Designing Non-constructability Tasks in a Dynamic Geometry Environment

Anna Baccaglini-Frank, Samuele Antonini, Allen Leung and Maria Alessandra Mariotti

Abstract This chapter highlights specific design features of tasks proposed in a Dynamic Geometry Environment (DGE) that can foster the production of indirect argumentations and proof by contradiction. We introduce the notion of *open construction problem* and describe the design of two types of problems, analysing their potential a priori, with the goal of elaborating on the potentials of designing problems in a DGE with respect to fostering processes of indirect argumentation. Specifically, we aim at showing how particular open construction problems, that we refer to as *non-constructability problems*, are expected to make indirect argumentations emerge.

Keywords Dynamic geometry environment · Indirect argumentation · Non-constructability problems · Proof by contradiction

1 Introduction

This chapter discusses the potential offered by specific tasks designed in a Dynamic Geometry Environment (DGE) of leading to the production of indirect argumentations and eventually to proof by contradiction. We will attempt to highlight the specific design features that characterize these tasks. We start by introducing two

A. Baccaglini-Frank (✉)
Department of Mathematics "G. Castelnuovo", "Sapienza" University of Rome, Rome, Italy
e-mail: baccaglinifrank@mat.uniroma1.it

S. Antonini
Università degli studi di Pavia, Pavia, Italy
e-mail: samuele.antonini@unipv.it

A. Leung
Department of Education Studies, Hong Kong Baptist Univerisity, Kowloon Tong, Hong Kong, SAR, China
e-mail: aylleung@hkbu.edu.hk

M.A. Mariotti
Università degli studi di Siena, Siena, Italy
e-mail: mariotti21@unisi.it

A. Leung and A. Baccaglini-Frank (eds.), *Digital Technologies in Designing Mathematics Education Tasks*, Mathematics Education in the Digital Era 8,
DOI 10.1007/978-3-319-43423-0_6

main aspects, emerging from previous studies: open problems, and open construction problems in particular, and explorations in a DGE. Then we characterize the design of two types of open construction problems and analyse their potential a priori. The main objective of this chapter is to elaborate on the potentials of designing problems of non-constructability in a DGE with respect to fostering processes of indirect argumentation. Such problems, indeed, withhold potential for fostering the emergence of argumentations referring to logical dependency between constructed properties and derived properties. Specifically, we aim at showing how non-constructability problems are expected to make indirect argumentations emerge. Our discussion is consistent with classical results coming from previous studies where the dragging strategies were described (Healy 2000; Hölzl 2001; Leung and Lopez-Real 2002; Arzarello et al. 2002), but aims at elaborating on them to support the didactic hypothesis that designing and solving non-constructability DGE tasks may offer a rich context for introducing proof by contradiction.

2 Indirect Argumentation

Studies in mathematics education have revealed students' specific difficulties with proof by contradiction (Thompson 1996; Antonini and Mariotti, 2008, 2006; Antonini 2004), at every school level. However, on the other hand, some studies underline that students spontaneously produce argumentations with a structure that is very similar to that of a proof by contradiction:

> The indirect proof is a very common activity ('Peter is at home since otherwise the door would not be locked'). A child who is left to himself with a problem, starts to reason spontaneously '… if it were not so, it would happen that…' (Freudenthal 1973, p. 629).

With the term "indirect argumentation" we intend, in line with Freudenthal, argumentations that are developed starting from the negation of what is to be supported. That is, for example,[1] of the type "…if it were not so, it would happen that…". The transition from indirect argumentation to a (direct or indirect) proof is beyond the scope of the analyses included.

Freudenthal concludes that "before the indirect proof is exhibited, it should have been experienced by the pupil" (Freudenthal 1973, p. 629) and, along the same lines, Thompson writes:

> If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure (Thompson 1996, p. 480).

[1]For a more articulated and refined analysis of argumentation supporting mathematical impossibility see Antonini (2010).

Assuming this last hypothesis proposed by Thompson, in this chapter we propose and analyse tasks that have the aim of fostering the production of indirect argumentations in Euclidean Geometry.

3 Open Construction Problems

The term ‘open problem’ is common in the mathematics education literature (Arsac 1999) to express a task that poses a question without revealing or suggesting the expected answer. When assigned an open problem, students are faced with a situation in which there are no precise instructions, but rather they are left free to explore the situation and make their own conclusions. Frequently reaching a solution involves processes of generation of conditionality after a mental and/or physical exploration of the problematic situation (Mariotti et al. 1997). In the literature, open problems have frequently been characterized by the presence of an explicit request to produce a conjecture (e.g., Boero et al. 1996; Olivero 2000; Arzarello et al. 2002; Boero et al. 2007). In these cases we will use the terminology *conjecturing open problem* (as in Baccaglini-Frank 2010, p. 84).

In Geometry, a conjecturing open problem can take the following form, as described in Mogetta et al. (1999):

> The statement is short, and does not suggest any particular solution method or the solution itself. It usually consists of a simple description of a configuration and a generic request for a statement about relationships between elements of the configuration or properties of the configuration (ibid, pp. 91–92).

Typically, when solving an open problem, the student must first advance one (or more) conjecture(s), as a culmination of what is referred to as the exploration phase or conjecturing phase (Baccaglini-Frank and Mariotti 2010), and then s/he is expected to engage in a proving phase that results in a proof supporting the validation of the conjecture (whether it turns out to be true or false). We will consider tasks that make use of conjecturing open problems that inquire about the *constructability* of a certain figure. Therefore we need to preliminarily discuss construction problems.

3.1 Construction Problems

Construction problems constitute the core of classic Euclidean Geometry. The use of specific artifacts, i.e. ruler and compass, can be considered at the origin of the set of axioms defining the theoretical system of Euclid’s *Elements*.

As stated by Heath (1956, in Arzarello et al. 2012),

> Euclidean Geometry is often referred to as ‘straight-edge and compass geometry’, because of the centrality of construction problems in Euclid’s work. Since antiquity geometrical constructions have had a fundamental theoretical importance in the Greek tradition (ibid, p. 98).

Accordingly, any geometrical construction corresponds to a theorem, which means that there is a proof that validates the construction procedure that solves the corresponding construction problem. As a matter of fact, the relationship between construction and theorems is very complex, and such complexity is witnessed by the discussions by the classical commentators of Euclid’s Elements (Heath 1956, p. 124 et seq.). Proclus distinguished between problems and theorems, “the former embracing the generation, division, subtraction or addition of figures, and generally the changes which are brought about in them, the latter exhibiting the essential attributes of each.” (Proclus, quoted by Heath, ibid, p. 125). While the ‘theoretical’ character of geometric constructions made them similar to theorems, the specificity of construction problems, as open problems, seemed to reclaim the need to maintain the distinction between the two types of statements. The distinction was to be further underlined by the expressions that Euclid put at the end respectively of a theorem and of a problem: in the case of a Theorem he wrote “that which was required to prove” and in the case of a construction he wrote “that which was required to do” (ibid, p. 126). However, the substantial unity of the Euclidean statements led some authors to use a unique term for both types of statements. In some of the later editions of the Elements we can find the term “Proposition” referring to any statement of the theory, followed or not by the specification of the theorem or by a problem (see for instance, Cametti 1755; Legendre 1802).

Thus in classic Euclidean Geometry the *theoretical nature* of a geometrical construction is clearly stated, in spite of the apparent practical objective, i.e. the accomplishment of a drawing following a certain construction procedure. We note that the “non-constructability” of a figure may become manifest in fundamentally two different ways: a figure may be non-constructible with certain (predefined) theoretical tools, mostly straightedge and compass; or non-constructability may derive from the non-existence of the figure of which one requires the construction, that is, from the contradiction that follows once its existence is assumed. Historically, there are many examples of the first case such as the trisection of an angle, doubling a cube or squaring the circle. The problems of constructability with straightedge and compass were solved definitively in the XIX Century with tools developed in analytic geometry and through algebraic extensions. The second case of non-constructability does not depend on the tools used to accomplish the construction because it is a consequence of the theoretical non-existence of the object. The latter is the context we will be working in throughout this chapter.

3.2 Conjecturing Open Problems of Constructability and Non-constructability

In this chapter we will be working with construction problems that involve the formulation of a conjecture, so they are *conjecturing open construction problems*. From the point of view of design, the main aspect we are interested in discussing here, it is useful to distinguish two subtypes within these problems based on whether the construction *is* or *is not* actually possible in Euclidean Geometry. If the construction *is* possible we will speak of *constructability problems*, while if the construction is *not* possible, of *non-constructability problems*. Clearly the solver initially does *not know* whether the construction problem s/he is addressing is a constructability or non-constructability problem, while the designer does.

We will be working, in particular, with non-constructability problems of two types: one in which the solver is asked whether a figure with described properties is constructible or not, and in either case s/he is required to provide an argumentation; second one in which steps of the construction of a figure are given and the solver is asked what kinds of figures of a specific type (e.g., of quadrilaterals) can/cannot the figure become, providing conjectures and argumentations in each case. In either case, the solver will probably attempt to construct the suggested or hypothesized figure. The solution can be provided either producing the construction procedure and its validation according the theory available (in this case Euclidean Geometry), or proving the fact that no construction procedure can be exhibited. This latter case, because of its very nature, may lead to an indirect argumentation, sowing seeds that may lead to a proof by contradiction. As a matter of fact a non-constructability statement expresses the fact that it is impossible to display a valid procedure for constructing a certain figure.

4 The DGE Dragging Phenomenon

Literature over the last 20 years has been filled with examples of how a DGE can be used for the exploration of open problems, and, more in general, in exploratory learning (e.g., Yerushalmy et al. 1993; Di Sessa et al. 1995). In particular, research has shown that a DGE impacts students' approach to investigating open problems in Euclidean Geometry, contributing particularly to students' reasoning during the conjecturing phase of open problem activities (e.g., Leher and Chazan 1998; Mariotti 2000; Arzarello et al. 2002; Leung 2008; Leung et al. 2013). The dynamic nature of the exploration in open problems is particularly evident in a DGE. Any DGE figure that has been constructed using specific primitives can be *acted upon* through dragging hence determining the phenomenon of *moving figures*. A Dragging Exploration Principle was proposed (Leung et al. 2013) to epitomize the DGE dragging phenomenon:

> During dragging, a figure maintains all the properties according to which it was constructed and all the consequences that the construction properties entail within the axiomatic world of Euclidean geometry (ibid, p. 458).

The perception of a *moving figure* in a DGE is the phenomenon on the screen that something about the figure changes while something is preserved under dragging. What is preserved under dragging (the *invariant*) becomes the identity of the object/figure in contrast with what changes that determines its *variation* and consequently its movement. "Dynamic geometry exteriorizes the duality invariant/variable in a tangible way by means of motion in the space of the plane." (Laborde 2005, p. 22). The invariants correspond to the properties that are preserved and allow the user to recognize the sequence of images as the same figure in movement. Perceiving and interpreting the interplay between variation and invariants under dragging is the core of the process of *discernment* in DGE whereby we recognize quite different objects as belonging to the same category (Leung 2008; Leung et al. 2013; Mariotti 2014).

In a DGE, it is possible to distinguish between two kinds of invariants appearing simultaneously as a dynamic-figure is acted upon and therefore "moves". First there are the invariants determined by the geometrical relations defined by the commands used to construct the figure which are called *direct invariants*. Second there are the invariants that are derived (*indirect invariants*) as a consequence within the theory of Euclidean Geometry (Laborde and Sträßer 1990). The relationship of dependency between these two types of invariants constitutes a crucial point in the process of exploration in a DGE, and the experience of dragging constructed figures allows the user to interpret what appears on the screen in terms of logical consequence between geometrical properties; in particular, derived invariants will be interpreted in terms of consequences of the direct invariants. Familiarity with explorations in a DGE will mean for a user to have high confidence of this kind of interpretation of images and transformations of images on the screen.

Solving constructability and non-constructability problems in a DGE presents specific visual features. Drawings realized with a straightedge and compass and theorems validating a construction statement have specific counterparts in a DGE. This can be described in terms of visual theorems (Davis, 1993).

> Briefly, a visual theorem is the graphical or visual output from a computer program—usually one of a family of such outputs—which the eye organizes into a coherent, identifiable whole and which is able to inspire mathematical questions of a traditional nature or which contributes in some way to our understanding or enrichment of some mathematical or real world situation (ibid, p. 333).

> It [visual theorem] is the passage from the mathematical iteration to the perceived figure grasped and intuited in all its stateable and unstateable visual complexities (ibid, p. 339).

Therefore a dynamic visual moving figure in a DGE stands for an interesting epistemic aspect of experimental mathematics where both a visual and theoretical dimensions are present. This duality has been discussed, for example, by Leung and Lopez-Real (2002):

> During any dragging episode, the boundary between exploring new geometrical situations and justifying a theorem is a blurred one …. The holistic nature of the dynamic visual representation in DGE allows variation in meaning when a DGE entity is observed (via dragging) from different points of view. Hence the dragging modality can be interpreted as a kind of "random access" to different cognitive modes (making conjecture, formulating proof) in the mind of the person who is interacting with DGE. This duality in interpretation in DGE … facilitates the acquisition of deeper insight into the task at hand that could lead to further generalization (ibid, p. 159).

Thus the user is let free to explore the possibility of realizing the requested properties. Different possible situations may occur leading to different possible exploration processes or strategies. As discussed by Sinclair and Robutti (2013), dynamic figures can be interpreted in two fundamentally different ways: one according to which the dynamic figure constitutes a "whole" whose behaviour is analysed all together; and a second way according to which the figure constitutes a (very large and discrete) set of static "examples". The authors remarked on how

> It is still unclear whether learners somehow naturally see the draggable diagrams as a series of examples or as one continuously changing object, and whether this depends on their previous exposure to the static geometric discourse of the typical classroom […] (ibid, p. 574).

The second modality described may be more present in explorations of non-constructability problems. Now let us consider a conjecturing open construction problem in a DGE that consists in asking to realize an image with a required set of properties. First let us look at the case in which a (robust[2]) construction is possible. The order of construction of the required properties may be important, in that it may not be possible to invert the order of robust construction of the properties and still reach the desired figure. For example, constructing a parallelogram with a right angle is not possible starting from a robust parallelogram. Instead, the solver needs to first construct a right angle and from there proceed to define the three other vertices and two sides of the parallelogram. So the possibility of realizing a figure with specific properties may be subordinated to selecting a certain order of construction of the properties. If the user starts with a robust parallelogram and then tries to impose a right angle in one of its vertices, all s/he can obtain is the right angle as a soft property.

In the case of impossibility of the construction, no matter in what order the solver chooses to construct the properties, s/he will not be able to generate a figure with the desired properties. However, the choice of which property to start constructing robustly may heavily influence the exploration. This issue is touched upon in the paper by Baccaglini-Frank et al. (2013) and will be further elaborated on in the present chapter, as it is key in capturing aspects of the didactical potential of the types of activities proposed in explorative learning contexts. Let us analyse examples of two paradigmatic types of tasks for non-constructability problems. We will give an a priori analysis of possible solution processes showing how indirect argumentations might emerge.

[2]The terminology "robust" and "soft" comes from Healy (2000) and refers to the fact that certain properties are or are not invariant under dragging.

5 Two Types of Non-constructability Task

5.1 Type One

The first type of non-constructability task can be given in the following form:

> Is it possible to construct a figure of type X with properties Y_1, Y_2, … Y_n? If so construct it robustly. If not explain why not.

Here "figure of type X" indicates a class of figures such as triangles, quadrilaterals, etc., and Y_i are properties of figures in Euclidean Geometry.

5.1.1 Example Task 1a

The task is formulated as follows: "Is it possible to construct a triangle with two perpendicular angle bisectors? If so, provide steps for a construction. If not, explain why not." The answer to the question posed by the problem is "No. A triangle with two perpendicular angle bisectors cannot be constructed".

Figure 1 depicts a robust construction of the triangle with soft angle bisectors. A proof by contradiction might go along the following lines (refer to Fig. 1).

On the one hand, let $\angle CDA$ be right and CD be the bisector of $\angle BCA$ and AD the bisector of $\angle CAB$. Then, passing to the angle measures, $\frac{1}{2}m\angle BCA + \frac{1}{2}m\angle BAC = 90°$, so $m\angle BCA + m\angle BAC = 180°$. On the other hand, $m\angle BCA + m\angle BAC < 180°$ because $\angle BCA$ and $\angle BAC$ are two angles of a triangle. Therefore, we have a contradiction, that is the conjunction between a proposition and its negation.

In a DGE the solver can choose to construct and reason in one of two fundamentally different ways as follows.

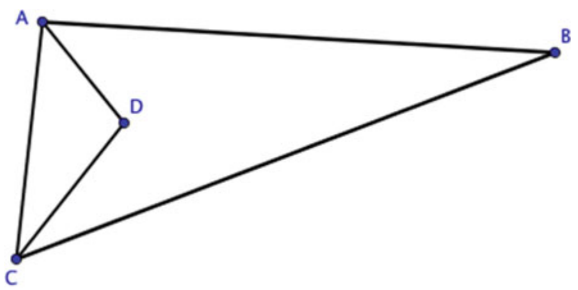

Fig. 1 Possible attempt at constructing a suitable triangle

(1) Construct the angle bisectors meeting at a right angle first, then construct the angles of which these are the bisectors and drag to try to "close" the triangle (see Fig. 2).

The solver drags the base points, but his/her attempt at constructing the triangle fails. At this point s/he may ask him/herself why s/he cannot close the triangle. S/he could discover that two sides of the triangle are parallel and so it is possible to construct other figures (like parallelograms, rhombuses …) but it is not possible to obtain a triangle. S/he could also continue to look for a particular configuration that satisfies the requirements and produce degenerate figures making the sides overlap in a single segment. The problem could move to accepting or not the obtained figure as in Fig. 2 as a triangle. If the solver accepts it as a triangle, what was requested has been constructed; otherwise s/he may conclude that the only way to obtain the requested figure is to make it degenerate, therefore excluding all together the possibility of constructing a triangle. In this second case, the argumentation supporting the conclusion may take an indirect form.

(2) Construct the triangle first, then the angle bisectors, and drag to force the angle at their intersection to become right (objective-property) (see Fig. 3).

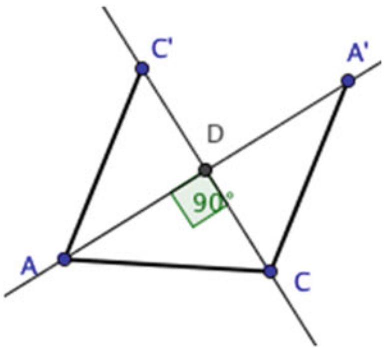

Fig. 2 Possible construction with robust angle bisectors intersecting at a robust right angle. One could drag to see if it is possible to make the lines AC′ and CA′ intersect to close the triangle

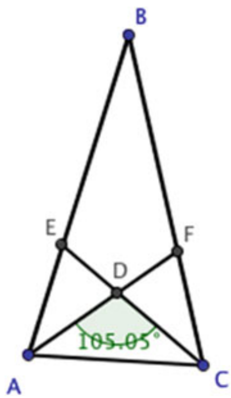

Fig. 3 Possible construction with a robust triangle and angle bisectors, but soft perpendicularity. One could drag to force the angle at the intersection to become right

The exploration leads to observing that it seems possible to make the angle a right angle, without however obtaining a single soft instantiation of this property. To check whether the angle is right, the solver may construct an additional element, for example the perpendicular line to the bisector CF through D or s/he might activate the measure of the angle. The fact that the angle can get closer and closer to a right angle can lead the student to thinking that a construction is in fact possible. This can lead him/her to trying to understand what the triangle should be like, and therefore it can lead to assuming that the figure *is already* properly constructed and searching for hypothetical additional properties to add to it (a typical process of analysis in Euclidean Geometry) in order to obtain a robust version of the desired figure. Starting from the assumption of having constructed the figure could lead to an argumentation like "if the triangle had perpendicular bisectors, then…" leading to indirect argumentations that can end in a contradiction or, at least, in properties that are unacceptable for the solver.

Different argumentations could be developed starting from the properties of the bisectors of the triangle, reaching a consequence that comes into conflict with the request of being perpendicular. For example, one can reach a conclusion that the angle between the bisectors has to be strictly greater than a right angle. Finally, theoretical considerations and the observation of the configurations emerging through dragging can lead to particular cases in which the properties are satisfied, but this happens only in degenerate cases in which the triangle collapses into a segment. Although obtaining the contradiction from a theoretical point of view is sufficient to prove that the triangle does not exist, from a cognitive point of view, we could have the necessity to see the consequence of the proposition $m\angle BCA + m\angle BAC = 180°$ which implies that sides BC and BA either coincide or are parallel. Then either B does not exist and so the initial triangle does not exist, or A, B, and C must be collinear, and so again the triangle cannot exist in a non-degenerate form. In other words, a determining difference of how this situation may be seen is how the figure degenerates. In one case the triangle can be seen to degenerate, breaking into an open figure (when BC and BA are seen as becoming parallel, see Mariotti and Antonini 2009), or it can be perceived as turning into a single line (for example, BC and BA are seen as collapsing onto the same line).

5.1.2 Example Task 1b

Task 1a can be given in a slightly different form: "Is it possible to construct a triangle with two perpendicular external angle bisectors? If so, provide steps for a construction. If not, explain why not." The answer to the question posed by the problem is "No. A triangle with two perpendicular external angle bisectors cannot be constructed".

Figure 4 depicts a robust construction of the triangle with soft external angle bisectors. A proof by contradiction might go along the following lines (refer to Fig. 4). Let ∠ADC be right and AD be the bisector of the external angle of the triangle in A(α), and CD be the bisector of the external angle of the triangle in C(γ).

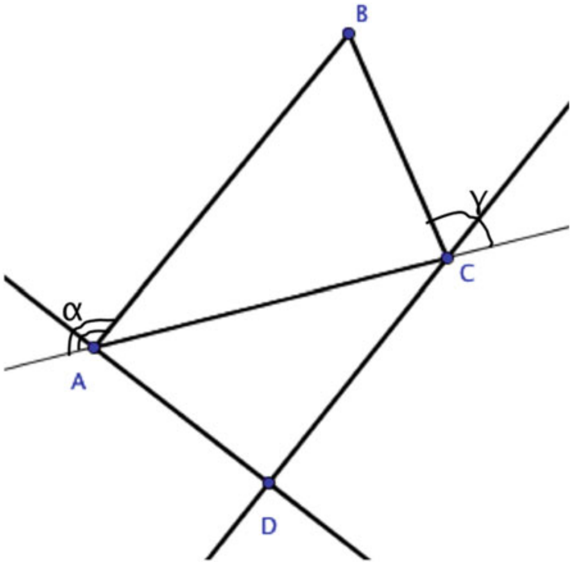

Fig. 4 Possible attempt at constructing a suitable triangle

Then, passing to the angle measures, $\gamma = 180° - \alpha + \angle ABC$. Considering the sum of the internal angles of triangle ACD,

$$\frac{1}{2}\alpha + \frac{180° - \alpha + \angle ABC}{2} = 180° - \angle ADC$$
$$\frac{1}{2}\alpha + \frac{180° - \alpha + \angle ABC}{2} = 180° - 90°$$
$$\frac{\alpha}{2} + 90° - \frac{\alpha}{2} + \frac{\angle ABC}{2} = 90°$$
$$\frac{\angle ABC}{2} = 0°.$$

Therefore, we have a contradiction, because $\angle ABC$ is not zero in a non-degenerate triangle.

In a DGE the solver can choose to proceed in one of two fundamentally different ways, as follows.

(1) Construct the external angle bisectors meeting at a right angle first, then construct the angles of which these are bisectors and try to drag B_1 to B_2 to “close” the triangle (see Fig. 5).
(2) Construct the triangle first, then the external angle bisectors, and drag to force the angle at the intersection D to become right (objective-property) (see Fig. 6).

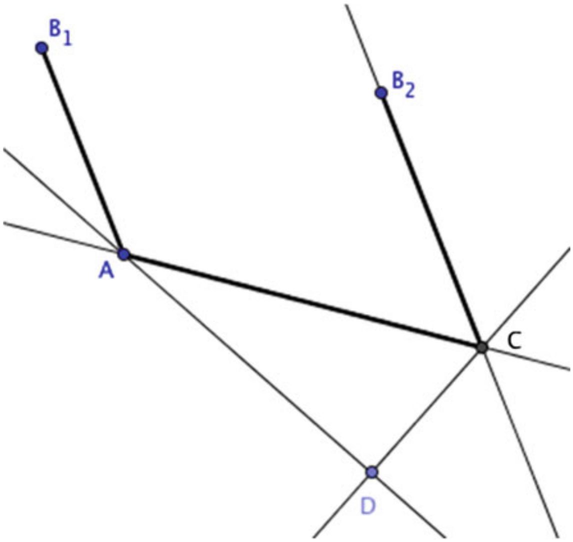

Fig. 5 Possible construction with robust external angle bisectors intersecting at a robust right angle

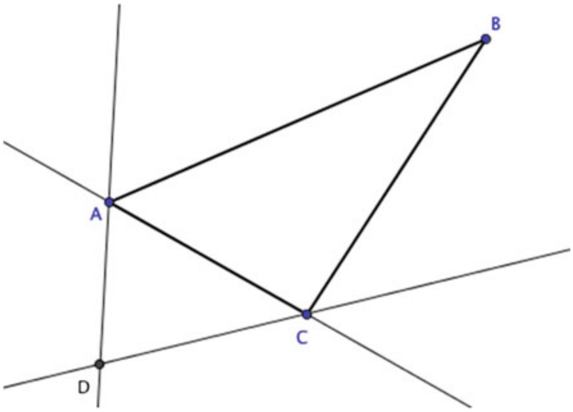

Fig. 6 Possible construction with robust external angle bisectors intersecting at a soft right angle

Summing up, in both cases the difficulties in trying to obtain what is requested can prompt a search for a number of different arguments, at different moments, and with different objectives:

- understanding, explaining, verifying failure (or the difficulties) of the search, and therefore explaining why the two properties cannot coexist at the same instant;
- identifying "when" the objective-property is satisfied;
- analysing the acceptability of the anomalous cases obtained.

Whatever the objectives, we expect both direct and indirect argumentations. Direct argumentations may stem from a certain property to identify consequences that can be incompatible with the second property. In other cases the solver may start from *the* triangle with all the desired coexisting properties, using processes of analysis and synthesis to see whether the construction is possible, or to motivate the impossibility of the construction. In both cases indirect argumentations may arise.

5.2 Type Two

The second type of non-constructability task can be given in the following form:

> Given the construction with steps S_1, S_2, … S_n, consider figure F originating from the steps. Which kinds of figures of type X is it possible for F to become? Make conjectures and explain.

Here S_i corresponds to a command in the DGE, F is a subset of elements originating from the construction which the solver's attention is called upon, and "figure of type X" indicates a class of figures such as triangles, quadrilaterals, etc.

First, the robust construction of a figure is required, the solver is asked to explore possible specifications of the original figure. Geometrically speaking, this will correspond to asking the solver to identify possible properties that can be consistently added to the construction properties that have been already realized. Let us discuss possible solutions in the following case.

Construct the following figure:

- a point P
- a line r through P
- the perpendicular to r through P
- a point C on the perpendicular
- point A symmetric to C with respect to P
- a point D on the semi plane opposite to C with respect to r
- line through D and P
- a circle with centre C and radius CP
- B as the 2nd intersection of the circle with the line through DP
- the quadrilateral ABCD.

Once the construction is achieved, an image appears like that in Fig. 7. What kinds of quadrilaterals can ABCD become?

As soon as the exploration begins, it will be easy to realize that ABCD can become a parallelogram (Fig. 8). Exploring this case, the solver can discover how to make ABCD into a robust parallelogram by only adding one new property to the construction, adding the following construction steps:

- a circle with centre in A and radius AP
- redefine D on this new circle.

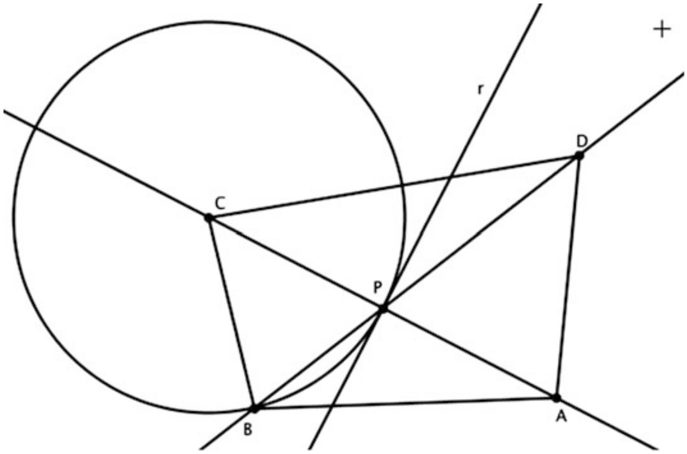

Fig. 7 Quadrilateral ABCD arising from the steps of the task

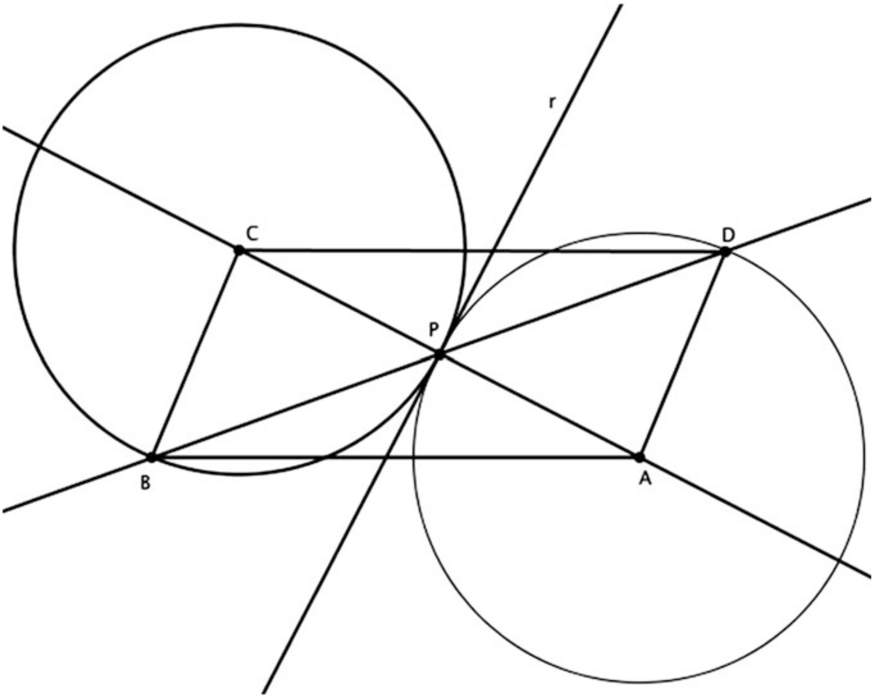

Fig. 8 The solver has discovered a way to transform ABCD into a robust parallelogram

The new properties are geometrically consistent with the previous properties and the construction is successfully achieved. At this point, the exploration could continue in two different directions. The solver decides either to come back to the original figure or to address the problem of constructing new quadrilaterals as subcases of the case of the parallelogram. In the latter case, a square would seem to be possible, since it is a particular kind of parallelogram.

The solver may notice that now the figure has new robust invariants PB congruent to PD; BC parallel and congruent to DA; BA parallel and congruent to CD; etc. The solver might attempt to obtain a figure that visually could be perceived as a square and to do this s/he may decide to see when ABCD has right angles, obtaining such configuration at specific instances. This is in fact possible, however the property is not sufficient for ABCD to be a square, but only a non-square rectangle (Fig. 9).

However, the solver may not grasp the theoretical reasons of such impossibility, therefore acknowledging the failure of his/her attempt. Instead s/he may search for another way of obtaining a square, identifying another property to add to the previous ones. For example, the solver may search for configurations in which the parallelogram has perpendicular diagonals. This happens only when the whole parallelogram collapses onto segment CA (Fig. 10).

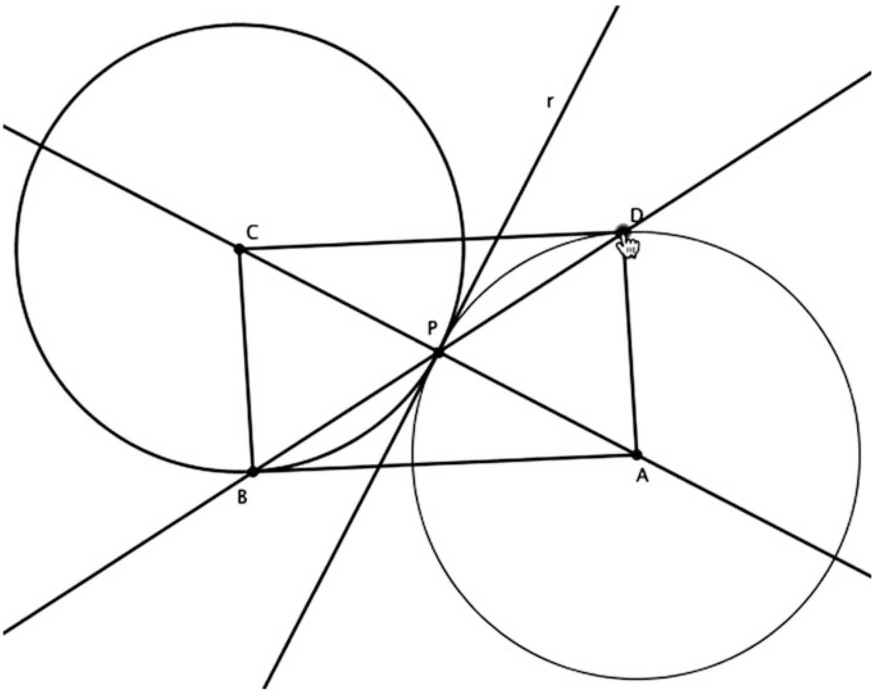

Fig. 9 The solver finds a position at which ABCD is a soft rectangle

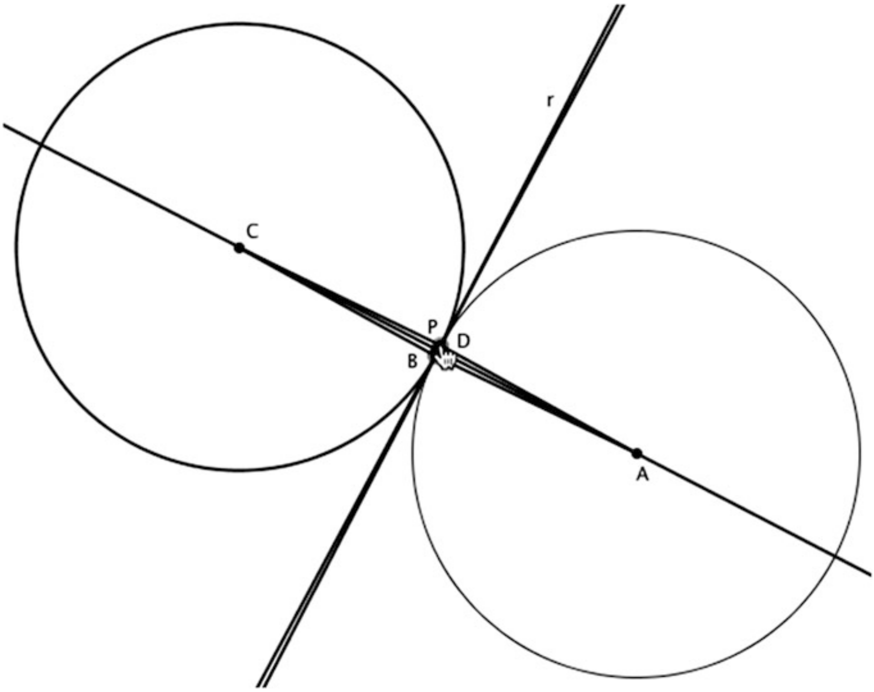

Fig. 10 ABCD has perpendicular diagonals only when the whole parallelogram collapses onto segment CA

The solver has found a particular configuration for which the desired property is visually verified. The solver may also become convinced that there is no other way to obtain the desired property (no matter where s/he places D on the circle). However s/he may now be concerned with which might be the other properties that the figure assumes when the desired property is visually verified. Why is this the case? Is the collapsed quadrilateral a square? These questions may trigger a rich production of argumentations.

Similarly, another issue could be the fact that in *no other* placement of D on the circle is it possible to obtain the desired configuration and thus a square. Finally, the solver may try to drag other points such as P or C to see if the desired property can be obtained for other—less awkward—configurations. In this case, more argumentations about why the configuration is not obtainable may follow.

Even if the solver had not constructed a robust parallelogram and started to explore the possibilities of obtaining a square by overlapping *r* and the line through PD, s/he would have ended up with a strange figure, such as the one below (Fig. 11), in which B and P coincide, thus new pressing questions might arise. Does this always happen? Why? These are more triggers for indirect argumentation processes. All these questions may originate further argumentation processes.

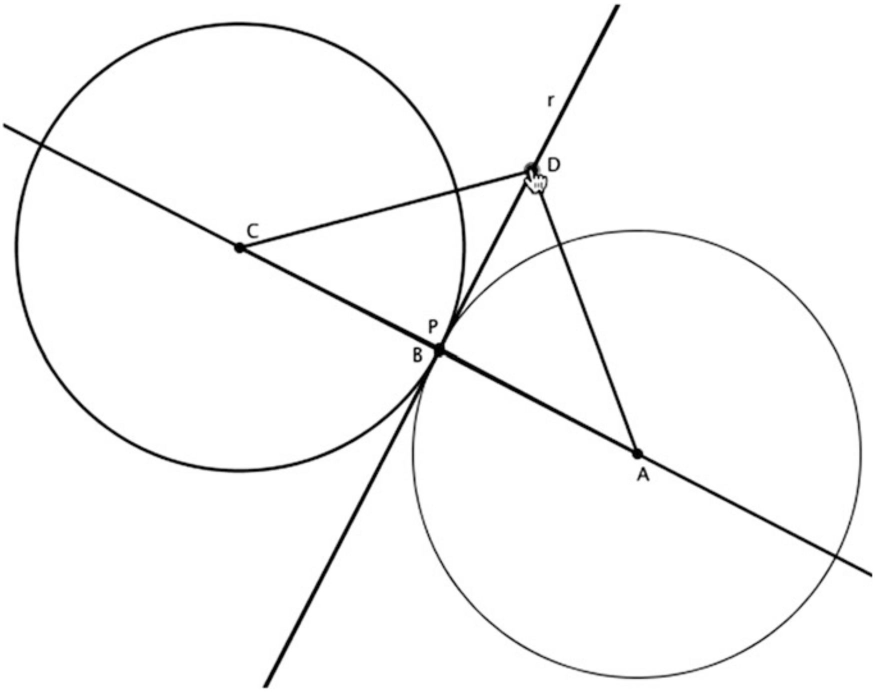

Fig. 11 Figure resulting from the solver's attempt of overlapping *r* and the line through PD

In summary, because of these two types of non-constructability task, it is likely that the arising argumentations will be indirect.

6 Nature of DGE Non-constructability Tasks and Design Considerations

In contrast to doodling with pen and paper to somehow draw what can appear to be impossible geometrical figures, in a DGE one cannot construct a "wrong" (Euclidean) figure! This makes a DGE a possible, and maybe even powerful, digital environment to explore and develop different types of argumentation in Euclidean Geometry. The visual robustness of DGE figures can force a certain desired property or condition into a visual anomaly which may produce experiential aspects that do not have immediate conceptual counterparts in the realm of Euclidean Geometry. The anomaly (here, possibly, a degenerated figure resulted from dragging to impose a condition) opens up a rich epistemic space for the solver to come up with logical argumentations to make sense out of it. Leung and Lopez-Real (2002) discussed a student exploration case about how to use a "biased DGE figure" (mentally projecting a condition on a robust DGE figure) to come up with a proof by contradiction and a related visual theorem under a drag-to-vanish strategy. Baccaglini-Frank et al. (2013) continued and expanded this discussion using tasks of the first type. In particular, they showed how a DGE can offer guidance in the solver's development of an indirect argumentation thanks to the potential it offers of both constructing chosen properties robustly. Therefore asking students to solve non-constructability tasks in a DGE can be conducive to developing their skills related to geometrical reasoning, proof and argumentation. Here we discuss the nature and considerations for this type of task design.

6.1 Task Nature

In Sect. 5, two task types were discussed:

1. Is it possible to construct a figure of type X with properties Y_1, Y_2, … Y_n? If so construct it robustly. If not explain why not.
2. Given the construction with steps S_1, S_2, … S_n, consider figure F originating from the steps. Which kinds of figures of type X is it possible for F to become? Make conjectures and explain.

"Is it possible?" is the common theme of these task types. Rather than the usual aiming to construct a DGE figure to ascertain a conjecture or to validate a theorem, an uncertainty is given as the main driving force for the task. In our experience we have noticed that the solver seems to initially be under the impression that a DGE *can* construct anything, possibly because of the Euclidean fidelity provided by a DGE. When the solver encounters a visual conflict with what s/he is expecting, or when s/he is unable to obtain an objective-property, s/he is prompted towards a dragging reasoning/discourse to re-solve the situation. The types of reasoning/discourse that the solver develops to re-solve the visual uncertainty are the main didactical goals of the task. In Sect. 5 the a priori analyses of the different tasks gave a glimpse of what possible dragging discourses can be developed due to the design of the tasks.

6.2 Visual Anomaly

The crux of this type of task design is to lead the solver to seeing a DGE phenomenon that does not seem to make sense at the first instant during a construction/dragging activity. That is, a cognitive conflict is created by a visual anomaly. Take, for instance, the case described in Figs. 10 and 11 in which B and P coincide, while the solver was expecting P to be the point of intersection of a square's (ABCD) diagonals. The anomaly forces the solver to combine concepts in Euclidean Geometry with her/his dragging strategies and the figures they lead to. In this case the solver would like to see a square (and may be seeing one mentally) but is forced by the DGE to recognize an isosceles right triangle which could be interpreted as "half of the square". In the task analyses in Sect. 5, we saw how this can happen for other figures (e.g., a triangle) that degenerated (e.g., into a line segment) when the solver was dragging in the attempt at realizing a desired condition/property. In these cases of degeneration, the anomaly seems to appear through a continuous dragging process that, in a certain sense, culminates with the generation of the anomaly: the objective property is obtained and in that instant something else is lost.

In general, visual anomalies can be generated when a certain condition is imposed on a construction, and the expected figure becomes something else. This can be the case when the solver chooses to robustly construct perpendicular angle bisectors and proceeds "backwards" to construct the sides of the triangle using reflections on the bisectors. The triangle's sides end up being robustly parallel, all of a sudden, and no matter how the solver drags, these sides will never intersect. As before, the solver may be mentally seeing a triangle, but actually with the DGE s/he will never be able to generate one. From these visual anomalies, the solver needs to resolve to Euclidean Geometry to explain the visual phenomena (Antonini and Mariotti 2010). Using the idea of figural concept (Fischbein 1993) as a "harmony" between a figural and conceptual component (Mariotti and Antonini 2009), an anomaly can be thought of as a break between the two components (figural and conceptual). It may be possible to restore the harmony within the figural concept by dragging to make a certain configuration vanish or degenerate, or by re-interpreting the obtained figure, rectifying the anomaly. This kind of solver-DGE interactive phenomenon should be typical in the solution of tasks designed as DGE non-constructible tasks.

We have been investigating the actual argumentations provided by students when solving tasks such as the ones analyzed in this chapter (Baccaglini-Frank et al. 2013), and we are currently working on associating specific types of dragging experiences and interpretation of the dynamic figures to the production of indirect argumentation. Our previous studies and preliminary results of our current research study suggest that specific types of dragging experiences and of interpretation of the dynamic figures seem to be associated to the production of indirect argumentation. Moreover, the non-constructability tasks analyzed in this chapter have proven to be particularly rich for gathering interesting data in this respect.

6.3 Didactical Reflections

We wish to conclude the chapter with some didactical reflections on the two types of task discussed. We find the two types of task to have different degrees of openness. The first type of task asks about the possibility of constructing a well-defined type of figure. The second type asks the solver to make conjectures on possible types of figures that might be obtainable given a certain (explicit) construction.

Though tasks of the first type are open, tasks of the second type appear to have a higher degree of openness, in that it is up to the solver to think of a particular configuration and then decide whether it is obtainable or not. Also, while in solving tasks of the first type the solver almost necessarily will explore (some form of) the impossibility of constructing a figure with the required properties; in solving tasks of the second type, the solver may concentrate on possible configurations that s/he encounters using wandering or guided dragging (Arzarello et al. 2002). This may occur for various reasons, for example: the solver is attracted to configurations s/he "recognizes", since it may be easier to "read" the figure interpreting it theoretically (ascending process[3]) as opposed to "impose something theoretical on the figure" (descending process); the search for ways to robustly impose a new condition on the figure and obtain a particular (possible) configuration may be time and energy consuming, and leave little time for the exploration of impossible cases; the solver thinks the teacher expects certain types of explorations from him/her because of the didactical contract, and such expectations may not include "impossible" cases since these might not be a typical aim of dynamic explorations; the student may not have developed a "mathematical eye" that allows him/her to attend to aspects that an expert mathematician would deem interesting (e.g., Hölzl 2001); etc.

Therefore, when proposing tasks of the second type, the teacher should consider the possible necessity of reformulating the task (maybe after some time or only for some students) in a more guided way, though maintaining the exploratory nature of the task. For example, in the case of the problem analysed in Sect. 5.2, the teacher might explicitly ask whether it is possible to obtain a square, thus making it clear for the students that "square" is a configuration considered interesting/relevant by the teacher and worth spending some time on.

On the other hand, we expect that tasks of the first type will relatively quickly put students in front of the fact that "it might not be that easy" to construct the desired figure, immediately opening the terrain to processes of argumentation. Moreover because the formulation of the task can guide the solver's attention to the contradictory properties, since these are stated explicitly in the task—although it is not stated that they are contradictory—some students may actualize processes of indirect argumentation.

[3]Ascending and descending processes are presented in Arzarello et al. (2002), referring to Saada-Robert (1989).

References

Antonini, S. (2004). A statement, the contrapositive and the inverse: Intuition and argumentation. In *Proceedings of the 28th PME, Bergen, Norway* (Vol. 2, pp. 47–54).

Antonini, S. (2010). A model to analyse argumentations supporting impossibilities in mathematics. In M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 153–160). Belo Horizonte, Brazil: PME.

Antonini, S., & Mariotti, M. A. (2006). Reasoning in an absurd world: Difficulties with proof by contradiction. In *Proceedings of the 30th PME Conference, Prague, Czech Republic* (Vol. 2, pp. 65–72).

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *Zentralblatt für Didaktik der Mathematik, 40*(3), 401–412.

Antonini, S., & Mariotti, M. A. (2010). Abduction and the explanation of anomalies: The case of proof by contradiction. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.). *Proceedings of the 6th Conference of European Research in Mathematics Education, Lyon, France, 2009* (pp. 322–331).

Arsac, G. (1999). Variations et variables de la démostration géométriques. *Recherches en Didactique de Mathématiques, 19*(3), 357–390.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM, 34*(3), 66–72.

Arzarello, F., Bartolini Bussi, M. G., Leung, A., Mariotti, M. A., & Stevenson, I. (2012). Experimental approaches to theoretical thinking: Artefacts and proofs. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education—The 19th ICMI study* (pp. 97–137). New York: Springer.

Baccaglini-Frank, A. (2010). Conjecturing in dynamic geometry: A model for conjecture-generation through maintaining dragging. *Doctoral dissertation*, University of New Hampshire, Durham, NH. ProQuest.

Baccaglini-Frank, A., Antonini, S., Leung, A., & Mariotti, M. A. (2013). Reasoning by contradiction in dynamic geometry. *PNA, 7*(2), 63–73.

Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating conjectures through dragging in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning, 15*(3), 225–253.

Boero, P., Garuti, R., & Mariotti, M. A. (1996). Some dynamic mental process underlying producing and proving conjectures. In *Proceedings of 20th PME Conference, Valencia, Spain* (Vol. 2, pp. 121–128).

Boero, P., Garuti, R., & Lemut, E. (2007). Approaching theorems in grade VIII. In P. Boero (Ed.), *Theorems in school: from history epistemology and cognition to classroom practice* (pp. 249–264). Sense Publishers.

Cametti, O. (1755). *Elementa Geometrie quae nova, et brevi methodo demostravit D. Octavianus Camettus*. Firenze.

Davis, P. (1993). Visual theorems. *Educational Studies in Mathematics, 24*, 333–344.

Di Sessa, A. A., Hoyles, C., & Noss, R. (1995). *Computers and exploratory learning*. Berlin, Germany: Springer.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*, 139–162.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, Holland: Reidel Publishing Company.

Healy, L. (2000). Identifying and explaining geometric relationship: Interactions with robust and soft Cabri constructions. In *Proceedings of the 24th Conference of the IGPME, Hiroshima, Japan* (Vol. 1, pp. 103–117).

Heath, T. L. (Ed.) (1956) *Euclid. The thirteen books of the elements* (vol. 1). Dover.

Hölzl, R. (2001). Using dynamic geometry software to add contrast to geometric situations—a case study. *International Journal of Computers for Mathematical Learning, 6*(1), 63–86.
Laborde, C. (2005). Robust and soft constructions: Two sides of the use of dynamic geometry environments. In *Proceedings of the 10th Asian Technology Conference in Mathematics* (pp. 22–35). Cheong-Ju, South Korea: Korea National University of Education.
Laborde, J. M., & Sträßer, R. (1990). Cabri-Géomètre: A microworld of geometry for guided discovery learning. *Zentralblatt für Didaktik der Mathematik, 22*(5), 171–177.
Legendre, A. M. (1802). *Elementi di geometria di Adriano M. Legendre per la prima volta tradotti in italiano*. Pisa: Tipografia della Società Letteraria.
Leher, R., & Chazan, D. (1998). *Designing learning environments for developing understanding of geometry and space*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning, 13*, 135–157.
Leung, A., & Lopez-Real, F. (2002). Theorem justification and acquisition in dynamic geometry: A case of proof by contradiction. *International Journal of Computers for Mathematical Learning, 7*, 145–165. Netherlands: Kluwer Academic Publishers.
Leung, A., Baccaglini-Frank, A., & Mariotti, M. A. (2013). Discernment in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.
Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics Special Issue, 44*, 25–53.
Mariotti, M. A. (2014). Transforming images in a DGS: The semiotic potential of the dragging tool for introducing the notion of conditional statement. In S. Rezat, et al. (Eds.), *Transformation—A fundamental idea of mathematics education* (pp. 155–172). New York: Springer.
Mariotti, M. A., Bartollni Bussi, M., Boero, P., Ferri, F., & Garuti, R. (1997). Approaching geometry theorems in contexts: From history and epistemology to cognition. In *Proceedings of the 21th PME Conference, Lathi, Finland* (Vol. 1, pp. 180–195).
Mariotti, M. A., & Antonini, S. (2009). Breakdown and reconstruction of figural concepts in proofs by contradiction in geometry. In F. L. Lin, F. J. Hsieh, G. Hanna, & M. de Villers (Eds.), *Proof and Proving in Mathematics Education, ICMI Study 19th Conference Proceedings* (Vol. 2, pp. 82–87).
Mogetta, C., Olivero, F., & Jones, K. (1999). Providing the motivation to prove in a dynamic geometry environment. In *Proceedings of the British Society for Research into Learning Mathematics* (pp. 91–96). Lancaster: St. Martin's University College, Lancaster.
Olivero, F. (2000). Conjecturing in open-geometric situations in Cabri-geometre: An exploratory classroom experiment. In C. Morgan, & K. Jones (Eds.), *BSLRM Annual Publication of Proceedings*.
Saada-Robert, M. (1989). La microgénèse de la representation d'un problem. *Psychologie Française*, *34*, 2/3.
Sinclair, N., & Robutti, O. (2013). Technology and the role of proof: The case of dynamic geometry. In A. J. Bishop, M. A. Clements, C. Keitel, & F. Leung (Eds.), *Third international handbook of mathematics education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
Thompson, D. R. (1996). Learning and teaching indirect proof. *The Mathematics Teacher, 89*(6), 474–482.
Yerushalmy, M., Chazan, D., & Gordon, M. (1993). Posing problems: One aspect of bringing inquiry into classrooms. In J. Schwartz, M. Yerushalmy, & B. Wilson (Eds.), *The geometric supposer, what is it a case of?* (pp. 117–142). Hillsdale, NJ: Lawrence Erlbaum Associates.