Engagement with Interactive Diagrams: The Role Played by Resources and Constraints

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Abstract Interactive textbooks appear to be the tools of choice in mathematics instruction in the foreseeable future. It is important, therefore, to establish the theoretical foundations of design that define student-textbook-teacher interactions. In our long-term research, we suggested, tested, and refined a semiotic framework that offers a set of terms helpful in analyzing how the designed features of interactive diagrams (IDs) function in these interactions. The present chapter summarizes key design decisions about resources and constraints of interactive texts according to various semiotic functions, and discusses the role of designed resources and constraints of the IDs in student engagement with interactive texts.

Keywords Task design · Interactive textbooks · Semiotic · Interactive diagrams · Examples · Representations

1 Introduction

Current technology makes possible the use of a variety of interactive tools and representations in interactive textbooks. Using technology to develop mathematical textbooks and tasks is an attempt to create new venues for engagement with mathematical meaning. An example of a digital interactive web textbook is the VisualMath eTextbook (Yerushalmy et al. 2002/2014). The interactive text in this book provides expositions and instructions in the form of interactive diagrams or of a link to another interactive diagram. The book also provides an implicit suggestion to use other related tasks, exercises, and tools selectable from a menu.

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We use the term interactive diagram (ID) to refer to a relatively small unit of interactive text in e-textbooks or another materials. The components of the ID are: the example being provided, its representations (verbal, visual, and other), and interactive tools. The text may be used for different purposes: an exposition, a task, an exercise, etc. Each ID is considered a task, according to Margolinas (2013): “a task is anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something. Tasks can also be anything that students decide to do for themselves in a particular situation. Tasks, therefore, are the mediating tools for teaching and learning mathematics” (p. 11). It is a challenge to design tasks that on one hand invite opportunities for active personal learning, and on the other satisfy the requirements of national standards or curricula.

In this chapter, we use the findings of our long-term research[1] to illustrate how the constraints and resources designed into IDs function in the process of developing mathematical knowledge.

2 The Semiotic Framework

In earlier studies, (e.g., Yerushalmy 2005; Naftaliev and Yerushalmy 2011, 2013; Naftaliev 2012) we conducted a semiotic analysis of IDs based on Kress and van Leeuwen’s (1996) visual social-semiotic theory. The semiotic analysis proposed three dimensions for defining the functionality of IDs.

The semiotic framework is characterized by three types of ID functions (Table 1) that address a variety of learning and teaching settings: *presentational*, *orientational*, and *organizational*.

2.1 Presentational Functions of IDs

Although examples in an ID are usually designed to be modified by the user, the example that initially appears in the diagram determines the nature of the *presentational* function of the example. Three types of examples are widely used in IDs: random, specific, and generic.

Specific examples present the exact data of the activity of which they are part. They serve as a dynamic illustration that helps analyze the situation without being able to change the information. Random examples are specific examples generated within given constraints, presenting different information at various times and for different users. In a generic example, the diagram is structured to be representative; it presents a situation that can be part of the given task, but it is not

[1]https://sites.google.com/site/interactivediagrams/.

Table 1 The semiotic framework: three types defining the functionality of IDs

Presentational function	Orientational function	Organizational function
Specific	Sketchy	Illustrating
Random	Accurate	Elaborating
Generic	Sketchy and accurate	Guiding

intended to present the specific data of the activity but to help learners become acquainted with the generic views of the example through a process of inquiry. Mason and Pimm (1984) noted that generic examples are transparent to the general case, allowing one to see the general through the particular: "A generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general; this is done by means of stressing and ignoring various key features, of attempting to structure one's perception of it" (p. 287). The art of designing generic examples consists of finding ways to place the focus on generality or representativeness, as elaborated by Goldenberg and Mason (2008). (Davydov 1972/1990) articulated the difficulty: "The real problem is precisely in finding a form for a concept in which the derivation of properties would be possible" (p. 35). Design that offers ways to systematically generate multiple and varied examples, and to preserve and reconstruct processes, provides the basis for conceptual construction of knowledge by generalizations and conjectures (Yerushalmy 1993). It is not usually the case, however, that the generic nature of the example is visible to the learner, and often the example remains a particular case. It is only when the viewer becomes aware of the generality in the specific example that its mission is achieved. The design of the setting of an example as an ID invites the viewer to activate the ID within given limitations in two ways: (a) interacting with components of the examples (the representations, as well as the linking and control tools), and (b) changing the example by generating similar or new ones.

2.2 *Orientational Functions of IDs*

The tone in which the text addresses the learner is subject to design decisions having to do with the *orientational* function. "Sketchiness" vs. "rigorousness" of the diagrams is an important factor in reader orientation. An example that appears in a diagram can have an accurate appearance and communicate in a strict, distant tone. For example, a graph drawn on paper indicating coordinate values and scale would be interpreted as a specific case. The example can adopt a non-authoritative tone. For example, it may not attempt to provide the complete picture, but rather to highlight important elements, so that it can be used as a plan for a variety of final products that share the same idea or structure. IDs can function both as sketches and as accurate diagrams (Table 2).

Table 2 Twenty seven types of interactive diagrams

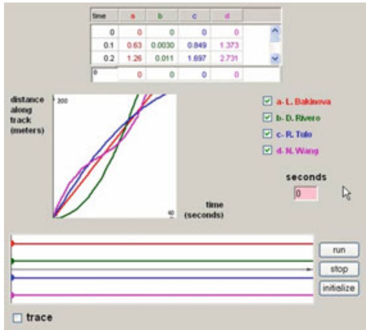
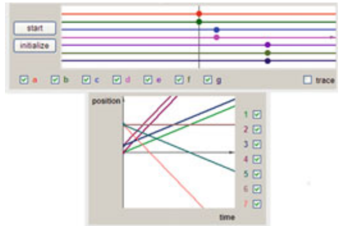
	Illustrating ID	Elaborating ID	Guiding ID
Specific example	Sketchy	Sketchy	Sketchy
	Accurate	Accurate	Accurate
	Sketchy/accurate	Sketchy/accurate	Sketchy/accurate
Random example	Sketchy	Sketchy	Sketchy
	Accurate	Accurate	Accurate
	Sketchy/accurate	Sketchy/accurate	Sketchy/accurate
Generic example	Sketchy	Sketchy	Sketchy
	Accurate	Accurate	Accurate
	Sketchy/accurate	Sketchy/accurate	Sketchy/accurate

2.3 Organizational Functions of IDs

The *organizational* function refers to the connection between all the components of the ID: representations, tools, examples, etc. IDs can be organized in three ways: illustrating, elaborating, and guiding[2] (narrating). The three types differ in their settings, each characterized by its own constraints and resources, and intended for a different aspect of inquiry. Illustrating IDs demonstrate the objective of the activity to the reader, usually by offering a single representation and relatively simple actions, such as viewing an animated example. For example, an illustrating ID might allow learners to manipulate rather than read a definition. Elaborating diagrams present occurrences relevant to the problem being explored while working on the task. They attempt to provide a means for students to engage in activities that lead to the formulation of a solution in different ways, and operate at a meta-cognitive level. For instance, the animated example that serves for illustration can also serve for elaboration when it is part of other tools and representations (Table 3). A guiding ID, similarly to an elaborating one, provides a means for learner exploration, but it is designed to also set boundaries for the available exploration options in such a way that it narrates the story to be learned by working on the task. Guiding IDs are designed to point students toward specific actions intended to support them in developing specific mathematical ideas. Although guiding IDs provide tools that promote inquiry, the tools are designed to limit the exploration and serve as boundaries while working on the task.

[2]In our earlier publications (e.g., 2005, 2009, 2011, 2011, 2013) we used the term "narrating ID".

Table 3 Comparative view of the three settings of IDs based on organizational, presentational, and orientational functions: modeling series

Organizational functions	Illustrating ID	Elaborating ID	Guiding ID
			
Presentational functions	Generic example	Generic example	Generic example with an exceptional case
Orientational functions	Sketchy	Sketchy/accurate	Sketchy

3 Design Decisions Based on the Three Organizational Functions

The above analysis classifies IDs into 27 types (Table 2), based on their combinations of features: three organizational functions (illustrating, guiding, and elaborating) × three types of examples (random, generic, and specific) × three types of orientational functions (accurate, sketch, and both).

To study the effect of organizational functions we generated different IDs based on the same content with different organizational functions. Three mathematical foci served as the pivotal points for the design of the series of the tasks:

1. Modeling series (analyzing properties of models and mathematizing outside mathematical phenomena) (Table 3).
2. Formulating mathematical phenomena series (writing expression for linear function) (Table 4).
3. Manipulating series (solving equations) (Table 5).

The tasks were assumed to be challenging and new for students and were designed to support investigation—in other words, formulating and exploring conjectures. These mathematical objectives of the school algebra curriculum are described in the literature (e.g., Schwartz 1999) as key actions undertaken with mathematical objects. Each objective included three comparable tasks, based on an ID of a different design type. In the modeling series, we designed settings of interactive diagrams that shared an example represented as an animation of multi-process motion (Table 3), and a task that required analyzing motion while paying attention to representations and models of path, speed, and pace.

In the formulating mathematical phenomena series, we chose a basic algebra task that required the writing of a symbolic expression to describe a given linear function graph (Table 4). In the manipulating series, we chose a task that focused on performing algebraic manipulations and required to create examples of equations that comply with certain constrains (Table 5).

The sequences chosen represent important issues in school algebra and were intended to reduce the effect of specific content in algebra on the conclusions of the study. Based on the assumption that the design of the ID establishes the context for a variety of learning and teaching settings, the focus of the study was to analyze how the designed constraints and resources of the IDs functioned in developing mathematical knowledge while solving unfamiliar tasks presented by multi-modal texts.

Table 4 Comparative view of the three settings of IDs based on the organizational, presentational, and orientational functions: formulating mathematical phenomena series

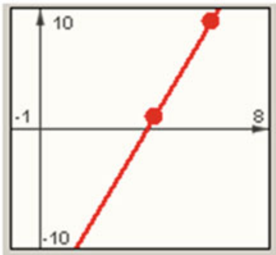
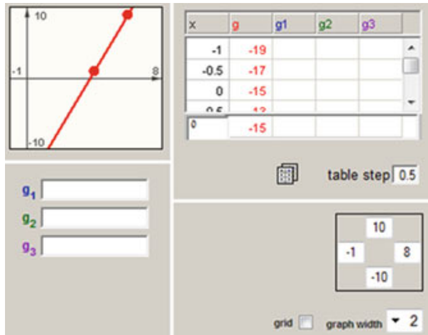
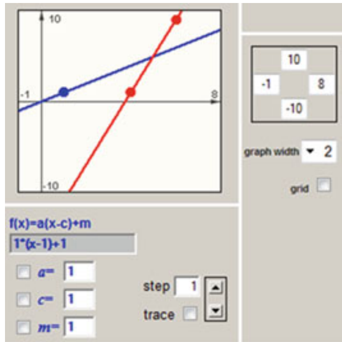
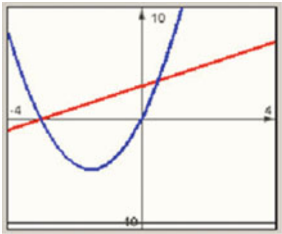
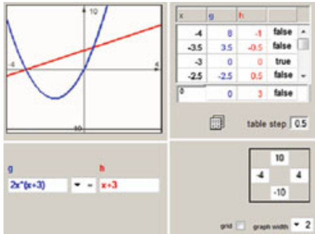
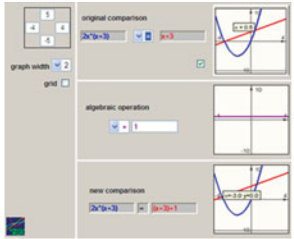
Organizational functions	Illustrating ID	Elaborating ID	Guiding ID
			
Presentational functions	Specific example	Specific example; The design of the ID offers ways to generate multiple and varied examples	Specific example; The design of the ID offers ways to systematically generate varied examples
Orientational functions	Sketchy/accurate	Sketchy/accurate	Sketchy/accurate

Table 5 Comparative view of the three settings of IDs based on organizational, presentational, and orientational functions: manipulating series

Organizational functions	Illustrating ID	Elaborating ID	Guiding ID
			
Presentational functions	Specific example	Specific example; The design of the ID offers ways to generate varied examples	Specific example; The design of the ID offers ways to systematically generate varied examples
Orientational functions	Sketchy/accurate	Sketchy/accurate	Sketchy/accurate

4 Mathematical Engagements with Interactive Texts Within Given Resources and Constraints

We illustrate the students' mathematical engagement with interactive texts concerning the modeling series, and the role played by the resources and constraints in the design of the text. We designed three ID settings (Tables 3, 4 and 5) over the course of three studies, as part of our broader research. The IDs share an example represented as an animation of multi-process motion, but they differ in their organizational functions. In these studies, we reported on the work of 14-year-old students from the same public school, who interacted with different IDs that had distinct organizational functions: illustrating (Naftaliev and Yerushalmy 2009), guiding (Naftaliev and Yerushalmy 2013), illustrating and elaborating (Yerushalmy and Naftaliev 2011).

4.1 Student Engagement with the Illustrating ID

The illustrating ID was simple to operate and provided the minimal control necessary for using the animation: at any time, users could freeze the positions on the track, continue the run, or initialize the race. The generic example provided the sketchy nature of the ID and the inclusion of only one representation were important resources and constraints in the mathematical engagement with the interactive text. In the following episode the students looked for ways to sidestep the design constraints of the ID. For example, Dan's sequence of static diagrams (Fig. 1) prompted him to mentally recreate and describe the entire motion process, pointing to changes of speed of the dots (or "runners") in correlation with changes in their relative positions:

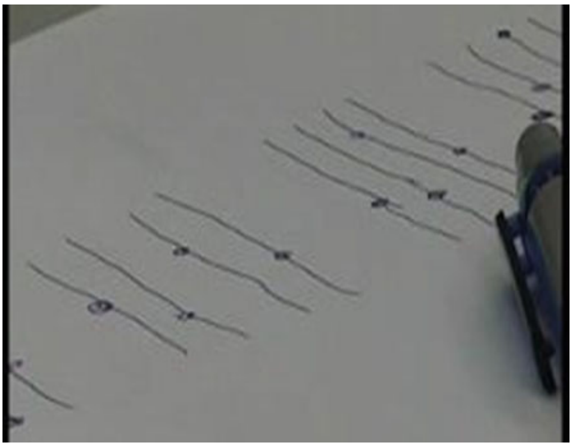


Fig. 1 Sequence of static diagrams

Dan: So in the end it shows that the pink that at first advanced, that began to gain acceleration, and the blue that passed him later, who also began to be the fastest; actually the black accelerated and passed him. The blue and the red started slowing a bit and the red continued at the same pace, and in the end passed the blue and came in third. The black was first. The pink came in second and the red came in third place. And the blue stayed… came in last, even though in the middle he started leading in the distances, and then the black began to pass everyone, and won actually.

Using paper and pencil, the ID was extended in a schematic way that served as a static model (Fig. 1), from which students were later able to describe the dynamic process without activating the ID. In sum, the influence of design constraints and resources on student engagement with the illustrating ID were: (a) the dynamic and sketchy nature of the ID presentation promoted comparative descriptions of the motion and made it easier for the students to distinguish between the runners, to address each one using colors, and to identify their relative progress; (b) the ID was designed as a generic example with an exceptional case that became pivotal in the description of the race; and (c) the inclusion of only one representation in the ID, and the minimal control needed to operate it stimulated looking for ways to bypass the designed constraints. To construct a picture of the motion process, the students resorted to complementing the ID using the representations and tools they created themselves.

4.2 Student Engagement with the Guiding ID

The guiding ID was designed around a known conflict concerning time-position graphs that describe motionless objects while time passes continuously. Analysis shows that a set of constraints and resources contributes to making the task an interesting challenge. Among these are the small number of animated representations in the ID, the partial linking between the various visual representations, the absence of representations and controls that could turn the sketchy nature of the representations into an accurate diagram, the absence of discrete information, and the exceptional example in a list of examples aimed at focusing on absence of motion over time.

We illustrate the students' mathematical engagements with the guiding ID by analyzing the following episode. The students, Lior and Daniel, started their work with the ID by activating it and asking questions such as why are there only six dots moving, whereas in the graph the dots move along seven lines:

Lior: If this doesn't move [points with the mouse at the static car] and here all are moving [points with his finger at the graphs] it doesn't make sense!

The students reflected on their current mathematical understanding with the guidance of the ID, which emphasized a well-known conflict, and they modified their understanding as they came across the various constraints and resources of the guiding ID. The students continued the discussion, reaching the conclusion that time goes on even when the dot is motionless: "Time does move, it [the dot] simply does not move at this time." The discussion took place without activating the animation in the diagram but based on its static mode, which students expanded using body movements and verbal descriptions. The students activated the animation to verify their hypotheses and conclusions.

The following conversation is the students' attempt to explain the correct result they arrived at regarding the static point:

Daniel: But here all the other five… Time moves and the position moves.

Lior: Because they actually move (points at the slanted graphs) but it—it doesn't move. This is the position and it doesn't move (the position dimension of the graph remains constant): it doesn't go up or down… only the time moves for it. This is it! The orange dot (on the linear constant graph) is the green (motionless) dot (in the animation)!

The students seemed to be able to conceptualize the motion or the absence of motion of the green dot, and focused on the analysis of the graphs and on the dimensions of position and time as they appeared on the axes of the graphs:

Lior: I understand. Usually when they move in place, if they move to here then this goes down (moves his hand to the left over the animation and then down on the graph).

Lior and Daniel: …and if they move here, then it moves upwards (Lior and Daniel move their hands left to right over the animation).

Lior: And if it goes neither here nor here [moves his hand to the left and to the right of the static dot] then it… It doesn't change position either up or down [performs a gesture up and down along the graph window], but its time does [moves his hand left to right along the constant line], like all of them, as if the time didn't pass, then it would have stayed at the beginning [marks the initial spot of the line on the Y-axis], but time does pass, so because of that there's the entire line [moves his hand along the constant line].

This conversation indicates progress in the understanding of the graphic representation and of the roles it has assumed in solving the problem. The students described the motion by imagining rather than executing the animation. They were able not only to describe the change in distance over time of a static dot in the graph, but also to identify two directions of motion in the animation, right and left, as increasing and decreasing lines in the distance-time graph.

According to the data, constraints and resources of the design affected student engagement with the guiding ID in the following ways: (a) Finding the invariance and the variability within the processes in the given set of examples was a foremost theme in all problem-solving processes (e.g., “And if it goes neither here nor here [moves his hand to the left and to the right of the static dot] then it… It doesn’t change position either up or down”); (b) The presence of a static object among the seven moving objects played a central role in triggering the exploration of the differences between the representation of motion by animation and its representation as change in the distance over time graph (“If this doesn’t move [points with the mouse at the static car] and here all are moving [points with his finger at the graphs] it doesn’t make sense!”); (c) The transformation of the graph into a meaningful representation for describing the motion process was made possible by using a combination of the two representations: the animation and the graph (e.g., “… if they move to here then this goes down. And if they move… (moves his hand to the left over the animation and then down on the graph)”). Most of the discussion in the interview excerpted above took place without activating the animation. Students used the animation less than we had expected, and carried out most of the conjecturing mentally. We found, however, that the animation played an important role in coping with the task. As a concrete representation, it drew the students’ attention to the conflicts concerning motionless objects over continuous time, and prompted questions and conjectures that were not raised in the task. Students used the animation also to check assumptions and to illustrate conclusions. (d) The animation and the graph were designed to be partially linked but they were not color-matched, and the identification process required extracting data from the animation and the graph in order to link them. The partial link ended up being an important component in the mental analysis that the students performed. (e) The possibility of viewing only the complete motion, without stopping or focusing on discrete events, was an important component in the development of meaning for the graph.

4.3 Student Engagement with the Elaborating ID

The elaborating ID provided four adjacent linked representations (as opposed to one in the illustrating ID and two in the guiding ID): a table of values that represented distance and time in 0.1 s intervals; a 2D graph of distance over time; and a 1D graph of traces of positions at each time unit, with an animation. The variety of linked representations and rich tools in the elaborating ID made possible various options for personal choices concerning how the ID was viewed, for example: a sketch, and/or an accurate diagram, discrete and/or continuous flow of information. It also provided a variety of opportunities, as demonstrated by the significant differences that were observed between the three interviewed groups. Below we present in some more detail only one approach, but note the paths followed by others.

At the beginning of the interview of the first group concerning the elaborating ID, the students activated the animation. A green dot in the animation attracted their attention because it stood out as different: at the beginning of its motion it was very slow, and yet it finished first. The students described the change in speed of the green dot: “initially the green [dot] is the slowest,” “the green [dot] begins to accelerate.” Without using the stop option, they described the change in the relative positions of the dots in the animation of the race and indicated the finishing positions. The students looked for a way to improve the interpretation, pressed various buttons such as Run, Stop, Initialize, and Traces, they checked boxes, and activated the animation with traces:

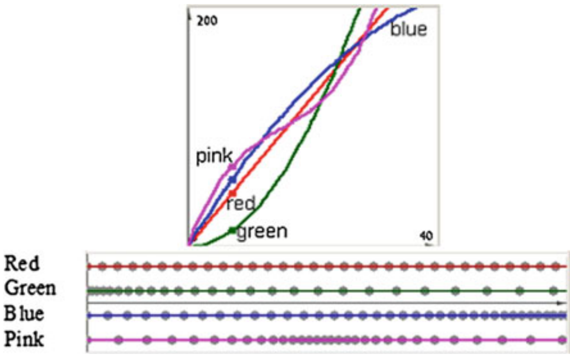
May: According to this, the red had reasonable speed, approximately the same speed all the way, the green at the beginning, he was, which means he was really slow at the beginning, and afterwards, when there are bigger spaces between the circles, then the…

Sara: He began to increase the pace. And here, he and the blue, he was faster at the beginning and towards the end he was really slow, you can see that the circles are really close together here, and the pink, so in the beginning he was fast, in the middle of the race he began to slow down, and at the end of the race he began to be fast.

They decided to integrate the two-dimensional graph and began to work with a combination of the two graphs, the 1D and the 2D, in a static state (Fig. 2). They focused on ascribing meaning to the 2D graph reflecting the speed of the motion, and used the description of speed that appeared in the 1D graph to attribute meaning to the 2D graph by analyzing the distance between the tracer dots: “the red is a straight line because its speed was uniform throughout the whole race, and the green is curved, curved because at the beginning it was slow.”

To verify the way in which speed was reflected on the graph, they decided to reactivate the animation. After observing the animation, they described the speed of the dots based on the shape of each graph:

Fig. 2 The 1D (traces) and the 2D graphs



May: Because you can see, if the line is straight like the red one, then you can see that it is a uniform pace.

Sara: In the beginning it () was not a uniform pace and then it () was a uniform pace, the blue () it has, at the beginning it had a uniform pace and then it curves and then it was the one in the last place. The pink () at the beginning it has uniform pace and then it has a drop in speed and then it has a little increase and then it has a uniform pace, that's why it came in second place.

The description showed the students' attempt to use meaningful items they created for themselves in previous work with the green graph  and the red graph  in order to ascribe meaning to similar items on other graphs: the part of the graph that is close to a straight line describes uniform pace, the part that "curves" describes a change of speed, and the part that looks like the beginning of the green graph describes deceleration.

Analyzing the work of the two other groups, we found different trajectories of mathematical engagement. The second group chose to start working with an animation and a graph, and gradually changed the students' concept image of the lines in the graph from motion paths to graphic representations of motion with changing rates of speed. In the beginning, the students treated the graph as a motion path. Next, they used the stop tool to monitor the change in the respective positions of the dots simultaneously across the two representations. The students proceeded examining a variety of examples in the diagram, some of which contradicted the erroneous assumption they made while examining other examples. The third group focused on the table of values and how it corresponded to the animation. The stop tool helped them match the discrete data in the table and in the animation. After analyzing the characteristics of the respective positions of the dots and the numbers in the table, they progressed from the incorrect assumption that the numbers in the table described time to considering the table values as a description of distance.

In sum, the constraints and resources affected learner engagement with the elaborating ID primarily as follows: (a) Work with the ID is characterized by a wide range of problem-solving choices that have some common elements but involve using different tools or different representations. (b) A carefully planned interactive example helped reveal its generic qualities in describing the various aspects of motion. The components of the example became significant in particular because they represented something exceptional for the students: the unique motion of the green dot relative to the other dots helped the students ascribe meaning to the 1D and 2D graph as descriptions of deceleration. Extracting the few pieces of information that were essential for developing the students' stories concerning the motion process in the task proved greatly useful for their work. (c) The link between the representations allowed students to ascribe meaning to unique situations in other representations. At times, the link between the representations was obvious, as, for example, in the case of the animation and the 1D graph. At other times, establishing the link between the representations using the tools included in a diagram (the stop tool, the clock, and the active line in the table) was another aspect of the process of motion description. (d) The combined use of discrete items of information that described the situation at moments when the action was stopped and of continuous information concerning the entire process was an important component in the students' engagement with the mathematics of motion.

5 Discussion

The present chapter focuses on students' ways of developing knowledge by identifying and defining relevant components of complex, new, digital environments. The following discussion summarizes: (a) the key design decisions concerning resources and constraints of the interactive text as reflected in various semiotic functions, and (b) the role played by ID resources and constraints in the development of mathematical knowledge when reading about and solving unfamiliar tasks. In the following summary and discussion of the conclusions we focus on (a) representations and tools for interacting with the representations and (b) examples and tools for interacting with the examples.

5.1 Key Design Decisions About Resources and Constraints

ID design must support a variety of purposes in the course of learning and teaching. The intentional design of what may be viewed as incomplete tools was based on the principle enunciated by Schwartz (1995) for designing software for mathematical inquiry: finding an "interesting middle," where the setting lets students "interact with and manipulate aspects of a subject that are complex enough to be interesting and simultaneously simple enough to be understood" (Schwartz, *ibid*., p. 180).

The IDs in our studies varied in the design choices of what was included in the given example and how it was represented and controlled. Designing constraints and resources with a *focus on representations* (graphs, tables, animations, etc.), and designing tools for interacting with these resources involves making decisions about (a) the number of representations, (b) the sketchy or accurate nature of the representation, (c) the kind of links to be made available between the representations, (d) the choices available for reducing the amount of information presented, (e) and the discrete or continuous nature of information. Similarly, designing constraints and resources with *focus on examples* and on the tools for interacting with the examples involves making decisions about: (a) the initial given set of example; (b) the tools available for changing an initial example by creating similar examples or new ones. Table 6 shows a comparative view of the resources and

Table 6 Comparative view of the resources and constraints in the modeling ID series, with a focus on representations and examples

	Illustrating ID	Elaborating ID	Guiding ID
Resources and constraints in the settings of representations			
Number of representations	1	4	2
Intended tone of the representation	Sketchy	Sketchy/accurate	Sketchy
Links between the representations	X	Linked	Partially linked
Options to change the amount of information presented in the representations	X	V	Partial
Mode of information presentation	Discrete/continuous	Discrete/continuous	Continuous
Resources and constraints in the settings of examples			
Initial set of given examples	Combination of the four types of accelerated motion	Combination of the four types of accelerated motion	Combination of the seven types of constant speed motion; static object among the seven objects
Tools for changing an initial example	X	X	X

constraints in the modeling ID series, with a focus on representations and examples that define each of the three organizational functions.

5.2 The Role that ID Resources and Constraints Play in Learners' Engagements: Representations, Tools, and Examples

We showed how ID resources contributed to the active reading of the mathematical text. The processes supporting learners' mathematical engagement with interactive texts position the learners as active readers who approach the text in an exploratory mode. Learners explore the interactive text, which becomes an instrument that assists them in their thinking and problem solving. Using the terms of Borasi et al. (1998), the exploration is characterized by negotiation, the reader and text shaping and being shaped by each other throughout the reading experience. Providing learners with opportunities for personal adaptations of the interactive mathematical text makes this exploratory negotiation more explicit because learners are encouraged to use various resources to interact with the text.

5.2.1 Representations and Tools for Interacting with Them

We begin by discussing the effect of the resources and constraints on engagement with the IDs in the three series (modeling, formulating mathematical phenomena, and manipulating; Tables 3, 4 and 5) with a *focus on representations* (graphs, tables, animations, etc.) and on the tools needed to interact with them. The illustrating IDs present the objective of the activity to the reader by offering a single representation and relatively simple actions for interaction with it. Students who worked with the illustrating IDs looked for ways to bypass the constraints built into the tool: they changed the representation of the data in the given example, expanded the given representation, or constructed new ones. We found that even the minimal interaction designed into the illustrating IDs can be helpful in consolidating relevant knowledge that is not yet adequately structured. The IDs can help present the parts of the less structured ideas to make them more coherent, meaningfully visible, and concrete in a problem-solving process. Following Vygotsky (1978) and Murata (2008) noted that the process of making students' ideas meaningfully visible and concrete helps students focus on core aspects of the problem and engage in their own meaning-making process. We found that personal attention paid to details in the ID presentation, awareness of these details, and the personal choices students made in the construction of additional details for the original presentation played an important role in students' work with illustrating IDs.

In the design of elaborating IDs (Tables 3, 4, 5 and 6), linked representations and rich tools for interacting with them played an important role. The learners' engagement with the elaborating IDs included use of linking tools and representations, and choosing the items considered to be significant in the example, the representations to work with, the order in which to work with the various representations, and whether to not to use the available tools. The differences between the engagement with illustrating and elaborating IDs reveal the pedagogical settings that elaborating IDs can support. The needs and choices of the students related to the exploration of a variety of unfamiliar representations and the interpretation of the links between them were addressed throughout students' engagement with the elaborating ID.

Guiding IDs are almost at the opposite pole from the elaborating IDs. Their design must be carefully adjusted to support development of knowledge and at the same time constrain students to the principal ideas of the task. It is possible to see the similarities in the constraints of the guiding and elaborating IDs with respect to the development of new mathematical ideas by students. Students in our studies started with a situation in which there was a mismatch between their knowledge and components of the IDs. The mismatch had to do both with the mathematical topics displayed in the IDs and with the resources and constraints designed into the IDs. The students spent time trying to learn the representations and tools that the IDs offered, to bypass the constraints of the IDs, and to expand their capabilities. Within the constraints of the IDs the students raised new ideas with respect to the tasks and tested them.

5.2.2 Examples and the Tools for Interacting with Them

Below we consider the role played by ID constraints and resources, with *focus on examples* and on the tools required for interacting with them. To do so, we refer to different settings of examples of IDs that invite viewers to activate them, with certain limitations, in two areas: changing given examples by generating similar examples or new ones, and interacting with components of the given examples.

The design of the three IDs in the modeling tasks supported interaction with components of the given examples but not the generation of new examples. The choice of types of motion was a key design decision. According to (Davydov 1972/1990), "the completeness and adequacy of the generalization depend on the breadth of the variations of the attributes that are combined, on the presence in the raw material of highly 'unexpected' and 'unusual' combinations of the common quality with the concomitant attributes or form of expression" (p. 6). Our intention was to include a repertoire of types of motion that would offer a general view of possible combinations of motion for the purpose of supporting the development of awareness of kinematic phenomena (such as rate of change, constant and changing speeds, two directions of movement, and different starting positions) and of their mathematical descriptions. Our findings show a process of development of a representative example that matches Davydov's argument (*ibid*.), according to which generalization is the result of a comparison of examples. The students revealed the salient features of the characteristics of motion by comparing special cases using the three IDs: they compared the motion in the animation, in the segments of the graphs, and in the value table. An important design feature of the example in the guiding ID was the motionless situation in the animation: one of the seven runners was represented by a steady dot, whereas all seven dots on the linked graphs were moving. Creating awareness of a conflicting image of this nature, which presents a difficult situation (a lack of information of the type described by Nemirovsky and Tierney (2001)), contributed to the ad hoc solution of the task of identifying the runners, and even more so, to understanding the Cartesian distance-time graphs.

The two other series of tasks, formulating mathematical phenomena and manipulating, differ from the modeling series in that they allow learners to change a given example by generating similar or new ones while getting feedback. Our studies concerning these two other series (e.g., Naftaliev and Yerushalmy 2011; Naftaliev 2012) highlight the importance of examples generated on the students' own initiative for clarifying the representativeness of the example as a carrier of the general meaning. The problem-solving process occurred when students were working with elaborating and guiding diagrams, which allowed examples to be entered freely. Problem solving was manifest when students used the given, specific example as a subject for comparison with the new example, which led students to conjectures and conflicts. It was also manifest when students generalized an example they created to produce an example space, in other words, a range of examples seen as instances of a generality based on that specific example (Goldenberg and Mason 2008). The students' strategy to solve the problem was comparison, and presumably it helped them perceive the structures, dependencies, and relationships that characterize any mathematical abstraction, as noted by Davydov (1972/1990).

Working with the guiding IDs, students treated the given example as a carrier of the general meaning and were able to reach a generalization in a process of systematic change and comparison. The variety of tools and representations offered in the elaborating IDs produced various strategies: students constructed, without guidance, various examples and initiated further inquiry that at times resulted in the systematic construction of examples, although the IDs provided no tools for systematic change. Initially, we hypothesized that illustrating IDs, which do not support the creation of new examples, would not be more helpful than a paper diagram because their design is similar to that of the paper diagram and offers only a limited choice of representations and tools. But we found that even the minimal interaction provided by the illustrating ID can be helpful in consolidating relevant knowledge that is not adequately structured yet. Students who worked with the illustrating IDs created their own examples only when it was required to do so by one of the tasks.

5.3 Concluding Remarks and Further Questions

Across our studies we found that similar tasks addressed with different IDs should be considered as different learning settings. The constraints and resources designed into the IDs contributed to making the tasks an interesting challenge for exploration by limiting the students' actions and by supporting guidance, while at the same time maintaining an open space for students' ideas. Based on our study, we find ourselves in a better position to raise relevant yet unresolved questions about the functions of IDs in teaching. In our past studies, we did not create a setting for observing teaching but rather focused on what we considered to be mostly spontaneous problem-solving processes. But the findings may have implications for teaching as a guiding mathematical inquiry, and the pedagogical situations that the tasks in the research appear to support should be studied further.

With regard to the related field of teacher learning, we adopt the approach suggested by Shulman (1986) and by Remillard and Bryans (2004), and assume that curricular resources are part of teacher learning and not intended to be used by students directly and independently of the teacher. The main purpose of teachers' engagement with curricular material is to guide student learning of subject matter during instruction. Teachers are taking further responsibility for their curricular resources, bringing Web materials to their textbooks, flexing and remixing their interactive textbooks. This raises questions about the decisions that educators should take as to whether and how they could use or import new IDs for different purposes in teaching and learning.

Studying such attempts is becoming increasingly important, especially given that the ability to use interactive textbooks and similar resources is growing, and that there are increasing opportunities for teachers to act as designers and authors of interactive materials. Teaching with an interactive textbook should be considered not merely a technological change but an attempt to create new paths for the construction of mathematical meaning. We conjecture that exploring IDs within a social semiotic framework could be useful for the professional development of teachers in analyzing curriculum materials and in examining the mathematical and pedagogical assumptions embedded in their design.

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