Revisiting Theory for the Design of Tasks: Special Considerations for Digital Environments

Marie Joubert

Abstract Teachers should and do design tasks for the mathematics classroom, with specific mathematical learning as the objective. Completing the tasks should require students to engage in dialectics of action, formulation and validation (Brousseau in *Theory of didactical situations in mathematics : didactique des mathematiques,* Dordrecht: Kluwer Academic Publishers, 1997) and to move between the pragmatic/empirical field and the mathematical/systematic field (Noss et al. in *Educational Studies in Mathematics, 33*(2), 203–233, 1997). In the classroom, students act within a *milieu*, and where computers are part of this *milieu*, particular considerations with respect to task design include questions about the mathematics the student does and the mathematics the computer does, and the role of feedback from the computer. Whilst taking into account the role of the computer, the design of tasks can also be guided by theoretical constructs related to obstacles of various kinds; ontogenic, didactical and epistemological (Brousseau in *Theory of didactical situations in mathematics : didactique des mathematiques,* Dordrecht: Kluwer Academic Publishers, 1997), and, whereas the first two should be avoided, the third should be encouraged. An example of a task taken from empirical research in an ordinary classroom is used to illustrate some of these ideas, also demonstrating how difficult and complex it is for many teachers to design tasks that use computer software in ways that provoke the sort of student activity that would be likely to lead to mathematical learning. Implications for teacher professional development are discussed.

Keywords Task design · Feedback · Modes of production · Pragmatic/empirical field · Mathematical/systematic field · Digital tools · Epistemological obstacles

M. Joubert (✉)
African Institute for Mathematical Sciences, Cape Town, South Africa
e-mail: marievjoubert@gmail.com

A. Leung and A. Baccaglini-Frank (eds.), *Digital Technologies in Designing Mathematics Education Tasks*, Mathematics Education in the Digital Era 8,
DOI 10.1007/978-3-319-43423-0_2

1 Introduction

This chapter outlines a theoretical approach to task design, where a ‘task’ is taken to mean what the teachers ask the students to do (Christiansen and Walther 1986; Joubert 2007; Monaghan and Trouche 2016). Activity is taken to mean what the students actually do and the focus is on tasks for digital environments. Although digital environments can change classrooms in significant ways, the theoretical approach described here is equally applicable in all mathematics classrooms.

The chapter is underpinned by three key assumptions. The first is that teachers should and do design or redesign tasks for their mathematics classrooms (Hoyles et al. 2013; Watson et al. 2013) and that tasks are adapted and tweaked by teachers to suit each context in which they give the task to students. The second assumption is that teachers want the tasks they set to provoke mathematical learning and have some understanding of the relationship between the task and the intended learning. This may seem obvious, but, as Brousseau (1997) argues, the detail of the intended learning is frequently not sufficiently well understood by teachers and as Watson et al. (2013), state:

> A distinct mathematical contribution can be made in understanding whether and how doing tasks, of whatever kind, enables conceptual learning (p. 10).

The third assumption is that students in mathematics classrooms do no more than is necessary to complete the task (Sierpinska 2004). The implication of these assumptions, taken together, is that teachers should have a clear idea of the sorts of activity their designed tasks will force the students to engage in and the mathematics they will learn by doing so, as well as factors that might hinder or prevent this activity taking place.

Digital technologies have the potential to contribute in significant ways to learning in mathematics, but designing tasks that use digital technologies well is complex and difficult (Joubert 2007; Laborde and Sträßer 2010) at least partly because of the ability of the computer to ‘do’ some mathematics and to provide mathematically relevant feedback. Importantly, the theoretical underpinnings of good task design are no less relevant when digital technologies are brought into the picture.

The chapter begins with a discussion of students’ mathematical activity, outlining the sorts of activity that might lead to mathematical learning; this is the kind of activity a task for mathematical learning should provoke. It goes on to discuss tools in the mathematics classroom, with a focus on computer software, emphasising how complex it is to really understand the role and potential of the software. The chapter then turns to the notion of obstacles, suggesting that some obstacles should be avoided because they get in the way of learning and some should be encouraged because they require students to engage in the sorts of activity that might lead to mathematical learning (as discussed previously). Obstacles provide a framework for the analysis of tasks, allowing us to predict likely student activity and hence mathematical learning. The chapter goes on to analyse of a teacher-designed task in which students work on computers. This has two purposes: first it illustrates the ideas outlined in the chapter and second it emphasises the importance of understanding the role and potential of digital technologies within classroom mathematics tasks. Finally the implications are discussed.

2 Students' Activity

Given the assumption that tasks in the mathematics classroom are designed to bring about student activity that might lead to mathematical learning, it seems important to understand how students engage with a task and what mathematical learning might look like. Of course, students come to a task with different backgrounds and the prior learning of students has been shown to be a key ingredient in determining how they react to different situations, as argued in detail by Roschelle (1997).

When students engage in a task, they tend to operate under a 'law of economy' Sierpinska (2004) and they find the easiest and shortest way to complete the task. Hillel (1992) explains that for many students, it is the end rather than the means that is important. and Mavrikis et al. suggest that 'students are inevitably focused on task completion, bypassing any need to mobilize structural reasoning or algebraic generalization' (2013, p. 2). However, as Brousseau (op. cit.) suggests, it is perhaps more likely that valuable and genuine mathematical learning will take place when students are invested in, and committed to, a problem situation. He contrasts this sort of motivation, mathematical motivation, with didactical motivation, where the student aims to solve the problem because this is what they were asked to do.

In completing the task, the students interact with the *milieu*, or the environment. In the case of tasks within digital environments, an important element of the *milieu* will be the device used, such as the computer, smartphone, tablet or graphical calculator. Together with the device, software ranging from networking capability, to generic software such as web browsers and spreadsheet packages, to mathematics-specific software such as computer-algebra systems (CAS) will be used. The interactions between the student and the *milieu* can be conceptualised as a dialogue between the student (or group of students) and the feedback from the *milieu* (Brousseau 1997). The importance of feedback should not be underestimated; as pointed out by Balacheff (1990):

> The pupils' behaviour and the type of control the pupils exert on the solution they produce strongly depend on the feedback given during the situation. If there is no feedback, then the pupils' cognitive activity is different from what it could be in a situation in which the falsity of the solution could have serious consequences (p. 260).

In the context of task design for digital environments, feedback from the *milieu* can be particularly powerful and important (see, for example, Bokhove and Drijvers 2010). Understanding what form the dialogue between the students and the *milieu* might take is crucial.

2.1 The Modes of Production

Brousseau (1997) uses the notion of 'modes of production' to describe the different types of dialectic interactions between students and the *milieu*; he suggests that as they work through a mathematical task, they will engage in all or some of the dialectics of action, formulation and validation. These are explained below.

2.1.1 Dialectic of Action

A dialectic of *action* involves the student constructing an initial solution to the problem straight away, informed by her current knowledge and using strategies that 'are, in a way, propositions confirmed or invalidated by experimentation in a sort of dialogue with the situation' (ibid, p. 9). For example, suppose the mathematical task a student is given requires them to produce a graph, perhaps on a graphing calculator, in a web-based graphing environment or in a specific graphing software package. If the student knows how to do this, and types the equation in, and the graph is produced on a big or small screen, this would be a dialectic of action. There may be an argument that, in some cases, the feedback from the *milieu* seems to have little or no role. However, as Brousseau argues, the student can be seen to be anticipating the results of her choices or strategies, and in this sense the *milieu* provides feedback, which can perhaps be seen as unrequested and as expected; it does not require the student to adapt her strategies. In the example given above, for example, the student would expect a graph to appear on the screen. She does not have to adapt her strategy. On the other hand, feedback may occur from time to time as the student works. For example, students may use a computer-based homework package, such as 'Mathspace' to check answers as they go along, or they may check answers to exercises in the back of the textbook or they could use self-checking methods such as multiplying out factorised functions. Very commonly, the teacher provides feedback either by checking the students' work as she walks around the classroom or by going through answers in a whole-class discussion. In these cases, the dialectical nature of the student interactions is clearer, and the feedback from the *milieu* can be seen as requested and as 'a positive or negative sanction relative to her action' (Brousseau 1997).

Whereas a negative sanction could encourage the students to formulate new strategies, it may also result in students simply trying something different but not using the feedback to inform the guess; this approach is sometimes called a trial and error approach (and is different to a trial and improvement approach which, it is argued below, requires some formulation). However, it is also possible that, although the task presents some mathematical challenges to the student (in other words she does not know exactly what to do and how to do it), she will choose to engage only in dialectics of action, perhaps by inventing easier ways to solve problems or perhaps by adopting a trial and error approach, which can be seen as didactically motivated rather than mathematically motivated (Sutherland 2007). It is possible that *all* dialectics between the student and the *milieu* in a given didactical situation are dialectics of action; if, for example, the student knows what to do and how to do it in order to complete the task. This would mean that, although she completes the task, she does not need to extend her mathematical knowledge or understanding to do so; ‘simple familiarity, even active familiarity … never suffices to provoke a mathematization’ (Brousseau 1997, p. 211). To a large extent this can be seen to depend on the demands of the task which in turn depends on the student’s previous knowledge and understanding, as argued above.

2.1.2 Dialectic of Formulation

On the other hand, some action dialectics are *necessary* within any didactical situation; and the suggestion is that it is when the action dialectics are motivated by the mathematics rather than the didactics, these actions may lead to a sequence of mathematical dialectical interactions which could include the development of hypotheses, alternative strategies, justifications and proofs, as described below. As suggested above, dialectics of *formulation* occur when students meet a difficulty or problem as they engage in mathematical activity; Brousseau explains that when a solution to a problem is inappropriate, the situation should feed back to the students in some way, perhaps by providing a new situation. The means that the student may become conscious of her strategies and begin to make suggestions. Brousseau includes in this category ‘classifying orders, questions etc. …’ (p. 61). He goes on to say that in these communications students do not ‘expect to be contradicted or called upon to verify … information’ (p. 61). Formulations necessarily include communication and have an explicit social dimension (Balacheff 1990) such as detailed descriptions and designations of phenomena, statements of properties or relationships, using and developing a language of a formal system. In making these formulations the students construct and acquire explicit models and language, which, as Christiansen and Walther (1986) argue, serves to make the learner conscious of strategies: ‘actions become conscious for the learner’ (p. 268).

A key activity in mathematics classrooms, and very often seen in classrooms where digital technologies are used, is ‘noticing’. In the context of research where digital technologies are used, Martin and Pirie (2003) provide an example. In their research, the teacher gives the students a printed sheet of a number of equations and asks them ‘to see what they can notice and find out about the graphs in relation to the printed equations’ (p. 176). When the didactical intention is that the student should make some observations, or to notice, it is likely that the students will begin to formulate choices; noticing requires the student to make conscious choices about what to notice, which aspects to attend to, which to suppress (Mason 1989) and how to express and articulate these. Whereas students can notice while remaining in action dialectics, noticing can also lead to conjecturing and engaging more in formulation dialectics.

Feedback has a variety of roles in formulation dialectics and can come from other people or from other elements of the *milieu*. As with action dialectics, the feedback can be immediate or delayed. For example, in the context of Brousseau's example game 'The Race to 20', feedback would be delayed until the effect of the chosen strategy can be seen (whether the student wins or not). Clearly, digital technologies are able to provide unique, usually immediate, feedback and this is discussed later in the chapter.

In the discussion above, the possibility of trial and error cycles of student behaviour was proposed. In these trial and error dialectics, the role of the feedback was seen to be only to inform the student that the strategy she had tried was incorrect. However, depending on the nature of the problem, it is possible that the feedback may also provide some clue for the student about how to improve her strategy and she may formulate a new strategy; this 'trial and improvement' or 'trial and refinement' approach is described by Sutherland (2007). She includes some cautionary remarks about the value of such approaches, arguing that, in some cases, trial and improvement may lead to a correct answer but perhaps not to the intended learning.

2.1.3 Dialectic of Validation

Both action and formulation involve manipulating 'moves in the game' or mathematical objects; validation however involves manipulating 'statements about the moves' (Sierpinska 2000, p. 6). Validation therefore takes place when an interaction intentionally includes an element of proof, theorem or explanation and is treated thus by the interaction partner, which could be the teacher, called the 'interlocutor' by Brousseau: 'this means that the interlocutor must be able to provide feedback…' (Brousseau 1997, p. 16). Brousseau argues that this interaction should be seen as a dialectic; this, he suggests, is due to the presence of the interlocutor. Examples of dialectics of *validation* include justification (perhaps of a procedure, a word, a language or a model), organising theoretical notions, 'axiomization' (ibid, p. 216), and a range of proofs.

Brousseau, while suggesting that all three modes of production are 'expected from students', (ibid, p. 62) argues that it is through situations of validation that genuine mathematical activities take place in the classroom. Romberg and Kaput (1999) echo Brousseau's sentiments in their vision of mathematics worth teaching: "Students will develop the habit of making and evaluating conjectures and of constructing, following and judging valid arguments" (p. 7). Brousseau suggests that situations of validation do not occur very often and are unlikely to occur spontaneously and it is probable that validation will not take place unless it is explicitly called for. "In order to obtain the latter [validation], one must organize a new type of didactical situation." (ibid, p. 13) The implication from Brousseau's 'interlocutor' (above) is that this interlocutor provides feedback. It is in discussion with this interlocutor that the individual develops his or her arguments; it is unlikely that feedback will come from any source in the *milieu* other than classmates or the teacher because of the need to convince someone else. Clearly, the teacher's role is crucial in encouraging students to engage in dialectics of formulation and particularly validation.

2.2 The Pragmatic/Empirical Field and the Mathematical/Systematic Field

It can be that the mathematical activity of the students relates only to the physical or concrete characteristics of the mathematical objects they work with. On the other hand, students may relate their activity to mathematical notions underpinning the objects. For example, on the one hand students may construct a tangent to a circle by rotating a straight line so that it touches the circle, and on the other hand they may use the geometrical property that the tangent is perpendicular to the radius to construct the tangent (Laborde et al. 2004).

A range of mathematics education theorists from a variety of theoretical backgrounds describe these different ways of working in different contexts; for Laborde (1998) the distinction is between 'spatio-graphical' and 'theoretical' and for Sfard (2000) it is between 'actual reality' and 'virtual reality'. While the varying theoretical perspectives imply variations on these ideas, the point for all of them is that it is useful to make the distinction and this chapter uses Noss et al. (1997) terminology 'pragmatic/empirical' and 'mathematical/systematic'. In addition to the perceived need to distinguish between the 'pragmatic/empirical' and 'mathematical/systematic' fields, there seems to be a consensus that a transition between the two fields is required for mathematical learning to take place (Dörfler 2000; Laborde 1998; Mason 1989; Sfard 2000). Brousseau, for example, describes different types of proofs, including contingent and experimental proofs and proofs by exhaustion. These, he suggests, relate to implicit models students hold and therefore they are likely to take place in the pragmatic/empirical field discussed above and not relate explicitly to theoretical mathematical knowledge. However, for 'mathematization' to take place, according to Brousseau, mathematical proofs which relate to the theoretical mathematics involved in the situation, are required. Similarly, Mariotti (2000) claims that the solution of geometry problems requires continuous moves between the two fields and Laborde (op. cit.) suggests that there is a need for interactions between images and concepts.

The implication, in terms of the relationship between the students' dialectics (particularly of formulation and validation), and student learning is that, where these dialectics remain in the pragmatic/empirical field, mathematical learning will be limited; however, transitions into the mathematical/systematic field are likely to lead to mathematical learning. This discussion about student activity has made frequent reference to the *milieu*. It now turns to a key component of the *milieu*; the tools used in the classroom.

3 Tools (Especially Digital Tools) in the Mathematics Classroom

It can be argued that mathematical activity, like all human activity, is mediated by tools (Vygotsky 1980). In the mathematics classroom, some tools are designed specifically for the teaching and learning of mathematics (Sutherland 2007) such as Dienes blocks or Cuisenaire rods, and can be seen to embody specific mathematical concepts. Other tools, such as matchsticks and mirrors are also frequently used in practical classroom work. These differ from those designed specifically for the teaching and learning of mathematics in that they do not embody mathematical concepts in the same way. The literature suggests that there is often an expectation or belief that these tools will support mathematical learning but that the effective use of tools is more complex and difficult than it first seems (McNeil and Jarvin 2007; Orton and Frobisher 1996; Pimm 1995; Resnick 1984; Sutherland 2007). Pimm (1995), for example, warns that objects cannot offer mathematical experience or understanding in themselves, particularly when the students direct all their attention to manipulating the equipment: ‘Pupils may end up *just* manipulating the equipment’ (p. 13). McNeil and Jarvin (2007) go further, suggesting that the use of such tools can actually hinder mathematical learning, stating that recent studies “have suggested that manipulatives may not only be ineffective, but also detrimental to learning and performance in some cases” (ibid, p. 312).

The point is that the introduction of all tools into the classroom is a complex issue and their use does not guarantee the desired mathematical learning, and that, as argued convincingly by Sutherland (op. cit.), it is important for teachers to understand the potential and the limitations of the tools and be clear about the role of the tool in the mathematical activity. It can be argued that this is particularly true for computer software, which has an ‘intrinsically cognitive character’ (Balacheff and Kaput 1996, p. 469) which means first that for some software used in the teaching and learning of mathematics, the software can perform some mathematical processes or ‘do the mathematics’ (Bokhove and Drijvers 2010; Hoyles and Noss 2003; Sutherland 2007) for the user and second that it can provide mathematically relevant feedback for the user (Bokhove and Drijvers 2011; Granberg and Olsson 2015).

In addition, the feedback is ‘quick and essentially unlimited…at ‘no cost’’ (Hillel 1992, p. 205). As argued above, built into the tools specifically designed for teaching and learning mathematics, is a set of epistemological assumptions. In the case of software, these assumptions are likely to be considerably more complex than with non-digital tools, because of the ability of the software to do (some of) the mathematics and to provide feedback.

3.1 Computers Can Do the Mathematics

The ability of software to perform mathematical operations may have the potential to make a significant contribution to the teaching and learning of mathematics. For example, its use can enable mathematical experiences which, for example, allow students to focus on patterns, to construct multiple representations and to interpret representations (Ainley and Pratt 2002; Balacheff and Kaput 1996; Condie and Munro 2007; Ruthven et al. 2004; Sutherland 2007). However, as Love et al. (1995) caution, it is common for students to avoid the mathematical ideas intended to be the focus of study by using the software in unintended ways.

It was argued above that teachers should be clear about the mathematics their students should do as they complete a task that has been set for them. When computers are used, however, working this out can be difficult, because much of the mathematics curriculum is directed towards the learning of techniques, such as finding areas of shapes, making calculations and plotting graphs. Completing the technique is frequently seen as the end point of the task (Love et al. 1995; Schwarz and Hershkowitz 2001). As Hoyles et al. put it, 'digital representations change the epistemological map' (2013, p. 1058). If computers are able to do the 'work' for the students, then what is the work of the students?

Monaghan and Trouche (2016) illustrate this idea. They contrast two tasks. In the first, students are required to sketch a cubic graph: $y = x^3 - x^2 + 2x - 1$. In the second, students are given three graphs of quadratic functions and asked to reflect them in the *x*-axis. Monaghan and Trouche discuss the fact that, for the first task, using a computer graphing package renders the task trivial but doing the task using pencil and paper would be more difficult; for the second using the same software would probably present a considerable challenge whereas the pencil and paper approach would probably not be difficult.

Working out the intended mathematical processing to be performed by each of the computer and the students relates to what the software is able to do and what opportunities it offers the students in terms of doing mathematics. It is perhaps the relationship between these that is particularly important; as Nevile et al. (1995) suggest, it is important to look 'at the software' in tandem with looking 'through the software' (ibid, p. 157). To understand this relationship, it is perhaps useful to think in terms of the 'epistemological domain of validity' of the software which 'refers to the knowledge and the relation to knowledge which is allowed by a piece of software' (Balacheff and Sutherland 1994, p. 138).

In a well-known example, Balacheff and Sutherland (1994) explain this notion; they compare the software *Logo* and *Cabri* in terms of what epistemological opportunities they offer students engaged in constructing a parallelogram. They point out that the environments differ significantly in terms of the mathematical concepts students need to mobilise in order to make the construction. To create a parallelogram in *Logo*, students need to use knowledge about the equality in length of opposite sides and about the relationship between internal and external angles. In *Cabri*, however, to construct the parallelogram, they need to understand that the parallelogram is completely determined either by one point and two directions or by three points. While it may be necessary for teachers to understand these relationships, it is sometimes very difficult, because of the complexity of much of the software available for use in mathematics classrooms (Bokhove and Drijvers 2010). Sutherland (2007) argues this case convincingly, suggesting that 'many of these tools are so powerful, have so much potential, that it is difficult for teachers to know where to get started' (p. 68). What this means is that, if the potential of the tool is not thoroughly analysed, then it is difficult for teachers to work out what mathematics it does and hence what mathematics they want the students to do.

The ability of software to do the mathematics also opens possibilities for tasks in which students are able to work inductively, from the products of the software's mathematical processing, to develop conjectures and theories. It can perhaps be argued that this is a more natural way to learn than the traditional deductive approaches in classrooms. However, as Goldenberg (1988) suggests, these approaches should be carefully thought out:

> Put simply, a wrong theory is the most likely result of the casual introduction of an inductive learning experience … into a curriculum that is not otherwise designed to make use of the questions such an experience raises (p. 144).

In these cases, an understanding of the way the feedback offered by the environment can support inductive working may also be very important in encouraging students to develop and test their theories. An understanding of feedback is perhaps crucially important in both these situations and more generally when computers are used.

3.2 Feedback

In the example about creating parallelograms in *Logo* and *Cabri* above, Balacheff and Sutherland also compare the feedback from the two environments. In both environments, the students use the computer feedback to make a decision about whether they have constructed what they intended to construct. In both cases, they are able to determine whether they are correct or not by looking at the feedback on the screen. However, the *Cabri* environment offers more; students are able to test the correctness of their parallelogram more thoroughly by making sure that it resists manipulation (dragging).

This highlights the important role played by the feedback and raises questions about the role of feedback in computer based learning environments (see also, for example, Bokhove and Drijvers 2012; Granberg and Olsson 2015; Monaghan and Trouche 2016). The general conception of computer feedback seems to be that it is beneficial in teaching and learning mathematics (e.g. Becta 2004). Feedback provides opportunities 'to quickly test ideas, to observe invariants … and, generally, to be bolder about making generalisations' (Hillel 1992, p. 205). Further, there is evidence (e.g. Hillel 1992) that with the quick availability of feedback, students are likely to engage with problems for longer, and with more persistence. However, there is also considerable evidence to suggest that the effect of feedback needs careful analysis. Säljö (1999), in his discussion about feedback, suggests that 'it can provoke active reflection on the part of the learner who has to consider alternatives, manage concepts and representations and so on…' (p. 154). The degree to which the reflection provoked by the feedback is active, however, can perhaps be seen to be, in part at least, dependent on the kind of feedback given by the software; as Balacheff and Sutherland (1994) point out in their example above, the feedback given by *Logo* provides a more fragile basis for students to decide whether the parallelogram they have constructed is indeed the one they intended to construct, whereas in *Cabri* the feedback can be seen to provide a more robust basis for such a decision.

Students sometimes interpret feedback in unexpected ways, which may have a negative or distracting impact on developing the understandings the teacher intended (Hoyles and Noss 2003; Sutherland 2007), and which may lead to the development of strategies which are at odds with the intended learning, such as trial and error, trial and improvement and 'intellectual passivity' (Hillel 1992, p. 217). This point is clearly made by Säljö:

> …what technologies provide are experiences, but they do not guarantee a specific interpretation of these experiences that would amount to learning what was intended. (1999, p. 158)

The important point, perhaps, is that students do not always use feedback in the ways in which it is intended and, further, teachers tend to overestimate students' ability to use this feedback (Laborde 2002).

Feedback from the computer can be seen to have a major influence on students' modes of production; action, formulation and validation. For example, as described by Noss et al. (1997) students used a microworld to test a prediction about the number of matchsticks in a given sequence. The feedback from the computer allowed them to evaluate their prediction; negative feedback led to further suggestions and, later, positive feedback led to the conclusion of the task. In these cases, the feedback can be seen as the 'dialectical partner' in the *milieu* in situations of formulation (and sometimes action). However, when the feedback allows the students to test a prediction, the students may not make the transition to the mathematical/systematic field, which, as argued above, is necessary for learning mathematics. In terms of Brousseau's modes of production, the feedback from the computer could be seen to 'validate' the students' predictions, and therefore the students do not need to engage in the dialectics of validation.

In the example quoted above, the students can be seen to have some expectations about the feedback. It is possible, however, that they have no expectations, and the feedback takes the role of 'oracle' (Sutherland and Balacheff 1999) in scenarios where the students explore. For example, Olivero (2003) describes how students, working with figures in *Cabri*, use 'wandering dragging' which she describes as 'moving the basic points on the screen randomly in order to discover configurations or regularities in the figures' (p. 98). The students did not seem to have any expectations about what would happen; they used the feedback to tell them. This exploring activity, which can be seen as an action dialectic, however, is not productive mathematically, unless it is accompanied by more focused explorations (which would include 'guided dragging' Olivero 2003), in which conjectures are made and tested (formulation dialectics) and include transitions between the pragmatic/empirical field and the mathematical/systematic field (which may lead to validations) (op cit). Further considerations on feedback provided by dynamic geometry environments can be found in Chapters "Designing Assessment Tasks in a Dynamic Geometry Environment", "Designing Non-constructability Tasks in a Dynamic Geometry Environment" and "The Planimeter as a Real and Virtual Instrument that Mediates an Infinitesimal Approach to Area" of this book.

Computer feedback can be seen, to a greater or lesser extent, as complementing the teacher's role. Balacheff and Kaput (1996) use the notion of the 'didactic directiveness' (p. 483) of feedback. They suggest that microworlds (such as dynamic geometry software) and tutoring systems represent two extremes of didactic directiveness; microworlds offer environments in which students can explore freely but tutoring systems give 'strong guiding feedback' (p. 484). What is useful about this way of conceptualising feedback is that it relates directly to the intended role of the software and to the degree to which teachers can devolve a problem to the students (in the sense of Brousseau). This highlights the importance of taking into account the contribution of the software as a 'third player' in the didactical situation; without it, the teacher is completely in charge of the degree of devolution to the students, whereas now the computer has some influence as well.

4 Designing Tasks

This chapter has set out a theoretical framing of the relationship between student activity in the mathematics classroom and mathematical learning. This theoretical position suggests that student activity that includes dialectics of action, formulation and validation with some movement between the pragmatic/empirical field and the mathematical/systematic field is desirable. However, as pointed out above, the assumption is that the students' agenda is to complete the task (they are not concerned about dialectics or movement between fields); and as a result a task should be designed to require the mathematical thinking and reasoning that are '*strictly necessary and sufficient to complete the task*' (Sierpinska 2000, p. 12 italics in the original) also forcing dialectics of action, formulation and validation and movement between the pragmatic/empirical and mathematical/systematic fields. This notion is discussed in detail by Brousseau, who thinks in terms of the need for students to adapt strategies in order to reach target knowledge; and he suggests that they will not do so unless the task forces them to do so.

4.1 Obstacles

Such an adaptation may take place, Brousseau suggests, if the students encounter and overcome 'obstacles' as they work through the task. He describes an obstacle as:

> a previous piece of knowledge which was once interesting and successful but which is now revealed as false or simply unadapted (p. 82).

Obstacles can take a variety of forms; Brousseau, for example, identifies obstacles of ontogenic origin, of didactical origin and of epistemological origin. The first of these relates to the 'student's limitations' (p. 86), and can also be described as 'developmental' limitations (Swan 2006). For example, the lack of prior learning can be seen as an obstacle to completing the task because the student is in some way not ready for the mathematics required to complete the task (Love and Mason 1992).

Obstacles of didactical origin 'seem to depend on a choice or a project in an educational system' (Brousseau 1997, p. 86) and are seen as obstacles for the students because of ill thought out presentation of subject matter, or 'the result of narrow or faulty instruction' (Harel and Sowder 2005, p. 34). The example Brousseau provides is of the way in which decimal numbers are taught by convention in French elementary schools and Harel and Sowder give the example of the didactical practice of teaching students to look for 'key words' in mathematics problems (such as 'altogether' signals that addition is required).

Both ontogenic and didactical obstacles can be seen to inhibit mathematical learning, and should and can be avoided (Brousseau 1997; Harel and Sowder 2005). The third type of obstacle, however, the epistemological obstacle, should not be avoided and is, in fact, key to the design of good tasks. Brousseau's epistemological obstacles are perhaps most clearly explained by Balacheff (1990, p. 264) who discusses how mathematical concepts are learnt through their use as tools in the process of problem solving with some content; this content is supported by the students' prior knowledge. However, although the old knowledge is a necessary basis for the content, it may cause problems for the students as they work through the problem; in other words it becomes an obstacle which causes the students to stumble. If, in order to overcome the obstacle, the students are required to construct the meaning of the new piece of knowledge, then this obstacle is an epistemological obstacle. Brousseau provides examples: $(a + b)^2 = a^2 + b^2$; $0.a = a$, $(0.2)^2 = 0.4$. Harel and Sowder (2005) suggest that 'MMB' (multiplication makes bigger) can often be seen as an epistemological obstacle.

It is also not always easy to distinguish between didactical and epistemological obstacles. For example, as Harel and Sowder (2005) suggest, many obstacles have elements of both. They provide examples and place them on a set of axes labelled 'epistemological obstacle' and 'didactical obstacle' as shown in Fig. 1 (taken from their diagram on p. 47).

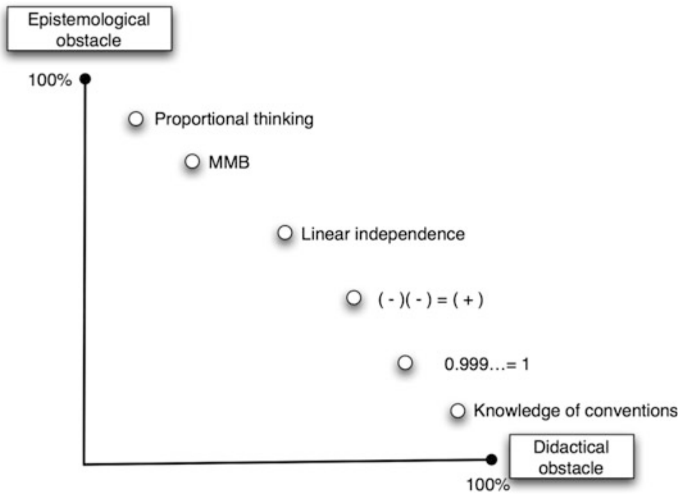

Fig. 1 Examples of obstacles as combinations of epistemological and didactical

The important thing about this idea is that the design of tasks needs to take these dimensions into account, in order to minimise the effects of didactical obstacles. Equally, an analysis of tasks should take both dimensions of obstacles into account and should be grounded in whatever is known about the prior learning and development of the students who complete the task.

The idea of obstacles is useful in thinking about relating the way a task is set up to the mathematical activity of the students, and in particular to the dialectics of action, formulation and validation. As students work on tasks, there will be stages when they know what to do and they do it; this is a dialectic of action. If they encounter an obstacle, however, they will be forced to change their approach and will develop new strategies, make suggestions to one another, conjecture and test; these are dialectics of formulation. If the obstacle has an epistemological dimension, then these formulations will be related to the intended learning.

The section below illustrates these ideas by analysing a task designed by a classroom teacher. The analysis brings together the expected student dialectics and movement between fields, the role of the computer and the obstacles discussed in this section. The example is given not as a ‘good’ example, as will be discussed later, but in order to provide something concrete to demonstrate not only the theoretical ideas above but also how complex it can be to design a computer-based task for the classroom.

5 An Example: Noticing

In my doctoral research, for which the overall aim was to explore how computers are used in ordinary mathematics classrooms, I observed a sequence of five or six lessons taught by four different teachers. All lessons were designed completely by the teachers themselves. In many lessons computers were used by the students. A careful analysis of all the tasks used in the computer-based lessons, framed by the notion of obstacles outlined above, reveals how the tasks were unlikely to provoke mathematical thinking. Although the students engaged with the task and did as they were asked, they were seldom encouraged to develop the sorts of reasoning that might be evidenced by the formulation and testing of hypotheses or by validation of these hypotheses.

I use one of the case studies to illustrate: an a priori analysis of a task given within the case study provides insight, perhaps, into the hypothetical learning trajectory of the students (Margolinas and Drijvers 2015).

The research took place in 2004 in an English classroom, with a class of about twenty students, of about 12 years of age. The task they were asked to complete had a number of questions. In each question the students were asked to use graphing software to draw a set of straight line graphs of the form $y = mx + c$ (or, in the last question, $y = c + mx$) and then write down what they noticed.

5.1 Intended Learning

The teacher's explicit learning intentions were that the students should make a connection between the graphs and their equations. The sorts of 'connections' it seems the teacher wanted the students to make were: to notice that the y-intercept is the same as the c value in the equation, to notice that there is a relationship between the m value and the gradient, perhaps that a bigger m yields a steeper graph or that graphs sloping one way have a positive m and those sloping the other way have a negative m value. In other words, she wanted them to move between the pragmatic/empirical and the mathematical/systematic fields.

The questions in the task had three parts: drawing the graphs and noticing (which includes finding the language needed to articulate features noticed) then reporting what had been noticed. A priori, drawing the graphs can be seen as a dialectic of action, with some limited feedback from the *milieu* as discussed below. Noticing might involve dialectics of formulation, with students making suggestions and perhaps hypothesising. However, there is nothing in the task that requires the students to engage in dialectics of validation.

To complete the task, the students needed first to construct a set of graphs using the software. In this task, creating the graphs can be seen as easy; the students had been given the equations of the graphs they were required to draw and all they needed to do was to type them in and press the **Enter** key. The graph is automatically drawn on the screen by the software. To create more graphs on the same page, the students only had to repeat the process.

The students had been told that they would be working on straight line graphs; it is likely that their expectations were that the on-screen graphs drawn would be straight lines. It is unlikely that the lines they produced would **not** be straight lines because to get a curved graph using this software the user has to use (for example) the 'squared' or 'cubed' key, which is a deliberate choice (so it is unlikely to happen by mistake), so unless they made a typing or copying error, the graphs will be as intended by the teacher. The task did not call for any discussion of the difference and similarities between the *equations* of the graphs, and nor did it call for any predictions about what the graphs might look like.

Next the students needed to write down 'what they noticed' about the set of graphs. Writing down 'what you notice' can be seen to have two distinct parts (which may take place simultaneously); the first is the noticing and the second is the writing down.

The question on the worksheet was 'Write down what you notice about the set of graphs'. The suggestion is that the way the question was phrased suggests a focus on the **set** of graphs, which may encourage working in the visual field alone because by looking students are able to notice many similarities and differences between the lines. The stated objective, however, was concerned with connecting **individual** graphs to their equations which would require transitions between the visual and theoretical fields. Having sets of graphs on the screen could therefore be seen to be a didactical obstacle, which may confuse the students and inhibit their learning. The fact that the students did not plot points to obtain the graphs could also be seen as a didactical obstacle because it means that they did not need to think about the equation, apart from typing it in.

5.1.1 Set 1

The first set of functions, taking the form $y = x + c$, were: $y = x$, $y = x + 1$, $y = x + 4$, $y = x - 2$ and $y = x - 3$. They were produced on the screen as shown in Fig. 2. Each equation below the graph page is a different colour and the lines drawn on the graph page are each the same colour as the corresponding equation.

When the graphs are produced using the software, the students may notice that each of the five lines is a different colour, corresponding to the text colour in the box below. They may also notice other things: that the blue and black graphs are close together, that all the lines run from the bottom left towards the top right, that the blue and black lines are the same distance apart as the purple and green lines, and so on.

It is likely that the teacher wanted the students to notice that the graphs are parallel. Difficulties involved in noticing the parallel-ness can be seen as an epistemological obstacle, but it does not, in itself, address the learning intentions of the task unless students realise that the fact of being parallel means that *something* is the same about the graphs, and then to look at the equations and notice that the

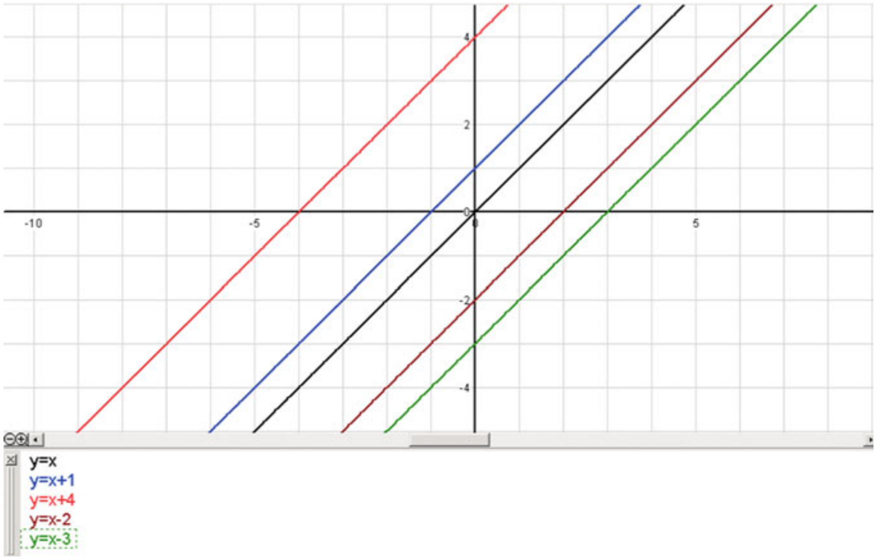

Fig. 2 Graphs for Question 1

m value is the same. However, here *m* is 1, so it is implied rather than being written down. It could be that the students notice that the graphs are parallel, but that they overcome the obstacles required to connect this parallel-ness to the (absent) *m* value is much less likely. Without the experience of this sort of work and with very little experience in working with straight line graphs, the suggestion is that these are ontogenic and didactical obstacles.

The teacher may also have hoped that the students would notice some connection between the intercepts and the value of *c*, perhaps that the *y*-intercept is equal to the value of *c* or that the *x*-intercept is the same as -*c*. This, however, assumes the prior knowledge and understanding that intercepts may be important, but it may be that the students did not have this prior knowledge.

It also requires a different way of looking at the graphs; instead of looking at what is the same about all the graphs, the students would need to focus on an individual line, read off the value of the intercept, note the colour of the line, find the equation in the same colour and notice that the *c* value in that equation is the same as the intercept. They would then need to hypothesise that the intercept is the same as the *c* value, and go and check that this is so for the other graph-equation pairs. Although this process may seem complex, it is, I suggest, not beyond the ability of the students, but I also suggest that it is more unlikely than noticing that the graphs are parallel.

5.1.2 Set 2

The second set of equations are of the form $y = mx$, where m is positive, and are shown in Fig. 3. Where in the previous question the m value is kept the same, here the m value varies but the c value is kept the same. The assumption is that the teacher thought that, by keeping one of the two variables in the equation constant, the students' attention would shift to the other.

In each of these functions, the c value in the equation $y = mx + c$ is zero, and therefore not included in the equation. Even though the students may notice that the lines all go through the origin, to link the value of this point to the absent c value would probably be more difficult and unlikely; this presents a didactical obstacle to their learning.

A second observation the students may make is that the lines are in the first and third quadrants (sloping from bottom left to top right). The teacher may have hoped that they would link this positive gradient to the positive m value in the equations but the suggestion is that, particularly with no negative m values with which to make a comparison, it is unlikely that the students would make the link.

The third observation the students may make is that the steeper the graph is, the higher the value of m is. This last observation is presumably what the teacher wanted them to notice, but it would require the students to have a language for steepness and to compare individual graphs by first noticing that one is steeper than another, then identifying which equation goes with which graph and then observing that the m value in the steeper graph is greater than the m value in the less steep graph. The suggestion is that, with all the graphs already on the screen, the students

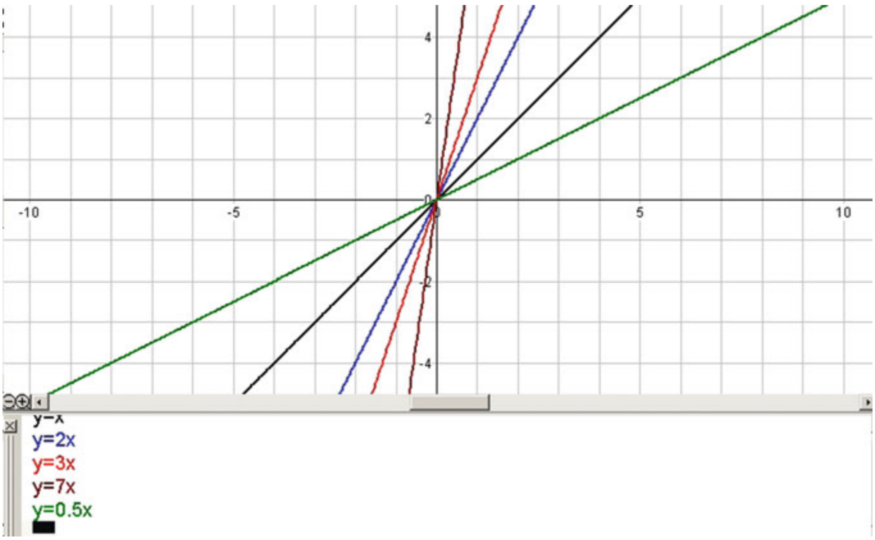

Fig. 3 Graphs for Question 2

would be unlikely to pick out individual lines, compare them and then find the equations. Further, with the question requiring them to notice things about the **set** of graphs, it is probably more likely that they would focus on the whole set, looking for what is the same about the lines rather than picking out individual lines.

Three further sets of graphs were given; again the students were required to draw them and then 'notice'. Overall, the intended learning for the task can be seen as learning to make the transition between the visual and theoretical fields (here between the graphical field and the algebraic) and hence to work out the connections between the graphical and algebraic representations of linear functions. However, there was no mechanism built into the task to encourage this way of working (such as making and a testing a prediction or conjecture, dialectics of formulation) and, further, having a whole set of graphs on the screen may have inhibited these transitions, as suggested above.

5.2 Discussion of the Task

This chapter is about the design of tasks, so the ways in which the students responded to the task are not discussed here. Suffice it to say that they responded very much in the way that is predicted by the analysis above. The interventions of the teacher are perhaps more relevant as she provided some further explanations and questioning during the lesson, thus in some way tweaking the design of the task. Both the teacher and the students reported that, for them, it had been a successful lesson. Certainly the students were engaged throughout and appeared to be working, but as the analysis predicts, their mathematical learning was very limited.

The example given above is not necessarily a badly designed task; as the discussion emphasised, the ontogenic obstacles got in the way of the students' learning, because they had very little prior knowledge about straight line graphs and it seems they did not have the experience to know that they should make connections between the equations of the graphs and what they looked like on the screen. However, students with some experience of working with graphs of straight lines may have benefited more in terms of mathematical learning.

On the other hand, it could be argued that the task is also not *well* designed, both in terms of the dialectics in which the students are forced to engage and movement between the pragmatic/empirical field and the mathematical/systematic field. As pointed out above, there is no requirement for students to engage in dialectics of validation. Also as discussed, there is nothing in the task that requires the students to move between fields by making connections between a graph and its equation.

The software is (almost) necessary to the task. It could be argued that in this case, its power is that it can produce on-screen graphs very quickly, doing some of the mathematics that students otherwise would have done and allowing them the time to look for patterns and connections. This exploits its ability to do the mathematics, but there is little in the way the task was designed that exploits the potential of the feedback from the software. It is perhaps not difficult to think of better ways the computer's power could have been used. One approach, as implicitly suggested above, might be to require the students to predict the shape of the graph before it is drawn by the computer, also justifying their predictions to their partners. This would overcome some of the didactical obstacles discussed above such as the difficulties of focusing on a graph rather than the set of graphs. It would further have encouraged dialectics of formulation and, importantly, validation.

More recent popular software, such as *GeoGebra*, could be used for a similar task. Commonly modern software for graphing includes functionality such as sliders, which allow users to change values incrementally. If *GeoGebra* had been used by this teacher with this class, the teacher may have designed the task differently to enable exploration of what changes and what stays the same as 'm' and 'c' are changed. However, improving the design of the task is not the point.

The point is that using computers in everyday or ordinary classrooms is perhaps not yet well enough understood by teachers, as I found in my research. For example, other tasks observed within the study suffered from similar difficulties. In particular, they did not capitalise on the potential of the computer to enable students to develop and test conjectures. Instead, there was an emphasis on the computer performing mathematical processes, leaving little mathematics for the students to perform. On the other hand, however, the teachers and the students seemed to think that the lessons were successful, and so did I to some extent. Students were actively engaged with the tasks they had been assigned and appeared to 'learn' what the teachers intended, although in fact this mathematical learning was limited by both the tasks and the teachers' interventions. It seems that we all fell into the trap described by Schoenfeld (1988) in his 'disasters of well taught mathematics courses'; the appearance of a successful mathematics lesson may be deceptive and it is not enough for students to be engaged in activities which *look like mathematics* unless they are learning mathematics (this is the purpose of mathematics teaching). It was through the careful analysis of the likely and actual mathematical activity of the students that shortcomings were revealed, particularly in terms of the students' mathematical learning.

6 Conclusion

The theoretical background for task design outlined in this chapter is probably already clearly understood by teachers and task designers, although sometimes tacitly. Making the theory explicit, however, might have value in providing a sort-of tick list against which designers of tasks can evaluate their tasks. For example, does the task force students to engage in dialectics of formulation and validation? Does it require the students to move between the visual and abstract fields? Crucially, what mathematics will the students learn?

The focus of this book is the use of digital technologies in mathematics task design; this chapter argues that the introduction of computers adds significant complexity to the work of task design, illustrating the point with the use of an example taken from empirical research. The tick list might now include questions such as 'What mathematics will the student do and what mathematics will the computer do?' and 'What is the role of feedback from the computer, and in which ways might this affect the dialectics between the student and the *milieu*?' This suggests that teachers need to analyse the likely and actual student activity as they work through the task. This sort of analysis is complex, particularly when computers are introduced into the *milieu,* and teachers often do not have the time and resources to develop the tools to help them make these analyses. This, perhaps, is a key role for the teacher professional development required for the effective embedding of computers in mathematics classrooms.

The arguments developed in this chapter imply that this professional development should include a focus on the intended mathematical learning of the students, the route the students might take through the task (and of the computer's role in determining the route), of the computer's feedback and the students' possible responses to the feedback. Importantly, it needs to raise awareness of the potential and role of the computer, emphasising the need to be clear about what mathematics the computer is intended to do, what mathematics the students are intended to do. It needs to make explicit the need to encourage dialectics of all three aspects, action, formulation and validation, in both the pragmatic/empirical and mathematical/systematic fields. Finally it should address the sorts of epistemological obstacles that can be built into the task to promote the learning of students and discuss the potential didactical obstacles that can be prevented. This sort of professional development will help teachers design tasks to exploit the power of computer software and, crucially, to plan their own interventions accordingly.

References

Ainley, J., & Pratt, D. (2002). Purpose and utility in pedagogic task design. *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education, 2*, 17–24.

Balacheff, N. (1990). Towards a problematique for research on mathematics teaching. *Journal for Research in Mathematics Education, 21*(4), 258–272. Retrieved from http://links.jstor.org/sici?sici=0021-8251(199007)21:4<258:TAPFRO>2.0.CO;2-U.

Balacheff, N., & Kaput, J. (1996). Computer-based learning environments in mathematics. In A. J. Bishop (Ed.), *International handbook of mathematics education* (pp. 469–502). Dordrecht: Kluwer.

Balacheff, N., & Sutherland, R. (1994). Epistemological domain of validity of microworlds: the case of Logo and Cabri-geometre. In R. Lewis & P. Mendelsohn (Eds.), *Lessons from learning*. Amsterdam: Elsevier Science BV. (N. Holland), IFIP.

Becta. (2004). What the research says about the use of ICT in maths. Retrieved from http://publications.becta.org.uk/display.cfm?resID=25808&page=1835.

Bokhove, C., & Drijvers, P. (2010). Digital tools for Algebra education: Criteria and evaluation. *International Journal of Computers for Mathematical Learning, 15*(1), 45–62. doi:10.1007/s10758-010-9162-x.

Bokhove, C., & Drijvers, P. (2011). Effects of a digital intervention the development of algebraic expertise. *Computers & Education*, (3), 3770–3777. Retrieved from http://www.sciencedirect.com/science/article/pii/S0360131511001886.

Bokhove, C., & Drijvers, P. (2012). Effects of feedback in an online algebra intervention. *Technology, Knowledge and Learning, 17*(1–2), 43–59. doi:10.1007/s10758-012-9191-8.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: didactique des mathematiques, 1970–1990*. Dordrecht: Kluwer Academic Publishers.

Christiansen, B., & Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 243–307). Dordrecht: Kluwer.

Condie, R., & Munro, B. (2007). *The impact of ICT in schools—a landscape review*. Report: Becta.

Dörfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms; Perspectives on discourse, tools, and instructional design* (pp. 99–131). Mahwah, New Jersey: Lawrence Erlbaum.

Goldenberg, E. P. (1988). mathematics, metaphors, and human factors: mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *Journal of Mathematical Behavior, 7*(2), 135–173.

Granberg, C., & Olsson, J. (2015). ICT-supported problem solving and collaborative creative reasoning: Exploring linear functions using dynamic mathematics software. *Journal of Mathematical Behavior, 37*, 48–62. doi:10.1016/j.jmathb.2014.11.001.

Harel, G., & Sowder, L. (2005). Advanced mathematical—thinking at any age: its nature and its development. *Mathematical Thinking and Learning, 7*(1), 27–50.

Hillel, J. (1992). The computer as a problem-solving tool; it gets a job done, but is it always appropriate? In J. P. Ponte, J. F. Matos, J. M. Matos, & D. Fernandes (Eds.), *Mathematical problem solving and new information technologies*. New York: Springer.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. J. Bishop (Ed.), *Second international handbook of mathematics education*. Dordrecht: Kluwer.

Hoyles, C., Noss, R., Vahey, P., & Roschelle, J. (2013). Cornerstone mathematics: Designing digital technology for teacher adaptation and scaling. *ZDM—International Journal on Mathematics Education, 45*(7), 1057–1070. doi:10.1007/s11858-013-0540-4.

Joubert, M. (2007). *Classroom mathematical learning with computers: The mediational effects of the computer, the teacher and the task.* University of Bristol.

Laborde, C. (1998). *Relationship between the spatial and theoretical in geometry—the role of computer dynamic representations in problem solving* (pp. 183–195). London: Chapman and Hall.

Laborde, C. (2002). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317.

Laborde, C., Kilpatrick, J., Hoyles, C., & Skovsmose, O. (2004). *The hidden role of diagrams in students' construction of meaning in geometry* (pp. 1–21). Netherlands: Kluwer.

Laborde, C., & Sträßer, R. (2010). Place and use of new technology in the teaching of mathematics: ICMI activities in the past 25 years. *ZDM, 42*(1), 121–133. doi:10.1007/s11858-009-0219-z.

Love, E., Burton, L., & Jaworski, B. (1995). *Software for mathematics education*. Lund: Chartwell-Bratt.

Love, E., & Mason, J. (1992). *Teaching mathematics: Action and awareness*. Milton Keynes: Open University.

Margolinas, C., & Drijvers, P. (2015). Didactical engineering in France; an insider's and an outsider's view on its foundations, its practice and its impact. *ZDM—Mathematics Education, 47*(6), 893–903. doi:10.1007/s11858-015-0698-z.

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics, 44*(1), 25–53.

Martin, L., & Pirie, S. (2003). Making images and noticing properties: The role of graphing software in mathematical generalisation. *Mathematics Education Research Journal, 15*(2), 171–186. doi:10.1007/BF03217377.

Mason, J. (1989). Mathematical abstraction as the result of a delicate shift of attention. *For the Learning of Mathematics, 9*(2), 2–8.

Mavrikis, M., Noss, R., Hoyles, C., & Geraniou, E. (2013). Sowing the seeds of algebraic generalization: Designing epistemic affordances for an intelligent microworld. *Journal of Computer Assisted learning, 29*(1), 68–84. doi:10.1111/j.1365-2729.2011.00469.x.

McNeil, N., & Jarvin, L. (2007). When theories don't add up: Disentangling he manipulatives debate. *Theory into Practice, 46*(4), 309–316. doi:10.1080/00405840701593899.

Monaghan, J., & Trouche, L. (2016). Tasks and digital tools. In J. Monaghan, L. Trouche, & J. M. Borwein (Eds.), *Tools and mathematics* (pp. 391–415). Switzerland: Springer.

Nevile, L., Burton, L., & Jaworski, B. (1995). *Looking at, through, back at: Useful ways of viewing mathematical software*. Lund: Chartwell-Bratt.

Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics, 33*(2), 203–233.

Olivero, F. (2003). *The proving process within a dynamic geometry environment*. *Advanced mathematical thinking* (Vol. PhD). PhD thesis. Graduate School of Education: University of Bristol.

Orton, A., & Frobisher, L. J. (1996). *Insights into teaching mathematics*. London: Cassell.

Pimm, D. (1995). *Symbols and meanings in school mathematics*. London: Routledge.

Resnick, L. B. (1984). *A developmental theory of number understanding*. Pittsburg: Learning Research and Development Center, University of Pittsburg.

Romberg, T., & Kaput, J. (1999). Mathematics worth teaching, mathematics worth understanding. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 3–19). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Roschelle, J. (1997). *Learning in interactive environments prior knowledge and new experience*. Exploratorium Institute for Inquiry: University of Massachusetts, Dartmouth.

Ruthven, K., Hennessy, S., & Brindley, S. (2004). Teacher representations of the successful use of computer-based tools and resources in secondary-school English, Mathematics and Science. *Teaching and Teacher Education, 20*(3), 259–275.

Säljö, R. (1999). Learning as the use of tools: A sociocultural perspective on the human-technology link. In *Learning with Computers: Analysing Productive Interaction Table of Contents*, pp. 144–161.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational Psychologist, 23*(2), 145–166.

Schwarz, B. B., & Hershkowitz, R. (2001). Production and transformation of computer artifacts toward construction of meaning in Mathematics. *Mind Culture and Activity, 8*(3), 250–267.

Sfard, A. (2000). Symbolizing mathematical reality into being—or how mathematical discourse and mathematical objects create each other. In *Symbolizing and communicating: Perspectives on Mathematical discourse, tools, and instructional design*, pp. 37–98.

Sierpinska, A. (2000). *The "theory of didactic situations": Lecture notes for a graduate course with samples of students' work. master in the teaching of mathematics*. Concordia University. Retrieved 2006 from http://alcor.concordia.ca/~sierp/TDSLecture%202.pdf.

Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the Learning of Mathematics, 24*(2), 7–15.

Sutherland, R. (2007). *Teaching for learning mathematics*. Maidenhead: Open University Press.

Swan, M. (2006). *Collaborative learning in mathematics: A challenge to our beliefs and practices*. In S. M. Wilson & J. Berne (Vol. 24). London: NRDC.
Vygotsky, L. S. (1980). *Mind in society*. Cambridge, Mass: Harvard University Press.
Watson, A., et al. (2013). Task design in mathematics education proceedings of ICMI study 22. In C. Margolinas (Ed.), T*ask Design in Mathematics Education Proceedings of ICMI Study 22*. Oxford.