Chapter 2
Equations and Phenomenology

In this section, we present the basic equations that are used to describe charged fluid flows, and the basic phenomenology of low-frequency turbulence. Readers interested in examining closely this subject can refer to the very wide literature on the subject of turbulence in fluid flows, as for example the recent books by, e.g., Pope (2000), McComb (1990), Frisch (1995) or many others, and the less known literature on MHD flows (Biskamp 1993; Boyd and Sanderson 2003; Biskamp 2003). In order to describe a plasma as a continuous medium it will be assumed collisional and, as a consequence, all quantities will be functions of space $\mathbf{r}$ and time t. Apart for the required quasi-neutrality, the basic assumption of MHD is that fields fluctuate on the same time and length scale as the plasma variables, say $\omega\tau_{\rm H} \simeq 1$ and $kL_{\rm H} \simeq 1$ (k and ω are, respectively, the wave number and the frequency of the fields, while $\tau_{\rm H}$ and $L_{\rm H}$ are the hydrodynamic time and length scale, respectively). Since the plasma is treated as a single fluid, we have to take the slow rates of ions. A simple analysis shows also that the electrostatic force and the displacement current can be neglected in the non-relativistic approximation. Then, MHD equations can be derived as shown in the following sections.

2.1 The Navier–Stokes Equation and the Reynolds Number

Equations which describe the dynamics of real incompressible fluid flows have been introduced by Claude-Louis Navier in 1823 and improved by George G. Stokes. They are nothing but the momentum equation based on Newton's second law, which relates the acceleration of a fluid particle[1] to the resulting volume and

[1] A fluid particle is defined as an infinitesimal portion of fluid which moves with the local velocity. As usual in fluid dynamics, infinitesimal means small with respect to large scale, but large enough with respect to molecular scales.

R. Bruno, V. Carbone, *Turbulence in the Solar Wind*, Lecture Notes in Physics 928, DOI 10.1007/978-3-319-43440-7_2

body forces acting on it. These equations have been introduced by Leonhard Euler, however, the main contribution by Navier was to add a friction forcing term due to the interactions between fluid layers which move with different speed. This term results to be proportional to the viscosity coefficients η and ξ and to the variation of speed. By defining the velocity field $\mathbf{u}(\mathbf{r}, t)$ the kinetic pressure p and the density ρ, the equations describing a fluid flow are the continuity equation to describe the conservation of mass

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\,\rho = -\rho \boldsymbol{\nabla} \cdot \mathbf{u}, \tag{2.1}$$

the equation for the conservation of momentum

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} \right] = -\nabla p + \eta \nabla^2 \mathbf{u} + \left(\xi + \frac{\eta}{3} \right) \nabla \left(\boldsymbol{\nabla} \cdot \mathbf{u} \right), \tag{2.2}$$

and an equation for the conservation of energy

$$\rho T \left[\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \nabla) s \right] = \boldsymbol{\nabla} \cdot (\chi \nabla T) + \frac{\eta}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \boldsymbol{\nabla} \cdot \mathbf{u} \right)^2 + \xi (\boldsymbol{\nabla} \cdot \mathbf{u})^2, \tag{2.3}$$

where s is the entropy per mass unit, T is the temperature, and χ is the coefficient of thermoconduction. An equation of state closes the system of fluid equations.

The above equations considerably simplify if we consider the incompressible fluid, where $\rho =$ const. so that we obtain the Navier–Stokes (NS) equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\left(\frac{\nabla p}{\rho} \right) + \nu \nabla^2 \mathbf{u}, \tag{2.4}$$

where the coefficient $\nu = \eta/\rho$ is the kinematic viscosity. The incompressibility of the flow translates in a condition on the velocity field, namely the field is divergence-free, i.e., $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. This condition eliminates all high-frequency sound waves and is called the incompressible limit. The non-linear term in equations represents the convective (or substantial) derivative. Of course, we can add on the right hand side of this equation all external forces, which eventually act on the fluid parcel.

We use the velocity scale U and the length scale L to define dimensionless independent variables, namely $\mathbf{r} = \mathbf{r}'L$ (from which $\nabla = \nabla'/L$) and $t = t'(L/U)$, and dependent variables $\mathbf{u} = \mathbf{u}'U$ and $p = p'U^2\rho$. Then, using these variables in Eq. (2.4), we obtain

$$\frac{\partial \mathbf{u}'}{\partial t'} + \left(\mathbf{u}' \cdot \nabla' \right) \mathbf{u}' = -\nabla' p' + Re^{-1} \nabla'^2 \mathbf{u}'. \tag{2.5}$$

The Reynolds number $Re = UL/\nu$ is evidently the only parameter of the fluid flow. This defines a Reynolds number similarity for fluid flows, namely fluids with the same value of the Reynolds number behaves in the same way. Looking at Eq. (2.5) it can be realized that the Reynolds number represents a measure of the relative strength between the non-linear convective term and the viscous term in Eq. (2.4). The higher *Re*, the more important the non-linear term is in the dynamics of the flow. Turbulence is a genuine result of the non-linear dynamics of fluid flows.

2.2 The Coupling Between a Charged Fluid and the Magnetic Field

Magnetic fields are ubiquitous in the Universe and are dynamically important. At high frequencies, kinetic effects are dominant, but at frequencies lower than the ion cyclotron frequency, the evolution of plasma can be modeled using the MHD approximation. Furthermore, dissipative phenomena can be neglected at large scales although their effects will be felt because of non-locality of non-linear interactions. In the presence of a magnetic field, the Lorentz force $\mathbf{j} \times \mathbf{B}$, where $\mathbf{j}$ is the electric current density, must be added to the fluid equations, namely

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right] = -\nabla p + \eta \nabla^2 \mathbf{u} + \left(\xi + \frac{\eta}{3} \right) \nabla\, (\boldsymbol{\nabla} \cdot \mathbf{u}) - \frac{1}{4\pi} \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}), \tag{2.6}$$

and the Joule heat must be added to the equation for energy

$$\rho T \left[\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \nabla) s \right] = \sigma_{ik} \frac{\partial u_i}{\partial x_k} + \chi \nabla^2 T + \frac{c^2}{16\pi^2 \sigma} (\boldsymbol{\nabla} \times \mathbf{B})^2, \tag{2.7}$$

where σ is the conductivity of the medium, and we introduced the viscous stress tensor

$$\sigma_{ik} = \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \boldsymbol{\nabla} \cdot \mathbf{u} \right) + \xi \delta_{ik} \boldsymbol{\nabla} \cdot \mathbf{u}. \tag{2.8}$$

An equation for the magnetic field stems from the Maxwell equations in which the displacement current is neglected under the assumption that the velocity of the fluid under consideration is much smaller than the speed of light. Then, using

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$$

and the Ohm's law for a conductor in motion with a speed $\mathbf{u}$ in a magnetic field

$$\mathbf{j} = \sigma\, (\mathbf{E} + \mathbf{u} \times \mathbf{B})\,,$$

we obtain the induction equation which describes the time evolution of the magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + (1/\sigma\mu_0)\nabla^2\mathbf{B}, \tag{2.9}$$

together with the constraint $\nabla\cdot\mathbf{B} = 0$ (no magnetic monopoles in the classical case).

In the incompressible case, where $\nabla \cdot \mathbf{u} = 0$, MHD equations can be reduced to

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\nabla P_{\text{tot}} + \nu\nabla^2\mathbf{u} + (\mathbf{b} \cdot \nabla)\,\mathbf{b} \tag{2.10}$$

and

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{b} = -\,(\mathbf{b} \cdot \nabla)\,\mathbf{u} + \eta\nabla^2\mathbf{b}. \tag{2.11}$$

Here P_{tot} is the total kinetic $P_k = nkT$ plus magnetic pressure $P_{\text{m}} = B^2/8\pi$, divided by the constant mass density ρ. Moreover, we introduced the velocity variables $\mathbf{b} = \mathbf{B}/\sqrt{4\pi\rho}$ and the magnetic diffusivity η.

Similar to the usual Reynolds number, a magnetic Reynolds number R_{m} can be defined, namely

$$R_{\text{m}} = \frac{\mathbf{c}_{\text{A}} L_0}{\eta}\,,$$

where $\mathbf{c}_{\text{A}} = \mathbf{B_0}/\sqrt{4\pi\rho}$ is the Alfvén speed related to the large-scale L_0 magnetic field $\mathbf{B_0}$. This number in most circumstances in astrophysics is very large, but the ratio of the two Reynolds numbers or, in other words, the magnetic Prandtl number $P_{\text{m}} = \nu/\eta$ can differ widely. In absence of dissipative terms, for each volume V MHD equations conserve the total energy $E(t)$

$$E(t) = \int_V (v^2 + b^2)\, d^3\mathbf{r}\,, \tag{2.12}$$

the cross-helicity $H_{\text{c}}(t)$, which represents a measure of the degree of correlations between velocity and magnetic fields

$$H_{\text{c}}(t) = \int_V \mathbf{v} \cdot \mathbf{b}\, d^3\mathbf{r}\,, \tag{2.13}$$

and the magnetic helicity $H(t)$, which represents a measure of the degree of linkage among magnetic flux tubes

$$H(t) = \int_V \mathbf{a} \cdot \mathbf{b}\, d^3\mathbf{r}\,, \tag{2.14}$$

where $\mathbf{b} = \nabla \times \mathbf{a}$.

The change of variable due to Elsässer (1950), say $\mathbf{z}^{\pm} = \mathbf{u} \pm \mathbf{b}'$, where we explicitly use the background uniform magnetic field $\mathbf{b}' = \mathbf{b} + \mathbf{c}_{\mathrm{A}}$ (at variance with the bulk velocity, the largest scale magnetic field cannot be eliminated through a Galilean transformation), leads to the more symmetrical form of the MHD equations in the incompressible case

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp (\mathbf{c}_{\mathrm{A}} \cdot \nabla)\, \mathbf{z}^{\pm} + \left(\mathbf{z}^{\mp} \cdot \nabla\right) \mathbf{z}^{\pm} = -\nabla P_{\mathrm{tot}} + \nu^{\pm} \nabla^2 \mathbf{z}^{\pm} + \nu^{\mp} \nabla^2 \mathbf{z}^{\mp} + \mathbf{F}^{\pm}, \qquad (2.15)$$

where $2\nu^{\pm} = \nu \pm \eta$ are the dissipative coefficients, and $\mathbf{F}^{\pm}$ are eventual external forcing terms. The relations $\nabla \cdot \mathbf{z}^{\pm} = 0$ complete the set of equations. On linearizing Eq. (2.15) and neglecting both the viscous and the external forcing terms, we have

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp (\mathbf{c}_{\mathrm{A}} \cdot \nabla)\, \mathbf{z}^{\pm} \simeq 0,$$

which shows that $\mathbf{z}^{-}(\mathbf{x} - \mathbf{c}_{\mathrm{A}} t)$ describes Alfvénic fluctuations propagating in the direction of $\mathbf{B_0}$, and $\mathbf{z}^{+}(\mathbf{x} + \mathbf{c}_{\mathrm{A}} t)$ describes Alfvénic fluctuations propagating opposite to $\mathbf{B_0}$. Note that MHD equations (2.15) have the same structure as the Navier–Stokes equation, the main difference stems from the fact that non-linear coupling happens only between fluctuations propagating in opposite directions. As we will see, this has a deep influence on turbulence described by MHD equations.

It is worthwhile to remark that in the classical hydrodynamics, dissipative processes are defined through three coefficients, namely two viscosities and one thermoconduction coefficient. In the hydromagnetic case the number of coefficients increases considerably. Apart from few additional electrical coefficients, we have a large-scale (background) magnetic field $\mathbf{B_0}$. This makes the MHD equations intrinsically anisotropic. Furthermore, the stress tensor (2.8) is deeply modified by the presence of a magnetic field $\mathbf{B_0}$, in that kinetic viscous coefficients must depend on the magnitude and direction of the magnetic field (Braginskii 1965). This has a strong influence on the determination of the Reynolds number.

2.3 Scaling Features of the Equations

The scaled Euler equations are the same as Eqs. (2.4) and (2.5), but without the term proportional to R^{-1}. The scaled variables obtained from the Euler equations are, then, the same. Thus, scaled variables exhibit scaling similarity, and the Euler equations are said to be invariant with respect to scale transformations. Said differently, this means that NS equations (2.4) show scaling properties (Frisch 1995), that is, there exists a class of solutions which are invariant under scaling transformations. Introducing a length scale ℓ, it is straightforward to verify that the scaling transformations $\ell \to \lambda \ell'$ and $\mathbf{u} \to \lambda^h \mathbf{u}'$ (λ is a scaling factor and h is a scaling index) leave invariant the inviscid NS equation for any scaling exponent h, providing $P \rightarrow \lambda^{2h} P'$. When the dissipative term is taken into account, a characteristic length scale exists, say the dissipative scale $\ell_{\rm D}$. From a phenomenological point of view, this is the length scale where dissipative effects start to be experienced by the flow. Of course, since ν is in general very low, we expect that $\ell_{\rm D}$ is very small. Actually, there exists a simple relationship for the scaling of $\ell_{\rm D}$ with the Reynolds number, namely $\ell_{\rm D} \sim L Re^{-3/4}$. The larger the Reynolds number, the smaller the dissipative length scale.

As it is easily verified, ideal MHD equations display similar scaling features. Say the following scaling transformations $\mathbf{u} \rightarrow \lambda^h \mathbf{u}'$ and $\mathbf{B} \rightarrow \lambda^{\beta} \mathbf{B}'$ (β here is a new scaling index different from h), leave the inviscid MHD equations unchanged, providing $P \rightarrow \lambda^{2\beta} P'$, $T \rightarrow \lambda^{2h} T'$, and $\rho \rightarrow \lambda^{2(\beta - h)} \rho'$. This means that velocity and magnetic variables have different scalings, say $h \neq \beta$, only when the scaling for the density is taken into account. In the incompressible case, we cannot distinguish between scaling laws for velocity and magnetic variables.

2.4 The Non-linear Energy Cascade

The basic properties of turbulence, as derived both from the Navier–Stokes equation and from phenomenological considerations, is the *legacy* of A. N. Kolmogorov (Frisch 1995).[2] Phenomenology is based on the old picture by Richardson who realized that turbulence is made by a collection of eddies at all scales. Energy, injected at a length scale L, is transferred by non-linear interactions to small scales where it is dissipated at a characteristic scale $\ell_{\rm D}$, the length scale where dissipation takes place. The main idea is that at very large Reynolds numbers, the injection scale L and the dissipative scale $\ell_{\rm D}$ are completely separated. In a stationary situation, the energy injection rate must be balanced by the energy dissipation rate and must also be the same as the energy transfer rate ε measured at any scale ℓ within the inertial range $\ell_{\rm D} \ll \ell \ll L$. From a phenomenological point of view, the energy injection rate at the scale L is given by $\epsilon_L \sim U^2/\tau_L$, where τ_L is a characteristic time for the injection energy process, which results to be $\tau_L \sim L/U$. At the same scale L the energy dissipation rate is due to $\epsilon_D \sim U^2/\tau_D$, where τ_D is the characteristic dissipation time which, from Eq. (2.4), can be estimated to be of the order of $\tau_D \sim L^2/\nu$. As a result, the ratio between the energy injection rate and dissipation rate is

$$\frac{\epsilon_L}{\epsilon_D} \sim \frac{\tau_D}{\tau_L} \sim Re \,, \tag{2.16}$$

that is, the energy injection rate at the largest scale L is Re-times the energy dissipation rate. In other words, in the case of large Reynolds numbers, the fluid

[2] The translation of the original paper by Kolmogorov (1941) can be found in the book edited by Kolmogorov (1991).

system is unable to dissipate the whole energy injected at the scale L. The excess energy must be dissipated at small scales where the dissipation process is much more efficient. This is the physical reason for the energy cascade.

Fully developed turbulence involves a hierarchical process, in which many scales of motion are involved. To look at this phenomenon it is often useful to investigate the behavior of the Fourier coefficients of the fields. Assuming periodic boundary conditions the αth component of velocity field can be Fourier decomposed as

$$u_\alpha(\mathbf{r},t) = \sum_{\mathbf{k}} u_\alpha(\mathbf{k},t)\exp(i\mathbf{k}\cdot\mathbf{r}),$$

where $\mathbf{k} = 2\pi\mathbf{n}/L$ and $\mathbf{n}$ is a vector of integers. When used in the Navier–Stokes equation, it is a simple matter to show that the non-linear term becomes the convolution sum

$$\frac{\partial u_\alpha(\mathbf{k},t)}{\partial t} = M_{\alpha\beta\gamma}(\mathbf{k})\sum_{\mathbf{q}} u_\gamma(\mathbf{k}-\mathbf{q},t)u_\beta(\mathbf{q},t), \tag{2.17}$$

where $M_{\alpha\beta\gamma}(\mathbf{k}) = -ik_\beta(\delta_{\alpha\gamma} - k_\alpha k_\beta/k^2)$ (for the moment we disregard the linear dissipative term).

MHD equations can be written in the same way, say by introducing the Fourier decomposition for Elsässer variables

$$z_\alpha^\pm(\mathbf{r},t) = \sum_{\mathbf{k}} z_\alpha^\pm(\mathbf{k},t)\exp(i\mathbf{k}\cdot\mathbf{r}),$$

and using this expression in the MHD equations we obtain an equation which describes the time evolution of each Fourier mode. However, the divergence-less condition means that not all Fourier modes are independent, rather $\mathbf{k}\cdot\mathbf{z}^\pm(\mathbf{k},t) = 0$ means that we can project the Fourier coefficients on two directions which are mutually orthogonal and orthogonal to the direction of $\mathbf{k}$, that is,

$$\mathbf{z}^\pm(\mathbf{k},t) = \sum_{a=1}^{2} z_a^\pm(\mathbf{k},t)\mathbf{e}^{(a)}(\mathbf{k}), \tag{2.18}$$

with the constraint that $\mathbf{k}\cdot\mathbf{e}^{(a)}(\mathbf{k}) = 0$. In presence of a background magnetic field we can use the well defined direction $\mathbf{B_0}$, so that

$$\mathbf{e}^{(1)}(\mathbf{k}) = \frac{i\mathbf{k}\times\mathbf{B_0}}{|\mathbf{k}\times\mathbf{B_0}|}; \qquad \mathbf{e}^{(2)}(\mathbf{k}) = \frac{i\mathbf{k}}{|\mathbf{k}|}\times\mathbf{e}^{(1)}(\mathbf{k}).$$

Note that in the linear approximation where the Elsässer variables represent the usual MHD modes, $z_1^\pm(\mathbf{k},t)$ represent the amplitude of the Alfvén mode while $z_2^\pm(\mathbf{k},t)$ represent the amplitude of the incompressible limit of the magnetosonic mode. From MHD equations (2.15) we obtain the following set of equations:

$$\left[\frac{\partial}{\partial t} \mp i\,(\mathbf{k}\cdot\mathbf{c}_{\mathrm{A}})\right] z_a^{\pm}(\mathbf{k},t) = \left(\frac{L}{2\pi}\right)^3 \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\delta} \sum_{b,c=1}^{2} A_{abc}(-\mathbf{k},\mathbf{p},\mathbf{q}) z_b^{\pm}(\mathbf{p},t) z_c^{\mp}(\mathbf{q},t). \tag{2.19}$$

The coupling coefficients, which satisfy the symmetry condition $A_{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) = -A_{bac}(\mathbf{p},\mathbf{k},\mathbf{q})$, are defined as

$$A_{abc}(-\mathbf{k},\mathbf{p},\mathbf{q}) = \left[(i\mathbf{k})^{\star}\cdot\mathbf{e}^{(c)}(\mathbf{q})\right]\left[\mathbf{e}^{(a)*}(\mathbf{k})\cdot\mathbf{e}^{(b)}(\mathbf{p})\right],$$

and the sum in Eq. (2.19) is defined as

$$\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}}^{\delta} \equiv \left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{p}} \sum_{\mathbf{q}} \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}},$$

where $\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}}$ is the Kronecher's symbol. Quadratic non-linearities of the original equations correspond to a convolution term involving wave vectors $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{q}$ related by the triangular relation $\mathbf{p} = \mathbf{k} - \mathbf{q}$. Fourier coefficients locally couple to generate an energy transfer from any pair of modes $\mathbf{p}$ and $\mathbf{q}$ to a mode $\mathbf{k} = \mathbf{p} + \mathbf{q}$.

The pseudo-energies $E^{\pm}(t)$ are defined as

$$E^{\pm}(t) = \frac{1}{2}\frac{1}{L^3}\int_{L^3} |\mathbf{z}^{\pm}(\mathbf{r},t)|^2 d^3\mathbf{r} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{a=1}^{2} |z_a^{\pm}(\mathbf{k},t)|^2$$

and, after some algebra, it can be shown that the non-linear term of Eq. (2.19) conserves separately $E^{\pm}(t)$. This means that both the total energy $E(t) = E^{+} + E^{-}$ and the cross-helicity $E_{\mathrm{c}}(t) = E^{+} - E^{-}$, say the correlation between velocity and magnetic field, are conserved in absence of dissipation and external forcing terms.

In the idealized homogeneous and isotropic situation we can define the pseudo-energy tensor, which using the incompressibility condition can be written as

$$U_{ab}^{\pm}(\mathbf{k},t) \equiv \left(\frac{L}{2\pi}\right)^3 \left\langle z_a^{\pm}(\mathbf{k},t) z_b^{\pm}(\mathbf{k},t)\right\rangle = \left(\delta_{ab} - \frac{k_a k_b}{k^2}\right) q^{\pm}(k),$$

brackets being ensemble averages, where $q^{\pm}(k)$ is an arbitrary odd function of the wave vector k and represents the pseudo-energies spectral density. When integrated over all wave vectors under the assumption of isotropy

$$Tr\left[\int d^3\mathbf{k}\; U_{ab}^{\pm}(\mathbf{k},t)\right] = 2\int_0^{\infty} E^{\pm}(k,t)dk,$$

where we introduce the spectral pseudo-energy $E^{\pm}(k,t) = 4\pi k^2 q^{\pm}(k,t)$. This last quantity can be measured, and it is shown that it satisfies the equations

$$\frac{\partial E^{\pm}(k,t)}{\partial t} = T^{\pm}(k,t) - 2\nu k^2 E^{\pm}(k,t) + F^{\pm}(k,t). \tag{2.20}$$

We use $\nu = \eta$ in order not to worry about coupling between $+$ and $-$ modes in the dissipative range. Since the non-linear term conserves total pseudo-energies we have

$$\int_0^{\infty} dk\, T^{\pm}(k,t) = 0,$$

so that, when integrated over all wave vectors, we obtain the energy balance equation for the total pseudo-energies

$$\frac{dE^{\pm}(t)}{dt} = \int_0^{\infty} dk\, F^{\pm}(k,t) - 2\nu \int_0^{\infty} dk\, k^2 E^{\pm}(k,t). \tag{2.21}$$

This last equation simply means that the time variations of pseudo-energies are due to the difference between the injected power and the dissipated power, so that in a stationary state

$$\int_0^{\infty} dk\, F^{\pm}(k,t) = 2\nu \int_0^{\infty} dk\, k^2 E^{\pm}(k,t) = \epsilon^{\pm}.$$

Looking at Eq. (2.20), we see that the role played by the non-linear term is that of a redistribution of energy among the various wave vectors. This is the physical meaning of the non-linear energy cascade of turbulence.

2.5 The Inhomogeneous Case

Equations (2.20) refer to the standard homogeneous and incompressible MHD. Of course, the solar wind is inhomogeneous and compressible and the energy transfer equations can be as complicated as we want by modeling all possible physical effects like, for example, the wind expansion or the inhomogeneous large-scale magnetic field. Of course, simulations of all turbulent scales requires a computational effort which is beyond the actual possibilities. A way to overcome this limitation is to introduce some turbulence modeling of the various physical effects. For example, a set of equations for the cross-correlation functions of both Elsässer fluctuations have been developed independently by Marsch and Tu (1989), Zhou and Matthaeus (1990), Oughton and Matthaeus (1992), and Tu and Marsch (1990), following Marsch and Mangeney (1987) (see review by Tu and Marsch

1996), and are based on some rather strong assumptions: (1) a two-scale separation, and (2) small-scale fluctuations are represented as a kind of stochastic process (Tu and Marsch 1996). These equations look quite complicated, and just a comparison based on order-of-magnitude estimates can be made between them and solar wind observations (Tu and Marsch 1996).

A different approach, introduced by Grappin et al. (1993), is based on the so-called "expanding-box model" (Grappin and Velli 1996; Liewer et al. 2001; Hellinger et al. 2005). The model uses transformation of variables to the moving solar wind frame that expands together with the size of the parcel of plasma as it propagates outward from the Sun. Despite the model requires several simplifying assumptions, like for example lateral expansion only for the wave-packets and constant solar wind speed, as well as a second-order approximation for coordinate transformation (Liewer et al. 2001) to remain tractable, it provides qualitatively good description of the solar wind expansions, thus connecting the disparate scales of the plasma in the various parts of the heliosphere.

2.6 Dynamical System Approach to Turbulence

In the limit of fully developed turbulence, when dissipation goes to zero, an infinite range of scales are excited, that is, energy lies over all available wave vectors. Dissipation takes place at a typical dissipation length scale which depends on the Reynolds number Re through $\ell_D \sim L Re^{-3/4}$ (for a Kolmogorov spectrum $E(k) \sim k^{-5/3}$). In 3D numerical simulations the minimum number of grid points necessary to obtain information on the fields at these scales is given by $N \sim (L/\ell_D)^3 \sim Re^{9/4}$. This rough estimate shows that a considerable amount of memory is required when we want to perform numerical simulations with high Re. At present, typical values of Reynolds numbers reached in 2D and 3D numerical simulations are of the order of 10^4 and 10^3, respectively. At these values the inertial range spans approximately one decade or a little more.

Given the situation described above, the question of the best description of dynamics which results from original equations, using only a small amount of degree of freedom, becomes a very important issue. This can be achieved by introducing turbulence models which are investigated using tools of dynamical system theory (Bohr et al. 1998). Dynamical systems, then, are solutions of minimal sets of ordinary differential equations that can mimic the gross features of energy cascade in turbulence. These studies are motivated by the famous Lorenz's model (Lorenz 1963) which, containing only three degrees of freedom, simulates the complex chaotic behavior of turbulent atmospheric flows, becoming a paradigm for the study of chaotic systems.

The Lorenz's model has been used as a paradigm as far as the transition to turbulence is concerned. Actually, since the solar wind is in a state of fully developed turbulence, the topic of the transition to turbulence is not so close to the main goal of this review. However, since their importance in the theory of dynamical systems, we spend few sentences abut this central topic. Up to the Lorenz's chaotic model, studies on the birth of turbulence dealt with linear and, very rarely, with weak non-linear evolution of external disturbances. The first physical model of laminar-turbulent transition is due to Landau and it is reported in the fourth volume of the course on Theoretical Physics (Landau and Lifshitz 1971). According to this model, as the Reynolds number is increased, the transition is due to a infinite series of Hopf bifurcations at fixed values of the Reynolds number. Each subsequent bifurcation adds a new incommensurate frequency to the flow whose dynamics become rapidly quasi-periodic. Due to the infinite number of degree of freedom involved, the quasi-periodic dynamics resembles that of a turbulent flow.

The Landau transition scenario is, however, untenable because incommensurate frequencies cannot exist without coupling between them. Ruelle and Takens (1971) proposed a new mathematical model, according to which after few, usually three, Hopf bifurcations the flow becomes suddenly chaotic. In the phase space this state is characterized by a very intricate attracting subset, a *strange attractor*. The flow corresponding to this state is highly irregular and strongly dependent on initial conditions. This characteristic feature is now known as the *butterfly effect* and represents the true definition of deterministic chaos. These authors indicated as an example for the occurrence of a strange attractor the old strange time behavior of the Lorenz's model. The model is a paradigm for the occurrence of turbulence in a deterministic system, it reads

$$\frac{dx}{dt} = P_{\mathrm{r}}(y - x)\,, \qquad \frac{dy}{dt} = Rx - y - xz\,, \qquad \frac{dz}{dt} = xy - bz\,, \tag{2.22}$$

where $x(t)$, $y(t)$, and $z(t)$ represent the first three modes of a Fourier expansion of fluid convective equations in the Boussinesq approximation, P_{r} is the Prandtl number, b is a geometrical parameter, and R is the ratio between the Rayleigh number and the critical Rayleigh number for convective motion. The time evolution of the variables $x(t)$, $y(t)$, and $z(t)$ is reported in Fig. 2.1. A reproduction of the Lorenz *butterfly* attractor, namely the projection of the variables on the plane (x, z) is shown in Fig. 2.2. A few years later, Gollub and Swinney (1975) performed very sophisticated experiments,[3] concluding that the transition to turbulence in a flow between co-rotating cylinders is described by the Ruelle and Takens (1971) model rather than by the Landau scenario.

After this discovery, the strange attractor model gained a lot of popularity, thus stimulating a large number of further studies on the time evolution of non-linear dynamical systems. An enormous number of papers on chaos rapidly appeared in literature, quite in all fields of physics, and transition to chaos became a new topic. Of course, further studies on chaos rapidly lost touch with turbulence studies

[3]These authors were the first ones to use physical technologies and methodologies to investigate turbulent flows from an experimental point of view. Before them, experimental studies on turbulence were motivated mainly by engineering aspects.

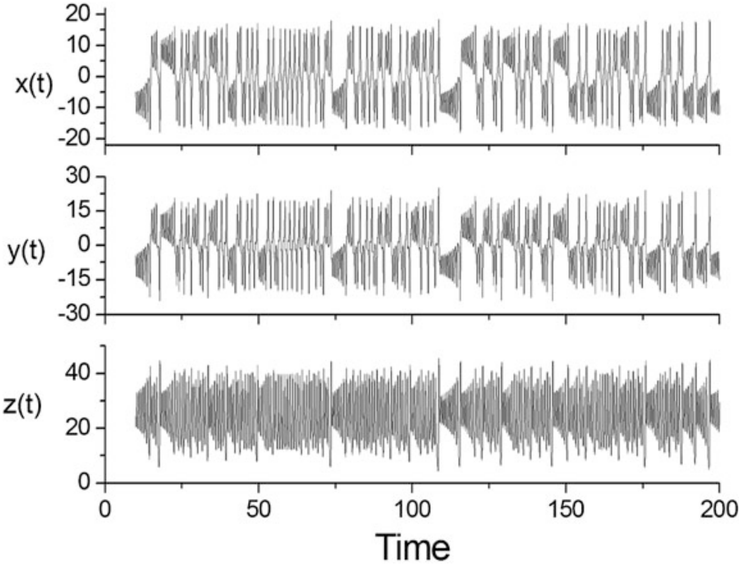

Fig. 2.1 Time evolution of the variables $x(t)$, $y(t)$, and $z(t)$ in the Lorenz's model [see Eq. (2.22)]. This figure has been obtained by using the parameters $P_r = 10$, $b = 8/3$, and $R = 28$

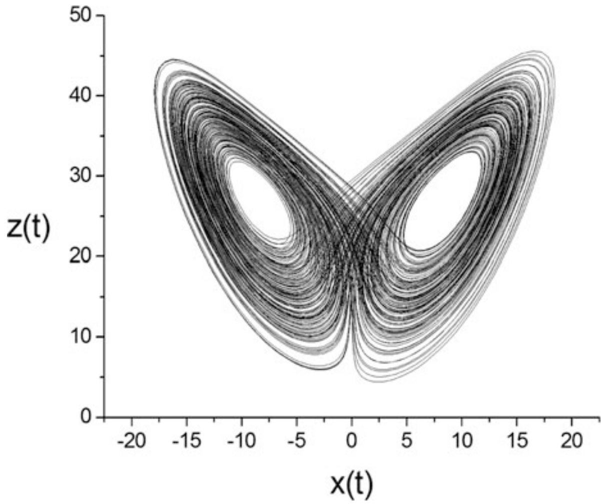

Fig. 2.2 The Lorenz *butterfly* attractor, namely the time behavior of the variables $z(t)$ vs. $x(t)$ as obtained from the Lorenz's model [see Eq. (2.22)]. This figure has been obtained by using the parameters $P_r = 10$, $b = 8/3$, and $R = 28$

and turbulence, as reported by Feynman et al. (1977), still remains … *the last great unsolved problem of the classical physics*. Furthermore, we like to cite recent theoretical efforts made by Chian et al. (1998, 2003) related to the onset of Alfvénic turbulence. These authors, numerically solved the derivative non-linear Schrödinger equation (Mjølhus 1976; Ghosh and Papadopoulos 1987) which governs the spatio-temporal dynamics of non-linear Alfvén waves, and found that Alfvénic intermittent turbulence is characterized by strange attractors. Note that, the physics involved in the derivative non-linear Schrödinger equation, and in particular the spatio-temporal dynamics of non-linear Alfvén waves, cannot be described by the usual incompressible MHD equations. Rather dispersive effects are required. At variance with the usual MHD, this can be satisfied by requiring that the effect of ion inertia be taken into account. This results in a generalized Ohm's law by including a $(\underline{j} \times \underline{B})$-term, which represents the compressible Hall correction to MHD, say the so-called compressible Hall-MHD model.

In this context turbulence can evolve via two distinct routes: Pomeau–Manneville intermittency (Pomeau and Manneville 1980) and crisis-induced intermittency (Ott and Sommerer 1994). Both types of chaotic transitions follow episodic switching between different temporal behaviors. In one case (Pomeau–Manneville) the behavior of the magnetic fluctuations evolve from nearly periodic to chaotic while, in the other case the behavior intermittently assumes weakly chaotic or strongly chaotic features.

2.7 Shell Models for Turbulence Cascade

Since numerical simulations, in some cases, cannot be used, simple dynamical systems can be introduced to investigate, for example, statistical properties of turbulent flows which can be compared with observations. These models, which try to mimic the gross features of the time evolution of spectral Navier–Stokes or MHD equations, are often called "shell models" or "discrete cascade models". Starting from the old papers by Siggia (1977) different shell models have been introduced in literature for 3D fluid turbulence (Biferale 2003). MHD shell models have been introduced to describe the MHD turbulent cascade (Plunian et al. 2012), starting from the paper by Gloaguen et al. (1985).

The most used shell model is usually quoted in literature as the GOY model, and has been introduced some time ago by Gledzer (1973) and by Ohkitani and Yamada (1989). Apart from the first MHD shell model (Gloaguen et al. 1985), further models, like those by Frick and Sokoloff (1998) and Giuliani and Carbone (1998) have been introduced and investigated in detail. In particular, the latter ones represent the counterpart of the hydrodynamic GOY model, that is they coincide with the usual GOY model when the magnetic variables are set to zero.

In the following, we will refer to the MHD shell model as the FSGC model. The shell model can be built up through four different steps:

(a) Introduce discrete wave vectors:
As a first step we divide the wave vector space in a discrete number of shells whose radii grow according to a power $k_n = k_0\lambda^n$, where $\lambda > 1$ is the inter-shell ratio, k_0 is the fundamental wave vector related to the largest available length scale L, and $n = 1, 2, \ldots, N$.

(b) Assign to each shell discrete scalar variables:
Each shell is assigned two or more complex scalar variables $u_n(t)$ and $b_n(t)$, or Elsässer variables $Z_n^{\pm}(t) = u_n \pm b_n(t)$. These variables describe the chaotic dynamics of modes in the shell of wave vectors between k_n and k_{n+1}. It is worth noting that the discrete variable, mimicking the average behavior of Fourier modes within each shell, represents characteristic fluctuations across eddies at the scale $\ell_n \sim k_n^{-1}$. That is, the fields have the same scalings as field differences, for example $Z_n^{\pm} \sim |Z^{\pm}(x + \ell_n) - Z^{\pm}(x)| \sim \ell_n^h$ in fully developed turbulence. In this way, the possibility to describe spatial behavior within the model is ruled out. We can only get, from a dynamical shell model, time series for shell variables at a given k_n, and we loose the fact that turbulence is a typical temporal and spatial complex phenomenon.

(c) Introduce a dynamical model which describes non-linear evolution:
Looking at Eq. (2.19) a model must have quadratic non-linearities among opposite variables $Z_n^{\pm}(t)$ and $Z_n^{\mp}(t)$, and must couple different shells with free coupling coefficients.

(d) Fix as much as possible the coupling coefficients:
This last step is not standard. A numerical investigation of the model might require the scanning of the properties of the system when all coefficients are varied. Coupling coefficients can be fixed by imposing the conservation laws of the original equations, namely the total pseudo-energies

$$E^{\pm}(t) = \frac{1}{2}\sum_n \left|Z_n^{\pm}\right|^2,$$

that means the conservation of both the total energy and the cross-helicity:

$$E(t) = \frac{1}{2}\sum_n |u_n|^2 + |b_n|^2\,; \qquad H_c(t) = \sum_n 2\Re e\left(u_n b_n^*\right),$$

where $\Re e$ indicates the real part of the product $u_n b_n^*$. As we said before, shell models cannot describe spatial geometry of non-linear interactions in turbulence, so that we loose the possibility of distinguishing between two-dimensional and three-dimensional turbulent behavior. The distinction is, however, of primary importance, for example as far as the dynamo effect is concerned in MHD. However, there is a third invariant which we can impose, namely

$$H(t) = \sum_n (-1)^n \frac{|b_n|^2}{k_n^\alpha}, \tag{2.23}$$

which can be dimensionally identified as the magnetic helicity when $\alpha = 1$, so that the shell model so obtained is able to mimic a kind of 3D MHD turbulence (Giuliani and Carbone 1998).

After some algebra, taking into account both the dissipative and forcing terms, FSGC model can be written as

$$\frac{dZ_n^\pm}{dt} = ik_n \Phi_n^{\pm *} + \frac{\nu \pm \mu}{2} k_n^2 Z_n^+ + \frac{\nu \mp \mu}{2} k_n^2 Z_n^- + F_n^\pm, \tag{2.24}$$

where

$$\begin{aligned}\Phi_n^\pm = & \left(\frac{2-a-c}{2}\right) Z_{n+2}^\pm Z_{n+1}^\mp + \left(\frac{a+c}{2}\right) Z_{n+1}^\pm Z_{n+2}^\mp + \\ & + \left(\frac{c-a}{2\lambda}\right) Z_{n-1}^\pm Z_{n+1}^\mp - \left(\frac{a+c}{2\lambda}\right) Z_{n-1}^\mp Z_{n+1}^\pm + \\ & - \left(\frac{c-a}{2\lambda^2}\right) Z_{n-2}^\mp Z_{n-1}^\pm - \left(\frac{2-a-c}{2\lambda^2}\right) Z_{n-1}^\mp Z_{n-2}^\pm,\end{aligned} \tag{2.25}$$

where[4] $\lambda = 2$, $a = 1/2$, and $c = 1/3$. In the following, we will consider only the case where the dissipative coefficients are the same, i.e., $\nu = \mu$.

2.8 The Phenomenology of Fully Developed Turbulence: Fluid-Like Case

Here we present the phenomenology of fully developed turbulence, as far as the scaling properties are concerned. In this way we are able to recover a universal form for the spectral pseudo-energy in the stationary case. In real space a common tool to investigate statistical properties of turbulence is represented by field increments $\Delta z_\ell^\pm(\mathbf{r}) = \left[\mathbf{z}^\pm(\mathbf{r}+\ell) - \mathbf{z}^\pm(\mathbf{r})\right] \cdot \mathbf{e}$, being $\mathbf{e}$ the longitudinal direction. These

[4] We can use a different definition for the third invariant $H(t)$, for example a quantity positive defined, without the term $(-1)^n$ and with $\alpha = 2$. This can be identified as the surrogate of the square of the vector potential, thus investigating a kind of 2D MHD. In this case, we obtain a shell model with $\lambda = 2$, $a = 5/4$, and $c = -1/3$. However, this model does not reproduce the inverse cascade of the square of magnetic potential observed in the true 2D MHD equations.

stochastic quantities represent fluctuations[5] across eddies at the scale ℓ. The scaling invariance of MHD equations (cf. Sect. 2.3), from a phenomenological point of view, implies that we expect solutions where $\Delta z_\ell^\pm \sim \ell^h$. All the statistical properties of the field depend only on the scale ℓ, on the mean pseudo-energy dissipation rates $\varepsilon^\pm$, and on the viscosity ν. Also, $\varepsilon^\pm$ is supposed to be the common value of the injection, transfer and dissipation rates. Moreover, the dependence on the viscosity only arises at small scales, near the bottom of the inertial range. Under these assumptions the typical pseudo-energy dissipation rate per unit mass scales as $\varepsilon^\pm \sim \left(\Delta z_\ell^\pm\right)^2 / t_\ell^\pm$. The time $t_\ell^\pm$ associated with the scale ℓ is the typical time needed for the energy to be transferred on a smaller scale, say the eddy turnover time $t_\ell^\pm \sim \ell / \Delta z_\ell^\mp$, so that

$$\varepsilon^\pm \sim \left(\Delta z_\ell^\pm\right)^2 \Delta z^\mp / \ell.$$

When we conjecture that both $\Delta z^\pm$ fluctuations have the same scaling laws, namely $\Delta z^\pm \sim \ell^h$, we recover the Kolmogorov scaling for the field increments

$$\Delta z_\ell^\pm \sim (\varepsilon^\pm)^{1/3} \ell^{1/3}. \tag{2.26}$$

Usually, we refer to this scaling as the K41 model (Kolmogorov 1941, 1991; Frisch 1995). Note that, since from dimensional considerations the scaling of the energy transfer rate should be $\varepsilon^\pm \sim \ell^{1-3h}$, $h = 1/3$ is the choice to guarantee the absence of scaling for $\varepsilon^\pm$.

In the real space turbulence properties can be described using either the probability distribution functions (PDFs hereafter) of increments, or the *longitudinal structure functions*, which represents nothing but the higher order moments of the field. Disregarding the magnetic field, in a purely fully developed fluid turbulence, this is defined as $S_\ell^{(p)} = \langle \Delta u_\ell^p \rangle$. These quantities, in the inertial range, behave as a power law $S_\ell^{(p)} \sim \ell^{\xi_p}$, so that it is interesting to compute the set of scaling exponent ξ_p. Using, from a phenomenological point of view, the scaling for field increments [see Eq. (2.26)], it is straightforward to compute the scaling laws $S_\ell^{(p)} \sim \ell^{p/3}$. Then $\xi_p = p/3$ results to be a linear function of the order p.

When we assume the scaling law $\Delta z_\ell^\pm \sim \ell^h$, we can compute the high-order moments of the structure functions for increments of the Elsässer variables, namely $\left\langle (\Delta z_\ell^\pm)^p \right\rangle \sim \ell^{\xi_p}$, thus obtaining a linear scaling $\xi_p = p/3$, similar to usual fluid flows. For Gaussianly distributed fields, a particular role is played by the second-order moment, because all moments can be computed from $S_\ell^{(2)}$. It is straightforward to translate the dimensional analysis results to Fourier spectra. The spectral property

[5]We have already defined fluctuations of a field as the difference between the field itself and its average value. This quantity has been defined as $\delta\psi$. Here, the differences $\Delta\psi_\ell$ of the field separated by a distance ℓ represents characteristic fluctuations *at the scale* ℓ, say characteristic fluctuations of the field across specific structures (eddies) that are present at that scale. The reader can realize the difference between both definitions.

of the field can be recovered from $S_\ell^{(2)}$, say in the homogeneous and isotropic case

$$S_\ell^{(2)} = 4\int_0^\infty E(k)\left(1 - \frac{\sin k\ell}{k\ell}\right) dk,$$

where $k \sim 1/\ell$ is the wave vector, so that in the inertial range where Eq. (2.42) is verified

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}. \tag{2.27}$$

The *Kolmogorov spectrum* [see Eq. (2.27)] is largely observed in all experimental investigations of turbulence, and is considered as the main result of the K41 phenomenology of turbulence (Frisch 1995). However, spectral analysis does not provide a complete description of the statistical properties of the field, unless this has Gaussian properties. The same considerations can be made for the spectral pseudo-energies $E^{\pm}(k)$, which are related to the second order structure functions $\left\langle \left[\Delta z_\ell^{\pm}\right]^2 \right\rangle$.

2.9 The Phenomenology of Fully Developed Turbulence: Magnetically-Dominated Case

The phenomenology of the magnetically-dominated case has been investigated by Iroshnikov (1963) and Kraichnan (1965), then developed by Dobrowolny et al. (1980) to tentatively explain the occurrence of the observed Alfvénic turbulence, and finally by Carbone (1993) and Biskamp (1993) to get scaling laws for structure functions. It is based on the Alfvén effect, that is, the decorrelation of interacting eddies, which can be explained phenomenologically as follows. Since non-linear interactions happen only between opposite propagating fluctuations, they are slowed down (with respect to the fluid-like case) by the sweeping of the fluctuations across each other. This means that $\varepsilon^{\pm} \sim \left(\Delta z_\ell^{\pm}\right)^2 / T_\ell^{\pm}$ but the characteristic time $T_\ell^{\pm}$ required to efficiently transfer energy from an eddy to another eddy at smaller scales cannot be the eddy-turnover time, rather it is increased by a factor $t_\ell^{\pm}/t_A$ ($t_A \sim \ell/\mathbf{c}_A < t_\ell^{\pm}$ is the Alfvén time), so that $T_\ell^{\pm} \sim (t_\ell^{\pm})^2/t_A$. Then, immediately

$$\varepsilon^{\pm} \sim \frac{[\Delta z_\ell^{\pm}]^2 [\Delta z_\ell^{\mp}]^2}{\ell \mathbf{c}_A}.$$

This means that both $\pm$ modes are transferred at the same rate to small scales, namely $\epsilon^+ \sim \epsilon^- \sim \epsilon$, and this is the conclusion drawn by Dobrowolny et al. (1980). In reality, this is not fully correct, namely the Alfvén effect yields to the fact that energy transfer rates have the same scaling laws for $\pm$ modes but, we cannot say anything about the amplitudes of ε^+ and ε^- (Carbone 1993). Using the usual scaling law for fluctuations, it can be shown that the scaling behavior holds $\varepsilon \to \lambda^{1-4h}\varepsilon'$.

Then, when the energy transfer rate is constant, we found a scaling law different from that of Kolmogorov and, in particular,

$$\Delta z_\ell^\pm \sim (\varepsilon \mathbf{c}_A)^{1/4} \ell^{1/4}. \tag{2.28}$$

Using this phenomenology the high-order moments of fluctuations are given by $S_\ell^{(p)} \sim \ell^{p/4}$. Even in this case, $\xi_p = p/4$ results to be a linear function of the order p. The pseudo-energy spectrum can be easily found to be

$$E^\pm(k) \sim (\varepsilon \mathbf{c}_A)^{1/2} k^{-3/2}. \tag{2.29}$$

This is the Iroshnikov–Kraichnan spectrum. However, in a situation in which there is a balance between the linear Alfvén time scale or wave period, and the non-linear time scale needed to transfer energy to smaller scales, the energy cascade is indicated as *critically balanced* (Goldreich and Sridhar 1995). In these conditions, it can be shown that the power spectrum $P(k)$ would scale as $f^{-5/3}$ when the angle θ_B between the mean field direction and the flow direction is 90° while, the same scaling would follow f^{-2} in case $\theta_B = 0°$ and the spectrum would also have a smaller energy content than in the other case.

2.10 Some Exact Relationships

So far, we have been discussing about the *inertial range* of turbulence. What this means from a heuristic point of view is somewhat clear, but when we try to identify the inertial range from the spectral properties of turbulence, in general the best we can do is to identify the inertial range with the intermediate range of scales where a Kolmogorov's spectrum is observed. The often used identity *inertial range* $\simeq$ *intermediate range*, is somewhat arbitrary. In this regard, a very important result on turbulence, due to Kolmogorov (1941, 1991), is the so-called "4/5-law" which, being obtained from the Navier–Stokes equation, is "*... one of the most important results in fully developed turbulence because it is both exact and nontrivial*" (cf. Frisch 1995). As a matter of fact, Kolmogorov analytically derived the following exact relation for the third order structure function of velocity fluctuations:

$$\left\langle (\Delta v_\parallel(\mathbf{r}, \ell))^3 \right\rangle = -\frac{4}{5} \epsilon \ell\,, \tag{2.30}$$

where $\mathbf{r}$ is the sampling direction, ℓ is the corresponding scale, and ϵ is the mean energy dissipation per unit mass, assumed to be finite and nonvanishing.

This important relation can be obtained in a more general framework from MHD equations. A Yaglom's relation for MHD can be obtained using the analogy of MHD equations with a transport equation, so that we can obtain a relation similar to the Yaglom's equation for the transport of a passive quantity (Monin and Yaglom 1975).

Using the above analogy, the Yaglom's relation has been extended some time ago to MHD turbulence by Chandrasekhar (1967), and recently it has been revised by Politano et al. (1998) and Politano and Pouquet (1998) in the framework of solar wind turbulence. In the following section we report an alternative and more general derivation of the Yaglom's law using structure functions (Sorriso-Valvo et al. 2007; Carbone et al. 2009a).

2.11 Yaglom's Law for MHD Turbulence

To obtain a general law we start from the incompressible MHD equations. If we write twice the MHD equations for two different and independent points x_i and $x_i' = x_i + \ell_i$, by substraction we obtain an equation for the vector differences $\Delta z_i^{\pm} = (z_i^{\pm})' - z_i^{\pm}$. Using the hypothesis of independence of points x_i' and x_i with respect to derivatives, namely $\partial_i (z_j^{\pm})' = \partial_i' z_j^{\pm} = 0$ (where ∂_i' represents derivative with respect to x_i'), we get

$$\partial_t \Delta z_i^{\pm} + \Delta z_\alpha^{\mp} \partial_\alpha' \Delta z_i^{\pm} + z_\alpha^{\mp} (\partial_\alpha' + \partial_\alpha) \Delta z_i^{\pm} = -(\partial_i' + \partial_i) \Delta P + \\ + (\partial_\alpha^{2\prime} + \partial_\alpha^2) \left[\nu^{\pm} \Delta z_i^{+} + \nu^{\mp} \Delta z_i^{-} \right] \tag{2.31}$$

($\Delta P = P_{\text{tot}}' - P_{\text{tot}}$). We look for an equation for the second-order correlation tensor $\langle \Delta z_i^{\pm} \Delta z_j^{\pm} \rangle$ related to pseudo-energies. Actually the more general thing should be to look for a mixed tensor, namely $\langle \Delta z_i^{\pm} \Delta z_j^{\mp} \rangle$, taking into account not only both pseudo-energies but also the time evolution of the mixed correlations $\langle z_i^{+} z_j^{-} \rangle$ and $\langle z_i^{-} z_j^{+} \rangle$. However, using the DIA closure by Kraichnan, it is possible to show that these elements are in general poorly correlated (Veltri 1980). Since we are interested in the energy cascade, we limit ourselves to the most interesting equation that describes correlations about Alfvénic fluctuations of the same sign. To obtain the equations for pseudo-energies we multiply Eq. (2.31) by $\Delta z_j^{\pm}$, then by averaging we get

$$\partial_t \langle \Delta z_i^{\pm} \Delta z_j^{\pm} \rangle + \frac{\partial}{\partial \ell_\alpha} \langle \Delta Z_\alpha^{\mp} (\Delta z_i^{\pm} \Delta z_j^{\pm}) \rangle = -\Lambda_{ij} - \Pi_{ij} + 2\nu \frac{\partial^2}{\partial \ell_\alpha^2} \langle \Delta z_i^{\pm} \Delta z_j^{\pm} \rangle \\ - \frac{4}{3} \frac{\partial}{\partial \ell_\alpha} (\epsilon_{ij}^{\pm} \ell_\alpha), \tag{2.32}$$

where we used the hypothesis of local homogeneity and incompressibility. In Eq. (2.32) we defined the average dissipation tensor

$$\epsilon_{ij}^{\pm} = \nu \langle (\partial_\alpha Z_i^{\pm})(\partial_\alpha Z_j^{\pm}) \rangle. \tag{2.33}$$

The first and second term on the r.h.s. of the Eq. (2.32) represent respectively a tensor related to large-scales inhomogeneities

$$\Lambda_{ij} = \langle z_\alpha^\mp (\partial'_\alpha + \partial_\alpha)(\Delta z_i^\pm \Delta z_j^\pm)\rangle \tag{2.34}$$

and the tensor related to the pressure term

$$\Pi_{ij} = \langle \Delta z_j^\pm (\partial'_i + \partial_i)\Delta P + \Delta z_i^\pm (\partial'_j + \partial_j)\Delta P\rangle. \tag{2.35}$$

Furthermore, In order not to worry about couplings between Elsässer variables in the dissipative terms, we make the usual simplifying assumption that kinematic viscosity is equal to magnetic diffusivity, that is $\nu^\pm = \nu^\mp = \nu$. Equation (2.32) is an exact equation for anisotropic MHD equations that links the second-order complete tensor to the third-order mixed tensor via the average dissipation rate tensor. Using the hypothesis of global homogeneity the term $\Lambda_{ij} = 0$, while assuming local isotropy $\Pi_{ij} = 0$. The equation for the trace of the tensor can be written as

$$\partial_t \langle |\Delta z_i^\pm|^2\rangle + \frac{\partial}{\partial \ell_\alpha}\langle \Delta Z_\alpha^\mp |\Delta z_i^\pm|^2\rangle = 2\nu \frac{\partial^2}{\partial \ell_\alpha}\langle |\Delta z_i^\pm|^2\rangle - \frac{4}{3}\frac{\partial}{\partial \ell_\alpha}(\epsilon_{ii}^\pm \ell_\alpha), \tag{2.36}$$

where the various quantities depends on the vector ℓ_α. Moreover, by considering only the trace we ruled out the possibility to investigate anisotropies related to different orientations of vectors within the second-order moment. It is worthwhile to remark here that *only* the diagonal elements of the dissipation rate tensor, namely $\epsilon_{ii}^\pm$ are positive defined while, in general, the off-diagonal elements $\epsilon_{ij}^\pm$ are not positive. For a stationary state the Eq. (2.36) can be written as the divergenceless condition of a quantity involving the third-order correlations and the dissipation rates

$$\frac{\partial}{\partial \ell_\alpha}\left[\langle \Delta z_\alpha^\mp |\Delta z_i^\pm|^2\rangle - 2\nu \frac{\partial}{\partial \ell_\alpha}\langle |\Delta z_i^\pm|^2\rangle - \frac{4}{3}(\epsilon_{ii}^\pm \ell_\alpha)\right] = 0, \tag{2.37}$$

from which we can obtain the Yaglom's relation by projecting Eq. (2.37) along the longitudinal $\ell_\alpha = \ell \mathbf{e}_r$ direction. This operation involves the assumption that the flow is locally isotropic, that is fields depends locally only on the separation ℓ, so that

$$\left(\frac{2}{\ell} + \frac{\partial}{\partial \ell}\right)\left[\langle \Delta z_\ell^\mp |\Delta z_i^\pm|^2\rangle - 2\nu \frac{\partial}{\partial \ell}\langle |\Delta z_i^\pm|^2\rangle + \frac{4}{3}\epsilon_{ii}^\pm \ell\right] = 0. \tag{2.38}$$

The only solution that is compatible with the absence of singularity in the limit $\ell \to 0$ is

$$\langle \Delta z_\ell^\mp |\Delta z_i^\pm|^2\rangle = 2\nu \frac{\partial}{\partial \ell}\langle |\Delta z_i^\pm|^2\rangle - \frac{4}{3}\epsilon_{ii}^\pm \ell, \tag{2.39}$$

which reduces to the Yaglom's law for MHD turbulence as obtained by Politano and Pouquet (1998) in the inertial range when $\nu \to 0$

$$Y_\ell^\pm \equiv \langle \Delta z_\ell^\mp |\Delta z_i^\pm|^2 \rangle = -\frac{4}{3}\epsilon_{ii}^\pm \ell. \tag{2.40}$$

Finally, in the fluid-like case where $z_i^+ = z_i^- = v_i$ we obtain the usual Yaglom's law for fluid flows

$$\langle \Delta v_\ell |\Delta v_i|^2 \rangle = -\frac{4}{3}\left(\epsilon \ell\right), \tag{2.41}$$

which in the isotropic case, where $\langle \Delta v_\ell^3 \rangle = 3\langle \Delta v_\ell \Delta v_y^2 \rangle = 3\langle \Delta v_\ell \Delta v_z^2 \rangle$ (Monin and Yaglom 1975), immediately reduces to the Kolmogorov's law

$$\langle \Delta v_\ell^3 \rangle = -\frac{4}{5}\epsilon \ell \tag{2.42}$$

(the separation ℓ has been taken along the streamwise x-direction).

The relations we obtained can be used, or better, in a certain sense they *might* be used, as a formal definition of inertial range. Since they are exact relationships derived from Navier–Stokes and MHD equations under usual hypotheses, they represent a kind of "zeroth-order" conditions on experimental and theoretical analysis of the inertial range properties of turbulence. It is worthwhile to remark the two main properties of the Yaglom's laws. The first one is the fact that, as it clearly appears from the Kolmogorov's relation (Kolmogorov 1941), the third-order moment of the velocity fluctuations is different from zero. This means that some non-Gaussian features must be at work, or, which is the same, some hidden phase correlations. Turbulence is something more complicated than random fluctuations with a certain slope for the spectral density. The second feature is the minus sign which appears in the various relations. This is essential when the sign of the energy cascade must be inferred from the Yaglom relations, the negative asymmetry being a signature of a direct cascade towards smaller scales. Note that, Eq. (2.40) has been obtained in the limit of zero viscosity *assuming* that the pseudo-energy dissipation rates $\epsilon_{ii}^\pm$ remain finite in this limit. In usual fluid flows the analogous hypothesis, namely ϵ remains finite in the limit $\nu \to 0$, is an experimental evidence, confirmed by experiments in different conditions (Frisch 1995). In MHD turbulent flows this remains a conjecture, confirmed only by high resolution numerical simulations (Mininni and Pouquet 2009).

From Eq. (2.37), by defining $\Delta Z_i^\pm = \Delta v_i \pm \Delta b_i$ we immediately obtain the two equations

$$\frac{\partial}{\partial \ell_\alpha}\left[\langle \Delta v_\alpha \Delta E \rangle - 2\langle \Delta b_\alpha \Delta C \rangle - 2\nu \frac{\partial}{\partial \ell_\alpha}\langle \Delta E \rangle - \frac{4}{3}(\epsilon_E \ell_\alpha)\right] = 0 \tag{2.43}$$

$$\frac{\partial}{\partial \ell_\alpha}\left[-\langle \Delta b_\alpha \Delta E \rangle + 2\langle \Delta v_\alpha \Delta C \rangle - 4\nu \frac{\partial}{\partial \ell_\alpha}\langle \Delta C \rangle - \frac{4}{3}(\epsilon_C \ell_\alpha)\right] = 0, \tag{2.44}$$

where we defined the energy fluctuations $\Delta E = |\Delta v_i|^2 + |\Delta b_i|^2$ and the correlation fluctuations $\Delta C = \Delta v_i \Delta b_i$. In the same way the quantities $\epsilon_E = \left(\epsilon_{ii}^+ + \epsilon_{ii}^-\right)/2$ and $\epsilon_C = \left(\epsilon_{ii}^+ - \epsilon_{ii}^-\right)/2$ represent the energy and correlation dissipation rate, respectively. By projecting once more on the longitudinal direction, and assuming vanishing viscosity, we obtain the Yaglom's law written in terms of velocity and magnetic fluctuations

$$\langle \Delta v_\ell \Delta E\rangle - 2\langle \Delta b_\ell \Delta C\rangle = -\frac{4}{3}\epsilon_E \ell \tag{2.45}$$

$$-\langle \Delta b_\ell \Delta E\rangle + 2\langle \Delta v_\ell \Delta C\rangle = -\frac{4}{3}\epsilon_C \ell. \tag{2.46}$$

2.11.1 Density-Mediated Elsässer Variables and Yaglom's Law

Relation (2.40), which is of general validity within MHD turbulence, requires local characteristics of the turbulent fluid flow which can be not always satisfied in the solar wind flow, namely, large-scale homogeneity, isotropy, and incompressibility. Density fluctuations in solar wind have a low amplitude, so that nearly incompressible MHD framework is usually considered (Montgomery et al. 1987; Matthaeus and Brown 1988; Zank and Matthaeus 1993; Matthaeus et al. 1991; Bavassano and Bruno 1995). However, compressible fluctuations are observed, typically convected structures characterized by anticorrelation between kinetic pressure and magnetic pressure (Tu and Marsch 1994). Properties and interaction of the basic MHD modes in the compressive case have also been considered (Goldreich and Sridhar 1995; Cho and Lazarian 2002).

A first attempt to include density fluctuations in the framework of fluid turbulence was due to Lighthill (1955). He pointed out that, in a compressible energy cascade, the *mean energy transfer rate per unit volume* $\epsilon_V \sim \rho v^3/\ell$ should be constant in a statistical sense (v being the characteristic velocity fluctuations at the scale ℓ), thus obtaining the scaling relation $v \sim (\ell/\rho)^{1/3}$. Fluctuations of a density-weighted velocity field $\mathbf{u} \equiv \rho^{1/3}\mathbf{v}$ should thus follow the usual Kolmogorov scaling $u^3 \sim \ell$. The same phenomenological arguments can be introduced in MHD turbulence (Carbone et al. 2009b) by considering the pseudoenergy dissipation rates per unit volume $\epsilon_V^\pm = \rho\epsilon_{ii}^\pm$ and introducing density-weighted Elsässer fields, defined as $\mathbf{w}^\pm \equiv \rho^{1/3}\mathbf{z}^\pm$. A relation equivalent to the Yaglom-type relation (2.40)

$$W_\ell^\pm \equiv \langle\rho\rangle^{-1}\langle \Delta w_\ell^\mp |\Delta w_i^\pm|^2\rangle = C\epsilon_{ii}^\pm \ell \tag{2.47}$$

(C is some constant assumed to be of the order of unit) should then hold for the density-weighted increments $\Delta\mathbf{w}^\pm$. Relation $W_\ell^\pm$ reduces to $Y_\ell^\pm$ in the case of constant density, allowing for comparison between the Yaglom's law for incompressible MHD flows and their compressible counterpart. Despite its simple phenomenological derivation, the introduction of the density fluctuations in the Yaglom-type scaling (2.47) should describe the turbulent cascade for compressible fluid (or magnetofluid) turbulence. Even if the modified Yaglom's law (2.47) is not an exact relation as (2.40), being obtained from phenomenological considerations, the law for the velocity field in a compressible fluid flow has been observed in numerical simulations, the value of the constant C results negative and of the order of unity (Padoan et al. 2007; Kowal and Lazarian 2007).

2.11.2 Yaglom's Law in the Shell Model for MHD Turbulence

As far as the shell model is concerned, the existence of a cascade towards small scales is expressed by an exact relation, which is equivalent to Eq. (2.41). Using Eq. (2.24), the scale-by-scale pseudo-energy budget is given by

$$\frac{d}{dt}\sum_n |Z_n^\pm|^2 = k_n Im\left[T_n^\pm\right] - \sum_n 2\nu k_n^2 |Z_n^\pm|^2 + \sum_n 2\Re e\left[Z_n^\pm F_n^{\pm *}\right].$$

The second and third terms on the right hand side represent, respectively, the rate of pseudo-energy dissipation and the rate of pseudo-energy injection. The first term represents the flux of pseudo-energy along the wave vectors, responsible for the redistribution of pseudo-energies on the wave vectors, and is given by

$$T_n^\pm = (a+c)Z_n^\pm Z_{n+1}^\pm Z_{n+2}^\mp + \left(\frac{2-a-c}{\lambda}\right) Z_{n-1}^\pm Z_{n+1}^\pm Z_n^\mp + \\ +(2-a-c)Z_n^\pm Z_{n+2}^\pm Z_{n+1}^\mp + \left(\frac{c-a}{\lambda}\right) Z_) Z_n^\pm Z_{n+1}^\pm Z_{n-1}^\mp . \tag{2.48}$$

Using the same assumptions as before, namely: (1) the forcing terms act only on the largest scales, (2) the system can reach a statistically stationary state, and (3) in the limit of fully developed turbulence, $\nu \to 0$, the mean pseudo-energy dissipation rates tend to finite positive limits $\epsilon^\pm$, it can be found that

$$\langle T_n^\pm \rangle = -\epsilon^\pm k_n^{-1}. \tag{2.49}$$

This is an exact relation which is valid in the inertial range of turbulence. Even in this case it can be used as an operative definition of the inertial range in the shell model, that is, the inertial range of the energy cascade in the shell model is defined as the range of scales k_n, where the law from Eq. (2.49) is verified.

References

B. Bavassano, R. Bruno, Density fluctuations and turbulent mach number in the inner solar wind. J. Geophys. Res. **100**, 9475–9480 (1995)

L. Biferale, Shell models of energy cascade in turbulence. Annu. Rev. Fluid Mech. **35**, 441–468 (2003). doi:10.1146/annurev.fluid.35.101101.161122

D. Biskamp, *Nonlinear Magnetohydrodynamics*. Cambridge Monographs on Plasma Physics, vol. 1 (Cambridge University Press, Cambridge, 1993)

D. Biskamp, *Magnetohydrodynamic Turbulence* (Cambridge University Press, Cambridge, 2003)

T. Bohr, M.H. Jensen, G. Paladin, A. Vulpiani, *Dynamical Systems Approach to Turbulence*. Cambridge Nonlinear Science Series, vol. 8 (Cambridge University Press, Cambridge, 1998)

T.J.M. Boyd, J.J. Sanderson, *The Physics of Plasmas* (Cambridge University Press, Cambridge, 2003)

S.I. Braginskii, Transport processes in plasma, in *Review of Plasma Physics*, ed. by M.A. Leontovich. Review of Plasma Physics, vol. 1 (Consultants Bureau, New York, 1965), pp. 201–311

V. Carbone, Cascade model for intermittency in fully developed magnetohydrodynamic turbulence. Phys. Rev. Lett. **71**, 1546–1548 (1993). doi:10.1103/PhysRevLett.71.1546

V. Carbone, L. Sorriso-Valvo, R. Marino, On the turbulent energy cascade in anisotropic magnetohydrodynamic turbulence. Europhys. Lett. **88**, 25001 (2009a). doi:10.1209/0295-5075/88/25001

V. Carbone, R. Marino, L. Sorriso-Valvo, A. Noullez, R. Bruno, Scaling laws of turbulence and heating of fast solar wind: the role of density fluctuations. Phys. Rev. Lett. **103**(6) (2009b). doi:10.1103/PhysRevLett.103.061102

S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover, New York, 1967)

A.C.L. Chian, F.A. Borotto, W.D. Gonzalez, Alfvén intermittent turbulence driven by temporal chaos. Astrophys. J. **505**, 993–998 (1998). doi:10.1086/306214

A.C.L. Chian, E.L. Rempel, E.E.N. Macau, R.R. Rosa, F. Christiansen, Alfvén turbulence driven by high-dimensional interior crisis in the solar wind, in *Solar Wind Ten*, ed. by M. Velli, R. Bruno, F. Malara. AIP Conference Proceedings, vol. 679 (American Institute of Physics, Melville, 2003), pp. 558–561

J. Cho, A. Lazarian, Numerical simulations of compressible MHD turbulence. Bull. Am. Astron. Soc. **34**, 1124 (2002)

M. Dobrowolny, A. Mangeney, P. Veltri, Properties of magnetohydrodynamic turbulence in the solar wind. Astron. Astrophys. **83**, 26–32 (1980)

W.M. Elsässer, The hydromagnetic equations. Phys. Rev. **79**, 183 (1950). doi:10.1103/PhysRev.79.183

R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures On Physics, Volume II: Mainly Electromagnetism and Matter* (Addison Wesley, Reading 1964)

P. Frick, D.D. Sokoloff, Cascade and dynamo action in a shell model of magnetohydrodynamic turbulence. Phys. Rev. E **57**, 4155–4164 (1998). doi:10.1103/PhysRevE.57.4155

U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995)

S. Ghosh, K. Papadopoulos, The onset of alfvénic turbulence. Phys. Fluids **30**, 1371–1387 (1987). doi:10.1063/1.866252

P. Giuliani, V. Carbone, A note on shell models for MHD turbulence. Europhys. Lett. **43**, 527–532 (1998). doi:10.1209/epl/i1998-00386-y

E.B. Gledzer, System of hydrodynamic type admitting two quadratic integrals of motion. Sov. Phys. Dokl. **18**, 216 (1973)

C. Gloaguen, J. Léorat, A. Pouquet, R. Grappin, A scalar model for MHD turbulence. Physica D **17**, 154–182 (1985). doi:10.1016/0167-2789(85)90002-8

P. Goldreich, S. Sridhar, Toward a theory of interstellar turbulence. 2: Strong alfvenic turbulence. Astrophys. J. **438**, 763–775 (1995). doi:10.1086/175121

J.P. Gollub, H.L. Swinney, Onset of turbulence in a rotating fluid. Phys. Rev. Lett. **35**, 927–930 (1975). doi:10.1103/PhysRevLett.35.927

R. Grappin, M. Velli, Waves and streams in the expanding solar wind. J. Geophys. Res. **101**, 425–444 (1996). doi:10.1029/95JA02147

R. Grappin, M. Velli, A. Mangeney, Nonlinear wave evolution in the expanding solar wind. Phys. Rev. Lett. **70**, 2190–2193 (1993). doi:10.1103/PhysRevLett.70.2190

P. Hellinger, M. Velli, P. Trávníček, S.P. Gary, B.E. Goldstein, P.C. Liewer, Alfvén wave heating of heavy ions in the expanding solar wind: hybrid simulations. J. Geophys. Res. **110**(9), 12109 (2005). doi:10.1029/2005JA011244

P.S. Iroshnikov, Turbulence of a conducting fluid in a strong magnetic field. Sov. Astron. **7**, 566 (1963)

A.N. Kolmogorov, The local structure turbulence in incompressible viscous fluids for very large reynolds numbers. Dokl. Akad. Nauk. SSSR **30**, 301–305 (1941)

A.N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers. Proc. R. Soc. Lond. A **434**, 9–13 1991. doi:10.1098/rspa.1991.0075. Published 8 July 1991; J.C.R. Hunt, O.M. Phillips, D. Williams (eds.) Kolmogorov's ideas 50 years on Turbulence and stochastic processes (English version)

G. Kowal, A. Lazarian, Scaling relations of compressible MHD turbulence. Astrophys. J. Lett. **666**, 69–72 (2007). doi:10.1086/521788

R.H. Kraichnan, Inertial range spectrum of hydromagnetic turbulence. Phys. Fluids **8**, 1385–1387 (1965)

L.D. Landau, E.M. Lifshitz, *Physique théorique.* Mécanique des fluides, vol. 6 (Editions MIR, Moscow, 1971)

P.C. Liewer, M. Velli, B.E. Goldstein, Alfvén wave propagation and ion cyclotron interactions in the expanding solar wind: one-dimensional hybrid simulations. J. Geophys. Res. **106**, 29261–29282 (2001). doi:10.1029/2001JA000086

M.J. Lighthill, The effect of compressibility on turbulence, in *Gas Dynamics of Cosmic Clouds*. IAU Symposium, vol. 2 (North-Holland, Amsterdam, 1955), pp. 121–130

E.N. Lorenz, Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130 (1963). doi:10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2

E. Marsch, A. Mangeney, Ideal MHD equations in terms of compressive elsässer variables. J. Geophys. Res. **92**(11), 7363–7367 (1987). doi:10.1029/JA092iA07p07363

E. Marsch, C.-Y. Tu, Dynamics of correlation functions with elsässer variables for inhomogeneous MHD turbulence. J. Plasma Phys. **41**, 479–491 (1989). doi:10.1017/S0022377800014033

W.H. Matthaeus, M.R. Brown, Nearly incompressible magnetohydrodynamics at low mach number. Phys. Fluids **31**, 3634–3644 (1988). doi:10.1063/1.866880

W.H. Matthaeus, L.W. Klein, S. Ghosh, M.R. Brown, Nearly incompressible magnetohydrodynamics, pseudosound, and solar wind fluctuations. J. Geophys. Res. **96**(15), 5421–5435 (1991). doi:10.1029/90JA02609

W.D. McComb, *The Physics of Fluid Turbulence*. Oxford Engineering Science Series, vol. 25 (Oxford University Press, Oxford, 1990)

P.D. Mininni, A. Pouquet, Finite dissipation and intermittency in magnetohydrodynamics. Phys. Rev. E **80**(2) (2009). doi:10.1103/PhysRevE.80.025401

E. Mjølhus, On the modulational instability of hydromagnetic waves parallel to the magnetic field. J. Plasma Phys. **16**, 321–334 (1976). doi:10.1017/S0022377800020249

A.S. Monin, A.M. Yaglom, *Statistical Fluid Mechanics – Mechanics of Turbulence*, vol. 2 (MIT Press, Cambridge, 1975)

D. Montgomery, M.R. Brown, W.H. Matthaeus, Density fluctuation spectra in magnetohydrodynamic turbulence. J. Geophys. Res. **92**(11), 282–284 (1987). doi:10.1029/JA092iA01p00282

K. Ohkitani, M. Yamada, Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully-developed model turbulence. Prog. Theor. Phys. **89**, 329–341 (1989). doi:10.1143/PTP.81.329

E. Ott, J.C. Sommerer, Blowout bifurcations: the occurrence of riddled basins and on-off intermittency. Phys. Lett. A **188**, 39–47 (1994). doi:10.1016/0375-9601(94)90114-7

S. Oughton, W.H. Matthaeus, Evolution of solar wind fluctuations and the influence of turbulent 'mixing', in *Solar Wind Seven*, ed. by E. Marsch, R. Schwenn. COSPAR Colloquia Series, vol. 3 (Pergamon Press, Oxford, 1992), pp. 523–526

P. Padoan, Å. Nordlund, A.G. Kritsuk, M.L. Norman, P.S. Li, Two regimes of turbulent fragmentation and the stellar initial mass function from primordial to present-day star formation. Astrophys. J. **661**, 972–981 (2007). doi:10.1086/516623

F. Plunian, R. Stepanov, P. Frick, Shell models of magnetohydrodynamic turbulence. Phys. Rep. **523**, 1–60 (2012)

H. Politano, A. Pouquet, von Kármán–Howarth equation for magnetohydrodynamics and its consequences on third-order longitudinal structure and correlation functions. Phys. Rev. E **57**, 21–25 (1998). doi:10.1103/PhysRevE.57.R21

H. Politano, A. Pouquet, V. Carbone, Determination of anomalous exponents of structure functions in two-dimensional magnetohydrodynamic turbulence. Europhys. Lett. **43**, 516–521 (1998). doi:10.1209/epl/i1998-00391-2

Y. Pomeau, P. Manneville, Intermittent transition to turbulence in dissipative dynamical systems. Commun. Math. Phys. **74**, 189–197 (1980). doi:10.1007/BF01197757

S.B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, 2000)

D. Ruelle, F. Takens, On the nature of turbulence. Commun. Math. Phys. **20**, 167 (1971)

E.D. Siggia, Origin of intermittency in fully developed turbulence. Phys. Rev. A **15**, 1730–1750 (1977). doi:10.1103/PhysRevA.15.1730

L. Sorriso-Valvo, R. Marino, V. Carbone, A. Noullez, F. Lepreti, P. Veltri, R. Bruno, B. Bavassano, E. Pietropaolo, Observation of inertial energy cascade in interplanetary space plasma. Phys. Rev. Lett. **99**(11) (2007). doi:10.1103/PhysRevLett.99.115001

C.-Y. Tu, E. Marsch, Transfer equations for spectral densities of inhomogeneous MHD turbulence. J. Plasma Phys. **44**, 103–122 (1990). doi:10.1017/S002237780001504X

C.-Y. Tu, E. Marsch, On the nature of compressive fluctuations in the solar wind. J. Geophys. Res. **99**(18), 21481 (1994)

C.-Y. Tu, E. Marsch, Energy spectrum transfer equations of solar wind turbulence, in *Solar Wind Eight*, ed. by D. Winterhalter, J.T. Gosling, S.R. Habbal, W.S. Kurth, M. Neugebauer. AIP Conference Proceedings, vol. 382 (American Institute of Physics, Woodbury, 1996), pp. 233–238

P. Veltri, An observational picture of solar-wind MHD turbulence. Nuovo Cimento C **3**, 45–55 (1980). doi:10.1007/BF02509190

G.P. Zank, W.H. Matthaeus, Nearly incompressible fluids. ii – magnetohydrodynamics, turbulence, and waves. Phys. Fluids **5**, 257–273 (1993)

Y. Zhou, W.H. Matthaeus, Transport and turbulence modeling of solar wind fluctuations. J. Geophys. Res. **95**(14), 10291–10311 (1990). doi:10.1029/JA095iA07p10291