \in Q, j \in S, S, Q \in \mathscr{A}, S \cap Q = \emptyset, \mathscr{A}_j \supset \mathscr{A}_i \setminus Q$,
$\beta \leq \max\{0, \min\{x_i, \frac{1}{2}(t^{\mathscr{B}}_{S,Q}(x) - t^{\mathscr{B}}_{Q,S}(x))\}\}$.

Lemma 2. *An imputation $x \in \bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}(N,v)$ if and only if $\varphi_{\bar{\mathscr{K}}}(x) = \{x\}$, i.e., x is an endpoint of $\varphi_{\bar{\mathscr{K}}}$.*

Proof. Let $x \in \bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}(N,v)$, then for all $Q,S \in \mathscr{A}$ with $Q \cap S = \emptyset$, $\min\{x(Q), \frac{1}{2}(t^{\mathscr{B}}_{S,Q}(x) - t^{\mathscr{B}}_{Q,S}(x))\} \leq 0$, hence $\beta = 0$ for all $i \in Q, j \in S$.

Let $x \notin \bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}(N,v)$, then there exist $Q,S \in \mathscr{A}$ with $Q \cap S = \emptyset$ such that $x(Q) > 0$ and $t^{\mathscr{B}}_{S,Q}(x) - t^{\mathscr{B}}_{Q,S}(x) > 0$. Then there exists $i \in Q$ such that $x_i > 0$, so $\max\{0, \min\{x_i, \frac{1}{2}(t^{\mathscr{B}}_{S,Q}(x) - t^{\mathscr{B}}_{Q,S}(x))\}\} > 0$ and $\beta > 0$ is possible. Since $\mathscr{A}$ is weakly mixed, there exists $j \in S$ such that $\mathscr{A}_j \supset \mathscr{A}_i \setminus Q$ and $\varphi_{\bar{\mathscr{K}}}(x) \neq \{x\}$. □

Let us construct a valuation for $\varphi_{\bar{\mathscr{K}}}$.

For each $k = 1, \ldots, m$, let $\mathscr{C}^k$ be the union closure of $\mathscr{B}^k$, $r(k) = |\mathscr{C}^k| = 2^{|\mathscr{B}^k|}$, for each imputation x of (N,v)

$$\theta^k(x) = (\theta^k_1(x), \ldots, \theta^k_{r(k)}(x))$$

be the vector of excesses $e(T,x,v) = v(T) - x(T)$, $T \in \mathscr{C}^k$ arranged in a non-increasing order,

$$\Psi^k(x) = \sum_{l=1}^{r(k)} 2^{r(k)-l} \theta^k_l(x).$$

Define $\Psi : X \to R^1$ by

$$\Psi(x) = \sum_{k=1}^{m} \Psi^k(x).$$

Lemma 3. *Let x be an imputation of (N,v), $y \in \varphi_{\bar{\mathscr{K}}}(x)$ then*

$$\Psi(x) - \Psi(y) \geq \|x - y\|,$$

where $\|x-y\| = \max_{i \in N} |x_i - y_i|$.

Proof. Let $y \in \varphi_{\bar{\mathscr{K}}}(x)$ $y_i = x_i - \beta$ $y_j = x_j + \beta$ $y_k = x_k$ for all $k \in N \setminus \{i,j\}$ $i \in Q \in \mathscr{B}^{k_0}$ $j \in S \in \mathscr{B}^{k_0}$. Then $\|x - y\| = \beta$.

Let $\theta_l^k(y) = e(T_l^k, y, v)$ for $l = 1, \ldots, r(k)$. Then

$$\Psi^k(x) \geq \sum_{l=1}^{r(k)} 2^{r(k)-l} e(T_l^k, x, v).$$

Consider two cases.

Case 1. $k \neq k_0$. Then $y(P) \geq x(P)$ for $P \in \mathscr{B}^k$ because $\mathscr{A}_j \supset \mathscr{A}_i \setminus Q$, hence $y(T) \geq x(T)$ for $T \in \mathscr{C}^k$. Then

$$\sum_{l=1}^{r(k)} 2^{r(k)-l} e(T_l^k, x, v) \geq \sum_{l=1}^{r(k)} 2^{r(k)-l} \theta_l^k(y),$$

hence $\Psi^k(x) \geq \Psi^k(y)$.
Case 2. $k = k_0$. If $t_{S,Q}(x) = e(T, x, v)$, then $T \in \mathscr{C}^{k_0}$, $T = T_p^{k_0}$ and without loss of generality we may assume that $\theta_p^{k_0}(y) < \theta_{p-1}^{k_0}(y)$.

By the definition of β, $t_{S,Q}(y) \geq t_{Q,S}(y)$, therefore, for $l < p$, $S \subset T_l^{k_0}$ iff $Q \subset T_l^{k_0}$, so $e(T_l^{k_0}, x, v) = e(T_l^{k_0}, y, v)$.
If $l = p$, then $e(T_l^{k_0}, x, v) - e(T_l^{k_0}, y, v) = \beta$.
If $l > p$, then $|e(T_l^{k_0}, x, v) - e(T_l^{k_0}, y, v)| \leq \beta$. Hence

$$\sum_{l=1}^{r(k_0)} 2^{r(k_0)-l} e(T_l^{k_0}, x, v) - \sum_{l=1}^{r(k_0)} 2^{r(k_0)-l} \theta_l^{k_0}(y) =$$

$$2^{r(k_0)-p}(e(T_p^{k_0}, x, v) - e(T_p^{k_0}, y, v)) + \sum_{l=P+1}^{r(k_0)} 2^{r(k_0)-l} e(T_l^{k_0}, x, v) - \sum_{l=1}^{r(k_0)} 2^{r(k_0)-l} \theta_l^{k_0}(y) \geq$$

$$\beta 2^{r(k_0)-p} - \beta \sum_{l=P+1}^{r(k_0)} 2^{r(k_0)-l} = \beta = \|x - y\|.$$

□

Theorem 7. *Let $\mathscr{A}$ be a weakly mixed collection of coalitions. Then for each (N, v) each maximal $\varphi_{\bar{\mathscr{K}}}$-sequence converges to a point of $\bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}(N, v)$.*

Proof. $\varphi_{\bar{\mathscr{K}}}$ is a lower hemicontinuous set-valued function because the functions $t^{\mathscr{B}}_{S,Q}(x)$ are continuous. The set of imputations is a compact metric space. By Lemma 3, $\varphi_{\bar{\mathscr{K}}}$ has a valuation. By Corollary 10.1.9 in [13], every maximal $\varphi_{\bar{\mathscr{K}}}$-sequence converges to an endpoint of $\varphi_{\bar{\mathscr{K}}}$. By Lemma 2, each endpoint of $\varphi_{\bar{\mathscr{K}}}$ belongs to $\bar{\mathscr{K}}^0_{\mathscr{A}\mathscr{B}}(N, v)$. □

Corollary 3. *Let $\mathscr{A}$ be a weakly mixed collection of coalitions. Then for each (N, v) each maximal $\varphi_{\bar{\mathscr{K}}}$-sequence converges to a point of $\bar{\mathscr{K}}_{\mathscr{A}\mathscr{B}}(N, v)$.*

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Occurrence of Deception Under the Oversight of a Regulator Having Reputation Concerns

Ayça Özdoğan

Abstract This paper studies deceptions conducted by agents in the presence of a regulator. The regulator is supposed to detect deviations from the "rightful" behavior through costly monitoring; thus she may not choose to be diligent in her job because of the associated costs. The goal is to understand the occurrence of deceptions when the interaction of the parties is not contractible, their behavior is not observable and the regulator has reputation concern for being perceived as diligent in a repeated incomplete-information setting. It is found that when the regulator faces a sequence of myopic agents, her payoff at *any* Nash equilibrium converges to the maximum payoff as the discount factor approaches to one for any prior belief on the regulator's type. This suggests that, contrary to the well-known disappearance of reputation results in the literature, the reputation of the regulator for being diligent persists in the long-run in any equilibrium. These findings imply that socially undesirable behavior of the agents could be prevented through reputation concerns in this repeated setting.

Keywords Regulation • Reputation • Repeated games • Inspection games • Short-lived agents

1 Introduction

This paper studies occurrence of repeated deceptions (e.g., misrepresentation of information) conducted by agents in regulated environments. In every period, the regulator is supposed to detect deviations from the "rightful" behavior through costly auditing and he may not be diligent in doing so because of the associated direct or opportunity costs. Our aim is to understand whether deceptions and socially undesirable behavior could be prevented when the regulator has reputation concerns for being diligent in settings where the behavior of the interacting parties is neither

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observable nor can be contracted upon. This situation is modeled as a dynamic game that is played by a regulator and different agents in each period with a particular payoff and signalling structure (to be discussed shortly). Employing the tools of repeated games and reputations, we find that when a patient long-lived regulator faces a sequence of myopic agents (short-lived players who play only once), reputation concerns help the regulator to prevent the undesirable behavior of the agents by inducing them to act rightfully and provide him the *maximum* payoff in *any* Nash equilibrium. This finding suggests that the reputation of the regulator for being diligent persists, i.e., it is sustainable in the long-run, which departs from the well-known disappearance of reputation results in the literature for games with the same payoff but different signalling structure.[1] Thus, the current paper contributes to the theory of reputations with its powerful findings regarding the attainable payoff being the maximum and the persistency of reputation in *any* Nash equilibrium.

Deception by an agent in the presence of a regulator who spends costly time and resources for auditing the agent is frequently observed. An investor may be engaged in fraud by misrepresenting its books to a regulatory agency; a bank may misreport the information about its financial health to a financial regulator; a tax payer may fill out false income statements to a tax authority; an athlete may be inclined to sports artifice through performance enhancing drugs; a regulated construction or mine company may neglect work safety precautions; an employee may not exert proper effort in a business owned by a principal overseas etc. The instances of financial frauds, corporate deceptions, and work accidents accompanied with lack of diligence in regulation[2] indicate that one crucial factor in the occurrence of these kinds of misbehavior may be related to regulator's reputation for being diligent or

[1]The pioneer work that shows the disappearance of noncredible reputations in a class of games that encompasses the same payoff structure of the current paper with different signalling technology is [6] (see Sect. 1.1 for further discussion).

[2]For instance, Bernard Madoff, a prominent investment manager, was arrested and found guilty to several offenses including securities fraud and making false statements with the Securities and Exchange Commission (SEC). He began the Ponzi scheme in the early 1990s, yet he was arrested in late 2008 even though the SEC had previously conducted several investigations since 1992. SEC has been criticized for failure to act on Madoff fraud. The SEC inspector confessed that "Despite several examinations and investigations being conducted, a through and competent investigation or examination was never performed" (see "SEC criticized for failure to act on Madoff" at http://business.timesonline.co.uk by Seib and "Madoff Explains How He Concealed the Fraud" at www.cbsnews.com). Another investment fraud charge was against Robert Allen Stanford in 2009. A report of investigation by the SEC Office of the Inspector General shows that the agency has been following Stanford's companies for much longer and reveals lack of diligence in the SEC enforcement (see the Report of Investigation at http://www.sec.gov/news/studies/2010/oig-526.pdf). The negligence of regulation may also have fatal consequences. For instance, there was a mine accident that took place in Soma, Turkey, which caused loss of 301 lives on May 13, 2014. In the response to a parliamentary question in the aftermath of the accident, it is understood that The General Directorate of Mining Affairs of Turkey—GDMA (that is connected to Ministry of Energy and Natural Resources), who is in charge of reviewing the conditions of mine fields, could only afford to audit less than one fourth all the mine fields annually. Yet, GDMA claimed that this particular mine had been reviewed many times. Although minor fees had been charged for infringement of some rules, an extensive audit of the field and mandatory safety measures had never been done and fatal mistakes went unnoticed according to the reports.

lack of it; and thus, reputation concerns of regulators may prevent or lessen the extent and severity of such bad outcomes. The goal of this paper is to analyze the effect of regulators' reputation concerns and sustainability of their reputations on preventing socially undesirable behavior. In settings where deceptions occur, the literature mostly investigates if the reputation concerns of the agents for having good behavior (i.e., intrinsic value for honesty) reduce the impact of opportunism that could be engaged by the agents. Here, we investigate whether the reputation concerns of the regulators alleviate the opportunistic behavior of the agents and try to understand if it can be used as a tool to achieve socially desirable behavior.

We propose a tractable model through a version of a simultaneous-move inspection game with unobservable actions and incomplete-information about the types of the regulator. The stage game can be described as: a privately informed agent wishes to misrepresent her/his information by sending false messages and a regulator aims to deter the deception via costly auditing. The auditing strategy of the regulator determines the probability of detecting the deception (i.e., the public signal) done by the agent. The regulator can be diligent or lazy in auditing the agent. If the agent is truthful, there won't be any detections regardless of the auditing strategy chosen by the regulator. There is a chance of finding out untruthfulness, i.e., the probability of detection is positive only when the regulator is diligent in auditing.[3] The payoff structure is constructed so that the best response of the agent when the regulator is lazy is to be truthful, whereas it is to be untruthful if the regulator is believed to be lazy. The best response of the regulator when the agent is truthful is to be lazy while it is to be diligent if the agent is expected to be untruthful. The reputation is modeled by introducing a behavioral type for the regulator.[4] The agent believes that the regulator could be a tough type who is always diligent in auditing the agent with a positive probability. The strategic type of the regulator establishes a false reputation by mimicking the tough counterpart. The stage game equilibrium analysis exhibits that if the belief that regulator is tough is above a threshold, the agent is truthful for sure; and, anticipating this, the regulator chooses to not to be diligent; whereas otherwise the parties choose mixed strategies.

We find that when a long-lived regulator faces a sequence of myopic agents who play only once but observe the history of previous public signals (whether there has been a detection or not in the previous play), any Nash equilibrium payoff of the strategic regulator converges to the maximum attainable payoff as the discount factor of the regulator approaches to one given any prior belief the agents may have about the regulator's type. So, the long-lived regulator guarantees a payoff that is more than her Stackelberg payoff in *any* Nash equilibrium, which was usually shown to be the amount of payoff a long-lived player can guarantee herself in

[3] An alternative formulation would be to have an agent who has the option to choose a proper action that generates a good signal from a set of actions and a regulator who can monitor the agent to check if she has chosen the proper action or not. There will be a bad signal only when the agent has chosen an improper action *and* the regulator monitors the agent. If the agent has chosen the proper action, the public signal generated is going to be always good regardless of the monitoring strategy chosen by the regulator.

[4] For an extensive overview of the reputation literature, we refer to [18].

earlier studies (see Sect. 1.1). This implies that the reputation to be perceived as diligent is sustainable in the long-run, contrary to the well-known disappearance of reputations result in the literature. This is a "very good" reputation result for the regulator that cannot be attained in the complete-information setting and implies that socially undesirable behavior of the agents can be prevented in the long-run thanks to reputation concerns of the regulator in the incomplete-information setting.[5]

These findings differ from the disappearance of reputations result in games that contain similar payoff structures with imperfect public monitoring which is established by Cripps et al. [6], who also showed that any Nash equilibrium of the incomplete-information game approaches to the equilibrium of the complete-information game (hence maximum payoff of the long-lived player is not necessarily attainable). The seemingly contrary results are due to the differences in the public monitoring technology. In [6], there is imperfect public monitoring with the properties of full-support and full-rank (identification). The public signals generated in our setting that helps to infer information about the regulator's behavior (if he is diligent in auditing or not) can only be observed when the agent is untruthful. Hence, our public monitoring violates both of the properties displayed in [6].

This particular signalling structure is also the key reason for the potential differences between the short-lived vs. long-lived agent cases (see footnote 5). Short-lived agents do not have an incentive to unravel the regulator's type as they only care about their own one-period payoffs. Thus, they give myopic best responses to their beliefs about the regulator's type updated based upon the public history of the play. If all the short-lived agents are truthful, no signal will be generated about the regulator's true type. However, a long-lived agent faces a tradeoff between short-run gains and expected continuation payoffs by being untruthful (with an incentive not only to have short-run payoff gains but also to unravel the regulator's type) and untruthful (as she fears from getting detected). This is the main motive why the regulator's reputation persists when she faces a sequence of short-lived agents; while it can disappear in the long-run when the agent is also long-lived.

1.1 Related Literature

There is a well-established literature on various variations of repeated games with incomplete-information and the theory of reputations. However, those studies do not tackle with the specific question and setting that we capture in the current one. This paper contributes to the literature on reputations suited for a particular environment, public signal, and payoff structure. Early literature studying the value

[5]These findings are likely to change when the regulator faces a long-lived agent having future objectives. Some other structures (such as stochastic replacement of the regulator or reform in the regulation system) would be needed to obtain recurrent reputation then. Cripps et al. [6] suggest that to obtain non-transient reputations other mechanisms should be incorporated into the model. One string of literature attains recurrent reputations by assuming that the type of the player is governed by a stochastic process through time. The reader is referred to [8, 12, 15, 17, 21, 23].

of reputations focuses on settings where a long-lived player faces a sequence of short-lived players who play once but observe the previous play. These studies derive lower bounds on the equilibrium payoffs to the sufficiently patient long-lived player given that there is a positive probability of a commitment type who always plays the strategy to which strategic player would most like to commit herself, i.e., the Stackelberg strategy.[6] Such one-sided reputation results have also shown to arise in settings that involve two long-lived players.[7] But, [6] shows that a long-lived player can maintain a permanent reputation for playing a commitment strategy in a game with imperfect public monitoring only if that strategy is an equilibrium of the complete-information stage game. Thus, the powerful results about the lower bounds on the long-lived player's average payoff are short-run reputation effects, where the long-lived player's payoff is calculated at the beginning of the game.[8] Ozdogan [20] also shows that, in two-sided incomplete-information games, the true types of both players will be revealed eventually for a class of stage games with imperfect public monitoring. The stage game proposed in the current paper fits into this class; however, there is uncertainty over the types of one player and the public signal technologies are different.

The rest of the paper is organized as follows: Sect. 2 presents the model. The findings of the dynamic setting with short-lived agents are provided in Sect. 3. Finally, Sect. 4 includes the proofs.

2 Model

The stage game is a simultaneous-move game with an agent (*she*) and a regulator (*he*). At the beginning of the game, the agent gets perfect information about the state of nature, n, which can be two values: high (H) or low (L). Then, she reports a message $m \in M = \{h', l'\}$ to the regulator. The agent has the incentive to deceive the uninformed regulator by strategically manipulating information through reporting false messages. The agent can either be *truthful* or *untruthful* about the information she has, thus the action set of the agent can be denoted by $A = \{T, U\}$ where $a \in A$. Reporting strategy of the agent is given by $\sigma_A \in \Delta(A)$ (simplex of the set A). With an abuse of notation, we also denote the probability that she is truthful by σ_A. We

[6]See [10] for games with perfect monitoring, [11] for games with imperfect public monitoring, and [13] for games with imperfect private monitoring.

[7]These studies include: [22] (for games of conflicting interests with asymmetric discount factors); [4] and [1] (for games with imperfect monitoring and asymmetric discount factors); [7] (for games of strictly conflicting interests with equal discount factors); [3] and [2] (for games of locally nonconflicting or strictly conflicting interests with equal discount factors); [5] (for games with equal discount factors where the commitment action is a dominant action).

[8]The results of [11] and [6] differ because [11] fixes the prior belief of being the commitment type and selects a threshold discount factor depending on this prior above which the player is sufficiently patient for their results to hold; whereas [6] fixes the discount factor while allowing the posterior belief to vary which eventually becomes so low that makes the required threshold discount factor (for [11]'s result to hold) to exceed way above the fixed discount factor.

assume that the payoff she receives with any reporting strategy is independent of the actual state of nature. The regulator is supposed to detect the deviations from the truthful behavior via costly auditing. He chooses to be *diligent* or *lazy* in auditing the agent. His choice generates different probabilities of eliciting information about agent's untruthfulness. The regulator's action set denoted by $R = \{D, L\}$, and $r \in R$. Probability of detecting untruthfulness (if the agent has chosen to be indeed untruthful) is referred as "auditing quality" and denoted by $\beta \in (0, 1)$ when the regulator chooses to be diligent; and it is zero when the regulator chooses to be lazy. There is no detection if the agent is indeed truthful regardless of the auditing strategy chosen. The strategy of the regulator is denoted by $\sigma_R \in \Delta(R)$ also showing the probability of choosing diligent with an abuse of notation.

Following [14, 16, 19], the uncertainty about the regulator's preferences are modeled by incorporating a behavioral type into the game. The regulator can be one of two types: *tough* or *strategic*. The tough regulator is committed to be diligent, whereas the strategic one maximizes his payoffs that are going to be described below. The regulator knows his true type (that is strategic) whereas the belief of the agent that the regulator is tough (i.e., the reputation of the regulator) is given by $\gamma \in (0, 1)$. The regulator has the imperfect technology that elicits information about agent's untruthfulness. The agent cannot observe the action chosen by the regulator. However, she can infer the action chosen by the regulator from the frequency of the detections. But for that to happen, the agent has to be untruthful with some probability. Publicly observed outcomes at the end of the period are whether the agent is detected or not.

Agent's payoff is normalized to zero when she is truthful (no matter what auditing strategy the regulator chooses). When she is untruthful, she pays a fine of l if it is detected and receives a gain of g if her untruthfulness is not detected and she can get away with her deception. We assume the following on the payoff structure of the agent:

Assumption 1. *The parameter values satisfy* $\frac{g}{g+l} < \beta$.

This assumption ensures that if the agent believes that the strategic regulator is diligent for sure, then her best response is to be truthful with probability one. The agent's behavior depends on her belief about the probability of detection. Let $\pi(\gamma, \sigma_R)$ be the expected probability of detection, i.e., $\pi \equiv \pi(\gamma, \sigma_R) = \gamma\beta + (1 - \gamma)\sigma_R\beta$. Then, the agent's problem is to choose $\sigma_A \in [0, 1]$ to maximize

$$\sigma_A.0 + (1 - \sigma_A)[(1 - \pi)g - \pi l] \tag{1}$$

There is a cutoff value of detection π^* (which is equal to $\frac{g}{g+l}$) that determines the optimal behavior of the agent. The agent chooses to be untruthful for sure if $\pi(\gamma, \sigma_R) < \pi^*$ and to be truthful for sure if $\pi(\gamma, \sigma_R) \geq \pi^*$. It is assumed that she is truthful when she is indifferent.

The regulator's payoff is normalized to zero if the agent is truthful and he does not put effort in auditing the agent. This is the maximum payoff the regulator could expect. And, given that the agent has been untruthful, his gain is d if there has been a detection of untruthfulness and his expected loss is f if the untruthfulness cannot be caught. Moreover, if the regulator chooses to be diligent in auditing then he incurs a cost of c. The expected payoff of the regulator can be summarized as: $u_R(T,L) = 0$, $u_R(T,D) = -c, u_R(U,D) = -e = \beta d - (1-\beta)f - c$, and $u_R(U,L) = -f$. The parameters have been chosen so that the expected utility of the regulator can be ranked as follows: $0 = u_R(T,L) > u_R(T,D) > u_R(U,D) > u_R(U,L)$. It should be noted that the minimum payoff the regulator obtains with these parameters becomes $-f < 0$. Under this construction, no matter what the regulator chooses, he prefers the agent to be truthful. Thus, he is willing to convince the agent for being tough to induce truthfulness. However, the regulator wants to be lazy if he thinks that the agent is truthful while he has an incentive to be diligent if he believes that the agent is going to be untruthful. To have this preference ordering for the strategic regulator, we need the following restriction on the parameter values.

Assumption 2. *The parameter values satisfy* $\frac{c}{d+f} < \beta < \frac{f}{d+f}$.

The stage game *minmax payoff profile* under this construction becomes *0* for the agent and *-e* for the regulator. In the complete-information game, when the regulator is known to be strategic, the unique Nash equilibrium is in mixed strategies, $\sigma_A^* = 1 - \frac{c}{\beta(d+f)}$ and $\sigma_R^* = \frac{g}{\beta(g+l)}$. The equilibria of the incomplete-information game is presented in Lemma 1.

Lemma 1. *The following strategy profile* (σ_A, σ_R) *constitute a Nash equilibrium,*

1. $\sigma_A = 1$ *and* $\sigma_R = 0$ *if* $\gamma \geq \gamma^*$,
2. $\sigma_A = 1 - \frac{c}{\beta(d+f)}$ *and* $\sigma_R = \frac{g-\gamma\beta(g+l)}{(1-\gamma)\beta(g+l)} = \frac{\pi^*-\gamma\beta}{(1-\gamma)\beta}$ *if* $\gamma < \gamma^*$,

where the cutoff value of the (prior) belief is $\gamma^* = \frac{g}{\beta(g+l)} \in (0,1)$.

This lemma says that if the prior belief that regulator is tough is above a threshold (which depends on the ratio of agent's expected gain and loss from untruthfulness and gets higher as her expected gain increases), then agent is truthful with probability one. Anticipating this, regulator chooses to be lazy with probability one. There is no circumstances where regulator chooses to be diligent with probability one in equilibrium. Moreover, equilibrium strategies are monotone in the prior belief. The proof is presented in Appendix A.1.

3 Dynamic Game with Short-Lived Agents

3.1 Strategies and Updating

The game begins at time $t = 1$ and it is infinitely repeated and the time is discrete. The regulator is the long-lived player with a discount factor $\delta \in (0,1)$ and the agents are short-lived (myopic) players who play only one period and thus only cares about her own payoff. At each period, the players simultaneously choose actions from their action sets. The type of the regulator is determined once and for all at time $t = 0$ with the common prior belief for tough type being $\gamma_0 \in (0,1)$. There is a finite public signal space denoted by $i_d \in I_d = \{0, 1\}$ which shows whether agent is detected (1) or not (0) with signal probabilities $\rho(i_d|a, r)$. It should be emphasized that $\rho(i_d = 1|T, r) = 0$ for any $r \in R$, $\rho(i_d = 1|U, L) = 0$ and $\rho(i_d = 1|U, D) = \beta \in (0, 1)$.

The reputational incentives affect the behavior only when the short-lived agents base their decision on the information about the past detections. That is why we suppose that each short-lived agent can observe whether a detection has occurred for the preceding agents or not. Let h^t denote the public history up to (not including) time $t \geq 1$ where h^1 denotes the null history. All players observe the history of public signals. We let $\bar{h}^t$ denote the private history of the long-lived regulator. This involves the public history of signals and his own past actions up to time t. A strategy for the long-lived player is a sequence of maps $\sigma_{R,t}(\bar{h}^t) \in \Delta(R)$ and a strategy for short-run player is $\sigma_{A,t}(h^t)$. Given a strategy profile σ, the prior belief γ_0, and a public history h^t that has positive probability under σ, we can find the conditional probability of long-lived strategic player action $\sigma_{R,t}(h^t)$ that depends on the public history. A Nash equilibrium is a strategy profile σ such that for each t and positive probability history h^t and $\bar{h}^t$, (1) $\sigma_{A,t}(h^t)$ is a best response for the short-lived player against $\sigma_{R,t}(h^t)$; and, (2) $\sigma_{R,t}(\bar{h}^t)$ is a best response of the strategic regulator against $\sigma_{A,t}(h^t)$. We restrict attention to public strategies and public equilibria.

We denote the posterior belief any agent holds at the beginning of period t to be $\gamma_{t-1}(h^t)$. As each agent is short-lived, their decision only depends on the updated reputation of the regulator, the strategic regulator's expected behavior and the resulting detection probability. Then, when $\gamma_{t-1}(h^t) \geq \gamma^*$, the agents will always be truthful and the strategic regulator will always be lazy; and consequently the payoffs will be zero for each party (this is the first-best for the regulator). When $\gamma_{t-1}(h^t) < \gamma^*$, the agent will be untruthful with some probability only if the strategic regulator is diligent with no more than probability $\frac{\pi^* - \gamma_{t-1}(h^t)\beta}{(1-\gamma_{t-1}(h^t))\beta}$ since from the perspective of the agent the probability of detection at $t = \{1, \ldots\}$ is

$$\pi(\gamma_{t-1}, \sigma_{R,t}(h^t)) \equiv \gamma_{t-1}\beta + (1 - \gamma_{t-1})\sigma_{R,t}(h^t)\beta. \tag{2}$$

Long-lived regulator's reputation depends on the signal I_d as well as the strategies of the players. It is imperative to remind that the Bayesian updating is possible only when the agent has chosen to be untruthful. When agent is truthful, the reputation of regulator does not change since no signal will be generated regarding his behavior. Given that the agent is untruthful with some strictly positive probability, the reputation after the signal $I_d \in \{0, 1\}$ has occurred can be calculated as follows:

$$\gamma_t = \begin{cases} \gamma_t^+ = \frac{\gamma_{t-1}\beta}{\pi(\gamma_{t-1}, \sigma_{R,t})} = \frac{\gamma_{t-1}\beta}{\gamma_{t-1}\beta + (1-\gamma_{t-1})\sigma_{R,t}\beta} & \text{if} \quad i_d = 1, \\ \gamma_t^- = \frac{\gamma_{t-1}(1-\beta)}{1-\pi(\gamma_{t-1}, \sigma_{R,t})} = \frac{\gamma_{t-1}(1-\beta)}{\gamma_{t-1}(1-\beta) + (1-\gamma_{t-1})[\sigma_{R,t}(1-\beta) + (1-\sigma_{R,t})]} & \text{if} \quad i_d = 0. \end{cases} \tag{3}$$

3.2 Bound on the Nash Equilibrium Payoff

The strategic regulator's incentives at date t balances the tradeoff between the short-run desire to give a best-reply to the current agent's expected behavior and the long-run objective of establishing a reputation for being tough. Note that in order to increase her reputation the strategic regulator has to choose L with some probability and hope for the agent to be untruthful and a detection is realized. We argue that when the agents are short-lived, the reputation helps the patient strategic regulator to achieve the first-best outcome payoff (if the discount factor is close to one); thus reputation is a useful tool in achieving the socially desirable outcome.

Specifically, the average payoff of the strategic regulator across all Nash equilibria converges to zero (the maximum payoff) as δ approaches to one for any prior belief $\gamma_0 > 0$ on regulator's tough type. We let $\hat{V}(\gamma_0, \delta)$ be the minimum payoff for the strategic regulator in any Nash equilibrium.[9]

Theorem 1. *For any prior belief* $\gamma_0 > 0$ *and any auditing quality* $\beta \in (0, 1)$*,* $\lim_{\delta \to 1} \hat{V}(\gamma_0, \delta) = 0$*, which is the maximum payoff attained by the regulator.*

This results suggests that no matter how good or bad auditing quality is, the event of subsequent detections that would result in the capture of all the surplus by the regulator is very likely if the regulator is sufficiently patient. This also says the slightest incomplete-information about the type of the regulator guarantees him to achieve his best outcome (more than the Stackelberg payoff of $-c$) in any Nash equilibrium (which is not possible in the complete-information case since then the short-lived agent's payoff must be pushed below her minmax payoff).

The proof uses a couple of lemmas whose proofs are relegated to the Appendix A.2. For the rest of this section and also for the proofs of the lemmas in the appendix, we fix an arbitrary Nash equilibrium in public strategies; and, each positive probability public history and posterior belief that are considered are with respect to this Nash equilibrium.

Before we present the lemmas that are used in the proof of Theorem 1, we would like to give some notation. Given an arbitrary Nash equilibrium, for each positive probability public history h^t, we let $v(h^t)$ denote the *expected continuation value* to the strategic regulator. If T has a positive probability under $\sigma_{A,t}(h^t)$ and D has a positive probability under $\sigma_{R,t}(h^t)$, then we define

$$v(h^t, T, D) \equiv (1-\delta)u_R(T, D) + \delta \sum_{i_d} \rho(i_d \mid T, D) v(h^t, i_d)$$

The definition for $v(h_t, \sigma_{A,t}(h^t), \sigma_{R,t}(h^t))$ is done in the natural way.

[9] The proof is along the lines with [9]. However, in their model, the long-lived agent's concern for differentiating himself from his bad counterpart results in the loss of all surplus and market collapse since the short-lived players choose not to participate the game.

The following lemma tells that if there is a positive probability history on which agent is truthful with probability one at some date, then the regulator must be lazy with probability one on that history and date.

Lemma 2. *Suppose that h^t is a positive probability history where $\sigma_{A,t}(h^t) = 1$. Then, $\sigma_{R,t}(h^t) = 0$.*

This result is the consequence of the fact that when the agent is truthful, being lazy and diligent generates the same distribution of public signals and therefore the same continuation payoffs $v(h^t, i_d = 0)$. The next lemma suggests that there is no time and positive probability history on which the regulator is diligent with probability one with respect to a Nash equilibrium.

Lemma 3. *There is no date t and positive probability history h^t at which $\sigma_{R,t}(h^t) = 1$. In other words, if h^t is a positive probability history, then $\sigma_{R,t}(h^t) < 1$.*

This is because as Lemma 2 says that when the agent is truthful, the regulator chooses to be lazy. And, the agents choose to be untruthful with some probability only if she expects the regulator to be lazy with some probability on a Nash equilibrium.

Lemma 4. *Let h^t be any positive probability history where $\gamma_{t-1}(h^t) < \gamma^*$. Then, there exists some $\tau \geq t$ and h^τ that appends after h^t for which $\sigma_{R,\tau}(h^\tau) > 0$.*

This lemma suggests that every Nash equilibrium continuation path starting from h^t must include play of diligence given that $\gamma_{t-1}(h^t) < \gamma^*$.

Lemma 5. *Suppose that h^t is a positive probability history on which detection occurs ($i_d = 1$) at time t given $\gamma \equiv \gamma_{t-1}(h^t) < \gamma^*$. Then, $0 < \sigma_{R,t}(h^t) \leq \frac{\pi^* - \beta\gamma}{\beta(1-\gamma)}$ and the smallest posterior belief after a detection has been observed $\gamma_t(h^t, i_d = 1)$ that is denoted by $\Gamma(\gamma)$ becomes*

$$\Gamma(\gamma) = \frac{\gamma\beta}{\pi^*} = \frac{\gamma\beta}{\gamma\beta + (1-\gamma)\beta\bar{\sigma}_R}$$

where $\bar{\sigma}_R \equiv \bar{\sigma}_R(\gamma) = \frac{\pi^* - \beta\gamma}{\beta(1-\gamma)}$.

This result is the consequence of the fact that a detection is possible only when the agent is untruthful and the regulator is diligent (with some probability). For the agent to be untruthful, the strategic regulator should be lazy with some probability. Thus, the regulator must be using a mixed action at that history. Thus, there is an upper bound on the diligence of the regulator for the agent to be untruthful. And, the smallest possible posterior after a detection occurs is calculated with respect to this upper bound.

Proof (Proof of Theorem 1). First, note that if the prior belief satisfies $\gamma_0 \geq \gamma^*$, then there is a unique Nash equilibrium on which the agents are always truthful and the regulator is always lazy. As all the short-lived agents are truthful with probability one, the reputation for being tough persists and the equilibrium payoffs are zero both for the strategic regulator and the short-lived agents.

Now, suppose that $\gamma_0 < \gamma^*$. Again, note that at any positive probability history h^t where $\gamma_{t-1}(h^t) \geq \gamma^*$, the agent is truthful and the regulator is lazy from then on. The reputation of the regulator does not change when the agent is truthful and again it is not optimal for the strategic regulator to be diligent in this situation. Then, the continuation payoff $v(h^t)$ of the regulator is zero on those histories.

Then, we consider a positive probability history h^t (with respect to a Nash equilibrium) for which $\gamma \equiv \gamma_{t-1}(h^t) < \gamma^*$ and the agent is untruthful with some probability (since otherwise, by Lemma 2, the strategic regulator is lazy and the stage game payoffs are all zero in those periods). This time, the reputation (posterior belief about the regulator's type) is updated according to expression (3). Recall that the agent is untruthful only if the expected probability of detection is $\pi(\gamma, \sigma_{R,t}(h^t)) \leq \pi^*$, which requires $\sigma_{R,t}(h^t)$ to be less than or equal to $\bar{\sigma}_R(\gamma) \equiv \frac{\pi^* - \gamma\beta}{(1-\gamma)\beta}$. Then, by Lemma 5, we define the smallest posterior probability of a tough regulator after a detection has been observed as: $\Gamma(\gamma) = \frac{\gamma\beta}{\pi^*}$ (where $\pi^* = \frac{g}{g+l}$). By Assumption 1, $\Gamma(\gamma) > \gamma$ for every $\gamma \in (0, \gamma^*)$, i.e., Γ is strictly increasing and it is continuous.

Following the footsteps of [9], we construct a decreasing sequence of cutoff beliefs p_n such that $p_1 = \gamma^*$ and $p_n = \Gamma^{-1}(p_{n-1})$ for $n > 1$ and use an induction on n to bound the payoffs attained in any Nash equilibria when γ exceeds p_n. For the induction hypothesis, suppose that there exists a lower bound $\hat{V}_n(\delta)$ with $\lim_{\delta \to 1} \hat{V}_n(\delta) = 0$ and $\hat{V}(\gamma, \delta) \geq \hat{V}_n(\delta)$ for all $\gamma > p_n$. Note that this holds for $n = 1$, i.e., for "$1\gamma > p_1 = \gamma^*$. We assume this holds for n and show that for $n + 1$. Now, fix $\gamma > p_{n+1}$. We want to show that $\lim_{\delta \to 1} \hat{V}_{n+1}(\delta) = 0$. As the maximum attainable payoff is zero, this will imply that $\lim_{\delta \to 1} \hat{V}(\gamma, \delta) = 0$.

It suffices to consider a Nash equilibrium in which the agent is untruthful in the first period with some probability, e.g., $\sigma_{A,t}(h^t) < 1$.[10] Since the agent has been untruthful with some probability, this implies that the strategic regulator is diligent with no more than $\bar{\sigma}_R(\gamma)$. Then this strategy must be a best-reply for the strategic regulator, i.e., the expected long-run payoff from being lazy should be no less than that of being diligent. Also, by Lemma 4, for any positive probability history where $\gamma < \gamma^*$, there exists a continuation history at which the regulator is diligent with some probability (but not with probability 1 by Lemma 3). Take h^t to be that history at which $\sigma_{R,t}(h^t) > 0$ (note that this is consistent with $\sigma_A(h^t) < 1$, if agent were to be truthful with probability one, by Lemma 2, the regulator would be lazy with probability 1). Thus, we obtain the following constraint at h^t:

[10]Take any Nash equilibrium in which the agents have been truthful with probability one until date $s > t$. In these periods, the reputation of the regulator will stay the same regardless of her behavior. Thus, it is a best response for the regulator to be lazy during these periods, thus her payoff is zero. Then the continuation play starting from date s is a Nash equilibrium with the same prior γ whose payoff can be no more than the original game.

$$(1-\delta)(-e)+\delta Z_D(\gamma) \leq (1-\delta)(-f)+\delta Z_L(\gamma) \equiv \hat{V}_{n+1}(\delta) \tag{4}$$

where $Z_D(\gamma)$ and $Z_L(\gamma)$ are the lower bounds on the expected continuation payoffs from choosing action D and L, respectively.[11] And, we can define $\hat{V}_{n+1}(\delta) := (1-\delta)(-f)+\delta Z_L(\gamma)$.

When there is no detection, in the worst case scenario, the posterior probability drops to $\gamma^- = \frac{\gamma(1-\beta)}{\gamma(1-\beta)+(1-\gamma)}$ at h^{t+1} and this happens with probability one when the regulator chooses to be lazy. Let the minimum continuation payoff for γ^- be $\hat{V}(\gamma^-,\delta)$. Then, $Z_L(\gamma) \equiv \hat{V}(\gamma^-,\delta)$. On the other hand, when the strategic regulator chooses to be diligent, there will be detection with probability β and the posterior probability after a detection occurs is at least $\Gamma(\gamma)$, which is at least p_n given that $\gamma > p_{n+1}$. Hence, we can derive the following lower bound for $Z_D(\gamma)$:

$$\beta\hat{V}_n(\delta)+(1-\beta)\hat{V}(\gamma^-,\delta) \leq Z_D(\gamma) \tag{5}$$

Then, combining (4) and (5) allows us to get:

$$(1-\delta)(-e)+\delta\beta\hat{V}_n(\delta)+\delta(1-\beta)\hat{V}(\gamma^-,\delta) \leq (1-\delta)(-f)+\delta\hat{V}(\gamma^-,\delta)$$
$$(1-\delta)(f-e)+\delta\beta\hat{V}_n(\delta) \leq \delta\beta\hat{V}(\gamma^-,\delta)$$

which implies that $\hat{V}_n(\delta) \leq \hat{V}(\gamma^-,\delta)$ since $(f-e)>0$.

As $\hat{V}_{n+1}(\delta) := (1-\delta)(-f)+\delta Z_L(\gamma) = (1-\delta)(-f)+\delta\hat{V}(\gamma^-,\delta)$ and $\hat{V}_n(\delta) \leq \hat{V}(\gamma^-,\delta)$, we conclude that

$$(1-\delta)(-f)+\delta\hat{V}_n(\delta) \leq \hat{V}_{n+1}(\delta) \tag{6}$$

By the induction hypothesis, the limit of the left-hand side of (6) is zero as δ approaches to one. Thus, $\lim_{\delta\to1}\hat{V}_{n+1}(\delta)=0$, which implies $\lim_{\delta\to1}\hat{V}(\gamma,\delta)=0$ as desired for $\gamma \equiv \gamma_{t-1}(h^t) < \gamma^*$. Then, following h^t, $\lim_{\delta\to1}\hat{V}(\gamma,\delta)=0$ for any $\gamma \equiv \gamma_{t-1}(h^t) > 0$ since the choice of p_n and $\gamma > p_n$ is arbitrary and $(p_n)_{n\in IN}$ is a decreasing sequence that converges to zero. The regulator gets a payoff possible different than zero only for finite number of periods up to h^t. Thus, we can conclude that $\lim_{\delta\to1}\hat{V}(\gamma_0,\delta)=0$ for any prior belief $\gamma_0 > 0$.

This completes the proof.

4 Conclusion

The goal of this paper is to analyze the effect of reputation concerns of regulators on preventing socially undesirable behavior that could be engaged by agents. In

[11]When the agent is truthful with probability $\sigma_{A,t}(h^t)$, choosing lazy is superior to diligent. Thus, the constraint involves an inequality rather than an equality.

a repeated incomplete-information setting where a long-lived regulator interacts with a sequence of short-lived agents who plays only once but observes the history of public signals, it is found that a patient regulator can guarantee herself the maximum payoff (which could be way more than the Stackelberg payoff level provided by earlier studies) in any Nash equilibrium. This suggests that, contrary to disappearance of reputation results in the literature, the regulator can sustain a reputation for being diligent in the long-run in any equilibrium. These results imply that socially undesirable behavior of the agents could be prevented thanks to the reputation concerns of the regulators in these situations.

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Appendix

A.1 The Proof of Lemma 1

The utility of agent by being truthful is $u_A(T, \sigma_R) = 0$. And, her expected utility by being untruthful is

$$u_A(U, \sigma_R) = \gamma[(1-\beta)g - \beta l] + (1-\gamma)\sigma_R[(1-\beta)g - \beta l] + (1-\gamma)(1-\sigma_R)g.$$

The agent's best response correspondence against the strategic regulator's strategy (which implies the cutoff detection probability) is given by:

$$\sigma_A \equiv BR_A(\sigma_R) = \begin{cases} 1 & \text{if} \quad \sigma_R > \frac{g-\gamma\beta(g+l)}{(1-\gamma)\beta(g+l)} \\ [0,1] & \text{if} \quad \sigma_R = \frac{g-\gamma\beta(g+l)}{(1-\gamma)\beta(g+l)} \\ 0 & \text{if} \quad \sigma_R < \frac{g-\gamma\beta(g+l)}{(1-\gamma)\beta(g+l)}. \end{cases} \tag{A.1}$$

From this, we can deduce the cutoff prior beliefs. The strategy of the regulator that makes the agent indifferent between being truthful and untruthful, $\sigma_R = \frac{g-\gamma\beta(g+l)}{(1-\gamma)\beta(g+l)}$, is greater than 0 if $\gamma < \gamma^* = \frac{g}{\beta(g+l)}$ and equals to 0 if $\gamma = \gamma^*$. If $\gamma > \gamma^*$, then $BR_A(\sigma_R) = 1$ for any value of σ_R, i.e., even if the strategic regulator is lazy for sure.

The expected utility of the regulator by choosing to be diligent is $u_R(\sigma_A, D) = (1-\sigma_A)[\beta d - (1-\beta)f] - c$, whereas his expected utility by choosing to be lazy is $u_R(\sigma_A, L) = -(1-\sigma_A)f$. Thus, regulator's best response is given by:

$$\sigma_R \equiv BR_R(\sigma_A) = \begin{cases} 1 & \text{if} \quad \sigma_A < 1 - \frac{c}{\beta(d+f)} \\ [0,1] & \text{if} \quad \sigma_A = 1 - \frac{c}{\beta(d+f)} \\ 0 & \text{if} \quad \sigma_A > 1 - \frac{c}{\beta(d+f)}. \end{cases} \tag{A.2}$$

The strategy of agent that makes regulator indifferent between choosing to be diligent and lazy, $\sigma_A = 1 - \frac{c}{\beta(d+f)}$, is greater than 0 if $\beta > \frac{c}{f+d}$ (which holds by assumption).

Case 1 $\gamma > \gamma^*$: In this case, $BR_A(\sigma_R) = 1$ for any σ_R. The unique fixed point of the best response correspondences is $\sigma_A = 1$ and $\sigma_R = 0$.

Case 2 $\gamma = \gamma^*$: The strategy that makes the agent indifferent between telling the truth and lying is $\sigma_R = 0$. For $\sigma_R > 0$, $BR_A(\sigma_R) = 1$. But, against $\sigma_A = 1$, $\sigma_R > 0$ cannot be a best response. Thus, the equilibrium strategies are $\sigma_A \in [1-\frac{c}{\beta(d+f)}, 1]$ and $\sigma_R(D) = 0$. As we have assumed that the agent is truthful for sure when she is indifferent $\sigma_A = 1$ and $\sigma_R = 0$ in this case.

Case 3 $\gamma < \gamma^*$: The unique intersection of the best response correspondences in this case is when $\sigma_A = 1 - \frac{c}{\beta(d+f)}$ and $\sigma_R = \frac{g-\gamma\beta(g+l)}{(1-\gamma)\beta(g+l)}$.

A.2 The Proofs of Lemmas 2–5

Proof (Proof of Lemma 2). Given that agent chooses $\sigma_{A,t}(h^t) = 1$ at h^t, choosing diligent or lazy generates the same distribution of public signals so that the continuation payoff $v(h^t, i_d)$ will be the same. Since the one-period utility $u_R(T, L) = 0 > u_R(T, L) = -c$, we conclude that $\sigma_{R,t}(h^t) = 0$ complying with the one-shot deviation principle.

Proof (Proof of Lemma 3). Suppose that there exists a Nash equilibrium with a positive probability history h^t at which $\sigma_{R,t}(h^t) = 1$. If agent is truthful with probability one at h^t, then the regulator would have one-shot deviation gain by switching to lazy since the distribution of public signals, the posterior belief, and thus the continuation payoff would not change regardless of regulator's action choice when the agent is truthful. And, the agent chooses to be untruthful with some probability on a Nash equilibrium only when the belief $\gamma_{t-1}(h^t)$ at $t-1$ is less than γ^* and the regulator chooses to be diligent by less than $\bar{\sigma}_R = \frac{\pi^*-\beta\gamma}{\beta(1-\gamma)} < 1$.

Proof (Proof of Lemma 4). Consider an arbitrary public history h^t that is reached with positive probability with respect to some Nash equilibrium. Suppose for a contradiction that for every $\tau \geq t$ and every h^τ that comes after h^t, $\sigma_{R,\tau}(h^\tau) = 0$. Given $\gamma_{t-1}(h^t) < \gamma^*$ and there will not be any detections after h^t for every $\tau \geq t$ and history since $\sigma_{R,\tau}(h^\tau) = 0$ by hypothesis, and thus the expected probability of detection is going to be less than π^* for every $\tau \geq t$ and h^τ. Thus, all the myopic agents are untruthful at every date and history starting h^t. Thus, the regulator's expected continuation payoff becomes $v(h^t) = -f$ which is less than the minmax payoff $-e$, providing us the desired contradiction.

Proof (Proof of Lemma 5). First, note that in order to observe a detection at time t, the agent must have been untruthful. For the agent to have chosen untruthfulness with some positive probability at h^t, the expected probability of detection must be lower than π^*. Given that the belief at the beginning of time t at h^t is $\gamma \equiv \gamma_{t-1}(h^t) < \gamma^*$, this requires that $\sigma_{R,t}(h^t) \leq \frac{\pi^*-\beta\gamma}{\beta(1-\gamma)}$, which is derived from (2). And, $\sigma_{R,t}(h^t)$ must be greater than zero because otherwise there would not be any detections. Lastly, it is easy to see from expression (3) that the smallest posterior is obtained when $\sigma_{R,t}(h^t) = \bar{\sigma}_R$ and equals to $\Gamma(\gamma) = \frac{\gamma\beta}{\pi^*}$.

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Bayesian Networks and Games of Deterrence

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Abstract The present paper analyzes possible bridges between two types of graphical models: on the one hand, Bayesian Networks, and on the other hand, Graphs of Deterrence associated with a particular category of qualitative games called matrix Games of Deterrence, in which players do not look for optimal outcomes but for acceptable ones. Three related-types of relations are scrutinized: implications and rebuttals; priors and hidden parts of the graph; probability and playability.

Keywords Antecedents • Bayes • Deterrence • Game • Graph • Graphical models • Implication • Matrix • Network • Path • Playability • Probability • Rebuttal • Solution • Tree

1 Introduction

Probabilistic graphical models [6, 8] and graphical models for Game Theory [4, 5] have been and are still the subject of intensive research, concerning in particular their capacity to provide diagnosis, predictive analysis, or decision-making. For instance, Bayesian Networks [7] are currently used in a variety of domains like education, sociology, psychology, medicine, intelligence analysis, etc. They determine connections between occurrences of different events or facts, through conditional probabilities complying with Bayes Rule. Thus, they support

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the development of inference schemes enabling to interpret a set of data and draw conclusions. On their side, Games of Deterrence—a particular category of qualitative games—dichotomize the outcomes sets of a game by differentiating only between acceptable and unacceptable outcomes. Beyond their original purpose which was to analyze nuclear deterrence strategies, these games have proved to be an appropriate tool for analyzing a variety of issues, like traffic control in the case of communication networks [1], road networks [13], or argumentation and logic [11, 12, 14, 15]. With respect to that last issue, matrix Games of Deterrence have already been used to model argumentation through a rebuttal process illustrated by a graph called Graph of Deterrence, associated in a one to one relation with the game under consideration. This association paves the way for the representation of inference schemes by Graphs of Deterrence, the difference with the standard representation being that edges of the graphs do not represent implications—be it conditional probabilistic ones—but rebuttals.

Bayesian Networks and Games of Deterrence are certainly not equivalent. Nevertheless, it will be seen in the sequel that Games of Deterrence can provide functionalities in terms of diagnosis which complement those stemming from standard Bayesian Networks analysis. Games of Deterrence also provide an alternative approach for determining the properties of highly complex inference schemes.

The present paper aims at paving the way for establishing relations between the two types of graphical models.

To that end, the second section will recall the core properties of Games and Graphs of Deterrence. The third section will briefly explore the reasons that make Games of Deterrence an appropriate tool for data analysis. The fourth section will analyze the relations between graphical models in which edges represent implications and graphical models in which edges represent rebuttals. The fifth section will show how the Graph of Deterrence approach may use priors in the Bayesian Networks to detect "hidden parts" of the graph. The sixth section will raise the issue of the relation between prior probabilities and strategies' playability.

2 Matrix Games of Deterrence Core properties

2.1 Acceptability Threshold

Games of Deterrence consider players which behavior corresponds to Herbert Simon's concept of bounded rationality [16]: in many circumstances of daily life, decision-makers do not try to optimize their choices, for a variety of reasons, like time pressure, incapacity to determine optimal solutions (even when they exist), or feeling that important information is missing. Instead, decision-makers are driven consciously or unconsciously by an acceptability threshold that pushes them to dichotomize the set of possible states of the world, distinguishing only between acceptable and unacceptable ones. Acceptable states are associated with outcome 1, while unacceptable states of the world are associated with outcome 0. A rational player always looks for an acceptable outcome.[1]

2.2 Positive Playability and Playability by Default

With respect to the acceptability threshold, Games of Deterrence analyze which strategies are playable and which are not (Fig. 1).

In the matrix game here below, each of the strategies e_1, e_2, and r_2 provides the player who selects it a 1, whatever the strategy selected by the other player. For this reason, these strategies are termed *safe*. A strategy which is not safe is called *dangerous*. This is here the case for r_1. So, intuitively in this game, playable strategies are safe, while dangerous strategies are not playable.[2]

To see if this is always the case, let us consider the game in Fig. 2 wherein Erwin's strategies are still safe, but the situation of Roger has changed.

Roger

		r_1	r_2
Erwin	e_1	(1,0)	(1,1)
	e_2	(1,1)	(1,1)

Fig. 1 Example 1

Roger

		r_1	r_2
Erwin	e_1	(1,0)	(1,0)
	e_2	(1,0)	(1,0)

Fig. 2 Example 2

[1] which does not mean of course that such outcome can always be reached.

[2] In such a case, playability, as introduced here, meets Rufus Isaacs' playability concept according to which in a qualitative differential game, a player's strategy is playable if it enables the player to reach his/her target, in which case the player under consideration will get an outcome equal to 1, against a 0 if the target is not reached [3].

Indeed, whatever his choice, he gets a 0. But as in a game a player must play, Roger has to select a strategy. Given that the two are equivalent, he can choose any one, and hence both are playable. But of course this kind of playability is different from the playability of Erwin's strategies.

Thus, a player's strategy is said to be:

- *positively playable* if its selection gives him/her a 1, provided the other player is rational
- *playable by default* if the player has no positively playable strategy.

On the whole, a player's strategy is termed *playable* if it is either positively playable or playable by default.

2.3 Deterrence

There are many cases, for instance, in geopolitics, where the objective of a player in the game is to prevent another player to play a given strategy, through making it not playable.

More precisely, given a strategic pair (e, r), e will be termed deterrent vis-à-vis r, iff the three following conditions are satisfied:

(a) e is playable
(b) implementation of (e, r) implies a 0 for Roger
(c) Roger has an alternative strategy r' which is positively playable.[3]

For instance, in Example 1, e_1 is deterrent vis-à-vis r_1. Indeed:

(a) e_1 being safe, is positively playable, hence playable
(b) implementation of strategic pair (e_1, r_1) implies a 0 for Roger
(c) Roger has an alternative strategy r_2, which is safe and hence positively playable.

On the opposite, in Example 2, the third condition is not satisfied. Hence Erwin cannot deter Roger from playing r_1.

It has been shown [9] that a strategy is playable if and only if there is no strategy of the other player which is deterrent vis-à-vis that strategy.

This means that playability and deterrence are like the two faces of the same coin: in other words, deterrence can be used as a tool for determining strategies' playability.

[3]This third condition is a direct implication of the proverb according to which "you have everything to fear from the one who has nothing to lose."

2.4 Playability System, Game Solutions, and Equilibria

More generally, in Games of Deterrence, solving the game means determining the strategies' playability. To that end, let:

- S_E and S_R be the strategic sets of Erwin and Roger, respectively
- $J(x)$ denotes the value of the positive playability index of strategy x
- j_E and j_R denote the values of the playability by default indices associated with Erwin and Roger, respectively[4]
- $a(e,\ r)$ and $b(e,\ r)$ denote the outcomes of Erwin and Roger, respectively, associated with implementation of strategic pair (e, r)
- e be a strategy of Erwin.

If e is safe, then $J(e) = 1$.
If e is dangerous, then $J(e) = (1 - j_E)(1 - j_R) \prod_{r\ \in\ S_R} [1 - J(r)(1 - a(e, r))]$
Furthermore, $j_E = \prod_{e\ \in\ S_E} [1 - J(e)]$
Similar definitions apply to the strategies r of Roger.

What precedes means that if Card $S_E = n$ and Card $S_R = p$, then, determining the playability properties of the players strategies requires solving a system of equations called the *playability system*, and comprised of $n + p + 2$ equations.

We shall call:

- *solution* of the Game of Deterrence any solution of the playability system
- *equilibrium* of the Game of Deterrence any pair of playable strategies.

It should be noticed from what precedes that a game may display several solutions, and that a solution may correspond to several equilibria.

Last, one will distinguish between solutions such that the values of playability indices:

- are integer numbers (*non-fuzzy solutions*).
- may take non integer values (*fuzzy solutions*).

2.5 Graph of Deterrence and Game Type

Whatever the case, one can associate with each matrix Game of Deterrence G a graph Γ called Graph of Deterrence, and defined as follows:

(a) Γ is a bipartite graph on $S_E \vee S_R$
(b) Given (e, r) ∈ S_E x S_R, there is an arc of origin e (resp. r) and extremity r (resp. e) iff b(e,r) = 0 (resp. a(e,r) = 0).

[4]Since a player's strategy is playable by default if the player under consideration has no positively playable strategy, there is no need to define a playability by default index of a strategy. It is enough to have such index at the player's level.

Now, every Graph of Deterrence can be broken down into parts that correspond to one of the following elementary types [9]:

- E-path defined as a path which root is strategy of Erwin
- R-path defined as a path which root is a strategy of Roger
- C-graph defined as a graph without roots.

The combination between these elementary types implies seven types of matrix Games of Deterrence: E, R, C, E/R, E/C, R/C, and E/R/C.[5]

Associating a graph with a matrix Game of Deterrence makes it easier in most cases to determine the playability system solution. For instance:

(a) in an E-type game, the only (positively) playable strategies of Erwin are the roots, while all strategies of Roger are playable by default [9]
(b) in a C-type game which graph is a circuit:

  (1) there is a unique non-fuzzy solution, in which all strategies are playable by default.
  (2) there are two fuzzy solutions: one for which all strategies are playable by default, and one in which the positive playability of all strategies equals 0.5.

For the following matrix game (Fig. 3), there is an associated Graph of Deterrence (Fig. 4).

More generally, in an E/R type game (like the one below) all strategies of odd rank are positively playable while all strategies of even rank are not playable (Fig. 4).

Roger

Erwin	r_1	r_2	r_3
e_1	(1,0)	(1,1)	(1,1)
e_2	(0,1)	(1,0)	(1,1)
e_3	(1,1)	(0,1)	(1,1)

Fig. 3 Example 3

[5]For instance, a game of type E is a game in which the corresponding graph contains only E-paths, while a game of type R/C is a game which graph contains R paths and C-graphs.

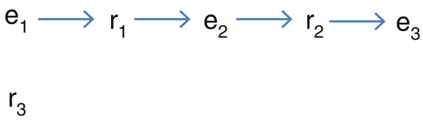


Fig. 4 Graph of Deterrence

Table 1 Bayesian Networks and Games of Deterrence

	Bayesian Networks	Games of Deterrence
Nature	Descriptive	Prescriptive
Tool	Implication	Rebuttal
Matrix	Probability	Playability

3 Games of Deterrence for Data Analysis

3.1 Bayesian Networks and Games of Deterrence

The main differences between Bayesian Networks (BNs), currently used for data analysis, and Games of Deterrence (GoDs) can be summarized in Table 1:

It must be noticed that there is a difference between the nature of the theory and its formalism. Several cases of formalism exist which have been developed in one field and imported by another field of completely different nature (such is the case, for instance, for the fields of Physics and Economy). In this respect it will be shown that the formalism of the prescriptive Games of Deterrence Theory can be applied to address issues like data analysis, usually represented through the formalism of descriptive Bayesian Networks Theory.

3.2 Propositional Logic and Games of Deterrence

To establish a relation between the two theories, a potential bridge already exists: Propositional Logic. Let us go back, for instance, to the Graph of Deterrence example of Fig. 3 and consider that e_1, e_2, e_3, r_1, r_2, and r_3 are evidences that can be true or false (binary logic). The existence of an edge of origin x and extremity y can be interpreted as the fact that if x is true, then y is false. So, one can re-interpret the E-path of the graph as the sequence of following implications: r_2 true implies e_3 false, e_2 true implies r_2 false, r_1 true implies e_2 false, e_1 true implies r_1 false. To this sequence it must be added that, e_1 having no parent, is considered true.[6] One can conclude that evidences e_1, e_2, and e_3 are true as well as evidence r_3, while evidences r_2 and r_3 are false.

[6]In other words, an evidence is considered true if it is rebutted by no other evidence.

Table 2 Propositional logic and Games of Deterrence

Propositional logic	Games of Deterrence
Proposition	Strategy
True	(Positively) playable
False	Not playable

In the above example, the presence of r_3 plays a core role for bridging binary Propositional Logic and non-fuzzy Games of Deterrence. Indeed, were r_3 not present, then all strategies of Roger would have been playable by default, and hence e_2 and e_3 would have been not playable. This would have meant in turn that, on the game side, no strategy of Roger would be deterred, which would translate on the side of Propositional Logic by the fact that evidences r_1 and r_2 would not be subject to rebuttal. So applying non-fuzzy Games of Deterrence to binary Propositional Logic requires that the game be of type E/R or E/R/C.[7] But such requirement generates no limitation, since one can always add a true proposition.[8] For instance, given any proposition P, the proposition P $\vee\neg$P is always true. When added to the initial set of propositions, the former is called *consistency condition*.

So, one can connect binary Propositional Logic with non-fuzzy matrix Games of Deterrence as indicated in Table 2.[9]

Starting from a data set, and structuring it by inserting appropriate rebuttal relations between its elements, enables an interpretation of this set as a Graph of Deterrence associated with a matrix Game of Deterrence, the strategies of which are in a one-to-one relation with the data. A core interest of such connection is that, to determine the status of data, evidences or hypotheses stemming from these data, one can use the results provided by Games of Deterrence, and in particular algorithms enabling to find the games solutions.

4 Transforming a Bayesian Network into a Graph of Deterrence

4.1 Implications and Rebuttals

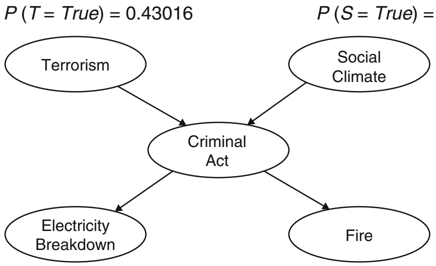
As noticed in Table 1, Bayesian Networks use edges that are (conditional) logical implications, while on the opposite, Graphs of Deterrence use edges that are rebuttals. So, transforming a Bayesian Network into a Graph of Deterrence requires translating implications of Bayesian Networks into rebuttals.

To that end, let us consider a pair of logical propositions $(P,\ Q)$ such that P implies Q.

[7] We shall see that things are different in the case of fuzzy games.

[8] and independent from the existing set of propositions.

[9] Beyond the specific case of Games of Deterrence, it must be noticed that extended research has already developed to bridge Logic and standard Game Theory [2, 17].



T	S	P(C|T,S)
True	True	0.50
True	False	0.20
False	True	0.10
False	False	0.001

C	P(B = True|C)
True	0.50
False	0.20

C	P(F = True|C)
True	0.65
False	0.30

T = Terrorism
S = Social Climate
C = Criminal Act
B = Electricity Breakdown
F = Fire

Fig. 5 Bayesian Network

If we consider that on the Graph of Deterrence, → means "rebuts," the above implication means that: $P \rightarrow \neg Q$. In other words $P \Rightarrow Q$ is equivalent to $P \rightarrow \neg Q \rightarrow Q$

4.2 Example: The IPC Scenario

Last week, a huge fire occurred at an oil field of the International Petroleum Company (IPC):

- Just before, there was an electricity breakdown in a part of IPC facilities which was not subject to fire.
- The investigators want to establish whether the cause is criminal or not.
- They consider two possible sources of a criminal act:
  (a) An underground terrorist structure (T) that might have developed in the area
  (b) The social climate (S), which has deeply deteriorated in the company: the employees have asked time and again that their salaries be increased, and IPC has constantly refused, arguing about the poor state of the market due to a global economic crisis.

Let us suppose that after an in-depth research of information, the Bayesian Network representation provides

- the evidences to be taken into consideration, and the possible causal relations between these evidences
- the prior probabilities for Terrorism and disruptive Social climate
- the conditional probabilities of a criminal act (C), an electricity breakdown (B), and a fire (F).

The Bayesian Network shown in Fig. 5 can be associated with a Graph of Deterrence (Fig. 6).

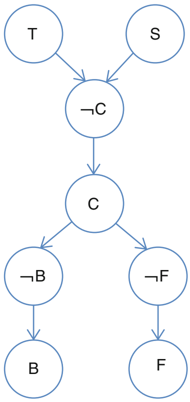


Fig. 6 Associated Graph of Deterrence

Table 3 Playability system

Rank	Playability equations
1	$J(T) = J(S) = 1$
2	$J(\neg C) = 0$
3	$J(C) = v$
4	$J(\neg B) = J(\neg F) = v(1 - v)$
5	$J(B) = J(F) = v - v^2 + v^3$
j_E	$1 - u = 0$
j_R	$1 - v = (1 - v + v^2)^2$

4.3 Solutions of the Associated Graph of Deterrence

Let:

- E be the player which strategies are the propositions associated with the nodes of the Bayesian Network
- R be the player which strategies are the negations of these propositions
- $u = 1 - j_E$ and $v = 1 - j_R$.

Furthermore, let us consider that the roots of the Bayesian Network are the roots of the Graph of Deterrence, without—momentarily—considering the issue of how the prior probabilities in the Bayesian Network translate in the Graph of Deterrence. One then gets the playability system (Table 3):

It can be easily seen that

- for strategies of odd rank, the value of the positive playability index decreases with the rank
- for strategies of even rank, the value of the positive playability index increases with the rank.

Likewise, by solving the equation corresponding to the expression of j_R one can establish that there are two possible values for v:

- $v = 0$, which corresponds to a non-fuzzy solution
- $v = 0.43016$ which is equal to the prior probability of T in the Bayesian Network representation.

In the non-fuzzy case, C, B, and F are not playable. This means that if there is an underground terrorist group, and a disruptive social climate within IPC, a criminal act, an electricity breakdown, and a fire cannot occur! So, to appropriately represent the Bayesian Network by a Graph of Deterrence, the non-fuzzy solution must of course be discarded.[10]

Switching to the fuzzy case, and starting again from the fact that the roots of the graph are—according to the Game of Deterrence interpretation—safe and hence positively playable strategies, it stems from the above that the playability indices associated with the strategies of odd rank (corresponding to the nodes of the Bayesian Network) decrease as one moves away from the roots.

Thus $J(T) = J(S) = 1$ implies $J(C) = 0.43016$ and $J(B) = J(F) = 0.32472$.[11]

In other words given the existence of an underground terrorist group and of a disruptive social climate within IPC, the playability of a criminal act is around 43 %, and the playability of a fire or of an electricity breakdown generated by such criminal act is a bit more than 32 %.

Given that the roots positive playability index takes the value 1, discarding, for instance, the social disruptive climate and considering only the existence of an underground terrorist group as a root, would not change the playability of a criminal act. Of course such result is not general, but depends on the structure of the graph.

5 The Hidden Parts of the Graph

5.1 Priors Effect

In the above example, the strategies playability indices values are derived from the assumption that the positive playability index of the roots equals 1. They do not take into account the various priors and conditional probabilities figuring in the Bayesian Network. Two different—albeit connected—reasons can explain that:

(a) the roots in the Graph of Deterrence are assumed (by definition) to have no antecedents

[10] This will be the case in general for games of type E, since in these games all strategies of odd rank, except roots, are not playable.

[11] This is exactly what playability is about (absence of deterrence).

(b) the playability properties do not stem from some extra source of information, like it is the case for the conditional probabilities associated with the nodes of the Bayesian Network, but only from the structure of the graph.

Now, the prior may simply describe the ratio between the number of occurrences of the event under consideration and the number of observations, without telling anything about the causes of such occurrence. So what the Games of Deterrence approach proposes, is to determine some structural properties of the causality process, thus making it easier to find the causes of the events.

In the IPC case, the prior probability of a terrorist environment as given on the Bayesian Network representation (Fig. 5) equals 0.43016. To take this prior into account through the Games of Deterrence representation, one must consider that the Graph of Deterrence associated with the Bayesian Network is not limited to the one given in Fig. 6. It should include some *hidden parts*, i.e., parts that do not appear directly through translating implications into rebuttals, but are the sources of a T positive playability index, different from 1.

Let us consider a simplified case of the IPC issue, in which the Graph of Deterrence is restricted to an E-path connecting evidence T and hypothesis C[12]:

$$T \rightarrow \neg T \rightarrow C$$

Let us assume furthermore that:

- the intelligence analyst in charge of the issue has nevertheless some doubt about the reality of T[13]
- this doubt, which in a Bayesian Network corresponds to a prior probability, can be expressed through associating a value of x smaller than 1 with the playability of strategy T in the associated Game of Deterrence.

In such a case, the playability system writes:

$$J(T) = x \tag{1}$$

$$J(\neg T) = [1 - J(T)]uv \tag{2}$$

$$J(C) = [1 - J(\neg T)]uv \tag{3}$$

$$1 - u = [1 - J(T)].[1 - J(C)] \tag{4}$$

$$1 - v = 1 - J(\neg T) \implies J(\neg T) = v \tag{5}$$

Then (1), (2), and (5) imply: $v = (1 - x)uv$

[12]This means that issues pertaining to social climate are not taken into consideration: this simplified case considers only one possible cause of a criminal act.

[13]Let us recall that T means the existence of some underground terrorist group.

Like in the global IPC case, two situations are to be considered:

- *Case 1:* $v = 0$ Then $J(\neg T) = J(C) = 0$ and $u = x$ (in other words, $j_E = 1 - x$). One then comes to the same disturbing result than in the full IPC case: the fact that an underground terrorist group may be active "prevents" the possibility that the IPC fire is of criminal origin. So once again the non-fuzzy solution has to be discarded.
- *Case 2:* $v \neq 0$
  Then

$$(1 - x)u = 1$$
$$\Rightarrow u = 1 \text{ and } x = 0$$

In such case, it stems from (2) and (3) that $J(\neg T) = v$ and $J(C) = v - v^2$. Now (4) implies that $1 - u = 1 - J(C) = 1 - v + v^2 = 0$, which has no real solution.

These results indicate that trying to connect a Bayesian Network with a Graph of Deterrence having the same roots, and associating with these roots a positive playability index which is not equal to 1, leads nowhere.

So to address the issue, one needs to consider that a root which positive playability index does not equal 1, is in fact a node which is not really a root, but has unknown antecedents that constitute the "hidden part" of the graph.[14] Then, assuming the existence of a correspondence between probability and playability, this alternative approach enables to go one step further, by providing information about the causes of the root's positive playability value, and hence of the prior probability value.

5.2 *Game Type and Antecedents*

Understanding these sources requires exploring the structure formed by the antecedents of T, starting by the game type.

Proposition 1. *A necessary condition for a Graph of Deterrence to be associated with a Bayesian Network is that the game type be E, R, or E/R.*

Proof. By definition a Bayesian Network is a directed acyclic graph, which means that the associated Graph of Deterrence contains no C-graph. Whence the conclusion. □

Now, as according to Proposition 1, three types of Games of Deterrence are possible, to determine some properties of the hidden parts of the graph requires that properties of strategies playability indices be refined with respect to the game type.

[14] This approach does not contradict the one developed in the example here above.

Proposition 2. *Given a strategy s such that J(s) is not an integer number, the global graph corresponds either to an E-type game or to an R-type game.*

Proof. It stems from Proposition 1 that the game is of type E, R, or E/R. In the latter case, the playability by default indices equal 0 and hence the indices of positive playability are all integer numbers, which contradicts the assumption. Whence the conclusion. □

Corollary 1. *Given a strategy s in the non-hidden part of the graph, such that J(s) is not an integer number, then the hidden part of the graph is of type E or of type R.*

Proof. As the property applies to the whole graph, it therefore applies to any part of the graph, and in a particular to its hidden part. □

For instance, in the case of the simplified IPC scenario, if the global graph corresponds to an E-type or an R-type game, the fact that J(T) = x is not an integer number means that the antecedents of T which form the hidden part of the graph, also belong either to an E-type or R-type game.

5.3 Going Further with an E-Type Game[15]

If one considers more specifically an E-path with 2n nodes ($2 \leq k \leq n$), it has been shown [10] that:

(a) $J(e_k) = \frac{v(1+v^{k-1})}{(1+v)}$ and $J(r_k) = \frac{v(1-v^k)}{(1+v)}$
(b) v is an increasing function of n.

It follows that:

- the value of the positive playability index of odd rank strategies decreases with the rank, while on the opposite, the value of the positive playability index of even rank strategies increases with the rank.
- if the above path is a simple part of an E/R type game, then $v = 1$, $J(e_k) = 1$, and $J(r_k) = 0$.
- if one goes back to the full IPC scenario, the fact that v is an increasing function of the number of nodes composing the graph means, for instance, that the positive playability of C ($J(C) = v$) increases with the number of nodes. This means in particular that, everything else being the same, if the number of descendants of C increases, so does the positive playability of C: in other words, the positive playability of C increases with the number of its possible consequences.

Now an E-path can also be a part of an E-type game such that on the corresponding Graph of Deterrence some of the nodes at least have more than one parent. Whence:

[15]This section focuses on E-type games, but of course similar results and conclusions would hold for R-type games.

Corollary 2. *Given:*

- *an E-type Game of Deterrence*
- *a node e_k of the associated graph with p parents $r^1_{k-1}, \ldots, r^p_{k-1}$, each one belonging to an E-path*
- *$e^1_{k-1}, \ldots, e^p_{k-1}$ the parents of $r^1_{k-1}, \ldots, r^p_{k-1}$, respectively.*
  Then, $J(e_n) \leq Min[J(e^1_{k-1}),\ \ldots, J(e^p_{k-1})]$

Proof. $J(e_k) = [1 - J(r^1_{k-1})]\ldots[1 - J(r^p_{k-1})]v$

If e_k had a single parent, say r^1_{k-1}, $J(e_k)$ would be equal to $[1 - J(r^1_{k-1})]v$, and hence the positive playability of e_k would be equal to or bigger than its value when it has p parents.

Now, when e_k has one parent, it has been seen that $J(e_k) \leq J(e_{k-1})$.

The same would apply if e_k had r^2_{k-1} as only parent: $J(e_k) \leq J(e^2_{k-1})$.

So, coming back to the case where e_k has p parents: $J(e_k) \leq Min[J(e^1_{k-1}), \ldots, J(e^p_{k-1})]$. □

This means in turn that, just like in an E path, $J(e_k)$ is a decreasing function of k.

Let us consider then a root of a Bayesian Network and the associated strategy s of player E in the corresponding Game of Deterrence. Let us assume furthermore that:

- J(s) is not an integer number
- The hidden part of the graph is an E-path
- $\neg s_{-1},\ s_{-1},\ \neg s_{-2},\ s_{-2},\ \ldots,\ \neg s_{-p},\ s_{-p}$ are the antecedents of s on this path.

Proposition 3. *If $v \neq 0$:*

(a) $J(s_{-p}) = \dfrac{v^{2p+1} - v + (1+v)J(s)}{v^{2p}(1+v)}$

(b) $J(\neg s_{-p}) = \dfrac{v^{2p} + v - (1+v)J(s)}{v^{2p-1}(1+v)}$.

Proof. (a) We shall proceed by recurrence.

$$J(s) = [1 - J(\neg s_{-1})]v$$
$$\Rightarrow J(\neg s_{-1}) = 1 - \frac{J(s)}{v}$$

Now,

$$J(\neg s_{-1}) = [1 - J(s_{-1})]v$$
$$\Rightarrow J(s_{-1}) = 1 - \frac{J(\neg s_{-1})}{v} = 1 - \frac{1 - \frac{J(s)}{v}}{v}$$
$$= \frac{v^2 - v + J(s)}{v^2} = \frac{v^3 - v + (1+v)J(s)}{v^2(1+v)}$$

Now let us assume that $J(s_{-(p-1)}) = \dfrac{v^{2p-1} - v + (1+v)J(s)}{v^{2p-2}(1+v)}$

$$J(\neg s_{-p}) = [1 - J(s_{-p})]v \Rightarrow J(s_{-p}) = 1 - \frac{J(\neg s_{-p})}{v}$$

$$J(s_{-(p-1)}) = [1 - J(\neg s_{-p})]v \Rightarrow J(\neg s_{-p}) = 1 - \frac{J(s_{-(p-1)})}{v}$$

$$J(s_{-p}) = 1 - \frac{1 - \frac{J(s_{-(p-1)})}{v}}{v} = \frac{v^2 - v + J(s_{-(p-1)})}{v^2}$$
$$= \frac{v^3 - v + (1+v)J(s_{-(p-1)})}{v^2(1+v)}$$

Replacing in the above expression $J(s_{-(p-1)})$ by its value, one gets

$$J(s_{-p}) = \frac{v^{2p+1} - v + (1+v)J(s)}{v^{2p}(1+v)}$$

(b) $J(\neg s_{-p}) = [1 - J(s_{-p})]v = (1 - \dfrac{v^{2p+1} - v + (1+v)J(s)}{v^{2p}(1+v)})v$
$$= \frac{v^{2p} + v - (1+v)J(s)}{v^{2p-1}(1+v)}.$$ □

Proposition 4.

(a) p, J(s), and v satisfy the following equation: $v^{2p} + v[1 - J(s)] - J(s) = 0$
(b) $v \le \frac{J(s)}{1-J(s)}$.

Proof.

(a) By Proposition 3, $J(\neg s_{-p}) = \frac{v^{2p}+v-(1+v)J(s)}{v^{2p-1}(1+v)}$ Now $\neg s_{-p}$ has a unique parent which is the root of the E-path. This means that $J(\neg s_{-p}) = 0$. Whence the conclusion.

(b) If v = 0, the result is obvious. So let us assume that $v \neq 0$. It stems from Proposition 3 that:

$$J(s_{-p}) - J(s_{-(p-1)}) = \frac{v^{2p} + v - (1+v)J(s)}{v^{2p-1}(1+v)} - \frac{v^{2p-1} - v + (1+v)J(s)}{v^{2p-2}(1+v)}$$
$$= \frac{(1-v)(-v + (1+v)J(s))}{v^{2p}}$$

As on an E-path, the positive playability of strategies of odd rank decreases with their rank, and as in the present case s_{-p} is by assumption the root of a path, $J(s_{-p}) \geq J(s_{-(p-1)})$

It follows that $-v + (1 + v)J(s) \geq 0$, which is equivalent to $v \leq \dfrac{J(s)}{1 - J(s)}$[16]

□

Besides the fact that the hidden part of the graph which is looked for is an E-path, let us assume, for instance, that:

- the non-hidden part of the graph is also an E-path in which s has three descendants
- $J(s) = 0.48$.

One can then establish that:

- the hidden part of the graph is an E-path comprised of two nodes (p=1)
- v = 0.48
- the condition $v \leq \dfrac{J(s)}{1 - J(s)}$ is satisfied.

To proceed to a more general analysis, let us consider $v^{2p} + v[1 - J(s)] - J(s) = 0$, as a polynomial equation in v, with p and J(s) being parameters. This equation can be represented by the following 3D graph (Fig. 7):

To this 3D graph corresponds the 2D representation shown in Fig. 8, in which p is now a parameter.

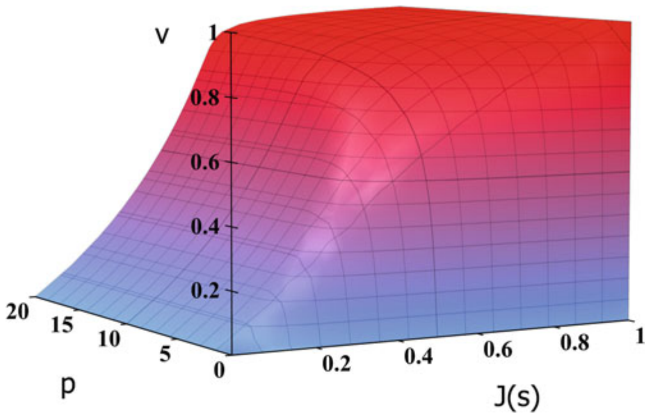


Fig. 7 v as a function of $J(s)$ and p

[16] The same conclusion can be obtained by comparing the positive playabilities of strategies of even rank, taking in account this time the fact that such positive playabilities are increasing with the rank [10].

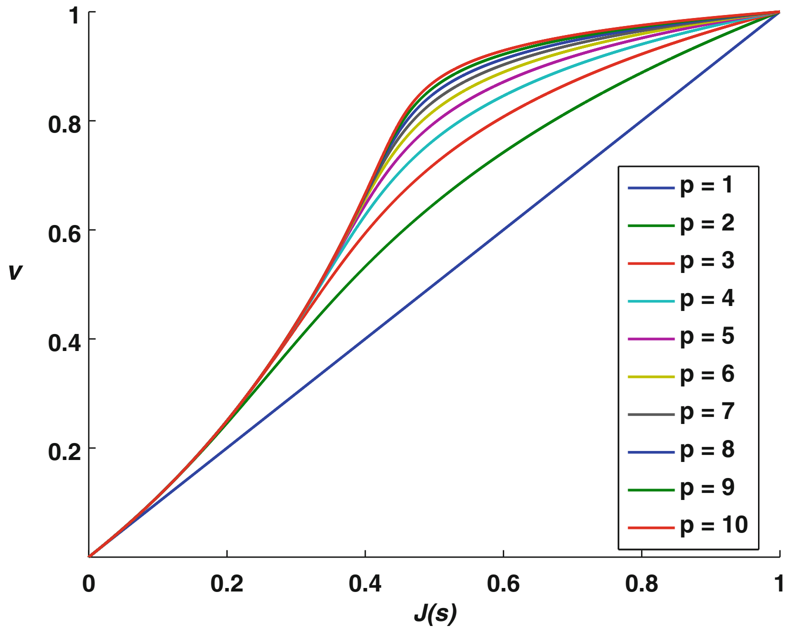


Fig. 8 v as a function of $J(s)$

Figure 8 shows that v is an increasing function of J(s) and of p. The possible variations of p have a small impact for values of J(s) that are near either 0 or 1. On the opposite this impact is more significant for values of J(s) that are near 0.5.

Another possible interpretation of these observations is that when one tries to reconstitute the hidden part of the graph, a number of solutions are possible a priori. But for the values of J(s) near the extremities of the interval [0,1], the selection of a particular value of p has little effect on v. In other words the length of the E-path which constitutes the hidden part of the graph can be diverse. This means that in terms of application to intelligence analysis, the sources of evidences may be structured according to different ways, and one can envisage not to take directly into account all possible sources or rebuttals of s.

Now, it seems reasonable to start by looking for the possible sources of s which distance to the latter is minimal, given of course that such sources may themselves have sources, and so on. The advantage of such selection is twofold:

- on the one hand, it focuses on sources that are either direct causes of s or causes that are not too far
- on the other hand, this focus translates into a simplification of the analysis.

5.4 Consequences for Intelligence Analysis

The translation of such procedure into practice which may serve as a guideline for the work of the intelligence analyst implies the following four stages:

(a) develop a list of possible sources
(b) classify the elements of that list, according to the assumed distance between these elements and s
(c) possibly determine the playability of the elements of the list, independently from J(s)
(d) compare the values found with $J(\neg s_p)$, and in case there is a too big difference, discard the corresponding sources.

To illustrate this, let us come back to the full IPC scenario in which the prior probabilities of T and S are $p\ (T = true) = 0.43016$ and $p\ (S = true) = 1$, and assume that:

- in the corresponding Graph of Deterrence, the positive playabilities of the roots take these values (i.e., $J(T) = 0.43016$ and $J(S) = 1$)[17]
- one looks for a hidden part of the graph that is an E-path.

The polynomial equation then writes $v^{2p} + 0.56984v - 0.43016 = 0$.

For p = 1, this equation has a unique solution (comprised between 0 and 1), which is: $v = 0.43016$.[18] This means that T has a source which is an antecedent located at a distance equal to 2. Consequently, in this case, the whole Graph of Deterrence (including the hidden part) is the one represented in Fig. 9 and the corresponding fuzzy solution of the playability system is

$$J(T_{-1}) = 1$$
$$J(\neg T_{-1}) = 0$$
$$J(T) = v = 0.43016$$
$$J(S) = 1$$
$$J(\neg C) = 0$$
$$J(C) = v = 0.43016$$
$$J(\neg B) = J(\neg F) = v - v^2 = 0.24512$$
$$J(B) = J(F) = v - v^2 + v^3 = 0.32472$$

[17]The justification of such assumption will be discussed in the next section.

[18]In other words if $p = 1$, then $v = J(T)$.

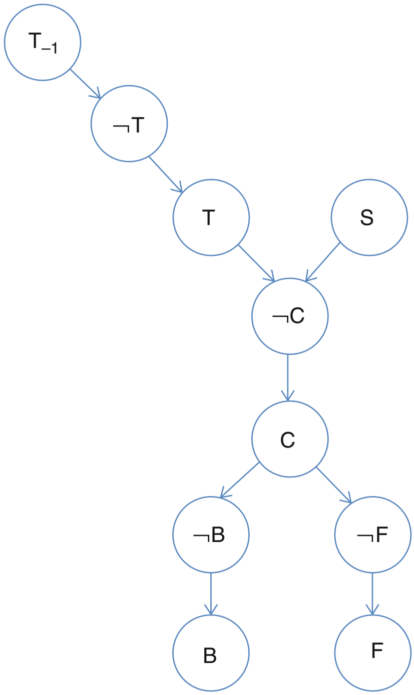


Fig. 9 The "augmented" Graph of Deterrence

One can easily check that this fuzzy solution satisfies the equation of the playability system defining the value of the playability by default index of player R:

$$1 - v = (1 - v + v^2)^2$$

As it can be seen, v and the positive playability indices of C, B, and F take the same values than in the graph with no hidden part. This is not a coincidence, but stems from the structures of both the Graph of Deterrence representation of the Bayesian Network and the hidden part of the graph. Indeed:

Proposition 5. *Given a Graph of Deterrence with two roots s' and s'' having a common child, adding a two-node path on the top of one root, for instance, s', makes the value of s' positive playability index equal to v, and keeps unchanged the values of v and of s' and s'' descendants positive playability indices.*

Proof. (a) Adding the two-node path on the top of s′, implies that the latter is then a strategy of rank 3 on the corresponding path. Consequently, its positive playability is now equal to v.

(b) Let s_{+1} be the common child of s′ and s″. The change in the rank of s′ and accordingly the change of its positive playability has no effect on the positive playability of s_{+1}, since the latter is still the child of s″ which status of root has not been modified by the introduction of the two-node path. It then follows that the same applies to all descendants of s_{+1}. □

Corollary 3. *Adding several two-node paths with s′ as the common last descendant, has the same effect on positive playabilities than adding only one two-node path.*

Proof. (a) On each of these two-node paths s′ is the child of a strategy of rank 2. This strategy is therefore characterized by an index of positive playability equal to 0. Consequently J(s′) = v

(b) As the positive playability of s′ is the same that in the case of a Graph of Deterrence with no hidden part, the positive playability of the other strategies remains the same. □

Proposition 5 and Corollary 3 imply that

(a) In terms of positive playability of strategies other than s' on the graph with no hidden path, there is an equivalence,—that will be called s'-equivalence—between that graph and the graphs with a hidden part composed of two-nodes paths as described here above
(b) The s'-equivalence does not depend on the number of two-node paths that constitute the hidden part of the graph.

Looking at the consequences for intelligence analysis, this means that:

(a) the existence of an underground terrorist group has direct sources which are established as true evidences
(b) the characteristics of the graph do not enable to determine the number of such direct sources[19]

Besides direct sources, the intelligence analyst may also have to look further back to detect indirect sources, i.e., sources corresponding to $p > 1$. For instance:

(a) for p = 2, v = 0.56984
(b) for p = 3 one gets v = 0.6373.

It can be shown that in both cases, the equation giving the value of player R's playability by default index is not satisfied, which means that there is no source corresponding to p = 2 or p = 3.

Now the existence of direct sources may depend on the value of J(T). For instance, everything else being the same, if keeping p = 1, the prior probability of T equals 0.75, the equation giving the playability by default index of player R is not satisfied. This means that in such case, the existence of an underground terrorist group may still have direct sources, but none of them has been established as a true evidence. So, the intelligence analyst has then to go further back.

[19]This of course might not be the case with another type of graph.

This does not imply of course that this cause is the only one: maybe other sources can be found when one looks for a hidden part which is, for instance, an E-tree, an R-path, etc.

6 Probabilities and Playability

In the above developments, s being a node of the Graph of Deterrence associated with a root of the associated Bayesian Network, it has been assumed that the value of J(s) is known, determined as a particular function of the prior probability associated with the Bayesian Network root. Now such assumption may a priori be questioned, with as a consequence, the fact that the developments made in the present paper may also be questioned: indeed if there is no function connecting probability and playability, what is the value of the results obtained? The answer lies in the existence of at least two different approaches to address that issue.

The first and most general one consists in comparing the laws of probabilities with the laws of playability, in order to possibly define playability as a function of probability. This requires an in-depth analysis based on specific Games of Deterrence tools, enabling to analyze multi-strategy games, i.e., games in which the players may select not only one strategy but also several [12]. The complexity and length of such analysis make it out of scope of the present paper.

The second approach looks at the relation between probability and playability, not in general, but just at the level of the prior probability. One possibility that can be thought of quite straightforwardly is to adopt a convention according to which the positive playability of s equals the prior probability of the Bayesian Network root associated with s. For instance, in the IPC case the prior probability pr (T) of the existence of an underground terrorist group is 0.43016. So the positive playability of the associated strategy T also equals 0.43016. Although there is presently no mathematical evidence of such equality, the convention can be justified as follows:

(a) The Graph of Deterrence provides a representation of a set of logical propositions which truth or falsity is assessed through probabilities.
(b) In that representation, the positive playability of a strategy equals 1 if the associated proposition is true, and 0 if it is false. Likewise the probability of the evidence associated with the proposition equals 1 if the evidence is true and 0 if it is false.
(c) It seems almost common sense that the positive playability of the strategy increases with the probability of occurrence of the corresponding node in the Bayesian Network.
(d) Therefore as both the probability and the playability are numbers comprised between 0 and 1, one may extend the property expressed in (b) to the whole interval [0,1].
(e) Whatever the relation between playability and probability based on the association between Bayesian Networks and Graphs of Deterrence, the various propositions and corollaries stated in the above sections of the present paper hold.

It should be noticed that the fact that the equality between the prior probability of a Bayesian Network root and the positive playability of the corresponding root of the non-hidden part of the Graph of Deterrence is valid, does not mean that it prevails at other nodes.

7 Conclusions

The aim of the present paper was to lay down early foundations of a data analysis approach that, instead of resorting to the classical Bayesian Networks, uses a particular category of qualitative games called Games of Deterrence. The reason for it is twofold: on the one hand, these games have proven their ability to deal with logical issues, in particular with inference schemes; on the other hand, this approach provides an alternative to the standard Bayesian Networks algorithms, when the number of data to be taken into consideration is high.

The first part of the paper recalled the core properties of matrix Games of Deterrence. The second part showed how these games can be used for data analysis. Then a connection was established between the graphs associated with Bayesian Networks and the graphs associated with matrix Games of Deterrence at both levels of the nodes and the edges. This was done in particular through expressing implications present in the Bayesian Networks under the form of rebuttals as they figure in the graphs deriving from matrix Games of Deterrence.

Then one has taken into account the fact that priors in Bayesian Networks might generate in Graphs of Deterrence what has been called "the hidden parts of the graph": the starting point was the assumption that if in the Bayesian Network, a prior probability is different from 1, it might translate in the Graph of Deterrence by the fact that the positive playability index of a root is also different from 1, meaning that in such case the node is not really a root. The validity of this assumption was established in the case of the IPC simplified scenario.[20]

Then, assuming momentarily that the translation problem was solved, and considering more specifically the cases where the hidden parts of the graph are E-paths, some properties of these hidden parts have been determined. In particular a relation has been established in terms of positive playability between a root of the non-hidden part of the graph and the root of the hidden part connected to the former, taking into consideration the distance between the two nodes.

This paves the way to future developments in at least three directions. The first one is to formalize the translation of a prior probability into a positive playability for the nodes of the Graph of Deterrence associated with roots of the Bayesian Network. The second direction concerns the generalization of the results found in the IPC case with a hidden part of the graph being an E-path, to other scenarios. The third direction concerns the extension of these results to the cases characterized by other kinds of hidden parts like E-trees, R-paths, or R-trees.

[20]No social climate being taken into account.

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A New Look at the Study of Solutions for Games in Partition Function Form

Joss Sánchez-Pérez

Abstract This chapter studies the structure of games in partition function and according to an axiomatic point of view, we provide a global description of linear symmetric solutions by means of a decomposition of the set of such games (as well as of a decomposition of the space of payoff vectors). The exhibition of relevant subspaces in such decomposition and based on the idea that every permutation of the set of players may be thought of as a linear map, allow for a new look at linear symmetric solutions.

Keywords Cooperative games in partition function form • Axiomatic solutions • Symmetric group

1 Introduction

There are a variety of economic and social contexts in which the activities of one group of agents affect payoffs of other groups. Consider, for instance, the issue of political alliances between different groups of countries. The benefit to each group will typically depend on the strength of the alliance between opposing groups of countries. In a similar way, the benefit to one group of agents from activities aimed at controlling pollution depends upon whether other agents are also engaged in similar pollution abatement exercises.

Given the widespread presence of externalities, it is important to study the distributional issue in environments with externalities. A game in partition function form, defined in [11], in which the worth of any coalition depends on how players outside the coalition are organized, provides an appropriate framework within which one can describe solution concepts for situations where externalities are considered. The advantage of this model is that it takes into account both internal factors (coalition itself) and external factors (coalition structure) that may affect cooperative

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outcomes and allows to go deeper into cooperation problems. Thus, it is closer to real life but more complex to analyze.

A number of solutions have been developed in the literature that deals with games in partition function form. Myerson [13] proposed the first value for this type of games and then Bolger [3] also provided a unique value. More recently, Albizuri et al. [1], Macho-Stadler et al. [12], Ju [8], Pham Do and Norde [14], and Hu and Yang [7] apply the axiomatic approach to characterize a value for these games.

Following an axiomatic methodology, this work provides a global description of linear symmetric solutions for games in partition function form by means of a decomposition of the set of such games. This decomposition, based on the idea that every permutation of the set of players may be thought of as a linear map,[1] allows for a new look at linear symmetric solutions.

There is one relevant fact that we must point out: the set of games in partition function form is much more than a vector space. Games in partition function form and abstract algebra become related after one recognizes that the notion of a permutation of players gives rise to a certain linear map. We then use this observation to attempt to turn the usual approach in the study of solutions for games in partition function form by analyzing such space from an algebraic viewpoint. In this sense, we can see if the algebraic structure of the space of games has anything to say about what constructs are significant from a game theoretic point of view.

Therefore, our primary goal is to show how certain tools from abstract algebra can be used to make sense of foundational ideas in games in partition function form, and how using these tools can in turn help us to obtain meaningful information concerning linear symmetric solutions in this type of games.

Roughly speaking, what we do is to compute a decomposition of the space of games in partition function form as a direct sum of three orthogonal subspaces: a subspace of "symmetric" games, another subspace which we call V, and the common kernel of all linear symmetric solutions. Although V does not have a natural definition in terms of well-known game theoretic considerations, it has a simple characterization in terms of vectors in $\mathbb{R}^n$ all of whose entries add up to zero. According to this decomposition, any linear symmetric solution when restricted to any such elementary piece is either zero or multiplication by a single scalar; therefore, all linear symmetric solutions may be written as a sum of trivial maps.

Once we have a global description of all linear and symmetric solutions, it is easy to understand the restriction imposed by the efficiency axiom. We then use such decomposition to provide, in a very economical way, a characterization for the class of linear symmetric solutions and the general expression for all linear, symmetric and efficient solutions.

As far as we know, there are not many works related to the approach proposed in this chapter to the study of topics in cooperative game theory. Kleinberg and Weiss [9] construct a direct sum decomposition of the null space of the Shapley value for games in characteristic function form (TU games). In [10], the authors followed

[1]The precise statement will be provided in Sect. 3.

the same line of reasoning to characterize the space of linear and symmetric values for TU games. More recently, Hernández-Lamoneda et al. [5] provide a complete analysis (providing a decomposition for the set of TU games) following the above scheme to study solutions for TU games. Finally, in [15], it is discussed about how to use a decomposition for the space of games in partition function form (for the particular cases with 3 and 4 players) to the characterization of solutions.

The chapter is organized as follows. We first recall the main basic features of games in partition function form in the next section. A decomposition for the space of games in partition function form is introduced in Sect. 3. We then present a description of linear symmetric solutions based in this decomposition, including characterizations of classes of solutions. Section 5 concludes the chapter and long proofs are relegated to an Appendix.

This introduction ends with a brief comment on the methods employed in the chapter. Although it is true that the characterization results could be proved without any explicit mention to the proposed decomposition, we believe, it sheds new light on the structure of the space of games in partition function form and their solutions. Part of the purpose of the present chapter is to share this viewpoint and encourage readers to contribute to these fields by going well beyond what we present here, we think we only scratch the surface on what could be done with tools from abstract algebra in games in partition function form.

To make this work as self contained as possible we have included an Appendix, with some facts and definitions for the sake of easier understanding of the proposed scheme.

2 Basic Definitions and Conventions

In this section we provide some concepts and notations related to n-person games in partition function form, as well as a brief subsection of preliminaries related to partitions of integers, since it is a necessary subject in subsequent developments.

Let $N = \{1, 2, \ldots, n\}$ be a fixed nonempty finite set, and let the members of N be interpreted as players in some game situation. Given N, let CL be the set of all coalitions (nonempty subsets) of N, $CL = \{S \mid S \subseteq N, S \neq \varnothing\}$. Let PT be the set of partitions of N, so

$$\{S_1, S_2, \ldots, S_m\} \in PT \ \text{ iff } \ \bigcup_{i=1}^{m} S_i = N,\ S_j \neq \varnothing\ \forall j,\ S_j \cap S_k = \varnothing\ \forall j \neq k$$

Additionally, for $Q \in PT$ and $i \in N$, Q^i denotes the member of Q where i belongs.

Also, let $EC = \{(S, Q) \mid S \in Q \in PT\}$ be the set of *embedded coalitions*, that is the set of coalitions together with specifications as to how the other players are aligned.

Definition 1. A function

$$w : EC \to \mathbb{R}$$

that assigns a real value, $w(S, Q)$, to each embedded coalition (S, Q) is called a game in partition function form. The set of games in partition function form with player set N is denoted by G, i.e.,

$$G = G^{(n)} = \{w \mid w : EC \to \mathbb{R}\}$$

The number $w(S, Q)$ represents the payoff of coalition S, given the coalition structure Q forms. In this kind of games, the worth of some coalition depends not only on what the players of such coalition can jointly obtain, but also on the way the other players are organized. We assume that, in any game situation, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w(N, \{N\})$ to divide among themselves. But we also anticipate that the actual allocation of this worth will depend on all the other potential worth $w(S, Q)$, as they influence the relative bargaining strengths of the players.

Given $w_1, w_2 \in G$ and $c \in \mathbb{R}$, we define the sum $w_1 + w_2$ and the product cw_1, in G, in the usual form, i.e.,

$$(w_1 + w_2)(S, Q) = w_1(S, Q) + w_2(S, Q) \quad \text{and} \quad (cw_1)(S, Q) = cw_1(S, Q)$$

respectively. It is easy to verify that G is a vector space with these operations.[2]

We assume that, in any game situation represented by w, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w(N, \{N\})$ to divide among themselves. But we also anticipate that the actual allocation of this wealth will depend on all the other potential worths, as they influence the relative bargaining strengths of the players. So we are interested in value functions:

Definition 2. A solution (on G) is a mapping

$$\varphi : G \to \mathbb{R}^n$$

If φ is a solution and $w \in G$, then we can interpret $\varphi_i(w)$ as the utility payoff which player i should expect from the game w.

Now, the group of permutations of N, $S_n = \{\theta : N \to N \mid \theta \text{ is bijective}\}$, acts on CL and on EC in the natural way; i.e., for $\theta \in S_n$:

$$\theta(S) = \{\theta(i) \mid i \in S\}$$
$$\theta(S_1, \{S_1, S_2, \ldots, S_l\}) = (\theta(S_1), \{\theta(S_1), \theta(S_2), \ldots, \theta(S_l)\})$$

[2]Notice that G can be identified with the set of real vectors indexed on the elements of EC, and so, $\dim G = |EC|$.

In a similar manner, S_n acts on the space of payoff vectors, $\mathbb{R}^n$:

$$\theta(x_1, x_2, \ldots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$$

Next, we define the usual linearity, symmetry, and efficiency axioms which are asked solutions to satisfy in the cooperative game theory framework.

Axiom 1 (Linearity). *The solution φ is linear if*

$$\varphi(w_1 + w_2) = \varphi(w_1) + \varphi(w_2) \quad \textit{and} \quad \varphi(cw_1) = c\varphi(w_1)$$

for all $w_1, w_2 \in G$ and $c \in \mathbb{R}$.

Axiom 2 (Symmetry). *The solution φ is said to be symmetric if and only if*

$$\varphi(\theta \cdot w) = \theta \cdot \varphi(w)$$

for every $\theta \in S_n$ and $w \in G$, where the game $\theta \cdot w$ is defined as $(\theta \cdot w)(S, Q) = w[\theta^{-1}(S, Q)]$.[3]

Axiom 3 (Efficiency). *The solution φ is efficient if*

$$\sum_{i \in N} \varphi_i(w) = w(N, \{N\})$$

for all $w \in G$.

The axiom of linearity means that when a group of players shares the benefits (or costs) stemming from two different issues, how much each player obtains does not depend on whether they consider the two issues together or one by one. Hence, the agenda does not affect the final outcome. Also, the sharing does not depend on the unit used to measure the benefits.

Symmetry means that player's payoffs do not depend on their names. The payoff of a player is only derived from his influence on the worth of the coalitions.

As we stated before, we assume that the grand coalition forms and we leave issues of coalition formation out of this work. Efficiency then simply means that the value must be feasible and exhaust all the benefits from cooperation, given that everyone cooperates.

[3] So, $\theta \cdot w$ is the partition function game which would result from w if we relabeled the players by permutation θ.

2.1 Partitions of Integers

A partition of a positive integer is a way of expressing it as the unordered sum of other positive integers, and it is often written in tuple notation. Formally,

Definition 3. $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_l]$ is a partition of n iff $\lambda_1, \lambda_2, \ldots, \lambda_l$ are positive integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Two partitions which only differ in the order of their elements are considered to be the same partition.

$\Pi(n)$ will denote the set of all partitions of n, and if $\lambda \in \Pi(n)$, $|\lambda|$ is the number of elements of λ.

In an integer partition, the order of the summands is unimportant, but for consistency, partitions of n will be written with the summands in a non-increasing order.

For instance, the partitions of $n = 4$ are $[1, 1, 1, 1]$, $[2, 1, 1]$, $[2, 2]$, $[3, 1]$, and $[4]$. Sometimes we will abbreviate this notation by dropping the commas, so $[2, 1, 1]$ becomes $[211]$.

If $Q \in PT$, there is a unique partition $\lambda_Q \in \Pi(n)$, associated with Q, where the elements of λ_Q are exactly the cardinalities of the elements of Q. In other words, if $Q = \{S_1, S_2, \ldots, S_m\} \in PT$, then $\lambda_Q = [|S_1|, |S_2|, \ldots, |S_m|]$.

For a given $\lambda \in \Pi(n)$, we represent by λ° the set of numbers determined by the λ_i's and for $k \in \lambda^\circ$, we denote by m_k^λ the multiplicity of k in partition λ. So, if $\lambda = [5, 4, 4, 4, 1, 1]$, then $|\lambda| = 6$, $\lambda^\circ = \{1, 4, 5\}$ and $m_1^\lambda = 2$, $m_4^\lambda = 3$, $m_5^\lambda = 1$.

If $[\lambda_1, \lambda_2, \ldots, \lambda_l] \in \Pi(n)$, for $k \geq 1$ we define $[\lambda_1, \lambda_2, \ldots, \lambda_l] - [\lambda_1, \lambda_2, \ldots, \lambda_k] = [\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_l]$. For example, $[4, 3, 2, 1, 1, 1] - [3, 1, 1] = [4, 2, 1]$.

For every $\lambda \in \Pi(n)\backslash\{[n]\}$ and every $k \in \lambda^\circ$, let $I_{\lambda,k}$ be a set such that

$$I_{\lambda,k} = \begin{cases} \lambda^\circ\backslash\{k\} & \text{if } m_k^\lambda = 1 \\ \lambda^\circ & \text{if } m_k^\lambda > 1 \end{cases}$$

Finally, we need to define certain sets which are used in the sequel.

Definition 4. Let A_n be a set of pairs, associated with all partitions $\lambda \in \Pi(n)$ and its elements, i.e.,

$$A_n = \{(\lambda, k) \mid \lambda \in \Pi(n), k \in \lambda^\circ\}$$

and similarly, define the set of triples

$$B_n = \{(\lambda, k, j) \mid \lambda \in \Pi(n)\backslash\{[n]\}, k \in \lambda^\circ, j \in I_{\lambda,k}\}$$

Example 1. If $n = 3$, then $\Pi(3) = \{[1, 1, 1], [2, 1], [3]\}$ and

$$A_3 = \{([111], 1), ([21], 1), ([21], 2), ([3], 3)\}$$

and

$$B_3 = \{([111], 1, 1), ([21], 1, 2), ([21], 2, 1)\}$$

3 Direct Sum Decompositions

In this section we compute a decomposition of the space of games G, as well as a decomposition for the space of payoff vectors, $\mathbb{R}^n$. Such decompositions are orthogonal direct sums of "elementary" subspaces and will provide information about solutions. In particular, any linear symmetric when restricted to any such elementary subspace is either zero or multiplication by a single scalar.[4]

We begin with the decomposition of $\mathbb{R}^n$, which is easier than the one for G. Recall that (in $\mathbb{R}^n$) every permutation $\theta \in S_n$ of the set of players may be thought of as a linear map by permuting the coordinates of any vector according to θ:

$$\theta(x_1, x_2, \ldots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$$

What we wish is to write $\mathbb{R}^n$ as a direct sum of subspaces, each invariant for all permutations[5] and in such way that the summands cannot be further decomposed (i.e., they are irreducible[6]).

For this, let

$$\Delta_n = \{(t, t, \ldots, t) \mid t \in \mathbb{R}\} \quad \text{and} \quad \Delta_n^{\perp} = \left\{ z \in \mathbb{R}^n \mid \sum_{i=1}^{n} z_i = 0 \right\}$$

Notice that Δ_n is a trivial subspace in the sense that every permutation acts as the identity transformation.

Proposition 1. *The decomposition of $\mathbb{R}^n$, under S_n, into irreducible subspaces is*

$$\mathbb{R}^n = \Delta_n \oplus \Delta_n^{\perp}$$

Proof. First, it is clear that $\Delta_n \cap \Delta_n^{\perp} = \{\mathbf{0}\}$.[7] We now prove that $\mathbb{R}^n = \Delta_n + \Delta_n^{\perp}$:

(i) If $z \in (\Delta_n + \Delta_n^{\perp})$, then $z \in \mathbb{R}^n$ since $(\Delta_n + \Delta_n^{\perp})$ is a subspace of $\mathbb{R}^n$.
(ii) For $z \in \mathbb{R}^n$, let $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ and z can be written as $z = (\overline{z}, \overline{z}, \ldots, \overline{z}) + (z_1 - \overline{z}, z_2 - \overline{z}, \ldots, z_n - \overline{z})$; and so, $z \in (\Delta_n + \Delta_n^{\perp})$.

Finally, since Δ_n is 1-dimensional, then it is irreducible. To check that $\Delta_n^{\perp}$ is also irreducible, it is an induction argument that can be found in [5].

[4]The formal statement will be found at the end of this section.

[5]Formally, if Y be a subspace of a vector space X, then Y is invariant (for the action of S_n) if for every $y \in Y$ and every $\theta \in S_n$, we have that $\theta \cdot y \in Y$.

[6]That is, a subspace Y is irreducible if Y itself has no invariant subspaces other than $\{0\}$ and Y itself.

[7]Here, $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$.

On the other hand, the group S_n acts naturally on the space of games in partition function form, G, via linear transformations. That is, each permutation $\theta \in S_n$ corresponds to a linear, invertible transformation, which we still call θ, of the vector space G, namely

$$(\theta \cdot w)(S, Q) = w[\theta^{-1}(S, Q)]$$

for every $\theta \in S_n$, $w \in G$ and $(S, Q) \in EC$.

Moreover, this assignment preserves multiplication in the sense that the linear map corresponding to the product of the two permutations $\theta_1\theta_2$ is the product (or composition) of the maps corresponding to θ_1 and θ_2, in that order.

Now, the decomposition of G is carried out in three steps:

Step 1.
For each $\lambda \in \Pi(n)$, define the subspace of games

$$G_\lambda = \{w \in G \mid w(S, Q) = 0 \text{ if } \lambda_Q \neq \lambda\}$$

Thus,

$$G = \bigoplus_{\lambda \in \Pi(n)} G_\lambda$$

Step 2.
Given $\lambda \in \Pi(n)$ and for $k \in \lambda^\circ$, inside G_λ define the subspace

$$G_\lambda^k = \{w \in G_\lambda \mid w(S, Q) = 0 \text{ if } |S| \neq k\}$$

Thus each G_λ has a decomposition $G_\lambda = \bigoplus_{k \in \lambda^\circ} G_\lambda^k$ and so, we get the following decomposition for G:

$$G = \bigoplus_{(\lambda,k) \in A_n} G_\lambda^k \tag{1}$$

Notice that each subspace G_λ^k is invariant under S_n and the decomposition is orthogonal with respect to the invariant inner product on G given by $\langle w_1, w_2\rangle = \sum_{(S,Q)\in EC} w_1(S, Q) \cdot w_2(S, Q)$.[8] Here, invariance of the inner product means that every permutation $\theta \in S_n$ is not only a linear map on G, but also an orthogonal map with respect to this inner product. Formally, $\langle\theta \cdot w_1, \theta \cdot w_2\rangle = \langle w_1, w_2\rangle$ for every $w_1, w_2 \in G$.

[8]This seems like the natural inner product to consider, since intuitively G can be identified with $\mathbb{R}^{EC}$.

Step 3.
Our next goal is to obtain a decomposition of each subspace of games G_λ^k into irreducible subspaces and so, we will get it for G.
The following games play an important role in describing the decomposition of the space of games. For each $(\lambda, k) \in A_n$, define $u_\lambda^k \in G_\lambda^k$ as follows:

$$u_\lambda^k(S,Q) = \begin{cases} 1 \text{ if } |S| = k \text{ and } \lambda_Q = \lambda \\ 0 \text{ otherwise} \end{cases}$$

Also, for each $(\lambda, k) \in A_n$ and for each $x \in \mathbb{R}^n$; define the game $x^{(\lambda,k)} \in G_\lambda^k$ as:

$$x^{(\lambda,k)}(S,Q) = \begin{cases} \sum\limits_{i \in S} x_i \text{ if } |S| = k \text{ and } \lambda_Q = \lambda \\ 0 \quad\quad \text{otherwise} \end{cases}$$

The next theorem provides us a decomposition of the space of games, into irreducible subspaces.

Theorem 1. *For each* $(\lambda, k) \in A_n \backslash \{([n], n)\}$,

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k \oplus W_\lambda^k$$

where

(i) $U_\lambda^k = \langle u_\lambda^k \rangle \simeq \Delta_n$
(ii) $V_\lambda^k = \bigoplus\limits_{j \in I_{\lambda,k}} \left\{ x_j^{(\lambda,k)} \mid x_j \in \Delta_n^\perp \right\}$, *in which every* $\left\{ x_j^{(\lambda,k)} \mid x_j \in \Delta_n^\perp \right\} \simeq \Delta_n^\perp$
(iii) W_λ^k *does not contain any summands isomorphic to either* Δ_n *or* $\Delta_n^\perp$

Moreover, the decomposition is orthogonal.

Proof. See Appendix.

Remark 1. It is important to mention that the previous theorem does not quite give us a decomposition of G_λ^k into irreducible summands. The subspace U_λ^k is irreducible and V_λ^k is a direct sum of irreducible subspaces. Whereas W_λ^k may or may not be irreducible (depending on λ and k), but as we shall see the exact nature of this subspace plays no role in the study of linear symmetric solutions since it lies in the kernel of any such solution.

Set $U = \bigoplus\limits_{(\lambda,k) \in A_n} U_\lambda^k$. This is a subspace of games, whose values $w(S,Q)$ depend only on the cardinality of S and on the structure of Q.[9] According to Theorem 1, U is the largest subspace of G where S_n acts trivially.

[9]This type of games may be thought of as its counterpart for symmetric games in TU games.

Let $V = \bigoplus_{(\lambda,k)\in A_n\setminus\{([n],n)\}} V^k_\lambda$ and $W = \bigoplus_{(\lambda,k)\in A_n\setminus\{([n],n)\}} W^k_\lambda$, then:

$$G = U \oplus V \oplus W$$

Notice that W will be non-zero as soon as $n > 3$.

Example 2. Consider the case $N = \{1, 2, 3, 4\}$. According to (1) in Step 2, $G^{(4)}$ decomposes:

$$G^{(4)} = G^1_{[1,1,1,1]} \oplus G^1_{[2,1,1]} \oplus G^2_{[2,1,1]} \oplus G^1_{[3,1]} \oplus G^3_{[3,1]} \oplus G^2_{[2,2]} \oplus G^4_{[4]}$$

4 A Description of Linear Symmetric Solutions

The decomposition results in the last section can be useful to provide a global description of linear symmetric solutions. In the following, we provide some implications based on such decompositions and the so-called Schur's Lemma (see Appendix for a precise statement).

The next result provides a good example of how the decomposition of G can be used to gain information about linear symmetric solutions.

Corollary 1. *If $\varphi : G \to \mathbb{R}^n$ is a linear symmetric solution, then $\varphi(w) = 0$ for every $w \in W$.*

Proof. Let $\varphi : G = U \oplus V \oplus W \to \mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$ be a linear symmetric solution. Suppose $Z \subset W$ is an irreducible summand in the decomposition of W (even while we do not know the decomposition of W as a sum of irreducible subspaces, it is known that such a decomposition exists). Let p_1 and p_2 denote orthogonal projection of $\mathbb{R}^n$ onto Δ_n and $\Delta_n^\perp$, respectively. Now, $\varphi : G \to \mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$ may be written as $\varphi = (p_1 \circ \varphi, p_2 \circ \varphi)$. Denote by $\iota : Z \to G$ the inclusion, then, the restriction of φ to Z may be expressed as $\varphi_{|Z} = \varphi \circ \iota = (p_1 \circ \varphi \circ \iota, p_2 \circ \varphi \circ \iota)$.

Finally, $p_1 \circ \varphi \circ \iota : Z \to \Delta_n$ and $p_2 \circ \varphi \circ \iota : X \to \Delta_n^\perp$ are linear symmetric maps; since Z is not isomorphic to either of these two spaces, thus Schur's Lemma says that $p_1 \circ \varphi \circ \iota$ and $p_2 \circ \varphi \circ \iota$ must be zero. Since this is true for every irreducible summand Z of W, φ is zero on all of W.

In other words, Corollary 1 is the statement that the common kernel of all linear symmetric solutions is W.

Remark 2. Notice that according to Theorem 1 and Corollary 1, in order to study linear symmetric solutions, one needs to look only at those games inside $U \oplus V$.

Remark 3. Also, from Theorem 1 we know that for every $(\lambda, k) \in A_n\setminus\{([n], n)\}$, G^k_λ contains exactly 1 copy of Δ_n and $|I_{\lambda,k}|$ copies of $\Delta_n^\perp$.

Example 3. For $n = 7$, the number of copies of $\Delta_n^\perp$ inside of G_λ^k for some particular values of λ and k:

G_λ^k		Number of copies
λ	k	of $\Delta_n^\perp$
[211111]	2	1
[2221]	2	2
[3211]	1	3
[322]	2	2
[322]	3	1
[421]	4	2
[7]	7	0

From the decomposition of G, given a game $w \in G$ we may decompose it relative to the above as $w = u + v + r$, where in turn $u = \sum a_{\lambda,k} u_\lambda^k$ and $v = \sum z_{\lambda,k,j}^{(\lambda,k)}$. This decomposition is very well suited to study the image of w under any linear symmetric solution. The reason being the following version of the well-known Schur's Lemma.

Theorem 2 (Schur's Lemma). *If φ is any linear symmetric solution*

$$\varphi : G = U \oplus V \oplus W \to \mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$$

then it satisfies:

(a) $\varphi(U) \subset \Delta_n$
(b) $\varphi(V) \subset \Delta_n^\perp$

Moreover,

- *for each $(\lambda, k) \in A_n$, there is a constant $\alpha\,(\lambda, k) \in \mathbb{R}$ such that, for every $u \in U_\lambda^k$,*

$$\varphi(u) = \alpha\,(\lambda, k) \cdot (1, 1, \dots, 1) \in \Delta_n$$

- *for each $(\lambda, k, j) \in B_n$, there is a constant $\beta\,(\lambda, k, j) \in \mathbb{R}$ such that, for every $z_{\lambda,k,j}^{(\lambda,k)} \in V_\lambda^k$,*

$$\varphi\left(z_{\lambda,k,j}^{(\lambda,k)}\right) = \beta\,(\lambda, k, j) \cdot z_{\lambda,k,j} \in \Delta_n^\perp$$

For many purposes it suffices to use merely the existence of the decomposition of the game $w \in G$, without having to worry about the precise value of each component. Nevertheless it will be useful to have it. Thus we give a formula for computing it.

Proposition 2. *Let $w \in G$. Then*

$$w = \sum_{(\lambda,k)\in A_n} a_{\lambda,k} u_\lambda^k + \sum_{(\lambda,k,j)\in B_n} z_{\lambda,k,j}^{(\lambda,k)} + r \tag{2}$$

where,

1. *$a_{\lambda,k}$ is the average of the values $w(S,Q)$ with S containing k players and Q with structure according to λ:*

$$a_{\lambda,k} = \frac{\sum\limits_{\substack{(S,Q)\in EC\\ |S|=k,\lambda_Q=\lambda}} w(S,Q)}{|(S,Q)\in EC : |S|=k, \lambda_Q=\lambda|}$$

2. *For every $(\lambda,k,j)\in B_n$, $z_{\lambda,k,j}\in \Delta_n^{\perp}$ is given by:*

$$\left(z_{\lambda,k,j}\right)_i = \sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}} \frac{j}{n} w(S,Q) - \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda,|Q^i|=j}} \frac{k}{n} w(S,Q)$$

3. *r may be computed as "the rest," i.e.,*

$$r = w - \sum_{(\lambda,k)\in A_n} a_{\lambda,k} u_\lambda^k - \sum_{(\lambda,k,j)\in B_n} z_{\lambda,k,j}^{(\lambda,k)}$$

Proof. See Appendix.

Example 4. Let $N=\{1,2,3\}$ and a game w described by

(S,Q)			$w(S,Q)$		
$\{1\}$	$\{2\}$	$\{3\}$	8	6	12
$\{1,2\}$	$\{3\}$		14	5	
$\{1,3\}$	$\{2\}$		26	0	
$\{2,3\}$	$\{1\}$		18	7	
$\{1,2,3\}$			33		

One then computes with the above formula the decomposition of such a game w as

$$w = \frac{26}{3}u_{[111]}^1 + 4u_{[21]}^1 + \frac{58}{3}u_{[21]}^2 + 33u_{[3]}^3 + z_{[111],1,1}^{([111],1)} + z_{[21],1,2}^{([21],1)} + z_{[21],2,1}^{([21],2)}$$

where

$$z_{[111],1,1} = \left(-\frac{2}{3}, -\frac{8}{3}, \frac{10}{3}\right), \; z_{[21],1,2} = (3, -4, 1) \;\text{ and }\; z_{[21],2,1} = \left(\frac{4}{3}, -\frac{20}{3}, \frac{16}{3}\right)$$

Recall that for the case $n = 3$, it turns out that $r = 0$.

Next, we show how to get characterizations of solutions easily by using the decomposition of a game given by (2) in conjunction with Schur's Lemma. We start by providing a characterization of all linear symmetric solutions $\varphi : G \to \mathbb{R}^n$ in the following:

Proposition 3. *The solution $\varphi : G \to \mathbb{R}^n$ is linear and symmetric if and only if it is of the form*

$$\varphi_i(w) = \sum_{(\lambda,k)\in A_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i, |S|=k \\ \lambda_Q=\lambda}} \gamma\,(\lambda, k) \cdot w(S, Q) + \sum_{(\lambda,k,j)\in B_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i, |S|=k \\ \lambda_Q=\lambda, |Q^i|=j}} \delta\,(\lambda, k, j) \cdot w(S, Q) \tag{3}$$

for arbitrary real numbers $\{\gamma(\lambda, k) \mid (\lambda, k) \in A_n\} \cup \{\delta(\lambda, k, j) \mid (\lambda, k, j) \in B_n\}$.

Proof. See Appendix.

Corollary 2. *The space of all linear and symmetric solutions on G has dimension $|A_n| + |B_n|$.*

Once we have such a entire characterization of all linear symmetric solutions, we can understand restrictions imposed by other conditions or axioms, for example, the efficiency axiom. The next result tells us how a linear symmetric solution behaves on every component of the decomposition of G in order to be an efficient solution.

Proposition 4. *Let $\varphi : G \to \mathbb{R}^n$ be a linear symmetric solution. Then φ is efficient if and only if*

(i) $\varphi_i(u_\lambda^k) = 0$*, for all* $(\lambda, k) \in A_n \backslash \{([n], n)\}$*; and*
(ii) $\varphi_i(u_{[n]}^n) = \frac{1}{n}$

Proof. First, $\left(U_{[n]}^n\right)^\perp$ is exactly the subspace of games w where $w(N, \{N\}) = 0$. Of these, those in V trivially satisfy $\sum_{i\in N} \varphi_i(w) = 0$, since (by Schur's Lemma) $\varphi(V) \subset \Delta_n^\perp$.

Thus, efficiency need only be checked in U. Since u_λ^k is fixed by every permutation in S_n, we have $\sum_{i\in N} \varphi_i(u_\lambda^k) = n\varphi_i(u_\lambda^k)$ and so, φ is efficient if and only if for $\lambda \neq [n]$, $n\varphi_i(u_\lambda^k) = u_\lambda^k(N, \{N\}) = 0$ and $n\varphi_i(u_{[n]}^n) = u_{[n]}^n(N, \{N\}) = 1$.

Recall that U is a subspace of games whose value on a given embedded coalition (S, Q) depends only on the cardinality of S and on the structure of Q. The next corollary characterizes the solutions to these games in terms of linearity, symmetry, and efficiency. It turns out that among all linear symmetric solutions, the egalitarian solution is characterized as the unique efficient solution on U. Formally,

Corollary 3. *There exists exactly one linear symmetric and efficient solution $\varphi : U \to \mathbb{R}^n$, and it is given by:*

$$\varphi_i(w) = \frac{w(N, \{N\})}{n}$$

for every $w \in U$.

In other words, all linear symmetric and efficient solutions coincide with the egalitarian solution when restricted to these type of games, U.

Now, using the decomposition of the space of games and the conditions provided in Proposition 4, we can obtain a characterization of all linear, symmetric, and efficient solutions.

Theorem 3. *The solution $\varphi : G \to \mathbb{R}^n$ satisfies linearity, symmetry, and efficiency axioms if and only if it is of the form*

$$\begin{aligned}\varphi_i(w) = &\tfrac{w(N,\{N\})}{n}\\ &+ \sum_{(\lambda,k,j)\in B_n} \delta(\lambda,k,j) \left[\sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}} jw(S,Q) - \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda,|Q^i|=j}} kw(S,Q) \right]\end{aligned} \tag{4}$$

for some real numbers $\{\delta(\lambda,k,j) \mid (\lambda,k,j) \in B_n\}$.

Proof. Let $\varphi : G \to \mathbb{R}^n$ be a linear, symmetric, and efficient solution; and $w \in G$. Then, applying Propositions 2 and 4, Corollary 1, and Schur's Lemma:

$$\begin{aligned}\varphi_i(w) &= \sum_{(\lambda,k)\in A_n} a_{\lambda,k}\varphi_i\left(u_\lambda^k\right) + \sum_{(\lambda,k,j)\in B_n} \varphi_i\left(z_{\lambda,k,j}^{(\lambda,k)}\right) + \varphi_i(r)\\ &= a_{[n],n}\varphi_i\left(u_{[n]}^n\right) + \sum_{(\lambda,k,j)\in B_n} \beta(\lambda,k,j)\cdot\left(z_{\lambda,k,j}\right)_i\\ &= \frac{w(N,\{N\})}{n}\\ &\quad + \sum_{(\lambda,k,j)\in B_n} \beta(\lambda,k,j) \left[\sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}} \frac{j}{n}w(S,Q) - \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda,|Q^i|=j}} \frac{k}{n}w(S,Q) \right]\end{aligned}$$

Finally, the result follows by setting $\delta(\lambda,k,j) = \frac{\beta(\lambda,k,j)}{n}$.

Bolger [2] showed that there are many linear symmetric and efficient solutions and provided a class of it, which actually is recovered from the general formula (4) by setting appropriate constants. More recently, equivalent expressions of linear symmetric (and efficient) solutions for games in partition function form have been obtained by Hernández-Lamoneda et al. [6].

Corollary 4. *The space of all linear, symmetric, and efficient solutions on G has dimension* $|B_n|$.

There are several values in the literature that satisfy the axioms considered in this work (linearity, symmetry, and efficiency) and therefore, all these values are of the form given by (4). For instance: [1, 3, 8, 12–14] and [7].

As an illustration of previous results, we use two particular values in the following example. We take the values characterized in [13] and [12] (denoted by φ^M and φ^{MS}, respectively).

Example 5. Consider the game w of Example 4. It has a decomposition $w = u + v + r$:

(S, Q)			$w(S, Q)$			$u(S, Q)$			$v(S, Q)$		
$\{1\}$	$\{2\}$	$\{3\}$	8	6	12	$\frac{26}{3}$	$\frac{26}{3}$	$\frac{26}{3}$	$-\frac{2}{3}$	$-\frac{8}{3}$	$\frac{10}{3}$
$\{1, 2\}$	$\{3\}$		14	5		$\frac{58}{3}$	4		$-\frac{16}{3}$	1	
$\{1, 3\}$	$\{2\}$		26	0		$\frac{58}{3}$	4		$\frac{20}{3}$	-4	
$\{2, 3\}$	$\{1\}$		18	7		$\frac{58}{3}$	4		$-\frac{4}{3}$	3	
$\{1, 2, 3\}$			33			33			0		

where, in turn, r is the zero game.[10]

The Myerson value of the game w is $\varphi^M(w) = (15, 5, 13)$. From Corollary 3, this value divides equally the worth of the grand coalition among the three players in the game u. Therefore $\varphi^M(u) = (11, 11, 11)$.

On the other hand, from Schur's Lemma the Myerson value when restricted to V provides an allocation in which the sum of the payoffs of all players are exactly zero. Indeed, $\varphi^M(v) = (4, -6, 2)$ and:

$$(15, 5, 13) = \varphi^M(w) = \varphi^M(u) + \varphi^M(v) + \varphi^M(r) = (11, 11, 11) + (4, -6, 2)$$

Similarly, for the value in Macho-Stadler et al.: $\varphi^{MS}(w) = \left(\frac{49}{4}, 6, \frac{59}{4}\right)$, $\varphi^{MS}(u) = (11, 11, 11)$ and $\varphi^{MS}(v) = \left(\frac{5}{4}, -5, \frac{15}{4}\right)$.

[10] That is, $r(S, Q) = 0$ for every $(S, Q) \in EC$.

5 Conclusion

This work has explored solutions for games in partition function form. We have noticed that the point of view we take in this chapter depends heavily on a decomposition of the space of games as a direct sum of "special" subspaces and characterizations of solutions follow from such decomposition in an very economical way. The space of games was decomposed as a direct sum of three orthogonal subspaces: $G = U \oplus V \oplus W$. U is a subspace of games whose values $w(S, Q)$ depend only on the cardinality of S and on the structure of Q. V does not have a natural definition in terms of well-known game theoretic considerations, but it has a simple characterization in terms of vectors all of whose entries add up to zero. And W which plays only the role of the common kernel of every linear symmetric solution.

Our approach suggests that in order to study linear symmetric solutions, one needs to look only at those games inside $U \oplus V$ and we presented a global description of all such solutions. Besides linearity and symmetry, we studied the efficiency axiom and provided the restrictions that this property imposed in the components of the decomposition of G. Once we understood those restrictions, we obtained characterizations of classes of solutions.

There are several open considerations regarding the analysis presented in this work. For instance, besides the axioms of linearity, symmetry, and efficiency, the consideration of a nullity axiom following our approach could be an interesting topic for further research. Another interesting issue to consider is the computational complexity of finding the solutions based on the decomposition of the game space with respect to the calculation by the standard approach without implementing the decomposition.

Although it is true that the characterization results could be proved without any explicit mention to the proposed decomposition, we believe, it sheds new light on the structure of the space of games in partition function form and their solutions. There is, however, much more work that could be done, and we encourage interested readers to consider how they might use these and other ideas to contribute to the understanding of games in partition function form and their solutions.

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Appendix

The point of view we take in this work depends heavily on a certain decomposition of G (and $\mathbb{R}^n$) as a direct sum of subspaces, as well as on the Schur's Lemma. A reference for this subject is [4]; nevertheless, we recall all the basic facts that we need.

The symmetric group S_n acts on G via linear transformations (i.e., G is a representation of S_n). That is, there is a group homomorphism $\rho : S_n \to GL(G)$, where $GL(G)$ is the group of invertible linear maps in G. This action is given by:

$$(\theta \cdot w)(S, Q) := [\rho(\theta)(w)](S, Q) = w[\theta^{-1}(S, Q)]$$

for every $\theta \in S_n$, $w \in G$ and $(S, Q) \in EC$.

The space of payoff vectors, $\mathbb{R}^n$, is also an S_n-representation:

$$\theta(x_1, x_2, \ldots, x_n) := [\rho(\theta)](x_1, x_2, \ldots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$$

In general, for any representation X of a finite group H, there is a decomposition

$$X = X_1^{\oplus a_1} \oplus X_2^{\oplus a_2} \oplus \cdots \oplus X_j^{\oplus a_j}$$

where the X_i are distinct irreducible representations. The decomposition is unique, as are the X_i that occur and their multiplicities a_i.

This property is called "complete reducibility" and the extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Theorem 4 (Schur's Lemma). *Let X_1,X_2 be irreducible representations of a group H. If $T : X_1 \to X_2$ is H-equivariant, then $T = 0$ or T is an isomorphism.*

Moreover, if X_1 and X_2 are complex vector spaces, then T is unique up to multiplication by a scalar $\lambda \in \mathbb{C}$.

The previous theorem is one of the reasons why it is worth carrying around the group action when there is one. Its simplicity hides the fact that it is a very powerful tool.

There is a remarkably effective technique for decomposing any given finite dimensional representation into its irreducible components. The secret is *character theory*.

Definition 5. Let $\rho : H \to GL(X)$ be a representation. The character of X is the complex-valued function $\chi_X : H \to \mathbb{C}$, defined as:

$$\chi_X(h) = Tr\,(\rho(h))$$

The character of a representation is easy to compute. If H acts on an n-dimensional space X, we write each element h as an $n \times n$ matrix according to its action expressed in some convenient basis, then sum up the diagonal elements of the matrix for h to get $\chi_X(h)$. For example, the trace of the identity map of an n-dimensional vector space is the trace of the $n \times n$ identity matrix, or n. In fact, $\chi_X(e) = \dim X$ for any finite dimensional representation X of any group.

Notice that, in particular, we have $\chi_X(h) = \chi_X(ghg^{-1})$ for $g, h \in H$. So that χ_X is constant on the conjugacy classes of H; such a function is called a *class function*.

Definition 6. Let $\mathbb{C}_{class}(H) = \{f : H \to \mathbb{C} \mid f$ is a class function on $H\}$. If $\chi_1, \chi_2 \in \mathbb{C}_{class}(H)$, we define an Hermitian inner product on $\mathbb{C}_{class}(H)$ by

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_1(h)} \cdot \chi_2(h) \tag{5}$$

The multiplicities of irreducible subspaces in a representation can be calculated via the previous inner product. Formally, if $Z = Z_1^{\oplus a_1} \oplus Z_2^{\oplus a_2} \oplus \cdots \oplus Z_j^{\oplus a_j}$, then the multiplicity Z_i (irreducible representation) in Z is $a_i = \langle \chi_Z, \chi_{Z_i} \rangle$.

Proof (of Theorem 1). First, $\left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n} \right\rangle$ and $\left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n^\perp} \right\rangle$ are the number of subspaces isomorphic to the trivial (Δ_n) and standard representation ($\Delta_n^\perp$) within G_λ^k, respectively.

We start by computing the number of copies of Δ_n in G_λ^k:

$$\left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n} \right\rangle = \frac{1}{n!} \sum_{\theta \in S_n} \chi_{G_\lambda^k}(\theta) \chi_{\Delta_n}(\theta) = \frac{1}{n!} \sum_{\theta \in S_n} \chi_{G_\lambda^k}(\theta)$$

Notice that $\chi_{G_\lambda^k}(\theta)$ is just the number of pairs $(S, Q) \in EC$ with $|S| = k$ and $\lambda_Q = \lambda$, that are fixed under $\theta \in S_n$.

Define

$$\{\theta\}_{(S,Q)} = \begin{cases} 1 \text{ if } \theta(S, Q) = (S, Q) \\ 0 \text{ otherwise} \end{cases}$$

Then,

$$\chi_{G_\lambda^k}(\theta) = \sum_{\substack{(S,Q) \in EC \\ |S| = k, \lambda_Q = \lambda}} \{\theta\}_{(S,Q)}$$

and so,

$$\left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n} \right\rangle = \frac{1}{n!} \sum_{\theta \in S_n} \sum_{\substack{(S,Q) \in EC \\ |S| = k, \lambda_Q = \lambda}} \{\theta\}_{(S,Q)} = \frac{1}{n!} \sum_{\substack{(S,Q) \in EC \\ |S| = k, \lambda_Q = \lambda}} \sum_{\theta \in S_n} \{\theta\}_{(S,Q)}$$

where,

$$\sum_{\theta \in S_n} \{\theta\}_{(S,Q)} = |\{\theta \in S_n : \theta(Q) = Q,\ \theta(S) = S\}|$$

Now, S_n acts on the set Q_λ and take $Q \in Q_\lambda$. The orbit of Q under S_n is

$$S_n Q = \{\theta(Q) \mid \theta \in S_n\} = Q_\lambda$$

and the isotropy subgroup of Q is

$$(S_n)_Q = \{\theta \in S_n \mid \theta(Q) = Q\}$$

By Lagrange theorem, we get

$$|S_n Q| = \frac{|S_n|}{|(S_n)_Q|} = |Q_\lambda| \quad \Rightarrow \quad |(S_n)_Q| = \frac{n!}{|Q_\lambda|}$$

Notice that $H = (S_n)_Q$ acts on Q and takes $S \in Q$ such that $|S| = k$. The orbit of S under H is

$$HS = \{hS \mid h \in H\} = \{T \in Q \mid |T| = k\}$$

Observe that $|HS| = m_k^\lambda$ and the isotropy subgroup of S is

$$H_S = \{h \in H \mid h(S) = S\} = \{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S\}$$

Again, by Lagrange theorem, we get

$$|HS| = \frac{|H|}{|H_S|} = \frac{|(S_n)_Q|}{|H_S|} = m_k^\lambda \quad \Rightarrow \quad |H_S| = \frac{|(S_n)_Q|}{m_k^\lambda} = \frac{n!}{|Q_\lambda|\, m_k^\lambda}$$

And therefore,

$$\left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n} \right\rangle = \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \frac{n!}{|Q_\lambda|\, m_k^\lambda} = \frac{1}{n!} |Q_\lambda|\, m_k \frac{n!}{|Q_\lambda|\, m_k^\lambda} = 1$$

Now, we compute the multiplicity of $\Delta_n^\perp$ in G_λ^k. Since $\mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$, then $\chi_{\mathbb{R}^n} = \chi_{\Delta_n} + \chi_{\Delta_n^\perp} \Rightarrow \left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle = \left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n} \right\rangle + \left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n^\perp} \right\rangle \Rightarrow \left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n^\perp} \right\rangle = \left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle - 1$.

Notice that $G^1_{[1,1,\ldots,1]} \simeq \mathbb{R}^n$ (as a representation for S_n). Let us compute

$$\left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle = \left\langle \chi_{G_\lambda^k}, \chi_{G^1_{[1,1,\ldots,1]}} \right\rangle = \frac{1}{n!} \sum_{\theta \in S_n} \chi_{G_\lambda^k}(\theta)\, \chi_{G^1_{[1,1,\ldots,1]}}(\theta)$$

$$= \frac{1}{n!} \sum_{\theta \in S_n} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \{\theta\}_{(S,Q)} \sum_{\substack{(S',Q')\in EC \\ |S'|=1 \\ Q'=Q_{[1,1,\ldots,1]}}} \{\theta\}_{(S',Q')}$$

$$= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \sum_{\substack{(S',Q')\in EC \\ |S'|=1, \lambda_{Q'}=[1,1,\ldots,1]}} \sum_{\theta \in S_n} \{\theta\}_{(S,Q)} \{\theta\}_{(S',Q')}$$

For $x \in S$:

$$\sum_{\theta \in S_n} \{\theta\}_{(S,Q)} \{\theta\}_{(S',Q')} = \{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S,\ \theta(x) = x\}$$

Without loss of generality, suppose $|S| = k = \lambda_1$ and take the case $m_k^\lambda = m_{\lambda_1}^\lambda = 1$. Here, $M = H_S = \{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S\}$ acts on $S \in Q \in \mathcal{Q}_\lambda$ and take $x \in S$. The orbit of x under M is

$$Mx = \{\theta(x) \mid \theta \in M\} = S$$

and the isotropy subgroup of x is

$$M_x = \{\theta \in M \mid\ \theta(x) = x\} = \{\theta \in S_n \mid \theta(Q) = Q,\ \theta(S) = S,\ \theta(x) = x\}$$

By Lagrange theorem,

$$|Mx| = \frac{|M|}{|M_x|} = \frac{|H_S|}{|M_x|} = k \quad \Rightarrow \quad |M_x| = \frac{|H_S|}{k} = \frac{n!}{k\,|\mathcal{Q}_\lambda|\,m_k^\lambda}$$

Thus,

$$\begin{aligned}\left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle &= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \left[k\frac{n!}{k\,|\mathcal{Q}_\lambda|\,m_k^\lambda} + m_{\lambda_2}^\lambda \lambda_2 \frac{n!}{m_{\lambda_2}^\lambda \lambda_2\,|\mathcal{Q}_\lambda|\,m_k^\lambda} + \cdots \right. \\ &\quad \left. +\ m_{\lambda_l}^\lambda \lambda_l \frac{n!}{m_{\lambda_l}^\lambda \lambda_l\,|\mathcal{Q}_\lambda|\,m_k^\lambda} \right] \\ &= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} |\lambda^\circ|\,\frac{n!}{|\mathcal{Q}_\lambda|\,m_k^\lambda} = \frac{1}{n!}\,|\mathcal{Q}_\lambda|\,m_k^\lambda \left[|\lambda^\circ|\,\frac{n!}{|\mathcal{Q}_\lambda|\,m_k^\lambda} \right] \\ &= |\lambda^\circ|\end{aligned}$$

Finally, without loss of generality, suppose $|S| = k = \lambda_1$ and take the case $m_k^\lambda > 1$. Following the same line of reasoning as above, we obtain

$$\left\langle \chi_{G_\lambda^k}, \chi_{\mathbb{R}^n} \right\rangle = \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k, \lambda_Q=\lambda}} \left[\begin{array}{c} k\frac{n!}{k|\mathcal{Q}_\lambda|m_k^\lambda} + k(m_k^\lambda - 1)\frac{n!}{k(m_k^\lambda-1)|\mathcal{Q}_\lambda|m_k^\lambda} + \\ m_{\lambda_2}^\lambda \lambda_2 \frac{n!}{m_{\lambda_2}^\lambda \lambda_2 |\mathcal{Q}_\lambda| m_k^\lambda} + \cdots + m_{\lambda_l}^\lambda \lambda_l \frac{n!}{m_{\lambda_l}^\lambda \lambda_l |\mathcal{Q}_\lambda| m_k^\lambda} \end{array} \right]$$

$$= \frac{1}{n!} \sum_{\substack{(S,Q)\in EC \\ |S|=k,\lambda_Q=\lambda}} (|\lambda^\circ|+1) \frac{n!}{|Q_\lambda|\, m_k^\lambda} = \frac{1}{n!} |Q_\lambda|\, m_k^\lambda \left[(|\lambda^\circ|+1) \frac{n!}{|Q_\lambda|\, m_k^\lambda} \right]$$

$$= |\lambda^\circ| + 1$$

In summary,

$$\left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n} \right\rangle = 1 \quad \text{and} \quad \left\langle \chi_{G_\lambda^k}, \chi_{\Delta_n^\perp} \right\rangle = |I_{\lambda,k}|$$

The next task is to identify such copies of Δ_n and $\Delta_n^\perp$ inside G_λ^k. For that end, define the functions $L_{\lambda,k} : \mathbb{R}^n \to G_\lambda^k$ by $L_{\lambda,k}(x) = x^{(\lambda,k)}$. These maps are isomorphisms between U_λ^k and Δ_n (similarly between $\left\{ x_j^{(\lambda,k)} \mid x_j \in \Delta_n^\perp \right\}$ and $\Delta_n^\perp$, for each $j \in I_{\lambda,k}$) since they are linear, S_n-equivariant, and $1-1$. Thus, inside of G_λ^k, we have the images of these two subspaces: $U_\lambda^k = L_{\lambda,k}(\Delta_n)$ and $V_\lambda^k = L_{\lambda,k}(\Delta_n^\perp)$.

Finally, the invariant inner product $\langle , \rangle$ gives an equivariant isomorphism, in particular must preserve the decomposition. This implies orthogonality of the decomposition.

Proof (of Proposition 2). We start by computing the orthogonal projection of w onto U. Notice that $\{u_\lambda^k\}$ is an orthogonal basis for U, and that $\left\| u_\lambda^k \right\|^2 = m_k^\lambda\, |Q_\lambda|$.

Thus, the projection of w onto U is

$$\sum_{(\lambda,k)\in A_n} \frac{\left\langle w, u_\lambda^k \right\rangle}{\left\langle u_\lambda^k, u_\lambda^k \right\rangle} u_\lambda^k$$

and so,

$$a_{\lambda,k} = \frac{\left\langle w, u_\lambda^k \right\rangle}{\left\langle u_\lambda^k, u_\lambda^k \right\rangle} = \frac{\sum_{|S|=k,\lambda_Q=\lambda} w(S,Q)}{|(S,Q) \in EC : |S| = k, \lambda_Q = \lambda|}$$

Now, for each $(\lambda, k) \in A_n$, we define $f^{\lambda,k} : G \to \mathbb{R}^n$ as

$$f_i^{\lambda,k}(w) = \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} w(S,Q)$$

where each $f^{\lambda,k}$ is S_n-equivariant and observe that $f^{[n],n}(w) = w(N,\{N\})(1,\ldots,1)$. Let $z \in \Delta_n^\perp$, then $f^{\lambda,k}(z_{\gamma,i,j}^{(\gamma,i)}) = 0$ if $\lambda \neq \gamma$ or $i \neq k$, whereas (by Schur's Lemma) $f^{\lambda,k}(z_{\lambda,k,j}^{(\lambda,k)}) = z_{\lambda,k,j} \in \Delta_n^\perp$.

Let $p : \mathbb{R}^n \to \Delta_n^{\perp}$ be the projection of $\mathbb{R}^n$ onto $\Delta_n^{\perp}$ given by

$$p_i(x) = x_i - \frac{1}{n}\sum_{j=1}^{n} x_j$$

This projection is equivariant, sends Δ_n to zero and it is the identity on $\Delta_n^{\perp}$.

Next, define $L^{\lambda,k} : G \to \Delta_n^{\perp}$ as $L^{\lambda,k} = p \circ f^{\lambda,k}$. Observe that

$$L^{\lambda,k}(w) = \sum_{j\in I_{\lambda,k}} z_{\lambda,k,j}$$

since by equivariance, $f^{\lambda,k}(U) \subset \Delta_n$ and $f^{\lambda,k}(W) = 0$. Moreover, $f^{\lambda,k}(z^{(\gamma,i)}_{\gamma,i,j}) = 0$ if $\lambda \neq \gamma$ or $i \neq k$. Then,

$$L_i^{\lambda,k}(w) = p_i(f^{\lambda,k}(w)) = \sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} w(S,Q) - \frac{1}{n}\sum_{l\in N}\sum_{\substack{(S,Q)\in EC\\ S\ni l,|S|=k\\ \lambda_Q=\lambda}} w(S,Q)$$

$$= \frac{n-k}{n}\sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} w(S,Q) - \frac{k}{n}\sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda}} w(S,Q)$$

The value of the component $\left(z_{\lambda,k,j}\right)_i$ follows from the fact that the last expression can be written as

$$\sum_{j\in I_{\lambda,k}}\left[\sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}}\sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}} \frac{j}{n} w(S,Q) - \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda,|Q^i|=j}} \frac{k}{n} w(S,Q)\right]$$

Proof (of Proposition 3). Let $\varphi : G \to \mathbb{R}^n$ be a linear symmetric solution. From Proposition 2, $w \in G$ decomposes as

$$w = \sum_{(\lambda,k)\in A_n} a_{\lambda,k} u_\lambda^k + \sum_{(\lambda,k,j)\in B_n} z^{(\lambda,k)}_{\lambda,k,j} + r$$

where by linearity,

$$\varphi_i(w) = \sum_{(\lambda,k)\in A_n} a_{\lambda,k}\varphi_i\left(u_\lambda^k\right) + \sum_{(\lambda,k,j)\in B_n} \varphi_i\left(z^{(\lambda,k)}_{\lambda,k,j}\right) + \varphi_i\left(r\right)$$

Now, $\varphi_i(r) = 0$ by Corollary 1 and Schur's Lemma implies

$$\varphi_i(w) = \sum_{(\lambda,k)\in A_n} a_{\lambda,k}\alpha(\lambda,k) + \sum_{(\lambda,k,j)\in B_n} \beta(\lambda,k,j)\cdot\left(z_{\lambda,k,j}\right)_i$$

for some constants $\{\alpha(\lambda,k) \mid (\lambda,k)\in A_n\} \cup \{\beta(\lambda,k,j) \mid (\lambda,k,j)\in B_n\}$.

Set $|(S,Q)_{\lambda,k}| = |(S,Q)\in EC : |S| = k, \lambda_Q = \lambda|$, then

$$\begin{aligned}
\varphi_i(w) &= \sum_{(\lambda,k)\in A_n} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} \sum_{\substack{(S,Q)\in EC\\ |S|=k,\lambda_Q=\lambda}} w(S,Q)\\
&\quad + \sum_{(\lambda,k,j)\in B_n} \beta(\lambda,k,j)\left[\sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}} jw(S,Q) - \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda,|Q^i|=j}} kw(S,Q)\right]\\
&= \sum_{(\lambda,k)\in A_n} \sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} w(S,Q) + \sum_{(\lambda,k)\in A_n} \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda}} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} w(S,Q)\\
&\quad + \sum_{(\lambda,k,j)\in B_n} \sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}} \frac{j}{n}\beta(\lambda,k,j)\, w(S,Q)\\
&\quad - \sum_{(\lambda,k,j)\in B_n} \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda,|Q^i|=j}} \frac{k}{n}\beta(\lambda,k,j)\, w(S,Q)
\end{aligned}$$

Observe that

$$\sum_{(\lambda,k)\in A_n} \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda}} w(S,Q) = \sum_{(\lambda,k,j)\in B_n} \sum_{\substack{(S,Q)\in EC\\ S\not\ni i,|S|=k\\ \lambda_Q=\lambda,|Q^i|=j}} w(S,Q)$$

and

$$\sum_{(\lambda,k,j)\in B_n} \sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\}\\ |T|=j}} w(S,Q) = \sum_{(\lambda,k)\in A_n} \sum_{\substack{(S,Q)\in EC\\ S\ni i,|S|=k\\ \lambda_Q=\lambda}} \sum_{j\in I_{\lambda,k}} m_j^{\lambda-[|S|]} w(S,Q)$$

Thus

$$\varphi_i(w) = \sum_{(\lambda,k)\in A_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} w(S,Q) + \sum_{(\lambda,k,j)\in B_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda,|Q^i|=j}} \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} w(S,Q)$$

$$+ \sum_{(\lambda,k)\in A_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i,|S|=k \\ \lambda_Q=\lambda}} \sum_{j\in I_{\lambda,k}} \frac{j}{n} m_j^{\lambda-[|S|]} \beta(\lambda,k,j)\, w(S,Q)$$

$$- \sum_{(\lambda,k,j)\in B_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i,|S|=k \\ \lambda_Q=\lambda,|Q^i|=j}} \frac{k}{n} \beta(\lambda,k,j)\, w(S,Q)$$

The result follows by setting $\gamma(\lambda,k) = \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} + \sum_{j\in I_{\lambda,k}} \frac{j}{n} m_j^{\lambda-[|S|]} \beta(\lambda,k,j)$ and $\delta(\lambda,k,j) = \frac{\alpha(\lambda,k)}{|(S,Q)_{\lambda,k}|} - \frac{k}{n}\beta(\lambda,k,j)$.

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A Model of Tacit Collusion: Nash-2 Equilibrium Concept

Marina Sandomirskaia

Abstract We examine an equilibrium concept for 2-person non-cooperative games with boundedly rational agents which we call Nash-2 equilibrium. It is weaker than Nash equilibrium and equilibrium in secure strategies: a player takes into account not only current strategies but also all profitable next-stage responses of the partners to her deviation from the current profile that reduces her relevant choice set. We provide a condition for Nash-2 existence in finite games and complete characterization of Nash-2 equilibrium in strictly competitive games. Nash-2 equilibria in Hotelling price-setting game are found and interpreted in terms of tacit collusion.

Keywords Nash-2 equilibrium • Secure deviation • Bertrand paradox • Hotelling model • Tacit collusion

JEL Classification: C72, D03, D43, D70, L13

1 Motivation

Many economic applications of game theory require modeling long-term interactions among players. The common examples are two or several oligopolists competing in Cournot, Bertrand, and Hotelling games. Somewhat similar is a repeated game between an employer and employee, a lender and a borrower, repeated common venture, etc. Intuitively many economists agree that quite often the realistic outcome of such game is a tacit collusion [35]; a sort of quasi-cooperative solution supported by *credible threats*. However, they disagree which formal game concept better describes this practical outcome.

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According to Nash any profile is an equilibrium if nobody can unilaterally increase her current payoff by changing own strategy when other players do not react. This approach proved to be quite fruitful when the influence of each participant to the whole situation is negligible. However, its common applications to oligopoly rise serious doubts. Say, undercutting one another in Bertrand duopoly is a failure of theory, the same can be said about absence of Nash equilibrium in Hotelling game with prices. In our opinion, the old theoretical battle between Cournot and Bertrand modeling is useless (see [1, 11, 28, 29, 33]). In one-shot version both are inefficient concepts for describing an interaction of few players from logic and realism viewpoints. By contrast, dynamic games and related supergames approach (see [4, 30]) look logically nice. However it is too computationally complicated, both for studying and for players themselves. Indeed, can a theorist believe that two traders optimize in infinitely dimensional space of all possible responses to all possible trajectories of their behavior? We prefer a bounded rationality concept: taking into account only the current and next step. This behavior displays moderate wisdom: not absolutely myopic as Nash concept and not infinitely wise as Folk theorem prediction.

Accounting for strategic aspects of interaction among players can be implemented as a generalization of the Nash equilibrium concept. Since the pioneer concept of perfect and proper equilibria (see [32] and [25], respectively) a number of further refinements have been done (for some of them, see [7, 14, 34]).

A rational player can take in account reactions of other players when she makes a decision whether to deviate from the current strategy or not. Related discussion on the iterated strategic thinking process can be found in [5]. The ideas of players' reflection are not new in non-cooperative games. Reflexive games with complicated hierarchy of reaction levels are developed in [6, 9, 27]. This approach with asymmetric rationality of participants leads to rather complicated computations for agents; however, some empirical studies support the approach of k-level rationality [6, 24]. We also must mention some farsighted solution concepts based on the idea of *k*-level rationality—they are the largest consistent set [8], non-cooperative farsighted stable set [26], and farsighted pre-equilibrium [22]. The reasonable degree of farsightedness is an open question.

Iterated thinking models aim to *transfer full dynamic game* with perfectly rational players *to one-shot setting* including limited rationality. Such a reducing can be caused by the lack of information about the true duration of the game. Indeed, in most economic activities agents "play" during some long period, but the circumstances affecting game rules may change exogenously at any moment. Thereby, though agents need some forecasts, these predictions might be neither too long-continued, not too "narrow." By this we mean that agents should assume some freedom of the opponents' behavior, since for certain prediction they must know the opponents' depth of thinking which actually may evolve during the course of the game [13].

One more approach introduces *security* as an additional motivation for players' behavior. Two (the most closed to our ideas) second-stage-foreseeing concepts that have been proposed: bargaining set based on the notion of threats and counter-threats (for cooperative games, see [2]) and equilibrium in secure strategies (EinSS, that differs in additional requirement: security of current profile, see [17, 18]). In fact, the latter paper introduces the idea of Nash-2 equilibrium as "threatening-proof profile" and now is independently studied by M. Iskakov and A. Iskakov under names: equilibrium contained by counter-threats [20] and equilibrium in threats and counter-threats [21].[1]

The idea of both Nash-2 equilibrium and EinSS is that players worry not only about own first-stage payoffs, but also about security against possible "threats" of the opponents, i.e., profitable responses that harm the player, and optimizing on secure set can bring additional stability, more equilibria. The main point of this paper is that for modeling oligopoly with 2-level rational agents, we actually do not need to additional security requirement. Nash-2 equilibrium concept means only absence of profitable deviations subject to the reaction of the opponent. The benefit of such weak concept is existence in most situations. The shortcoming is typical multiplicity of equilibria. So, to select among these equilibria in a specific application of NE-2, we need some additional game-specific considerations to predict a unique solution.

In the sequel, Sect. 2 defines Nash equilibrium, equilibrium in secure strategies and Nash-2 equilibrium in terms of deviations, threats, and security, it illustrates the concepts with Prisoner's Dilemma. It also provides a condition for existence of Nash-2 equilibrium in a two-person finite game. In Sect. 3 we give the complete characterization of Nash-2 equilibria in the class of strictly competitive games. In Sect. 4 we apply Nash-2 equilibrium approach to the classical Hotelling linear city model and show that Nash-2 concept can provide a strategic explanation for possible collusion between farsighted firms.

2 Basic Notions and Equilibrium Concepts

Consider a 2-person non-cooperative game in the normal form

$$G = (i \in \{1, 2\}; s_i \in S_i; u_i : S_1 \times S_2 \to \mathbb{R}),$$

where s_i, S_i, and u_i are the strategies, the set of all available strategies, and the payoff function, respectively, of player i, $i = 1, 2$. Henceforth, in this paper we will deal only with pure strategies.

Let us give the formal definition of the Nash equilibrium in terms of deviations. This will help us to explain our modification of this equilibrium concept more clearly.

[1] We will accurately refer to existence results during the further exposition in case of some intersection.

Definition 1. A *profitable deviation* of player i at strategy profile $s = (s_i, s_{-i})$ is a strategy s'_i such that $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$. A strategy profile s is a *Nash equilibrium* (NE) if no player has a profitable deviation.

Note that inequality is strict so players cannot deviate to a profile with the same payoff. The condition on the deviation is rather weak in the sense that too many deviations are allowed. So a game can have no stable profiles. Another shortcoming is that there are games where a Nash equilibrium exists but doesn't seem a reasonable outcome.

In [17] authors propose a refinement with the notion of security. Here we slightly reformulate the main definitions from [17].

Definition 2. A (credible) *threat* of player i to player $-i$ at strategy profile s is a profitable deviation s'_i of player i such that

$$u_{-i}(s'_i, s_{-i}) < u_{-i}(s_i, s_{-i}).$$

The strategy profile s is said to *pose a threat* from player i to player $-i$. A strategy profile s is *secure* for player i if s poses no treats from other players to i.

Definition 3. A profitable deviation s'_i of player i at s is *secure* if for any threat s'_{-i} of player $-i$ at profile (s'_i, s_{-i}) player i is not worse off:

$$u_i(s'_i, s'_{-i}) \geq u_i(s_i, s_{-i}).$$

Definition 4. A strategy profile is an *equilibrium in secure strategies* (EinSS) if

- it is secure,
- no player has a profitable secure deviation.

The crucial suggestion of our paper is that the condition of security may be omitted. This case was mentioned in [18] and such profiles were named threatening-proof. However, these profiles do not satisfy the security condition and the authors didn't study such profiles as equilibria and added the explicit condition of the profit maximization on the set of threatening-proof profiles (solution in objections and counter objections).

We argue that a threatening-proof profile itself is stable enough be viewed as a possible equilibrium concept whereas any additional condition should be motivated by additional information about the specific game modeled. Moreover, we can simplify the definition since we need not the notion of threats to characterize a deviation as secure. The reflexive idea of accounting the responses of the opponent seems to be sufficient for our purposes. Thus, secure deviation in fact matters only on 2-level rationality.

Definition 5 (Alternative: Profitable Secure Deviation). A profitable deviation s'_i of player i at $s = (s_i, s_{-i})$ is *secure* if for any profitable strategy s'_{-i} of player $-i$ at (s'_i, s_{-i}) player i is not worse off:

$$u_i(s'_i, s'_{-i}) \geq u_i(s_i, s_{-i}).$$

Definition 6. A strategy profile is a *Nash-2 equilibrium* (NE-2) if no player has a profitable secure deviation.

Here we consider only equilibria in pure strategies, but one can consider mixed extension. Mixed strategies (see [12]) is an alternative approach to deal with non-existence of pure NE, but we expect that a motivated solution in pure strategies can also be useful for economic applications.

Thus, NE-2 concept seems to be a reasonable outcome in quasi-repeated games (with possibility of response). Such 2-stage rationality is one of the possible compromises between zero rationality of NE and infinite rationality of infinite repetition. Obviously, NE-2 may be not secure. Indeed, when NE-2 includes a threat from one player to another nobody actualizes her threats because they are not secure deviations for her.

Example 1 (Prisoner's Dilemma). Consider the classical non-repeated Prisoner's dilemma.

	Cooperate	Defect
Cooperate	(1,1)	(−1,2)
Defect	(2,−1)	(0,0)

The unique NE is defection for both players. EinSS is the same and moreover it is the unique secure profile in the game. But one can easily check that in addition to this low-profit equilibrium the mutual cooperation is NE-2, and the profit $(1, 1)$ is more desirable for both players. So we observe the strategic motivation for cooperative solution without explicit modeling repeated game structure; NE-2 is an appropriate description of tacit collusion in such situations whereas NE and EinSS are not.

Example 2 (The Bertrand Duopoly). Consider the classical Bertrand model with two firms producing a homogeneous product and demand is non-elastic. Let m_c be the constant marginal costs (equal for both firms), p_M being the monopoly price level, and p_1 and p_2 being the price levels of firms 1 and 2, respectively. Well known is that the unique NE in this model is equal prices $p_1 = p_2 = m_c$ for both firms, that leads to zero profits. It is just the situation of Bertrand paradox, criticized as a bad description of the real-life behavior. It can't be resolved by using a concept of equilibrium in secure strategies because EinSS here coincide with NE. However, applying the concept of NE-2 allows to obtain an equilibrium with any price level $p_1 = p_2 \in [m_c, p_M]$ yielding positive firms' profits. In particular, the highest of the NE-2 profiles establishes the monopoly price level. This outcome is exactly what can be regarded as a tacit collusion between the firms (explicit cooperation is not permitted).

Thus, from the Nash-2 point of view, the classical paradox can be resolved without changing the model or its timing. *Though we have in mind a repeated game, its simple one-shot form is sufficient for modeling*. As far as we can judge, the problem of choosing the appropriate outcome out of the NE-2 set is deeply connected with the problem of stability and failure of collusion in a long-run perspective. One of possible approaches is to introduce the financial power of firms affecting the collusion stability, for instance, see [36].

Proposition 1 ([18]). *Any NE is an EinSS. Any EinSS is a NE-2.*[2]

This claim just follows from definitions and the question rises: is EinSS the most "natural" refinement of NE-2? Some refinement would be good for predictions because NE-2 set in some games appears to be large (even continuum, see Sect. 4). We can regard it as an analogue of the bargaining set in cooperative games. The result of such approach looks like a tacit collusion between agents. Maximization of joint payoff is one of the ways to choose the concrete equilibrium from the rich set of NE-2, but not the unique one and actually depends on the specific game under consideration. Alternative way is, for instance, to restrict the set of admissible strategies to the set of NE-2 profiles and then play NE, in such a reduced game.

In the definition of profitable and secure deviation a player takes care about all profitable responses of the opponent. An interesting modification would be to take into account only the opponent's best responses. Papers [3, 15] introduce the similar concepts of equilibrium: cooperative equilibrium and equilibrium in double best responses. One can expect they to be a very close concept to NE-2. Indeed, for some games double best response provides the same equilibrium set, but not always. At least because of computational reasons for analytic solution it is sometimes more practical to apply NE-2 concept as more simple one.

Existence of Nash-2 Equilibrium in Finite Games

Now we show that NE-2 equilibrium really exists for most finite games and fails to exist only in degenerate (in some sense) cases. One way to show the existence of Nash-2 equilibria follows from Propositions 2 and 8[3] in [20] and is based on some intermediate equilibrium concept. We will not go into any details here not to multiply definitions.

Below we present an alternative *direct* formulation, which is intuitively clear. For this purpose we introduce the notion of secure cycle.

Definition 7. A path of profiles $\{(s_i^t, s_{-i}^t)\}_{t=1,\ldots,T}$ is called a *secure path* of length T if each its arc $(s_i^t, s_{-i}^t) \to (s_i^{t+1}, s_{-i}^{t+1}) = (s_i^{t+1}, s_{-i}^t)$ is a secure profitable deviation from s_i^t to s_i^{t+1} for some player i. This path is called a *secure cycle* if it is closed: $(s_i^1, s_{-i}^1) = (s_i^T, s_{-i}^T)$, minimum of such a T is called a *length of cycle*.

[2] Authors formulated this result in terms of threatening-proof profile.

[3] Authors formulated this result in terms of equilibrium contained by counter-threats.

Using this notion one can easily check the following theorem providing the criterion for absence of NE-2 in finite game. By *finite game* we mean a game with finitely many players, each of which has a finite set of pure strategies.

Proposition 2. *The finite 2-person game in normal form does not have a Nash-2 equilibrium in pure strategies if and only if*

1. *it contains at least one secure cycle of finite length,*
2. *for any profile which is not in secure cycle, there exists a finite secure path from it to some profile which is in secure cycle.*

Let us, without loss of generality, assume that in a secure cycle player 1 deviates at odd steps while player 2 deviates at even ones. It helps us to formulate an important observation that secure cycles are very special: all nodes where player 1 deviates (say, odd ones) should have *same* payoff for this player ($u_1(s_1^{2t+1}, s_2^{2t+1}) = u_1(s_1^{2t+3}, s_2^{2t+3})\ \forall t$), the same is true for even nodes and player 2: ($u_2(s_1^{2t}, s_2^{2t}) = u_2(s_1^{2t+2}, s_2^{2t+2})\ \forall t$)—see Example 4 (game "Heads or Tails").

Corollary. *Whenever a finite 2-person game does not have pure NE-2, any random perturbation of payoffs (that breaks at least one equality in the secure cycle condition above) immediately yields pure NE-2 existence.*

This perturbation can be, for instance, adding an infinitesimal epsilon to the payoff at one profile included in the secure cycle.

Thereby, in essence, we have proven that NE-2 exists "almost always" without strictly defining this notion. Proposition 2 hence aims to demonstrate the existence of NE-2 but not the optimal algorithm of finding it in arbitrary game.

Now we turn to some particular cases.

3 Nash-2 Equilibrium in Strictly Competitive Games

In this section we deal with the class of strictly competitive games for which NE and EinSS in pure strategies coincide and therefore often fail to exist. By contrast NE-2 concept typically provides existence and even a wide range of equilibria.

Definition 8. A two-person game G is *strictly competitive* if for every two strategy profiles s and s'

$$u_i(s) \geq u_i(s') \quad \Longrightarrow \quad u_{-i}(s) \leq u_{-i}(s').$$

Examples of strictly competitive games are zero-sum games, constant-sum games.

Moreover, when we confine ourselves to pure strategies, strictly competitive games are equivalent to zero-sum games. Still, to compare our propositions with the next one, we prefer the terminology of strictly competitive games.

Proposition 3 ([17]). *Any EinSS in a strictly competitive game is a NE.*

Now let us provide a necessary and sufficient conditions characterizing NE-2 profiles in terms of guaranteed payoffs. In order to apply NE-2 concept to strictly competitive games, let us introduce the notation.

Denote the guaranteed gain of player 1 by

$$\underline{V_1} = \max_{s_1} \min_{s_2} u_1(s_1, s_2).$$

When maxmin is attained on strategy profile $s^I = (s_1^I, s_2^I)$, we denote the corresponded gain of player 2 by $\overline{V_2} = u_2(s^I)$.

By analogy we denote the guaranteed gain of player 2 by

$$\underline{V_2} = \max_{s_2} \min_{s_1} u_2(s_1, s_2)$$

and the corresponding gain of player 1 by $\overline{V_1}$. The interval $[\underline{V_i}, \overline{V_i}]$ being called further *attainable interval.*

Theorem 1 (Necessary Condition of NE-2 in SC Games). *If strategy profile s is a NE-2 in a strictly competitive game, then payoffs belong to the attainable interval:*

$$u_i(s) \in [\underline{V_i}, \overline{V_i}], \quad i = 1, 2.$$

Proof. Let us consider a NE-2 $\hat{s} = (\hat{s}_1, \hat{s}_2)$. Assume that $u_1(\hat{s}_1, \hat{s}_2) > \overline{V_1}$. It means that $u_2(\hat{s}_1, \hat{s}_2) < \underline{V_2}$.

On the other hand, consider another strategy s_2^{II} of player 2 that guarantees him/her at least $\underline{V_2}$. It is easy to see that the deviation s_2^{II} of player 2 from strategy profile $\hat{s}$ is profitable and secure. Thus, $\hat{s}$ is not NE-2.

Theorem 2 (Sufficient Condition of NE-2 in SC Games). *If a strategy profile* $\hat{s} = (\hat{s}_1, \hat{s}_2)$ *in a strictly competitive game is such that for each player* $i = 1, 2$ *the payoff is strictly inside the attainable interval:*

$$u_i(\hat{s}) \in (\underline{V_i}, \overline{V_i}),$$

then s is a NE-2.

Proof. Assume that player 1 has a profitable deviation s_1^* at $\hat{s}$: $u_1(s_1^*, \hat{s}_2) > u_1(\hat{s}_1, \hat{s}_2)$. Lets us show that it is not secure. Consider the strategy s_2^I of player 2. Then

$$u_1(s_1^*, s_2^I) \le \max_{s_1} u_1(s_1, s_2^I) = \overline{V_1}.$$

So $u_1(s_1^*, s_2^I) \le \overline{V_1} < u_1(\hat{s}_1, \hat{s}_2)$ and, thus, deviation s_1^* is not secure.

Thus, for NE-2 existence it is sufficient that the game would be "rich" enough, i.e., have intermediate profiles in which each players gets the payoff inside the attainable interval.

Now let us consider the boundary profile when one player gets exactly her lower guaranteed gain. Next two theorems complete the full classification of NE-2 in class of strictly competitive games.

Theorem 3 (Criterion 1 of NE-2). *Assume a strictly competitive game* $\underline{V_i} < \overline{V_i}$*, and a strategy profile* $s^* = (s_i^*, s_{-i}^*)$ *that brings minimal payoff* $u_i(s^*) = \underline{V_i}$ *for some player i. Then* s^* *is NE-2 if and only if for any strategy* $s_i \in \widetilde{S_i} \equiv \{s_i : \min_{s_{-i}} u_i(s_i, s_{-i}) = \underline{V_i}\}$ *bringing the same payoff under optimal partner's behavior,* s_i *yields the same payoff under current behavior:*

$$u_i(s_i, s_{-i}^*) = \underline{V_i}.$$

Proof. Consider NE-2 profile $s^* = (s_i^*, s_{-i}^*)$, for which $u_i(s^*) = \underline{V_i}$. Assume that there exists a strategy $s_i \in \widetilde{S_i}$ such that $u_i(s_i, s_{-i}^*) > \underline{V_i}$. Then the deviation s_i at s^* is profitable and secure. This proves the necessity.

Sufficiency. If $u_i(s^*) = \underline{V_i}$, then $u_{-i}(s^*) = \overline{V_{-i}}$. Assume that for any $s_i \in \widetilde{S_i}$ $u_i(s_i, s_{-i}^*) = \underline{V_i}$. Let us show that no player has a profitable secure deviation.

Consider any profitable deviation s_{-i} of player $-i$ at s^*. It means that $u_{-i}(s_i^*, s_{-i}) > u_{-i}(s_i^*, s_{-i}^*)$. Chose a strategy s_i of player i that minimizes $u_{-i}(s_i, s_{-i})$. Then

$$u_{-i}(s_i, s_{-i}) \le \max_{s_{-i}} \min_{s_i} u_{-i}(s_i, s_{-i}) = \underline{V_{-i}} < \overline{V_{-i}} = u_{-i}(s^*).$$

So the deviation s_{-i} is not secure.

Consider now a profitable deviation s_i of player i at s^*. If $s_i \notin \widetilde{S_i}$, then there exists a strategy s_{-i} such that $u_i(s_i, s_{-i}) \le \underline{V_i} = u_i(s^*)$.

The deviation s_i is secure only if $\min_{s_{-i}} u_i(s_i, s_{-i}) = \underline{V_i}$, i.e., $s_i \in \widetilde{S_i}$. By the hypothesis for all $s_i \in \widetilde{S_i}$ $u_i(s_i, s_{-i}^*) = \underline{V_i} = u_i(s^*)$, that means that the deviation s_i is not profitable. This completes the proof.

Theorem 4 (Criterion 2 of NE-2). *Assume a strictly competitive game with degenerate admissible interval* $\underline{V_i} = \overline{V_i} = V_i^*$*, and a strategy profile* $s^* = (s_i^*, s_{-i}^*)$*, such that* $u_i(s^*) = V_i^*, \quad i = \overline{1, 2}$*. A strategy profile* s^* *is NE-2 if and only if for any* $s_i \in \widetilde{S_i} = \{s_i : \min_{s_{-i}} u_i(s_i, s_{-i}) = V_i^*\}$ *equality* $u_i(s_i, s_{-i}^*) = V_i^*$ *holds for both* $i = 1, 2$.

Proof. The proof of necessity is the same as in Theorem 3. Let us prove the sufficiency. Consider a profitable deviation s_i of player i at s^*: $u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$. Then $s_i \notin \widetilde{S_i}$. This means that there exists a strategy s_{-i} such that $u_i(s_i, s_{-i}) < V_i^*$. Thereby, the deviation s_i is not secure.

If we restrict the class of our strictly competitive games with continuous payoff functions and connected strategy space we immediately obtain the existence theorem for such type of games. Recall the definition of a path-connected space.

Definition 9. The topological space X is said to be *path-connected* if for any two points $x, y \in X$ there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Example. convex set in $\mathbb{R}^n$.

One can easily prove the following theorem:

Theorem 5 (NE-2 Existence in Continuous SC Games). *Let G be a two-person strictly competitive game. Assume that strategy sets S_1 and S_2 are compact and path-connected, payoff functions u_1 and u_2 are continuous. Then there exists a pure NE-2 in G.*

Thus, we have shown very mild conditions for NE-2 existence in competitive games. Now, to show the difference between NE and NE-2, consider a few examples of concrete games.

Example 3 (NE $\neq$ NE-2).

	L	R
T	1	−1
B	0	0

This is a very degenerate example in the sense that $\underline{V} = \overline{V} = 0$. However, the game has unique NE and two NE-2 providing the same zero profit to both players. Namely, strategy profile (B,R) is NE and NE-2, and profile (B,L) is NE-2, but not NE. So both boundary profiles here are NE-2.

Example 4 (Heads or Tails: NE-2 Does Not Exist).

	L	R
T	1	−1
B	−1	1

This is an example of a game in which $\underline{V} = -1$, $\overline{V} = 1$, and it is “poor” in the sense that no intermediate profile exists. In this game any boundary profile fails to be NE-2.

Example 5 (SC Game with an Intermediate Profiles[4]).

	L	R
T	(2/3, 1/3)	(−1, 2)
C	(1/2, 1/2)	(1, 0)
B	(1, 0)	(0, 1)

Here $\underline{V_1} = 1/2$, $\overline{V_1} = 1$, $\underline{V_2} = 0$, $\overline{V_2} = 1/2$. NE-2 set consists of two strategy profiles: boundary profile (C,L) and intermediate profile (T,L) with profits (1/2, 1/2) and (2/3, 1/3), respectively. Thereby, not every boundary profile is NE-2.

Note that the assumption of strict competitiveness is essential. For instance, we can slightly relax this requirement and look at a unilaterally competitive game (where only one player harms her partner by improving her payoff, see [23]). Then the statement of Theorem 1 need not hold.

Example 6 (UC Game).

	L	C	R
T	(−1, 3)	(2, −1)	(1, 2)
B	(1, 0.5)	(0, 1)	(2, 0)

Here $\underline{V_1} = 0$, $\overline{V_1} = 1$, $\underline{V_2} = 0.5$, $\overline{V_2} = 1$.

The profile (T,R) is NE-2 (not unique) with profits (1, 2). However, related payoff is not in the admissible interval: $2 \notin [\underline{V_2}, \overline{V_2}]$.

4 The Hotelling Price Game with Symmetric Locations of Firms

Let us compare the concepts NE-2 and EinSS in the simple version of the Hotelling price game (with exogenous locations). Consumers are uniformly distributed along the unit line. Two firms producing the homogeneous product are located equidistant from the ends of the interval and at distance $d \in [0; 1]$ from each other. Production costs are zero for both firms. Transportation costs are linear, one unit per unit of distance, being covered by consumers. The demand is absolutely inelastic that means that irrespectively of its price a unit quantity of the product must be consumed by a buyer from each point of the interval. A buyer chooses a firm with a lower final price (including the delivery cost).

[4]Examples 5 and 6 have been proposed by Nikolay Bazenkov and Alexey Iskakov in private collaboration.

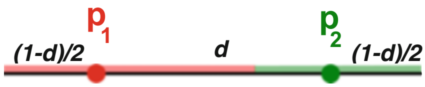

Fig. 1 Linear city model with symmetric location of firms

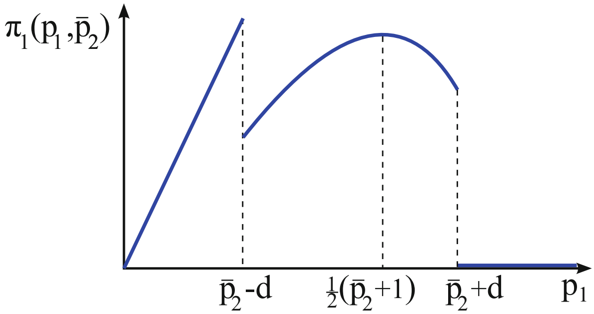

Fig. 2 Gain function $\pi_1(p_1, \bar{p}_2)$ with fixed $\bar{p}_2$

In case when no firm proposes inadequately high price (high enough to drop out), the market is divided into two parts, presented in Fig. 1.

Here firm 1 assigns price p_1 that is lower than p_2, so more consumers prefer to buy from firm 1. The interval of consumers who buy from firm 1 is marked with red color, whereas the firm 2's consumers are marked with green one.

Let location d has already been chosen and now is unchangeable. Now we look only on the price-setting game, the location problem being beyond our consideration. The price strategy of firm i is to propose price $p_i \in [0, \infty)$, $i = 1, 2$. The profit functions π_i, $i = 1, 2$, are given by:

$$\pi_i(p_i, p_{-i}) = \begin{cases} p_i(1 + p_{-i} - p_i)/2, & \textit{if} \;\; |p_i - p_{-i}| \le d, \\ p_i, & \textit{if} \;\; p_i < p_{-i} - d, \\ 0, & \textit{if} \;\; p_i > p_{-i} + d, \end{cases} \tag{1}$$

When $\bar{p}_2$ is fixed, the discontinuous shape of firm's 1 profit function $\pi_1(p_1, \bar{p}_2)$ is presented in Fig. 2.

Simplifying a theorem from [10], we claim that pure Nash equilibrium need not exist for all locations.

Theorem 6 (See [10]). *Consider the Hotelling price-setting game* $H = \{i \in \{1, 2\}, p_i \in \mathbb{R}, \pi_i(p_1, p_2) : \mathbb{R}^2 \to \mathbb{R}\}$, *where profits* π_i *are given by (1).*

- *For* $d \in [\frac{1}{2}, 1]$ *a unique NE exists: equilibrium prices are* $p_1^* = p_2^* = 1$ *and equilibrium profits are* $\pi_1 = \pi_2 = 1/2$.
- *For* $d = 0$ *the unique NE is* $p_1^* = p_2^* = 0$. $\pi_1 = \pi_2 = 0$.
- *For* $d \in (0, \frac{1}{2})$ *NE does not exist.*

Applying EinSS allows to obtain new type of equilibria even in cases when NE doesn't exist. The complete solution is contained in[19].

Theorem 7 (See [19]). *In the Hotelling game H there exists a unique EinSS for all locations*

- *For* $d \in [\frac{1}{2}; 1]$ *the EinSS is* $p_1^* = p_2^* = 1$. $\pi_1 = \pi_2 = 1/2$.
- *For* $d \in [0; \frac{1}{2})$ *the EinSS is* $p_1^* = p_2^* = 2d$. $\pi_1 = pi_2 = d < 1/2$.

It follows from proof in [19] that all profiles with high profits (we mean prices p_i higher then $(p_{-i} + 1)/2$) are excluded as non-secure. But these situations are obviously the most profitable for players if they are rational enough to collude, i.e., to *reject sharply undercutting* prices or locally decreasing them. Applying the NE-2 concept we obtain all these "collusive outcomes" as a reasonable equilibria, in addition to strong competition.

The simulation in Figs. 3, 4, 5, 6 demonstrates various outcomes depending on the parameters. Yellow areas are NE-2.

The following theorem characterizes the shape of the equilibria sets presented in the figures above.

Theorem 8. *In the Hotelling price-setting game H the boundary curves of the NE-2 area have the following form:*

Red: $|p_1 - p_2| = d$
Green: $p_1 = (p_2 + 1)/2$ *and vice versa.*
Pink: $2(p_1 - d) = p_2(1 + p_1 - p_2)$ *and vice versa.*

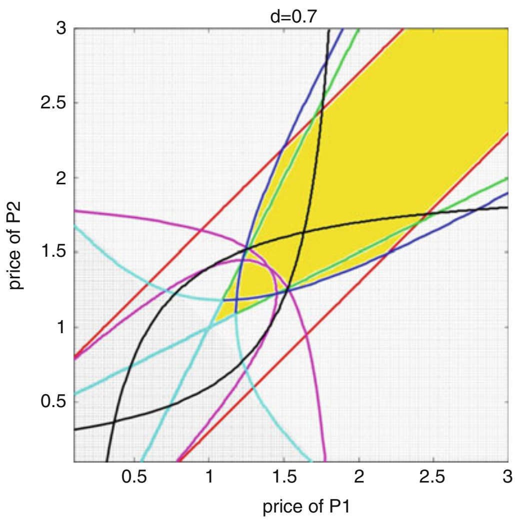

Fig. 3 $d = 0.7$. (1, 1) is NE

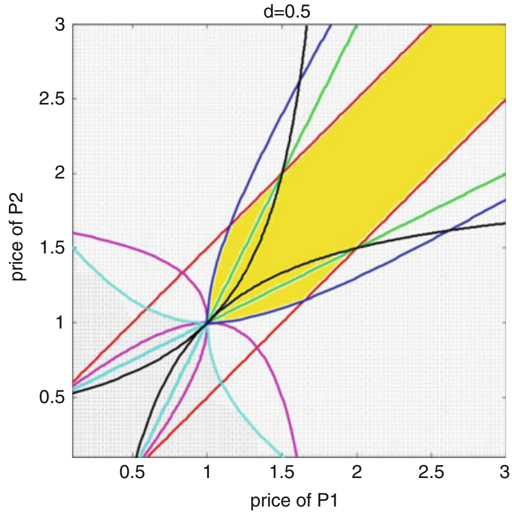

Fig. 4 $d = 0.5.(1, 1)$ is NE

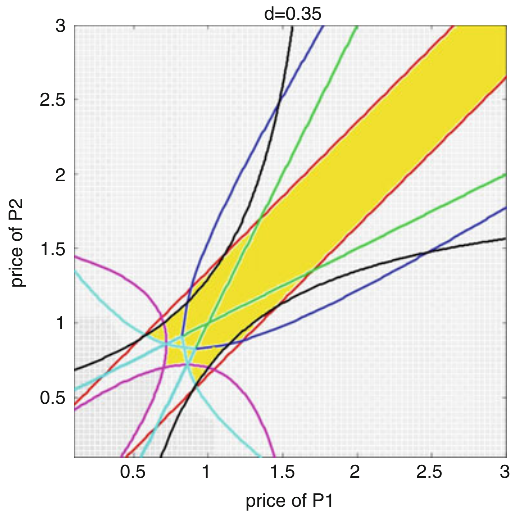

Fig. 5 $d = 0.35$. (2d, 2d) is EinSS

Dark blue: $p_1 = \frac{1+p_2}{2} + \sqrt{\left(\frac{1+p_2}{2}\right)^2 - 2d - p_2(1-p_2)}$ *and vice versa.*

Light blue: $p_2 = \frac{1+p_1}{2} - \sqrt{\left(\frac{1+p_1}{2}\right)^2 - 2d - p_1(1-p_1)}$ *and vice versa.*

Black: $p_2 = 2\left(1 - \frac{1-d}{p_1}\right)$ *and vice versa.*

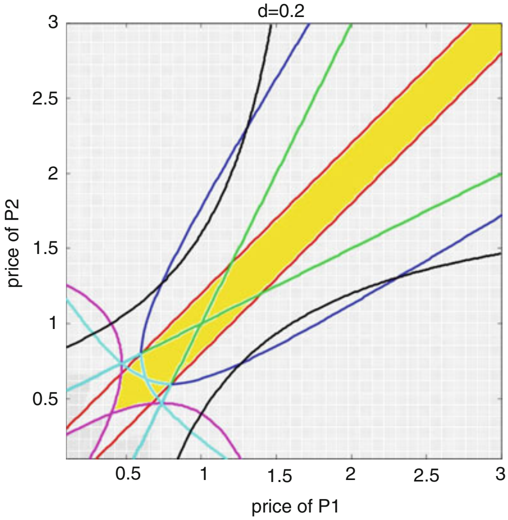

Fig. 6 $d = 0.2$. (2d, 2d) is EinSS

Proof is in Appendix.

Comparing the outcomes under various locations, we note that too close locations of firms are not good for them under EinSS logic, because they get low revenue. However, under NE-2 where security is not required, the location doesn't affect profit too much.

An interesting feature of NE-2 concept is that there exist asymmetric price equilibria under symmetric locations and costs. We can also observe that NE-2, being insecure, provides higher profits to firms than secure equilibria (EinSS). Actually, *any* profit level can be achieved because of inelastic demand and absence of choke-price in the model. This possibility has already been mentioned in the original paper of Hotelling [16]. Such non-secure profiles with higher profits can be treated as a tacit collusion, whereas EinSS may be regarded as a tough competition.

5 Concluding Remarks

This paper aims to shade some light on the process of rational iterated thinking with limited computational abilities of agents. The main idea is that when players have the same level of computational abilities to predict the opponents reaction, then they are not able to make a prediction with certainty and should take into consideration all possible profitable responses.

The first attempt to develop such an approach demonstrates intuitively clear interpretation of equilibria obtained for the class of strictly competitive games and for widely known models of Prisoners' Dilemma, Bertrand duopoly, and Hotelling spatial competition. The second level of farsightedness in these models plays a role of tacit communication between agents and they can reach better payoffs at some

Nash-2 equilibria. Existence in pure strategies makes such equilibria rather tractable and fruitful for further applications.

Problem of multiplicity of Nash-2 equilibrium can be solved in many ways depending on the concrete game framework. The paper [31] offers several ways to deal with the equilibrium set. In particular, introducing the measure of Nash-2 feasibility and its modifications seems to be a promising method for comparison it with empirical distributions.

There are at least two ways of future elaboration of Nash-2 equilibrium concept: applying it to the range of other models for which Nash equilibrium fails to provide indisputable answer (some examples are contained in [31]) and worthwhile extending the concept to n-person games.

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Appendix

Proof of Theorem 8.

1. For any $d \in [0, 1]$: $|p_1 - p_2| \leq d$. If it is not the case, then one firm gets all the market and it is profitable and secure for another firm, for instance, to undercut. Hereinafter we assume this condition to be held.

   It is to be noted that for any firm undercutting is never a profitable secure deviation. Hence we can test on security only deviations that preserve sharing the market.
2. If both firms propose prices $p_i \geq (p_{-i} + 1)/2$, $i = 1, 2$, then (p_1, p_2) is NE-2. Consider the profitable deviation $p_i' = p_i - \varepsilon$ of the firm i with $\varepsilon \in (0, 2p_i - p_{-i} - 1)$. It is not secure because of the firm $-i$ profitable deviation $p_{-i}' = p_i' - \varepsilon$.
3. Assume now that for both firms: $p_i \leq (p_{-i} + 1)/2$, $i = 1, 2$.

   If at least for one firm $i = 1$ or $i = 2$: $2(p_{-i} - d) < p_i(1 + p_{-i} - p_i)$, then (p_1, p_2) is not NE-2. Indeed, the firm $-i$ has the profitable secure deviation $p_{-i}' = p_{-i} + 0$ to slightly increase it's price.

   If for both firms $i = 1, 2$: $2(p_{-i} - d) > p_i(1 + p_{-i} - p_i)$, then (p_1, p_2) is NE-2. The deviation $p_i' = p_i + \varepsilon$ is profitable for the firm i for $\varepsilon \in (0, p_{-i} + 1 - 2p_i)$. The undercutting $p_{-i}' = p_i' - d$ for the firm $-i$ is profitable, so the initial deviation p_i' of the firm i is not secure.
4. Consider the remaining case: $p_i \geq (p_{-i} + 1)/2$ and $p_{-i} \leq (p_i + 1)/2$. In spite of symmetry let $i = 1$. There are two possibilities for the profile (p_1, p_2) not to be NE-2.

Let the firm 1 has a profitable secure deviation p_1': $p_1 > p_1' > 1 + p_2 - p_1$. The firm 2 shouldn't benefit from undercutting: $2(p_1' - d) \le p_2(1 + p_1' - p_2)$.

The boundary of the area is given by the system of equations:

$$\begin{cases} p_1' = 1 + p_2 - p_1, \\ 2(p_1' - d) = p_2(1 + p_1' - p_2). \end{cases}$$

This yields the black curve $p_2 = 2\left(1 - \frac{1-d}{p_1}\right)$.

Also if p_1' is small enough ($p_2 > (p_1' + 1)/2$) then p_1' should remain profitable for the firm 1 even the firm 2 maximum decreases its price $p_2' = 1 + p_1' - p_2 + 0$: $p_1(1 + p_2 - p_1) \le p_1'(1 + p_2' - p_1')$.

The system

$$\begin{cases} 2(p_1' - d) = p_2(1 + p_1' - p_2), \\ p_2' = 1 + p_1' - p_2, \\ p_1(1 + p_2 - p_1) = p_1'(1 + p_2' - p_1'), \end{cases}$$

leads to the equation of the dark blue boundary:

$$p_1 = \frac{1 + p_2}{2} + \sqrt{\left(\frac{1 + p_2}{2}\right)^2 - 2d - p_2(1 - p_2)}.$$

Another possibility is that the firm 2 has a profitable secure deviation $p_2' = 1 + p_1 - p_2 - 0$. It should remains profitable in case of decreasing price by the firm 1, and undercutting shouldn't give benefits to the firm 1. Similarly to the above these two conditions result in the equation of the light blue curve

$$p_2 = \frac{1 + p_1}{2} - \sqrt{\left(\frac{1 + p_1}{2}\right)^2 - 2d - p_1(1 - p_1)}.$$

This completes the proof.

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Strong Coalitional Structure in an Open Vehicle Routing Game

Nikolay Zenkevich and Andrey Zyatchin

Abstract In the chapter it is investigated a special case of one-product open vehicle routing game, in which there is a central warehouse or wholesaler, several customers, who are considered to be players. Each player is placed in a node of the transportation network and is characterized by demand and distance to the warehouse. For such a problem a coalitional transportation game (CTG) is formalized. In such a game each customer (player) should rent a track to deliver goods from the central warehouse. It is assumed that all tracks have the same capacity. The players tend to minimize their transportation costs and totally supply their demands. A player may rent a vehicle alone, or chose a coalition of players to cooperate. In cooperation the players of coalitions find the shortest path form the central depot to all the player of coalition. Transportation costs are allocated between players according to the Nash arbitration scheme. Strong equilibrium which is stable against deviations of any coalition of players is found in a CTG. A computation procedure for strong equilibrium construction is proposed. Implementation of procedure is illustrated with a numerical example.

Keywords Strong equilibrium • Cooperative game • Vehicle routing game

1 Introduction

Cooperation in logistics and supply chain management has received attention of researches and business in recent years [12, 17, 20]. There are several fields of operation for such cooperation, such as production, transportation, and inventory management. Cooperation in these fields is based on resources share and allows companies to increase performance in costs, quality, reliability, and adaptation. When a company seeks to minimize operational costs the following two questions appear: with whom to cooperate and how to allocate benefits between participants

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in cooperation. As a result, there are several decision makers in such a problem, and they tend not only to minimize cost, but also to create a cooperation which is stable in terms of deviations of any subset of players [21–23, 25]. In this chapter we focus on cooperation in minimizing transportation costs. The method for constructing such agreement should be transparent for anyone and simple enough for implementation.

Classical statement of a transportation problem introduces one decision maker, who delivers goods from several supply points to the several demand points. In this case routes include only one node. Another statement is known as vehicle routing problem (VRP) with one supply node and several demand points. Here a route may include several demand nodes. The VRP was originally introduced by Dantzig and Ramser [8]. Later a variety of special cases of VRP appeared. For example, just-in-time requirements introduce time windows—periods when a customer should be visited and served [19, 27, 28]. This class of VRP is known as the vehicle routing problem with time windows. The pick-up and delivery distribution scheme (VRPPD) represents a VRP where goods can be delivered to and taken from a customer [1, 26]. The petrol station replenishment problem requires vehicle capacity to be an integer between 1 and 4, according to the number of counterparts of a gasoline track [7]. Desrochers, Lenstra, and Savelsbergh provide wider classification of VRP extensions [9]. Clarke and Wright proposed one heuristic for solving the VRP [6], and later publications explored other heuristics for different VRP varieties [5, 11, 32]. Toth and Vigo suggest a unified classification for testing solutions to the VRP [30].

Our research is based on open vehicle routing problem, with one decision maker and which could be described as follows: there is a wholesaler in a geographic region and several customers, who buy goods from the wholesaler. Each customer is characterized with demand and inventory of the wholesaler is enough to meet demand of all the customer in the region. Wholesaler's warehouse we call a depot. Transportation of goods from the depot to customers is performed by vehicles of the same capacity. The problem is single-product. Every customer should be visited once and totally supplied. Distances between any two customers and a customer and the depot are known. The problem is to find such routes, which minimizes the total transportation costs. The routes should be open, i.e., they start at the depot and ends at a customer's point without coming back to the depot.

We introduce a special case of one-product open vehicle routing game, in which customers are decision makers. Each player is placed in a node of the transportation network and is characterized by demand and distance to the warehouse. For such a problem a coalitional transportation game (CTG) is formalized. In such a game each customer (player) should rent a track to deliver goods from the central warehouse. It is assumed that all tracks have the same capacity. The players tend to minimize their transportation costs and totally supply their demands. A player may rent a vehicle alone, or chose a coalition of players to cooperate. In cooperation the players of coalitions find the shortest path from the central depot to all the players of coalition. Transportation costs are allocated between players according to the Nash arbitration scheme.

In this article we introduce new approach to form coalitional structure and payoff functions on the base of game in normal form [10, 13–16]. Strong equilibrium is used as optimality principle [18, 33, 34]. For couple coalitional structure the same result was found in Andersson et al. [2], Talman and Yang [29].

This chapter consists of three sections. The first section is introduction and literature review. In the second section we formalize a transportation network in terms of graph theory and define the vehicle routing game by explaining sets of players, their strategies, and payoff functions. In the third section we propose a method to construct strong equilibrium and a numerical example for a vehicle routing game with 12 players.

2 Game-Theoretic Model of Vehicle Routing Game

In this chapter we define a transportation network G and introduce game-theoretic model of VRP.

2.1 Transportation Network

Consider a finite set of nodes $V \subset R^2$ on a coordinate plane R^2. Denote $v = |V|$ number of elements in the set V. Let function γ: $V \times V \to R^1_+$ to be Euclidian distance $\gamma(x, y)$ between nodes $x, y \in V$.

The set V and function $\gamma(x, y)$ define no oriented graph $Z = (V, E)$, where $V = \{x\}$—is a set on nodes in the graph and $E = \{(x, y)\} = V \times V$ is a set of edges. A fixed node $a \in V$, which corresponds to the position of a central warehouse, we call *a depot*.

Let's define a transportation network $G(a)$ on a graph $Z = (V, E)$ as a collection $G(a) = \langle V, \gamma; a\rangle$ [31, 35].

Consider a set of nodes $X = \{x_1, x_2, \dots, x_l\}$, $x_i \in V$ and $\pi_X = (k_1, \dots, k_l)$—permutation of numbers $1, \dots, l$ nodes from the set X.

Definition 1. A route r of serving nodes form a set $X = \{x_1, \dots, x_l\}$ in order π_X is oriented simple chain, which starts at the node a:

$$r = r_{X,\pi_X} = (a, x_{k_1}, x_{k_2}, \dots, x_{k_l}),$$

where $1 \le l < v$, $x_{k_i} \in V$, $x_{k_i} \ne x_{k_j}$, $x_{k_i} \ne a$, $k_i = 1, \dots, l$.

Obviously, for any set of nodes $X = \{x_1, \dots, x_l\}$ there are $l!$ different routes. Define a set of all routes as $R_0 = R_0[G(a)]$.

Definition 2. Two different routes $r^1 = r_{X,\pi_X} = (a, x_1^1, x_2^1, \ldots, x_{l_1}^1) \in R_0$ and $r^2 = r_{Y,\pi_Y} = (a, y_1^2, y_2^2, \ldots, y_{l_2}^2) \in R_0$ are said to be non-intersecting, and write $r^1 \cap r^2 = \emptyset$, if $X \cap Y = \emptyset$. We consider only non-intersecting routes in this paper.

Definition 3. Length of a route r_{X,π_X}, $X = \{x_1, \ldots, x_l\}$, $\pi_X = (k_1, \ldots, k_l)$ is the following value:

$$L(r_{X,\pi_X}) = \gamma(a, x_{k_1}) + \gamma(x_{k_1}, x_{k_2}) + \ldots + \gamma(x_{k_{l-1}}, x_{k_l}). \tag{1}$$

Definition 4. The shortest route for a set of nodes X is the route $r_X^{\min}$, which brings the minimal value of the length of the route in (1) on all possible permutations π_X of a set$X$:

$$r_X^{\min} = \{(a, x_{\bar{k}_1}, x_{\bar{k}_2}, \ldots, x_{\bar{k}_l}) | L(a, x_{\bar{k}_1}, x_{\bar{k}_2}, \ldots, x_{\bar{k}_l}) = \min_{\pi_X} L(a, x_{k_1}, x_{k_2}, \ldots, x_{k_l})\}.$$

2.2 *Coalitional Transportation Game Formulation*

Define a CTG of $n = v - 1$ players on the transportation network $G(a)$. Denote a set of players as $N = \{1, \ldots, n\}$. Assume that any player $i \in N$ is located at the node $x_i \in V$, $x_i \neq a$, $i = 1, \ldots, n$. For each player $i \in N$ demand $d_i = d\,(x_i) \geq 0$ is known, where $d\,(a) = 0$, $d\,(x)$, $x \in V$—is given demand function.

Assume that goods are shipped from a depot to players by a logistics company. The logistics company possesses a fleet of T vehicles of the same capacity D. Demand of each player is less or equal to capacity and the logistics company has enough facilities to meet all demand:

$$d_i \leq D, \quad i \in N, \quad n \leq T. \tag{2}$$

Assume that transportation cost $C(r)$ on a route $r \in R_0$ is proportional to the length of a route:

$$C(r) = \alpha L(r),$$

where α—is transportation cost for a unit of distance.

Let $S \subseteq N$ is a coalition of player and $s = |S|$ is number of players in this coalition. The set V is finite, so for any coalition $S \subseteq N$ which is located in a set $X = \{x_i\}_{i=1}^s$, it is the shortest route $r_S^{\min}$. Denote $C(r_S^{\min})$ transportation costs for the route $r_S^{\min} = r_X^{\min}$. As a result on the set of coalitions $\{S\}$, $S \subseteq N$ we define a function of transportation costs c: $S \to R^1$:

$$c(S) = C(r_S^{\min}).$$

Denote $c_i = c(\{i\})$ transportation cost of a single coalition, and $r_i = (a, x_i)$ a route of a single coalition $i \in N$.

Assume that transportation costs $c(S)$ are allocated between players of the coalition S according to the Nash arbitration scheme, i.e., transportation costs of a player $i \in S$ have the following form:

$$\phi_i(S) = c_i - \frac{\sum_{j \in S} c_j - c(S)}{s}. \quad (3)$$

Here a value $\sum_{j \in S} c_j - c(S)$ could be described as a payoff of the coalition S. The coalition S could also be characterized by marginal payoff of the following form:

$$\psi(S) = \frac{\sum_{j \in S} c_j - c(S)}{s}, \quad S \subseteq N, \quad (4)$$

as a result expression (3) could be introduced in the following form:

$$\phi_i(S) = c_i - \psi(S), i \in S.$$

Definition 5. Coalition S is said to be feasible, if:

$$\sum_{i \in S} d_i \leq D.$$

Denote a set of all possible coalitions as $\hat{S}$.

Definition 6. Coalition S is said to be essential, if:

$$\psi(S) \geq 0.$$

Denote a set of all essential coalitions as $\tilde{S}$. Each single coalition is essential, since:

$$\psi(\{i\}) = 0, \quad i \in N. \quad (5)$$

Example 1 (Feasible and Essential Coalitions). Consider a transportation network with 12 players. Coordinates of nodes location and demands of players are represented in Table 1. Assume a vehicle capacity is $D = 5$ units, $\alpha = 1$, $T = 10$.

Locations of players on a coordinate plane are given in Fig. 1, where players are numerated and the depot is marked by a star.

Consider a coalition $S = \{10, 11\}$. When the players of the coalition rent vehicles separately, their transportation costs are $c_{10} = 10{,}44$ and $c_{11} = 7{,}07$ correspondently (Fig. 2).

The total costs are $\sum_{j=10}^{11} c_j = 17{,}51$. Coalition $S = \{10, 11\}$ is feasible, since total demand of players in the coalition is 4 units (see Table 1) and capacity of a vehicle is 5. The shortest path $r_S^{\min}$ of the coalition S is shown in Fig. 3.

Table 1 Coordinates of nodes in a transportation network and demands of players

Node	Coordinates $x_i = (\xi_i, \eta_i)$		Demand (units)
	ξ	η_i	
Depot	19	45	No
Player 1	18	46	1
Player 2	20	47	1
Player 3	23	47	1
Player 4	24	45	2
Player 5	23	43	2
Player 6	20	41	2
Player 7	18	39	1
Player 8	14	40	1
Player 9	12	40	2
Player 10	9	42	1
Player 11	12	44	3
Player 12	13	46	1

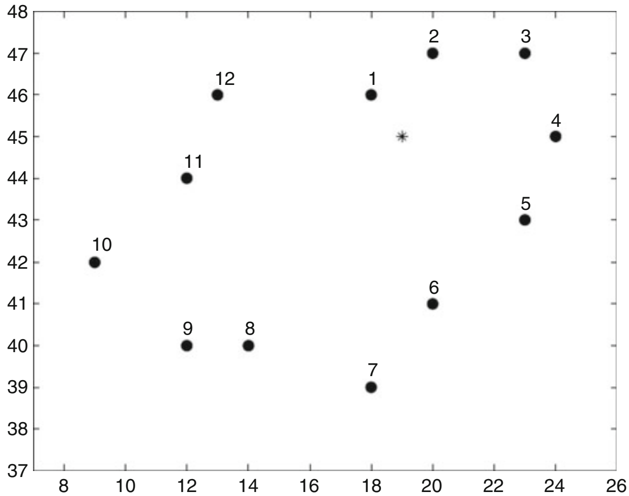

Fig. 1 Locations of nodes of a transportation network

Coalition S is essential since $c(S) = 10,68$, payoff of the coalition is

$$\sum_{j \in S} c_j - c(S) = 17,51 - 10,68 = 6,83,$$

and marginal payoff $\psi(S)$ of the coalition is $3,42 > 0$. Transportation costs of players 10 and 11, defined in (3), are the following: $\phi_{10}(S) = 7,02 < c_{10}$, $\phi_{11}(S) = 3,66 < c_{11}$.

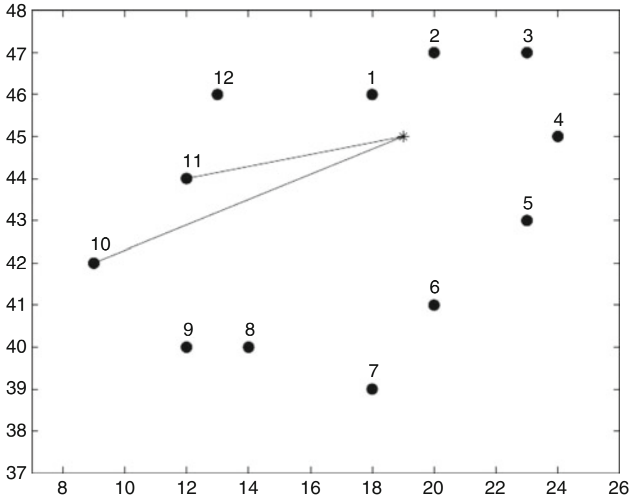

Fig. 2 Routes of players 10 and 11 in single coalitions

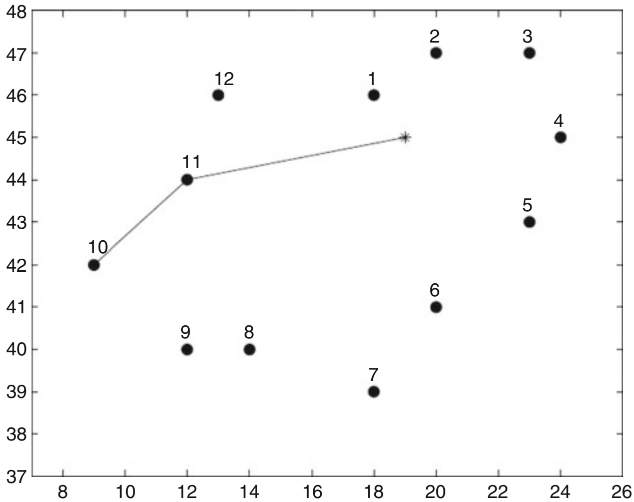

Fig. 3 The shortest path of the coalition of players 10 and 11

Consider a coalition $S_2 = \{2, 11\}$. In this case transportation costs for single coalitions are the following: $A_2 = 2,24$, $c_{11} = 7,07$. The coalition S_2 is feasible, since the total demand of players 2 and 11 is 4 units (see. Table 1). The coalition S_2 is not essential, since the total transportation costs of the coalition are $c(S_2) = 10,78$ and payoff of the coalition is

$$\sum_{j \in S} c_j - c(S) = 9,31 - 10,78 = -1,47 < 0.$$

A strategy h_i of a player $i \in N$ in a transportation game is a such a feasible coalition $h_i \in \hat{S}$, as $i \in h_i$. The set of all strategies of a player $i \in N$ denote H_i. Assume all players choose strategies simultaneously and independently. As a result a profile $h = (h_1, \dots, h_n)$, $h_i \in H_i$ is formed. The set of all profiles denote H.

Definition 7. Coalitional fragmentation is such set of coalitions $\bar{S} = \{\bar{S}_j\}_{j=1}^J$, as:

$$\bigcup_j \bar{S}_j = N, \quad \bar{S}_i \bigcap \bar{S}_j = \emptyset, \quad \textit{for any} \quad i \neq j.$$

Definition 8. For any profile $h = (h_1, \dots, h_n)$ define the following multistep procedure $\mu(h)$, which constructs a coalitional fragmentation $\bar{S} = \{\bar{S}_j\}_{j=1}^J$:

Step 1: $N_1 = N$; $i_1 = 1 = \min\{j | j \in N_1\}$

$$\bar{S}_1 = \begin{cases} h_1, & \textit{if } h_i = h_{i_1},\ i \in h_{i_1} \\ \{1\}, & \textit{else} \end{cases},$$

Step 2: Consider a set $N_2 = N_1 \backslash \bar{S}_1$. If $N_2 = \emptyset$, then $J = 1$ and $\bar{S} = \{\bar{S}_1\}$, else find a number $i_2 = \min\{j | j \in N_2\}$ and determine a set

$$\bar{S}_2 = \begin{cases} h_{i_2}, & \textit{if } h_i = h_{i_2},\ i \in h_{i_2} \\ \{i_2\}, & \textit{else} \end{cases},$$

...

Step k: Consider a set $N_k = N_{k-1} \backslash \bar{S}_{k-1}$. If $N_k = \emptyset$, then $J = k - 1$ and $\bar{S} = \{\bar{S}_j\}_{j=1}^J$, else find a number $i_k = \min\{j | j \in N_k\}$ and determine a set

$$\bar{S}_j = \begin{cases} h_{i_j}, & \textit{if } h_i = h_{i_j},\ i \in h_{i_j} \\ \{i_j\}, & \textit{else} \end{cases}$$

Multistep procedure $\mu(h)$ for any profile $h = (h_1, \dots, h_n)$ determine a unique coalitional fragmentation $\bar{S} = \{\bar{S}_j\}_{j=1}^J$ and the procedure is independent of numeration order of players, then $\mu(h) = \bar{S}$.

Example 2 (Coalitional Fragmentation Determination). Consider the transportation network from the previous example and a profile $h = (h_1, \dots, h_i, \dots, h_{12})$, where each player chooses a coalition of two players of the following form:

$$h_1 = \{1, 2\},\ h_2 = \{2, 3\},\ \dots,\ h_i = \{i, i + 1\},\ \dots,\ h_{12} = \{12, 1\}.$$

Then we implement procedure $\mu(h)$ and the result is coalitional fragmentation, which consists of single coalitions:

$$\bar{S} = \{\bar{S}_j\}_{j=1}^{12}, \bar{S}_j = \{j\}.$$

Assume now the players choose strategies of the following form: $h_1 = h_2 = h_3 = \{1, 2, 3\}, h_4 = \{1, 2, 3, 4\}, h_5 = \{5, 6, 7\}, h_6 = \{5, 6\}, h_7 = \{5, 7\}, h_8 = h_9 = h_{10} = \{8, 9, 10\}, h_{11} = h_{12} = \{11, 12\}$. Then we implement procedure $\mu(h)$ and the result is the following coalitional fragmentation:

$$\bar{S} = \{\bar{S}_j\}_{j=1}^{7},$$

where $\bar{S}_1 = \{1, 2, 3\}$, $\bar{S}_2 = \{4\}$, $\bar{S}_3 = \{5\}$, $\bar{S}_4 = \{6\}$, $\bar{S}_5 = \{7\}$, $\bar{S}_6 = \{8, 9, 10\}$, $\bar{S}_7 = \{11, 12\}$.

Define payoff function $K_i(h)$ of a player i as follows:

$$K_i(h) = \psi(\bar{S}_j),$$

where $i \in \bar{S}_j$, $\bar{S}_j \in \mu(h)$.

Assume for all non-intersecting coalitions $S_i, S_j \in \hat{S}$ which consist of two or more players the following properties are true:

$$\psi(S_i) \neq 0, \quad \psi(S_i) \neq \psi(S_j), \quad S_i \neq S_j, \quad S_i, S_j \in \hat{S}. \tag{6}$$

As a result we have constructed a transportation game $\Gamma = \Gamma(a)$ in normal form

$$\Gamma(a) = \langle G(a), N, \{H_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle,$$

where $G(a)$ is transportation network, $N = \{1, 2, \ldots, n\}$ is set of players, H_i is set of strategies, and K_i is payoff function of a player $i \in N$.

3 Strong Equilibrium

Define a strategy of coalition $S \subseteq N$ as ordered collection of strategies of players from the coalition, i.e., $h_S = (h_i)$, $i \in S$. The set of all strategies of coalition S denote H_S. Strategy of supplemental coalition $N \backslash S$ for a coalition S denotes $h_{-S} = (h_i)$, $i \in N \backslash S$ [24].

Definition 9. A profile $h^1 = (h_1^1, \ldots, h_n^1) \in H$ is said to be strong equilibrium in a transportation game $\Gamma(a)$, if for any coalition $S \subseteq N$ and strategy $h_S \in H_S$ there is a player $i_0 \in S$, for which the following inequality holds:

$$K_{i_0}(h_S^1, h_{-S}^1) \geq K_{i_0}(h_S, h_{-S}^1). \tag{7}$$

The set of all strong equilibria in a transportation game $\Gamma(a)$ denote SE [3, 4].

Definition 10. Let $h^1 = (h^1_1, \dots, h^1_n) \in H$ is strong equilibrium in transportation game $\Gamma(a)$, then $\bar{S} = \mu(h^1)$ is said to be strong equilibrium coalitional fragmentation.

3.1 Strong Equilibrium Determination

To determine strong equilibrium we assume the condition (6) holds and find coalitional fragmentation $\bar{S}_0$ according to the following multistep procedure M:

Step 1. Assume $N_1 = N$ and determine a set $\hat{S}_1$ of feasible coalitions of players from the set N_1. Find a coalition $\bar{S}_1 \in \hat{S}_1$ such as $\bar{S}_1 = \arg\max_{S \in \hat{S}_1} \psi(S)$.

Step 2. Determine a set of players $N_2 = N_1 \backslash \bar{S}_1$. If $N_2 = \emptyset$, then $J = 1$ and $\bar{S}_0 = \{\bar{S}_1\}$, else determine a set $\hat{S}_2$ of feasible coalitions of players from the set N_2. Find a coalition $\bar{S}_2 \in \hat{S}_2$ such as $\bar{S}_2 = \arg\max_{S \in \hat{S}_2} \psi(S)$.

Step k. Determine a set of players $N_k = N_{k-1} \backslash \bar{S}_{k-1}$. If $N_k = \emptyset$, then $J = k - 1$ and $\bar{S}_0 = \{\bar{S}_j\}_{j=1}^{J}$, else determine a set $\hat{S}_k$ of feasible coalitions of players from the set N_k. Find a coalition $\bar{S}_k \in \hat{S}_k$ such as $\bar{S}_k = \arg\max_{S \in \hat{S}_k} \psi(S)$.

In case of need single coalitions are ordered by numbers of players.

As a result of implementation of the procedure M in finite number of steps we get coalitional fragmentation $\bar{S}_0 = \{\bar{S}_j\}$. According to the procedure M the set $\bar{S}_0$ consists of essential non-intersecting coalitions, and $\bigcup_{j=1}^{J} \bar{S}_j = N$.

Theorem 1. *The profile $h^1 = (h^1_1, \dots, h^1_n)$, where $h^1_i = \bar{S}_j$, $i \in \bar{S}_j$, $\bar{S}_j \in \bar{S}_0$ constitutes strong equilibrium in a transportation game $\Gamma(a)$.*

Proof. Provide proof by induction by number of elements s in a coalition S.

Let $s = 1$, $S = \{i_0\}$ and $i_0 \in \bar{S}_j$. Then

$$\psi(\bar{S}_j) = \max_{U \in \hat{S}_j} \psi(U) = K_{i_0}(h^1_{i_0}, h^1_{-i_0}) \geq K_{i_0}(h_{i_0}, h^1_{-i_0}) = \begin{cases} c_{i_0}, h_{i_0} \neq \bar{S}_j \\ \psi(\bar{S}_j),\ h_{i_0} = \bar{S}_j \end{cases}$$

and the theorem will be proved.

Consider a case $s = 2$, $S = \{i_0,\ i_1\}$.

Let $S \in \bar{S}_j$. Then for any $i \in S$ the following inequality holds:

$$\psi(\bar{S}_j) = \max_{U \in \hat{S}_j} \psi(U) = K_i(h^1_S, h^1_{-S}) \geq K_i(h_S, h^1_{-S}) = \begin{cases} \psi(S),\ h_i = S,\ i = i_0, i_1 \\ \psi(\bar{S}_j),\ h_i = \bar{S}_j,\ i = i_0, i_1 \\ c_i,\ else \end{cases}$$

and the theorem will be proved.

Let $S \notin \bar{S}_j$, i.e., $i_0 \in \bar{S}_k$, $i_1 \in \bar{S}_j$, $\bar{S}_k \neq \bar{S}_j$ and let $k < j$, then $S \in \hat{S}_k$. According to the determination of $\bar{S}_k$ the following inequality holds:

$$\psi(\bar{S}_k) = \max_{U \in \hat{S}_k} \psi(U) = K_{i_0}(h^1_S, h^1_{-S}) \geq K_{i_0}(h_S, h^1_{-S}) = \begin{cases} \psi(\bar{S}_k),\ h_i = \bar{S}_k,\ i = i_0, i_1 \\ c_{i_0},\ else \end{cases}$$

and the base of induction is proved.

Let's prove the theorem for the coalitions of size s.

If a coalition S is infeasible, then it includes a feasible subcoalition $U \subset S$ of size $|U| < s$ and by induction assumption the theorem is proved.

Assume S is a feasible coalition. Then

$$K_i(h_S, h^1_{-S}) = \psi(U), i \in U, U \subset S.$$

If $|U| < s$, then the statement of the theorem is true by induction assumption. Let $U = S$. If $S \subset \bar{S}_j$, the inequality (7) is true for all $i \in S$ by determination of the set $\bar{S}_j$, else each player $i \in S$ belongs to a different set $i \in \bar{S}_{k_i}$. Consider $i_0 \in \bar{S}_k$, where $k = \min_{i \in S} k_i$. Then $S \in \hat{S}_k$ and the inequality (7) holds, since

$$K_{i_0}(h^1_S, h^1_{-S}) = \psi(\bar{S}_k) = \max_{U \in \hat{S}_k} \psi(U) \geq \psi(U) = K_{i_0}(h_S, h^1_{-S}), i \in U.$$

By induction assumption the theorem is proved. □

Example 3 (Strong Equilibrium in a Transportation Network). Consider a transportation game $\Gamma(a)$, defined on a transportation network from Examples 1 and 2. In this game there are 4095 coalitions, 456 feasible coalitions, and 247 essential coalitions.

The first five, the 247th and the 456th elements of the set $\hat{S}_1$ are represented in Table 2.

Table 2 Elements of $\hat{S}_1$

	Coalition, S	Marginal payoff, $\psi(S)$	Total demand of the coalition, $\sum_{i \in S} d_i$
1	{8, 9, 10}	4,47	4
2	{7, 8, 9, 10}	4,10	5
3	{10, 11, 12}	3,89	5
4	{8, 9, 10, 12}	3,71	5
5	{10, 11}	3,42	4
…	…	…	…
247	{12}	0	1
…	…	…	…
456	{4,11}	−2,48	5

Table 3 Strong equilibrium coalitional fragmentation, $\bar{S}_0 = \{S_j\}_{j=1}^6$

Step, j	Coalition, $\bar{S}_j$	The shortest route, $r_{S_j}^{\min}$	Marginal payoff, $\psi(\bar{S}_j)$
1	$\{8, 9, 10\}$	(a, x_8, x_9, x_{10})	4,47
2	$\{11, 12\}$	(a, x_{12}, x_{11})	2,41
3	$\{3, 4, 5\}$	(a, x_5, x_4, x_3)	1,67
4	$\{6, 7\}$	(a, x_6, x_7)	1,62
5	$\{1\}$	(a, x_1)	0
6	$\{2\}$	(a, x_2)	0

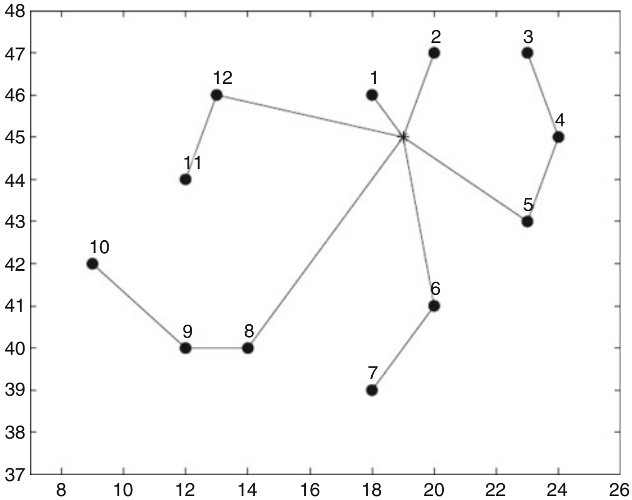

Fig. 4 Shortest routes corresponding to the strong equilibrium

The result of the determination of coalitional fragmentation according to the procedure M is represented in Table 3. By the theorem the profile $h^1 = (h_1^1, \ldots, h_{12}^1)$, where $h_1^1 = \bar{S}_5$, $h_2^1 = \bar{S}_6$, $h_3^1 = h_4^1 = h_5^1 = \bar{S}_3$, $h_6^1 = h_7^1 = \bar{S}_4$, $h_8^1 = h_9^1 = h_{10}^1 = \bar{S}_1$, $h_{11}^1 = h_{12}^1 = \bar{S}_2$, constitutes strong equilibrium in the transportation game $\Gamma(a)$. The shortest routes for coalitions from strong equilibrium are shown in Fig. 4.

4 Conclusion

A special case of the VRP is considered in the paper. It is assumed that all the customers are the players and they make decision about which coalition to form to rent and share a vehicle to deliver goods from the central warehouse.

In the proposed model capacities of all vehicles are equal. In general case the same model could be extended for transportation game with different capacities. Moreover we may consider players to be a logistics company, which order transportation service. In this case transportation costs correspond to cost of fuel consumed on a route.

In this paper we assume each route corresponds to a vehicle, i.e., shipment from the depot to all routes start simultaneously. Obviously, the procedure to determine strong equilibrium could be implemented in the case when the number of vehicles is less, then amount of routes, even in case with a single vehicle. In such a case all the routes are performed one by one.

The proposed approach could also be implemented for a transportation network, in which distance is determined by real geographic data.

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