Supporting Variation in Task Design Through the Use of Technology

Christian Bokhove

Abstract This chapter describes a digital intervention for algebraic expertise that was built on three principles, crises, feedback and fading, as described by Bokhove and Drijvers (*Technology, Knowledge and Learning*. *7*(1–2), 43–59, 2012b). The principles are retrospectively scrutinized through Marton's Theory of Variation, concluding that the principles share several elements with the patterns of variation: contrast, generalisation, separation and fusion. The integration of these principles in a digital intervention suggests that technology has affordances and might be beneficial for task design with variation. The affordances in the presented technology comprise (i) authoring features, which enable teacher-authors to design their own contrasting task sequences, (ii) randomisation, which automates the creation of a vast amount of tasks with similar patterns and generalisations, (iii) feedback, which aids students in improving students' learning outcomes, and (iv) visualisations, which allow fusion through presenting multiple representations. The principles are demonstrated by discussing a sequence of tasks involving quadratic formulas. Advantages and limitations are discussed.

Keywords Task · Design · Sequence · Crisis · Feedback · Fading · Variation

1 Introduction

In recent years task design in mathematics education has become more and more important, culminating in specific conferences and a separate ICMI study devoted to this topic. One challenge in task design is that tasks are often only described vaguely. Furthermore, Schoenfeld (2009) advises on having more communication between designers and researchers. In this way educational research and design can be bridged, as the communities involving task design are naturally overlapping and diverse. One particular focus concerns the observation that tasks are not single

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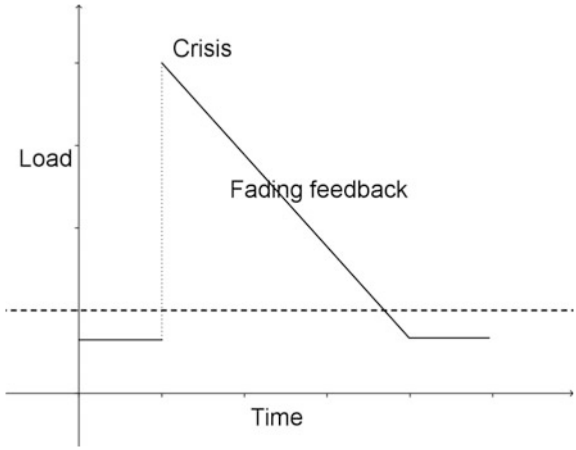
events, but are often embedded in a sequence of tasks. It is suggested that the design of sequences of near-similar tasks deserves specific attention. In such sequences it is possible to 'vary' specific parts of tasks over the course of a sequence. One example might be task sequences in which the problem formulation remains constant but the numbers used increase the complexity of the task. This approach has been used previously in an earlier article (Bokhove and Drijvers 2012b), whereby the complexity of tasks first increases, and then—with the help of feedback—decreases. In one sense this can be seen as an adaptation of the 'variation' "Watson and Mason (2006) used when coining the term 'micromodelling' to describe 'learners' response to exercises in which dimensions of variation have been carefully controlled, because the aim is to promote generalization of the dimensions being varied in the exercise, and thence to focus on mathematical relationships between dimensions." (p. 104). Furthermore, certainly in using the term 'variation', it draws similarities with Marton's suggestion of 'Variation Theory'. This theory, promoted by Marton and colleagues (Marton and Booth 1997; Marton and Trigwell 2000; Marton and Tsui 2004; Marton and Pang 2006) and extended by Watson and Mason (2002, 2005) proposes that learners must experience variation in the critical aspects of a concept, within limited space and time, in order for the concept to be learnable. The aim of this chapter is to first describe a digital intervention for acquiring, practicing and assessing algebraic expertise (Bokhove 2011), then go into Marton's concept of Variation, and then demonstrate how these principles tie into the idea of 'variation'. It demonstrates how patterns of variation can be used to frame task design to further the discovery of mathematical knowledge. I will specifically emphasize the role and affordances of technology in operationalizing the idea of variation in this specific algebra intervention, supplementing literature that used the lens of variation for other digital environments (e.g. dynamic geometry environments, Leung 2008; Leung et al. 2013). I will conclude with thoughts on what added value, theoretically and didactically, such an approach might have.

2 Digital Intervention for Algebraic Expertise

The starting point for this chapter is an online intervention which was designed at Utrecht University (Bokhove 2011). The intervention was part of a study called 'Algebra met Inzicht' [Algebra with Insight] and was made in the Digital Mathematical Environment (DME http://www.fi.uu.nl/dwo/en).[1] The DME is a digital learning and assessment environment for mathematics in secondary and higher education, in which interactive teaching methods and feedback play a central role. Within the DME, students can work at any time on modules that have been

[1]An English translation of part of the module can be found at http://www.fi.uu.nl/dwo/soton. Log in as guest, and choose 'Demo for 22nd ICMI study'. Java is needed.

Fig. 1 Proposed model for crises, feedback and fading



selected for them and receive feedback on their answers. Teachers can view the students' work and adapt modules and activities to meet the class' needs. The DME intervention in the current study consists of a paper-and-pencil pre-test, four digital modules, a digital diagnostic test, and a final digital test and, finally, a paper-and-pencil post-test, and aims to address algebraic expertise. It was deployed in fifteen 12th grade classes from nine Dutch secondary schools (N = 324), involving eleven mathematics teachers. The schools were spread across the country and showed a variation in school size and pedagogical and religious backgrounds. The participating classes consisted of pre-university level 'wiskunde B' students (comparable to grade 12 in Anglo-Saxon countries). As this chapter focuses on the design and sequencing of the tasks, I refer to different articles for more details of the set-up of the study and the actual effects of the digital intervention (Bokhove and Drijvers 2012a, b). The three main design principles behind the design of (sequences of) algebra tasks are crises, feedback and fading. The cohesive argument behind the three principles is depicted in Fig. 1.

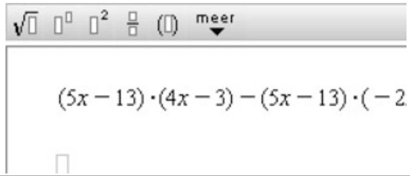
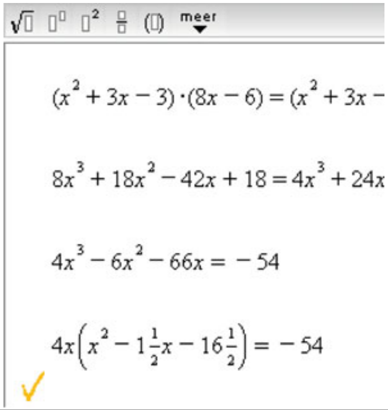
I propose that near-similar tasks and repetitive exercises are interspersed by *intentional crises* i.e. tasks that are hard or impossible to solve with skills and knowledge that are available. In other words, the 'load' of the task is too high. I will not go into the word 'load' in detail. There is a vast body of knowledge connected to the term Cognitive Load Theory (Sweller 1988). There also is, rightly so, criticism (De Jong 2010). For the purpose of this model, we will only assume that knowledge that isn't known (novice) potentially will bear a larger load than known knowledge (expert). The intentional crises might be overcome by *providing feedback*. To avoid a dependency on feedback for the summative assessments *feedback is faded* during the course of the sequence of tasks. The three elements will first be explained more extensively.

3 Describing the Three Principles for the Digital Intervention

I will first elaborate on the three elements of the model: crises, feedback and fading. The first element, *crises*, builds on the idea which poet John Keats[2] so eloquently described in the early 19th century as 'failure is the highway to success.' In subsequent centuries this idea that what goes wrong contributes to better learning has been used by several scholars. The principle, for example, seems to underpin Piaget's (1964) concepts of *equilibrium* and *disequilibrium*. These essentially say that, whenever the child's experience/interaction with the environment yielded results that confirmed her mental model, he or she could easily assimilate the experience. When the experience resulted in something new and unexpected, the result was disequilibrium, with a child in some cases experiencing confusion or frustration. Eventually, the child changes his or her cognitive structures to accommodate the new experience and moves back into equilibrium. Tall (1977) refers to *cognitive conflicts*: "one of the distinguishing factors in catastrophe theory is the existence of discontinuities, or sudden jumps in behaviour when certain paths are taken." (p. 6). In his 'levels of thinking' Van Hiele (1985) discerns structure and insight. According to Van Hiele, there can be a '*crisis of thinking*', which has a link to the Vygotskian zone of proximal development. The common ground between the two is that there is a need for challenge. Recently, drawing on cognitive psychology, Kapur (2010) has used the term *productive failure* and cites Clifford (1984): "However, allowing for the concomitant possibility that under certain conditions letting learners persist, struggle, and even fail at tasks that are complex and beyond their skills and abilities may in fact be a productive exercise in failure requiring a paradigm shift". The difference with my own work (Bokhove and Drijvers 2012b) seems to be whether crises are an inherent part of learning when solving open problems, or actually embedding tasks that could *intentionally* cause a crisis, would be a good thing. It is proposed that intentional crisis tasks are added to sequences of near-similar tasks, for example in the way depicted in Table 1, which illustrates the way in which crisis items are integrated within the current study's intervention. The general structure of such a sequence would then be: first appropriate pre-crisis items, then the item that intends to intentionally cause a crisis (for some students), and then some post-crisis items. The subsequent question then becomes how students can address this crisis. It is suggested that this is done through the second design principle: feedback. Feedback is an integral part of assessment *for* learning, so-called *formative* assessment. Black and Wiliam (1998) define assessment as being 'formative' only when feedback from learning activities is actually used to modify teaching to meet the learner's needs. From this it is clear that feedback plays a pivotal role in the process of formative assessment. Hattie and Timperley (2007) conducted a meta review of the effectiveness of different types of feedback. The

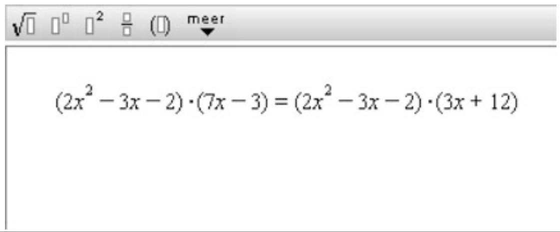
[2]It is attributed to Keats but he probably used a different wording.

Table 1 Sequence of items illustrating crises and feedback (Bokhove 2014)

1	Tasks: "Solve the following equation:"	**Pre-crisis items** In the first few items students are confronted with equations they have experience with. Students may choose their own strategy. Many students choose to expand brackets as that is the strategy that they have used often: work towards the form $ax^2 + bx + c = 0$ and use the Quadratic Formula. There is some limited feedback on the task
2	$(4x - 3) \cdot (4x - 1) = (4x - 3) \cdot 2$	
3	$(x - 4) \cdot (3x - 5) = (x - 4) \cdot (-2x + 1)$	
4	$(x - 4) \cdot (2x - 5) = (x - 4) \cdot (-3x + 3)$	
5	**Opgave 1.5** Los de volgende vergelijking op:	
6		**Crisis item** Some students will be confronted with a crisis if they try to use their conventional strategy of expanding the expression. The yellow tick at the bottom of the screen denotes that the equation is algebraically equivalent to the initial one, but that it is not the final answer. This is accompanied by a partial score for an item and some feedback on correctly rewriting the expression. Although these students showed good rewriting skills, in the end they are not able to continue, as they do not master the skill to solve a third order equation. There is some limited feedback on the task

(continued)

Table 1 (continued)

7		
8	$(x^2 - 3x - 2) \cdot (6x - 3) = (x^2 - 3x - 2) \cdot (4x + 12)$	**Post-crisis items** After the crisis item students are offered help by providing a feedback, an instructional screencast, and buttons to get hints ('tip'), the next step in the solution ('stap') or a worked solution ('losop'). These features have in common that they provide feedforward information at the task level and self-regulation (Hattie and Timperley 2007)
9	$\sqrt{3x+3} \cdot (2x+4) = \sqrt{3x+3} \cdot (6x-5)$	
10	$(4x+4) \cdot \sqrt{-2+3x} = \sqrt{3x-2} \cdot (7x-5)$	
11	$(-5 + {}^2\log(x-2)) \cdot (6x-6) = (-5 + {}^2\log(x-2)) \cdot (3x+14)$	
12	$(4x - 13) \cdot (3x - 3) = (4x - 13) \cdot (-3x + 2)$	
13	$(-4x + 5) \cdot (8x - 5) = (-4x + 6) \cdot (3x + 14)$	

feedback effects of hints and corrective feedback are deemed best. A meta review of feedback in computer-based learning environments suggested that elaborated feedback, providing an explanation, had large effect sizes for mathematics (Van der Kleij et al. 2015). However, one challenge while providing feedback is that one must make sure students do not overly rely on this feedback, as eventually students often will need to pass a summative test on their own. Assuming that students finally have to pass an exam themselves, it makes sense to address this over-reliance on feedback. In a follow-up paper on his productive failure Kapur (2011) notes that scaffolding implies help to overcome failure (Pea 2004). As a design principle it is therefore proposed that initially a lot of feedback is provided to foster learning, but the amount is decreased towards the end, to facilitate transfer. Using scaffolding this way is based on the concept of *fading* (Renkl et al. 2004). Formative scenarios (Bokhove 2008) are a variation of this concept, starting off with a lot of feedback, and providing a gradually decreasing level of feedback.

Figure 2 shows how this principle was implemented in the intervention. At the start feedback is provided for all intermediate steps of a solution. The subsequent part of the intervention concerns self-assessment and diagnostics: the student performs the steps without any feedback and chooses when to check his or her solution by clicking a “check” button. Feedback is then given for the whole of the exercise.

Finally, students get a final exam with no means to see how they performed, no feedback is given. Just as is the case with a paper test, the teacher will be able to check and grade the exam (in this case automatically) and give students feedback on their performance. A student needs to be able to accomplish tasks independently, without the help of a computer. An implicit advantage of implementing feedback in a sequence of tasks is that teachers and designers have to think upfront about possible student responses (Bokhove 2010). Together the three principles propose

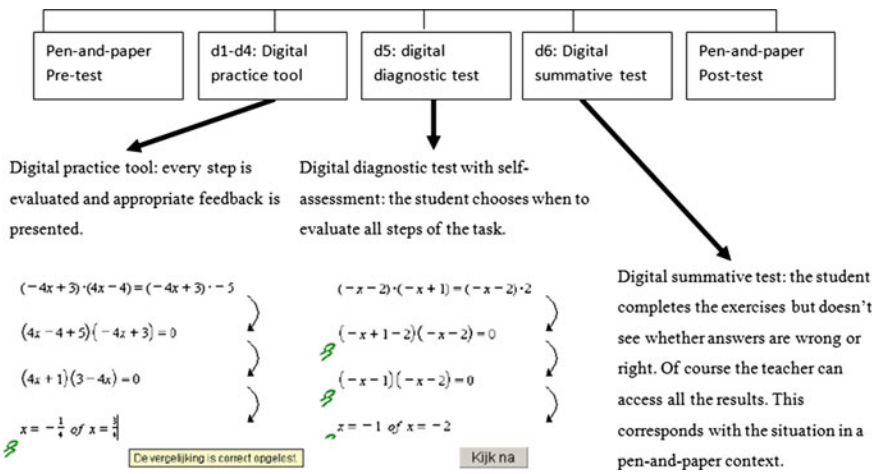


Fig. 2 Outline of fading feedback in formative scenarios (Bokhove and Drijvers 2012b)

embedding *variation* in a sequence of tasks. To demonstrate how the principles might relate to Variation Theory it is first necessary to unpick this term.

4 Unpacking Variation Theories

The idea that invariant structures during changing phenomena often denote the presence knowledge acquisition is an essential part of phenomenology. This is expressed in, among others, the Theory of Variation (e.g. Marton and Booth 1997; Gu et al. 2004). By using variation certain constraints, and associated freedom, give rise to the 'dimensions of possible variation' and 'ranges of permissible change' which are usually at the heart of task design (Mason and Johnston-Wilder 2006). One way of integrating different tasks into pedagogic situations is to make use of 'learner generated examples' and the shared 'example space' (Watson and Mason 2005) or the 'outcome space' (Marton and Booth 1997).

Fan and colleagues (Fan et al. 2004) showed that the "Two Basics" (Basic Knowledge and Basic Skills) in the Mathematics Curriculum of Mainland China can develop into meaningful learning. (Gu 1981) systematically analysed and synthesized the concepts of teaching with variation. He identified and illustrated the two forms of variations: '*conceptual variation*' and '*procedural variation*'. Conceptual variation has as starting point that concepts can be understood from multiple perspectives. Variation is created in several ways. The first way, standard concept variation, is by varying the concept in a standard way via inducing concepts by varying visual and concrete instances. The main purpose of using this variation is to help students establish the connection between concrete experience and abstract concepts. The second way, non-standard concept variation, highlights the essence of a concept by contrasting the concept with a non-standard example. This stresses the teaching strategy that examples should not only be the 'normal' ones, but also the non-standard ones. Finally, the third way of non-concept variation uses non-concepts, for example counterexamples, to reinforce a concept. Procedural variation concerns progressively unfolding mathematics activities. In procedural variation students can arrive at solutions to a problem and form connections among different concepts step by step from multiple approaches. This type of variation is also created in several ways. The first way addresses the formation of concepts and the process of unfolding concepts. The second way uses scaffolding for problem solving. Multiple variations (analyses) of the configuration of a problem do not only help students clarify the *process* of solving the problem and the *structure* of the problem, but also are an effective way of experiencing problem solving and enhancing the competency of problem solving (Gu 1994, as cited in Gu et al. 2004). The third way establishes a system of mathematics experience. Variations make up a system of (hierarchical) experiences and strategies that are internalized into the cognitive structure. The two forms of variation are closely linked to each other, forming a hierarchical system of experiencing process through forming concepts or solving stages of problems. It is this mechanism, where conceptual variation is static and procedural variation is dynamic, that connects processes with previous and new knowledge. Chinese scholars have called this distance between previous and new knowledge 'potential distance' (Gu 1994, as cited in Gu et al. 2004). Effective teachers can judge this distance perfectly, balancing a short distance where new knowledge is acquired, and a long distance which is useful for developing students' exploring competency.

In another vein, the four patterns of variation articulated by Marton in his Theory of Variation (Marton et al. 2004) might provide the best framework for my purpose:

1. Contrast. "… In order to experience something, a person must experience something else to compare it with…"
2. Generalisation. "… In order to fully understand what "three" is, we must also experience varying appearances of three…"
3. Separation. "… In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant."
4. Fusion. "…if there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously" (p. 16).

These patterns of variation can bring about discernment and awareness. The focal point of variation is the procedure or form in which problems are proposed. It is *carefully designed* such that only their *non-fundamental elements of knowledge and skills* are changed in a variety of ways. By comparing and differentiating, students struggle to identify invariant properties. In the next section I will argue how these four functions are apparent in the task design for the intervention and what role technology plays in facilitating the four functions.

5 Linking the Patterns of Variation to the Intervention

To demonstrate how the four patterns of variation can be used to design tasks, I take one of the elements of the intervention described previously, namely the sequence with quadratic equations in Table 1. Obviously, this concerns only one mathematical topic, and then only one sub-topic within that topic, but the use of variation can potentially be used for any topic. What variables can we discern? The first variable is the appearance of the equation: basically there are quadratic Eqs. (2, 3, 4, 5, 12 and 13), third order Eqs. (6, 7 and 8) and equations with square roots and logarithms (9, 10 and 11). Almost all the equations have a variation of the pattern AB = AC (2 to 12), only 13 does not have this pattern and tries to confront any new-found assumptions of equations in the sequence always having the pattern AB = AC. In my view it possible to describe this sequence of tasks in terms of functions of variation.

When it comes to *contrast*, the third order equations of 6, 7 and 8 were specifically added to provide a non-standard variation. Although they conform to the AB = AC pattern, one of the correct strategies that might have been used in 2 to 5, expanding the brackets, will not be efficient for third order equations.

$$(x-4)\cdot(2x-5)=(x-4)\cdot(-3x+3) \tag{4}$$

$$\left(x^2+3x-3\right)\cdot(8x-6)=\left(x^2+3x-3\right)\cdot(4x+12) \tag{6}$$

A second contrast might be provided by 13 as this equation, although quadratic and visually similar, does not conform to the AB = AC pattern. The amount of contrast in the sequence could further be expanded by including equations with no solutions. Variation is also provided for *generalisation* by presenting the pattern AB = AC in numerous ways, with square roots, with logarithms or even just with all terms at one side of the equality sign.

$$\sqrt{3x+3}\cdot(2x+4)=\sqrt{3x+3}\cdot(6x-5) \tag{9}$$

$$\left(-5+{}^{2}\log(x-2)\right)\cdot(6x-6)=\left(-5+{}^{2}\log(x-2)\right)\cdot(3x+14) \tag{11}$$

A further expansion, not in this intervention, could be implemented by not solely presenting factored equations but also the expanded expressions presented in the format $ax^2+bx+c=0$. *Separation*, vary an aspect while other aspects remain invariant, is apparent in these examples as well. For example, it is important to note that the coefficients in the items are randomized, something which shall be presented as one of the affordances of technology. Another example was already demonstrated through generalisation: the pattern AB = AC is invariant while the appearance is varied. Potentially, more elements can be varied or kept invariant, which I will demonstrate when I present some affordances of technology in this regard. Finally, *fusion* is obtained by providing multiple representations e.g. through a graphical representation of the equation. Presenting graphs next to equations further emphasizes a crucial aspect of 'solving an equation', namely that it generates coordinates for intersection points of two graphs. This might contribute to the understanding that indeed, for example, Eqs. (4) and (6) have a different number of intersection points and therefore that the Quadratic Formula might not be appropriate. This aspect was under-utilized in this specific study. In my opinion this shows that a sequence of tasks can be designed in such a way that all functions of variation are met. It will often be the case that the four functions are inter-twined rather than distinct functions. In the next section I will describe how technology managed to support these functions.

6 Affordances of Technology for Variation

Technology used in this study supported variation in several ways: through authoring, randomisation, feedback and visualizations.

6.1 *Authoring*

A prerequisite for any task design is that it's possible to *author* one's own tasks. With pen-and-paper it is evident that this is possible. Digital systems, however, often come with their own non-customizable content (e.g. materials from publishers) or have limited authoring capabilities. If the materials provided already contain aspects of variation then this could serve as appropriate content, but ideally technology would cater for authoring. The DME provides an authoring environment, as depicted in Fig. 3.

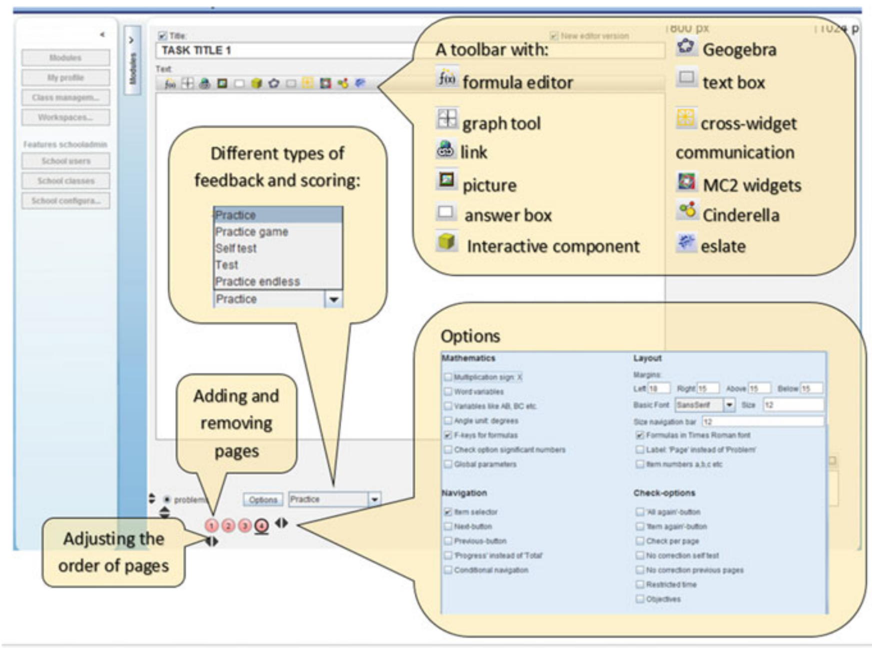


Fig. 3 Overview of the editing window (adapted from: Abels et al. 2013). This is an extended version of the original DME authoring interface, which was part of the MC-squared project, a FP7 EU project which aimed to author creative digital books for mathematics. See http://www.mc2-project.eu/

The main editor allows authors to add and remove pages to a digital book, adjust their order, add different types of feedback and scoring, and—most importantly — add a variety of elements to the pages of the book, ranging from basic static texts to complex, interactive widgets. Potentially, such a feature might allow the authoring of sequences of tasks with variation. In the case of the current intervention these authoring facilities were used to author randomisation, specific *crisis* items and integrate representations.

6.2 Randomisation

A specific example of authoring can be done by using the randomization features in the DME. By 'varying' coefficients within equations it is possible to address the 'separation' function of variation as we 'vary an aspect while other aspects remain invariant'. We can take the quadratic example from 4 to demonstrate this.

$$(x-4)\cdot(2x-5)=(x-4)\cdot(-3x+3) \qquad (4)$$

One starting point could be that this is a specific case of a general equation, which can be authored in the DME, as depicted in Fig. 4.

As working with parameters complicates solving equations the 'possible answer' button allows authors to solve the equation with parameters in place. The solution can be copied to an 'answer model'. The authoring environment also has a box 'Variables for random parameters' in which the values for the parameters can be defined. For example, to get the equation from 4 one would enter b = 1, c = −4, d = 2, f = −5, g = −3 and h = 3, and these values would be substituted into the respective parameters in Fig. 4. Potentially these parameters can be any random whole integer. One interesting observation to make is that, as expected with a carefully designed variational sequence, thought has to go into this randomisation. Often full randomisation is not desirable. In this specific example it is obvious that variables with a zero value might influence the original pedagogical intentions of the equation. It might even mean equations suddenly do not yield any solutions.

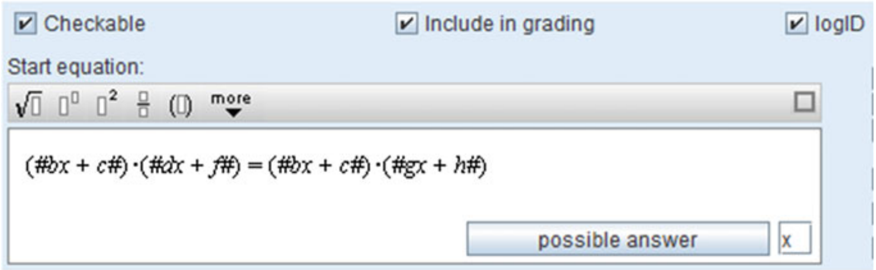


Fig. 4 Using parameters to get variation

With the quadratic equation $ax^2 + bx + c = 0$ one initially would want there to be solutions, even well rounded solutions, for example to practise factoring. Just as with designing sequences of variational tasks one wants to 'control' the values of parameters, but also benefit from the technological affordance of randomisation. This might be typified as *semi-randomisation* for variation: parameters are random but the author/designer carefully chooses the range of values parameters may take. In the final version of the intervention variables were defined as b = 1…5; $c = -3, -4, -5, -6, -7$; $d = 1\ldots4$; $f = -5, -3, -1$; $g = -3, -2, -1$ and $h = 1\ldots4$ whereby 1…5 denotes all whole numbers from 1 to 5. This means there are potentially 5 × 5 × 4 × 3 × 3 × 4 = 3600 different equations a student can get, but they all adhere to the AB = AC pattern because they have been authored that way. If one would choose to expand the terms in the equation with the parameters in place, this would be a useful way to change the appearance of the equation but still ensure there are suitable (and nicely rounded) solutions. In principle we could even start with a third order equation and in the early cases simply choose our parameters in such a way that only lower order terms are generated. Likewise, we could argue that linear terms are nothing more than quadratic terms $ax^2 + bx + c$ formatted with $a = 0$. Anecdotally, in projects in which these features were used, some designers really enjoyed the process of determining appropriate parameters. The feature of randomisation allows a 'scaffolding' of variation by using a similar 'generalized' template for all the tasks, and varying the parameters. Randomisation is not only restricted to algebraic topics parameters in the DME can also be used for other domains, for example generating random coordinates in geometry tasks.

6.3 Feedback

In static paper-and-pencil cases a carefully designed sequence with variation hinges on assumptions about student responses to the tasks. Ideally, sequences are designed in such a way that (most) students can make them. The previously mentioned 'potential distance' should not be too large i.e. students should be enabled to overcome any problems. This, of course, is even more important when an *intentional* crisis is implemented in the sequence of tasks. Normally, in a classroom setting, I would assume this would be part of teacher feedback. If I go with the desire to implement crisis items for variation, and would want to address these by providing feedback, one affordance of technology might be the provision of automatic feedback to overcome such crises (Van der Kleij et al. 2015). I specifically see the design of feedback as an important part of the task design as well (also see Bokhove 2010), allowing not only to design for variation but also the feedback that might scaffold the sequence of tasks. The DME allows authors to design such feedback in several ways. The first feature is the 'built in' feedback

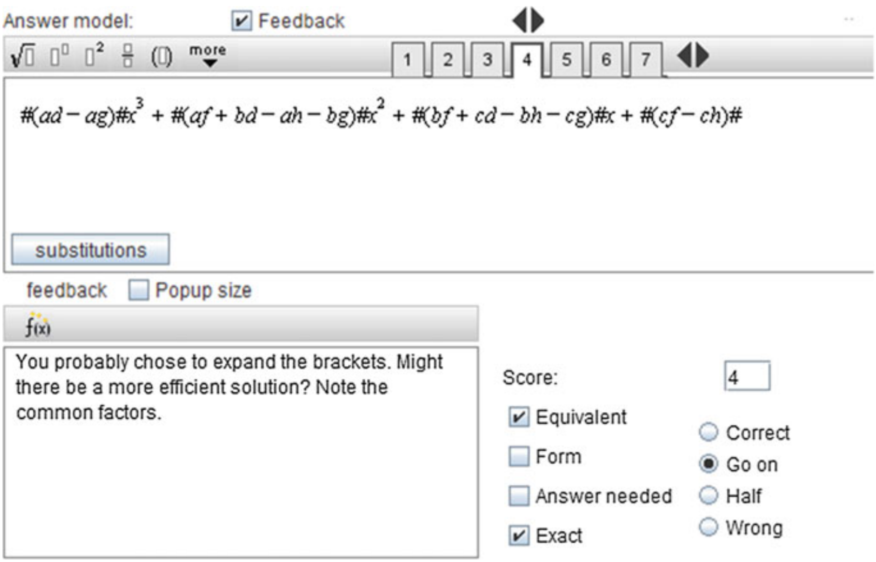


Fig. 5 Feedback might be customized

features which indicate whether a next step in a solution to an equation is (partially) correct. This can be accompanied by a score as well. A second concerns the feature of being able to author custom feedback.

In the context of variation in the previous examples, feedback could be provided to help students notice the AB = AC pattern. Figure 5 shows how for Eq. (7) from Table 1, feedback might be customized to indicate the common factors. There also is a rule-based feedback provision with buttons to get hints ('tip'), the next step in the solution ('stap') or a worked solution ('losop'). Feedback can also be moderated at a more general level by choosing several 'modes': a practise mode gives full feedback, a self-test mode asks students to evaluate their answer when they're ready and an exam mode 'mimics' a summative test setting. This feature is used for the previously mentioned fading of feedback.

6.4 Visualisations

To discern several critical aspects in one go, fusion, the task designer can incorporate several representations in the sequence of tasks. Figure 6 shows how two representations can be authored in the DME. This feature becomes particularly powerful if these representations can be provided for several variational tasks.

This feature can also be 'linked' in that graphs, equations and other representations like a balancing scale can interact: change one representation and the other representation(s) change(s) as well, as demonstrated in Fig. 7.

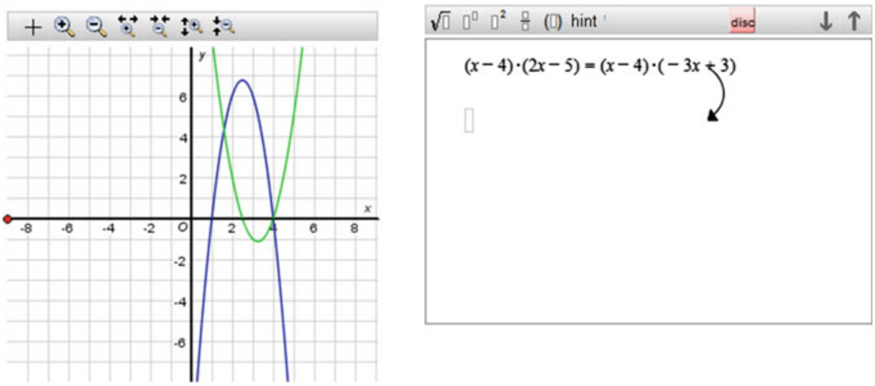


Fig. 6 Multiple representations for one equation

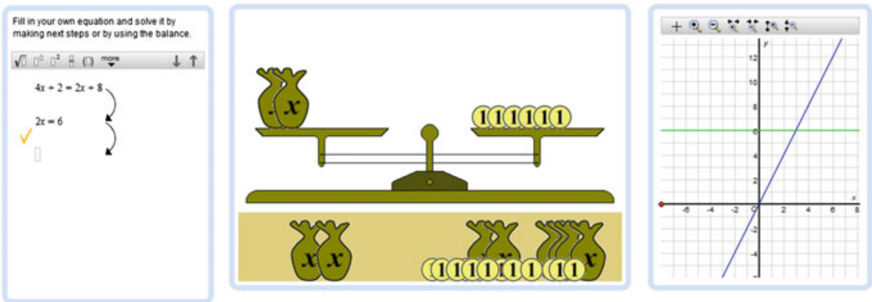


Fig. 7 Three linked representations: equation, scales and graphs

Teachers or students can create an equation (left side of the figure). The equation that is created can be represented as a pair of scales (middle part of the figure). A third representation is provided by the graph. Several critical aspects are experienced together: the algebraic notation of the equation, the fact that ‘solving an equation’ corresponds to using a model of ‘balancing scales’ and also that it represents the intersection of two—in this case linear—graphs.

7 A Student at Work

Let’s look at one student named Pauline while utilizing the environment. In the first task the student has to get acquainted with the digital environment. The pre-crisis items pose no problem for most students, including Pauline. On arriving at the crisis item 1.6 students exhibit three behaviours, roughly corresponding with the ones already observed in the pre-test: (i) students solve the equation correctly, (ii) students recognize the pattern AB = AC of the equation but subsequently make

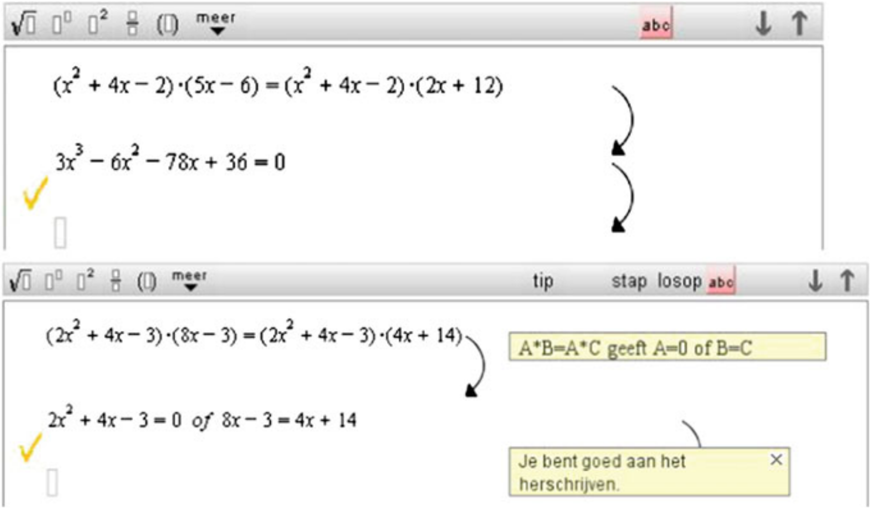


Fig. 8 Pauline's digital work. *Left* crisis item *left*. *Right* post-crisis item

mistakes (for example by losing solutions in the process), and (iii) students do not recognize AB = AC and expand the expressions, getting stuck with an equation of the third power. The design of this sequence, utilizing the affordances of the technology, hypothesized that the crisis item serves as a variational element in the sequence of tasks. The left side of Fig. 8 shows that our case student Pauline demonstrates the third type of behaviour. At this point in the sequence feedback is still restricted to correct or incorrect. In addition, students are allowed to choose their own strategies, even when they aren't efficient or might lead to problems. In the post-crisis items, the feedback correct/incorrect is supplemented by custom feedback, buttons for hints and worked examples, and a movie clip demonstrating the correct solution. From the log-files of the online environment—all student work is stored—it becomes clear that Pauline fails at the crisis-item, but succeeds at the post-crisis item with feedback.

This example demonstrates how the elements of technology use and three design principles (crises, feedback and fading) can be combined with a carefully designed sequence of tasks with variation.

8 Conclusion and Discussion

This chapter sought to demonstrate how three principles, crises, feedback and fading, which were the basis of a digital intervention for algebraic expertise, were retrospectively scrutinised through the lens of Variation Theory with its functions of contrast, generalisation, separation and fusion. I contend that the principles share several elements with this lens. Crisis items primarily aim to *contrast* with the standard procedure students tend to use. By intentionally causing a crisis this contrast is emphasized. This is also emphasized by using varying appearances of the task, aimed at instilling *generalisation* in the students. By carefully varying certain aspects only, and leaving other aspects invariant, *separation* is obtained. Finally, *fusion* is obtained by providing multiple representations e.g. through a graphical representation of the equation. The functions of variation can be supported by technology. In this specific intervention this was done through four features: the feature of being able to author one's own tasks, the feature of using randomisation for these tasks, the feature of authoring and providing feedback, and the feature of being able to use multiple representations. The task designer can really utilize these features to implement a carefully designed sequence of tasks with appropriate variation. In this example the main focus was on algebra, but the DME also implements other domains, like geometry and statistics. It might be beneficial if educators, teachers, designers and researchers alike can adopt these principles when designing and implementing sequences of (near-similar) tasks. There are several advantages to this approach. Firstly, the 'lens' of Variation Theory enables authors, teachers and designers alike to realise the importance of careful task design. Making good instructional materials is an art and should not be taken lightly. Guiding principles for their design can facilitate the creation of quality materials. Secondly, as a consequence of thinking more carefully about task design with technology, using these aspects of variation contributes to more effective learning by students. We must keep in mind that we are talking about web-based tools, basically digital books, which incorporate these features. Their presence greatly enhances the quality of instructional resources. They can incorporate sensible design of task sequences, a vast set of tasks through randomisation, relevant feedback and multiple representations.

There are, however, some points of discussion. One concerns when we are actually talking about variation, as variation depends on variant and invariant elements of the tasks. In other words, the variation needs to be observed. It would be hard to argue that the level of discernment does not also depend on prior knowledge or the difficulty of the task. What can be a simple task for one year-eight student can prove to be difficult for another student, even when at first sight they seem fairly similar. Also, the way in which a crisis is overcome can differ: some students learn from repeating near-similar tasks, others seem to recognize 'a pattern' immediately and apply this to new tasks. Given this diversity, it is important to field-test and evaluate sequences of tasks, again combining the power of teaching, researching and designing. In my opinion, there sometimes also is the wrong assumption to classify certain tasks as 'more creative' and other tasks as 'less creative'. This too depends on the background and context of the learner: a wonderful, new and creative task can become a repetitive task the second time around. In this respect, variation is context-dependent, and this further emphasizes the importance of being able to flexibly change and author materials. One could even go as far to say that the predicate 'near-similar' applies to almost *all* tasks in education: if a student has seen a task before, even the elaborate, creative ones, it becomes part of the cognitive structure. It fits the literature which says the acquisition of concepts and procedures is inextricably linked (e.g. Star 2005;

Rittle-Johnson et al. 2015; or 'deep learning' by Ohlsson 2011). The already cited work by Fan et al. (2004) and my work with Fan on algorithms (Fan and Bokhove 2014) suggests that this might 'explain' some contradictory observations that Asian countries address memorisation and understanding. Variation certainly seems to be a powerful way to combine both, and in this light I think this chapter describes already powerful design principles through the lens of variation.

References

Abels, M., Boon, P., & Tacoma, S. (2013). *Designing in the digital mathematics environment*. Retrieved from http://www.fisme.uu.nl/wisweb/dwo/DWO_handleidingen/2013-10-11manual_DMEauthoringtool.pdf.

Black, P., & Wiliam, D. (1998). Inside the black box: Raising standards through classroom Assessment. *Phi Delta Kappan, 80*(2), 139–149.

Bokhove, C. (2008, June). *Use of ICT in formative scenarios for algebraic skills*. Paper presented at the 4th Conference of the International Society for Design and Development in Education, Egmond aan Zee, The Netherlands.

Bokhove, C. (2010). Implementing feedback in a digital tool for symbol sense. *International Journal for Technology in Mathematics Education, 17*(3), 121–126.

Bokhove, C. (2011). *Use of ICT for acquiring, practicing and assessing algebraic expertise*. Utrecht: Freudenthal Institute, Utrecht University.

Bokhove, C., & Drijvers, P. (2012a). Effects of a digital intervention on the development of algebraic expertise. *Computers & Education, 58*(1), 197–208. doi:10.1016/j.compedu.2011.08.010.

Bokhove, C., & Drijvers, P. (2012b). Effects of feedback in an online algebra intervention. *Technology, Knowledge and Learning., 7*(1–2), 43–59. doi:10.1007/s10758-012-9191-8.

Bokhove, C. (2014). Using crises, feedback and fading for online task design. *PNA, 8*(4), 127–138.

Clifford, M. M. (1984). Thoughts on a theory of constructive failure. *Educational Psychologist, 19*(2), 108–120.

De Jong, T. (2010). Cognitive load theory, educational research, and instructional design: Some food for thought. *Instructional Science, 38*(2), 105–134. doi:10.1007/s11251-009-9110-0.

Fan, L., & Bokhove, C. (2014). Rethinking the role of algorithms in school mathematics: a conceptual model with focus on cognitive development. *ZDM-International Journal on Mathematics Education, 46*(3). doi:10.1007/s11858-014-0590-2.

Fan, L., Wong, N. Y., Cai, J., & Li, S. (Eds.). (2004). *How Chinese learn mathematics: Perspectives from insiders*. Singapore: World Scientific.

Gu, L. (1981). *The visual effect and psychological implication of transformation of figures in geometry*. Paper presented at Annual Conference of Shanghai Mathematics Association, Shanghai, China.

Gu, L. (1994). Theory of teaching experiment: The methodology and teaching principle of Qingpu [in Chinese]. Beijing, China: Educational Science Press.

Gu, L., Huang, R., & Marton, F (2004) Teaching with variation: A Chinese way of promoting effective Mathematics learning. In L. Fan, N. Y. Wong, J. Cai & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (2nd ed.). Singapore. World Scientific Publishing.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research, 77*(1), 81–112.

Kapur, M. (2010). Productive failure in mathematical problem solving. *Instructional Science, 38*(6), 523–550.

Kapur, M. (2011). A further study of productive failure in mathematical problem solving: Unpacking the design components. *Instructional Science, 39*(4), 561–579.

Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning, 13*, 135–157.

Leung, A., Baccaglini-Frank, A., & Mariotti, M. A. (2013). Discernment in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460. doi:10.1007/s10649-013-9492-4.

Marton, F., & Booth, S. (1997). *Learning and Awareness*. Mahwah: Lawrence Erlbaum.

Marton, F., & Pang, M. (2006). On some necessary conditions of learning. *Journal of the Learning Sciences, 15*(2), 193–220.

Marton, F., & Trigwell, K. (2000). Variatio est Mater Studiorum. *Higher Education Research and Development, 19*(3), 381–395.

Marton, F., Runesson, U., & Tsui, A. B. M. (2004). The space of learning. In F. Marton & A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3–40). Mahwah: Lawrence Erlbaum Associates, Inc. Publishers.

Marton, F., & Tsui, A. (Eds.). (2004). *Classroom discourse and the space for learning*. Marwah: Erlbaum.

Mason, J., & Johnston-Wilder, S. (2006). *Designing and using mathematical tasks*. London: QED Publishing.

Ohlsson, S. (2011). *Deep learning: How the mind overrides experience*? Cambridge: Cambridge University Press.

Pea, R. D. (2004). The social and technological dimensions of scaffolding and related theoretical concepts of learning, education, and human activity. *Journal of the Learning Sciences, 13*(3), 423–451.

Piaget, J. (1964). Development and learning. In R. E. Ripple & V. N. Rockcastle (Eds.), *Piaget Rediscovered* (pp. 7–20). New York: Cornell University Press.

Renkl, A., Atkinson, R. K., & Große, C. S. (2004). How fading worked solution steps works—a cognitive load perspective. *Instructional Science, 32*(1/2), 59–82.

Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review, 27*(4), 587–597.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253–286. doi:10.1177/0895904803260042.

Schoenfeld, A. H. (2009). Bridging the cultures of educational research and design. *Educational Designer, 1*(2).

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36*(5), 404–411.

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science, 12*(2), 257–285. doi:10.1016/0364-0213(88)90023-7.

Tall, D. (1977). Cognitive conflict and the learning of mathematics. In *Proceedings of the First Conference of The International Group for the Psychology of Mathematics Education*. Utrecht: PME. Retrieved from http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1977a-cog-confl-pme.pdf.

Van der Kleij, F. M., Feskens, R. C. W., & Eggen, T. J. H. M. (2015). Effects of feedback in a computer-based learning environment on students' learning outcomes: A meta-analysis. *Review of Educational Research*. Advance online publication. doi:10.3102/0034654314564881.

Van Hiele, P. M. V. (1985). *Structure and Insight: A theory of mathematics education*. Orlando: Academic Press.

Watson, A., & Mason, J. (2002). Student-generated examples in the learning of mathematics. *Canadian Journal of Science, Mathematics and Technology Education, 2*(2), 237–249.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah: Erlbaum.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91–111.