Designing Innovative Learning Activities to Face Difficulties in Algebra of Dyscalculic Students: Exploiting the Functionalities of *AlNuSet*

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Abstract In this chapter I discuss students' difficulties in algebra, considering in particular those students affected by *developmental dyscalculia* (DD) (Butterworth in *Handbook of Mathematical Cognition*. Hove, UK: Psychology Press, 2005; Dehaene in *The number sense: How the mind creates mathematics*. New York, Oxford University Press, 1997). Focusing on algebraic notions such as *unknown*, *variable*, *algebraic expression*, *equation* and *solution of an equation*, I will describe possible processes of meaning making in students with low achievement in mathematics, or even diagnosed with DD including adult learners. This involves considering algebra not only in its syntactic aspects but also in its semantic ones. The assumption on which the work is based, is that some difficulties in learning algebra could be due to the lack of meaning attributed by the students to the algebraic notions. Basing the analyses on studies both in the domain of cognitive psychology and in the domain of mathematics education, I will show how students with DD can make sense of the algebraic notions considered above, thanks to tasks designed within AlNuSet exploiting its semiotic multi-representations based on visual, non-verbal and kinaesthetic-tactile systems. AlNuSet (Algebra of Numerical Sets) is a digital artifact for dynamic algebra, designed for students of lower and upper secondary school.

Keywords Algebra · Developmental dyscalculia · AlNuSet · Task · Variable · Solution of an equation · Algebraic expression

1 Introduction

For a significant percentage of students the current teaching of algebra does not seem to be sufficient for helping them effectively develop the necessary skills and knowledge to master this domain of knowledge (Sfard and Linchevsky 1992; Kieran 2006). Here I will focus on students manifesting low achievement in mathematics, or even diagnosed with developmental dyscalculia (DD) (Butterworth 2005; Dehaene 1997).

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The need to deal with different cognitive demands and in particular with those of students having learning difficulties in mathematics is discussed in the mathematics education research, in the cognitive psychology and in the literature on learning disorders. In Italy, students with learning disorders are estimated to be between 3 % and 5 %, and recent data disclosed by the Italian Ministry of Education, indicate that only 0.9 % of the school population has obtained diagnoses, so the number of certified students with learning disorders is likely to increase. A conscious use of specific teaching strategies suitable for students diagnosed with learning disorders, and in particular with developmental dyscalculia, is also important for those students who, although uncertified, show learning difficulty profiles very similar to those of dyscalculic students.

Students' difficulties in algebra seem to be due to lack of meaning developed for algebraic notions (Arzarello et al. 1994). Having acquired meanings of algebraic notions seems to be very important also in order to have better control over algebraic manipulations (Radford 2005; Robotti and Ferrando 2013). Recent studies in mathematics education indicate that the construction of mathematical knowledge, as a cognitive activity, is supported by the sensory-motor system activated in suitable contexts.

According to Arzarello's definition of *semiotic bundle* (2006), in addition to the standard semiotic resources used by students and teachers (e.g. written symbols and speech), I consider other important resources, such as graphic and extra-linguistic modes of expressions. These can be particularly useful both for teachers in designing effective tasks and for students in learning algebra.

Thus, the construction of meaning in mathematical activities is based on a rich interplay between three different types of semiotic sets: speech, gestures and written representations (Arzarello 2006). In this respect, Radford (2005) underlines that the understanding of the relationship between body, actions carried out through artifacts (objects, technological tools, etc.), and linguistic and symbolic activity is essential in order to understand human cognition in general, and mathematical thinking in particular.

The design and use of tasks for pedagogic purposes is at the core of mathematics education (Artigue and Perrin-Glorian 1991). Tasks generate activities, which afford opportunity to encounter mathematical concepts, ideas, strategies, and also to use and develop mathematical thinking. Following Mason and Johnston-Wilder's idea (2006), I mean by "task", what students are asked to do.

To understand how tasks are linked to one others in order to support teaching, it is important to understand the nature of the transformation of knowledge from implicit knowledge-in-action (see Vergnaud 1982) to knowledge which is formulated, formalized, memorized, related to cultural knowledge, and so on. This work is often undertaken by using a textbook and/or other resources designed by outsiders. I will show how *AlNuSet* provides teachers with a new and innovative environment to design tasks, which also support inclusive education.

Therefore, it is my belief that to make algebraic notions explicit, ensuring that students grasp the meaning of the algebraic notions used, teachers need artifacts, which make available new semiotic representations of the algebraic objects. For this reason, I will examine the software AlNuSet and analyse its potential in designing

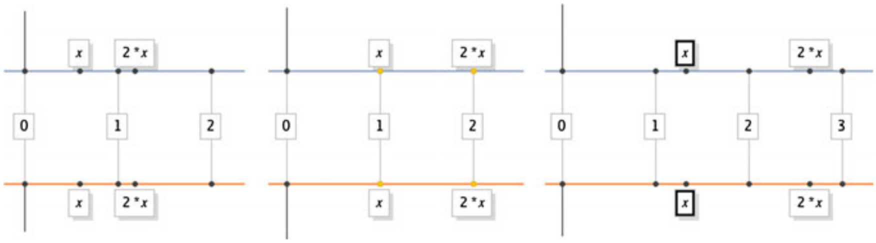

Fig. 1 The algebraic expressions containing x move accordingly with x

activities that take into account students' difficulties in algebra, as described in the literature, and, at the same time, that are aimed at engaging all students in the class, as much as possible (Baccaglini-Frank and Robotti 2013). In particular, I report on a case study with a 26 year old DD student, which shows that tasks in AlNuSet can be designed to support the construction of algebraic notions using in particular the visual non-verbal and kinaesthetic channels of access to information.

In the following section I will present a short description of the software.

2 Description of AlNuSet

AlNuSet was developed in the context of the ReMath (IST - 4 - 26751) EC project and it was designed for students of lower and upper secondary school (from age 12–13 to age 16–17). It was developed by the research group of the ITD (Istituto per le Tecnologie Didattiche)- CNR (Centro Nazionale di Ricerca) of Genoa (Italy) to which the author belongs.

AlNuSet is made up of three tightly integrated components: the Algebraic Line, the Symbolic Manipulator, and the Functions component (for more details see www.alnuset.com). Since this paper concerns tasks in which only the first component is used, I will describe only the Algebraic Line component.[1]

The main characteristic of the Algebraic Line component is the possibility of representing an algebraic variable as a mobile point on the line, namely, a point that can be dragged along the line thanks to the mouse. The point can be labelled with a letter (Fig. 1). By dragging the mobile point along the line, the letter associated to the point assumes the values of numerical set instantiated. This new visuo-spatial approach, which exploits dynamic representations, allows making explicit the notion of *variable* as a mobile point on the line that can assume all values within the numerical set instantiated. Therefore, by dragging the mobile point on the line, all algebraic expressions containing such a variable, move accordingly (Fig. 1).

[1]For a detailed description of algebraic activities developed within the Manipulator component, which allows the teacher to approach algebraic manipulation in an innovative way, see Robotti and Ferrando (2013).

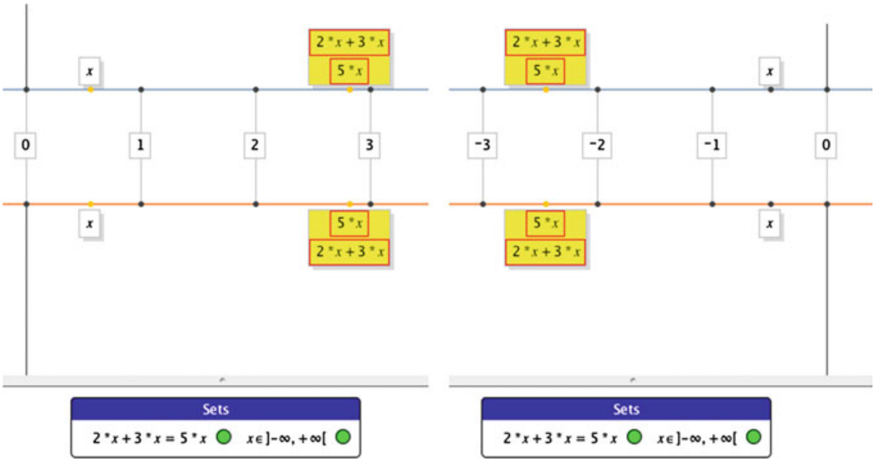

Fig. 2 Variable and expressions on the line. By dragging *x* along the line, it is possible to verify the equivalence of 2x + 3x and 5x because they belong to the same post-it for all values of x

This feature has transformed the number line into an algebraic line where it is possible to operate with algebraic expressions and propositions in a quantitative and dynamic way.

This visuo-spatial approach to algebra allows the student to handle dynamic representations as new semiotic representations of algebraic objects on the Algebraic Line. This makes dynamic algebra possible and it supports students in the conceptualization of algebraic objects. The most important new semiotic representations available in the Algebraic Line of AlNuSet that are involved in the tasks presented in this paper are:

- the yellow square named "post-it": two expressions belonging to the same post-it can be connected to the notions of *equation* seen as equivalence between expressions (see Figs. 2 and 3a);
- the colour of the dot associated to a proposition (equality/inequality) and/or to the truth set built by the user (see Figs. 2 and 3a, b): the colour match between the two dots can be used to validate the constructed numerical set as the truth set of the proposition.

In the Algebraic Line it is possible to explore equations, inequalities and systems of equations and of inequalities. Their solution sets are visualized in a specific window, labelled "Sets", and they are associated to a coloured dot: green if the instantiated value of x belongs to the set, red otherwise. This way, dragging the mobile point along the line, the colour of the dot changes depending on the value of x.

Mediation provided by AlNuSet is profoundly different from mediation offered by other software used for the traditional teaching of algebra: new dynamic representations, based on a visuo-spatial approach, offer the possibility of reifying

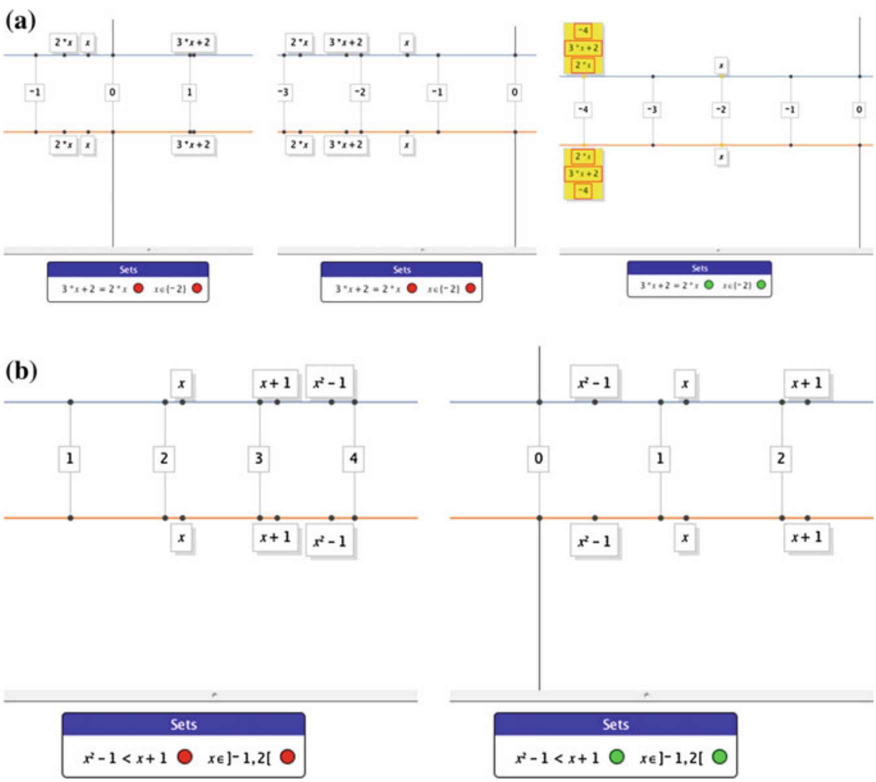

Fig. 3 **a** By dragging *x* along the line, it is possible to explore the solution of the equation $3x + 2 = 2x$. The truth set of the equation is the value −2. **b** By dragging *x* along the line, it is possible to explore the solution of the inequality $x^2 - 1 > x + 1$. The truth set of the inequality is]−1, 2[

semiotic representations and of constructing meaning for algebraic notions linking the semantic and symbolic nature of algebraic objects.

Several studies (Chaachoua et al. 2012; Chiappini et al. 2010; Robotti 2013; Leung and Bolite-Frant 2015) have shown the educational potential of this software indicating how the new approach described above can be effective in fostering understanding of the basic mathematical concepts (fractions, expressions, equations,…).

3 Developmental Dyscalculia and Algebra

According to Butterworth (2005), Developmental Dyscalculia (DD) is a *learning disability* that affects the acquisition of knowledge about numbers and arithmetic: "DD children have problems with both knowledge of facts and knowledge of arithmetical procedures […], although Temple (1991) has demonstrated, using case studies, that the knowledge of facts and grasp of procedures and strategies are dissociable in the developmental dyscalculia" (Butterworth 2005, p. 459).

Thus, it is known that students with DD have severe difficulties in arithmetic, that is, in the areas of mathematics that depend on quantity (Butterworth 2003).

However, there are also areas of mathematics that do not depend so much on manipulating quantities—algebra, geometry and topology, for example. It may be that students with DD can in fact become proficient in these areas, even though their arithmetic is poor. As a matter of fact, some studies on dyscalculic learners showed that there is a dissociation between the recovery ability of arithmetic facts, which is compromised, and algebraic manipulation, which is intact (Hittmair-Delazer et al. 1995; Dehaene 1997). Dehaene analysed the mathematical performances of dyscalculic subjects: they presented difficulties in simple calculations such as $2 \cdot 3$, 7–3, 9:3, $5 \cdot 4$, but they were able to transform and simplify algebraic expressions such as:

$$\frac{a \cdot b}{b \cdot a} = 1$$
$$a \cdot a \cdot a = a^3$$

and they were able to judge the non equivalence between algebraic expressions such as:

$$\frac{d}{c} + a = \frac{d + c}{c + a}$$

These results have been interpreted as evidence for the existence of two independent processing levels of mathematics: a formal-algebraic level and an arithmetic-numeric level (Dehaene 1997). Moreover, neuroimaging results, focusing on the algebraic transformations, have highlighted how the visual-spatial areas of the brain are activated at the expense of language. For example, it has been shown that in solving equations, the expressions are manipulated mentally by means of a visual elaboration rather than of a verbal one (Landy and Goldstone 2010).

Those results help us highlight, from a neuro-scientific perspective, the difficulties of students with DD in algebra. In the next paragraph I will describe the nature of such difficulties from a strictly didactical point of view.

4 Difficulties in Algebra of Students with DD

Research in mathematics education characterizes as semantic difficulties in algebra, the difficulties to give meaning to algebraic notions (Sfard and Linchevsky 1992; Thomas and Tall 2001; Arzarello et al. 1994) among the major difficulties encountered by students. Now, the question is: what about difficulties of students with DD in algebra?

In my research, I have identified some main algebraic difficulties of students with DD (Robotti and Ferrando 2013; Robotti 2013):

– constructing meaning for the algebraic symbols, e.g. giving the correct meaning to the expressions $a \cdot a$ and $2 \cdot a$;
– recovering skills of an arithmetical nature, e.g. recovering number facts, for instance the times tables.

Of course arithmetical difficulties influence algebraic performance, but these are of a different nature. Students with DD could have difficulties in developing and using new skills for the rules of algebraic transformation. Indeed, in general they find it hard to efficiently use long and short term memory. Therefore they have difficulties in memorizing and recovering facts:

– algebraic rules; for example, factorization formulas used to factor algebraic expressions, such as

$$(a+b)(a-b) = a^2 - b^2$$

or the quadratic formula for finding the roots of a quadratic equation,

$$x_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}$$

– algebraic facts, such as, $a \cdot a = a^2$.

Once some main algebraic difficulties for students with DD are identified, the question is: How can we didactically intervene in effective ways?

In the following section I present some research results, which allowed me to answer this question.

5 Some Results from Research in Neuroscience, Cognitive Psychology and Mathematics Education

Research in cognitive psychology has identified four basic channels of access to and production of information: the visual-verbal channel (verbal written code), the visual non-verbal channel (visual-spatial code), the auditory channel (verbal oral code), and the kinaesthetic-tactile channel (Mariani 1996).

Italian research has indicated that most students with specific learning difficulties (or disabilities), not only in mathematics, encounter the greatest difficulties in using the visual-verbal channel and this impacts their development for preferring different channels (Stella and Grandi 2011). The importance of these different channels of access to and production of information shifts the focus from simply “being able or not” to solve a certain task, to different paths and strategies adopted by the individual (whether successful or not) for approaching the task. This allows to explain mathematical difficulties not only in terms of “lacking abilities” but also in terms of necessity to use certain preferred modalities that lead the student to access, elaborate and/or produce information in a certain way.

Moreover, various studies in cognitive science point to a correlation between mathematical achievement, working memory (Raghubar et al. 2010; Mammarella et al. 2010, 2013; Szucs et al. 2013), and non verbal intelligence (DeThorne and Schaefer 2004; Szucs et al. 2013). These findings suggest that non-verbal intelligence may partially depend on spatial skills (Rourke and Conway 1997) and these can be potentially important in mathematical achievement, where explicit or implicit visualization is required.

My colleagues and I (e.g. Robotti and Baccaglini-Frank 2016) have found other theoretical stances advanced in mathematics education that are in line with the idea that means of access to and production of information, different from the visual-verbal one, can be very important in learning. Some studies in this domain have stressed the important role of bodily actions, gestures, language and the use of technological artifacts in students’ elaborations of mathematics (Arzarello 2006; Nemirovsky 2003; Núñez 2000) and, in particular, of algebra (Arzarello and Robutti 2001). According to Arzarello’s notion of *semiotic bundle* (2006) the construction of meaning in mathematical activities, is based on a rich interplay among three different types of semiotic sets: speech, gestures and written representations (from sketches and diagrams to mathematical symbols). These constitute a semiotic bundle, which dynamically evolves over time.

Thus, important research questions, developed in math education, are related to our understanding of the relationship between body, actions carried out through artifacts (objects, technological tools, etc.), and linguistic and symbolic activity (Radford 2005).

According to Radford, research on the epistemological relationship between these three main sources of knowledge formation is essential in order to understand human cognition and mathematical thinking, in particular. For this reason, he underlines, from a semiotic point of view, the importance of revisiting cognition in such a way that leads to thinking of cognitive activity as something that is not confined to mental activity alone.

Arzarello (2006) refers also to the discoveries in neuropsychology underlining aspects of cognition. His aim is to put semiotic representations in relation with mental ones, in mathematics. He remarks (2006) that a major result of neuroscience is that “conceptual knowledge is embodied, that is, it is mapped within the sensory-motor system” (Gallese and Lakoff 2005, p. 456). The sensory-motor system of the brain is multimodal. This means that imagining and doing use a shared neural substrate. Moreover, “sensory modalities like vision, touch, hearing, and so on are actually integrated with each other and with motor control and planning” (Gallese and Lakoff 2005, p. 456).

Thus, the paradigm of multimodality seems to be crucial in order to alleviate the difficulties of students with DD in maths: “the understanding of a mathematical concept, rather than having a definitional essence, spans diverse perceptuo-activities, which become more or less active depending on the context. […]. Learning a different approach for what appears to be the "same" idea, far from being redundant, often calls for recruiting entirely different perceptuo-motor resources." (Nemirovsky 2003, p. 108).

A consequence of this approach is that "not only the usual transformations and conversions (in the sense of Duval) from one register to the other must be considered as the basic producers of the mathematical knowledge. Its essence consists, rather, of the multimodal interactions among the different registers within a unique integrated system, composed of different modalities: gestures, oral and written language, symbols, and so on (Arzarello and Edwards 2005; Robutti 2005). Also the symbolic function of signs is absorbed within such a picture." (Arzarello 2006, p. 284).

According to these considerations, the design of tasks is essential as the context in which students are asked to work. In this sense I consider AlNuSet an effective context to foster understanding of algebraic concepts both for students with difficulties and for students with dyscalculia.

In addition to the standard semiotic resources used by students and teachers in teaching and learning algebra (e.g. written symbols and speech), other important resources are considered in the case study which I will treat. In particular, dynamic representations available in AlnuSet (such as the point which can be dragged along the line), symbols (such as the post-it or the coloured dots), and, more generally, extra-linguistic modes of expressions, which turned out to be particularly useful for the student with DD involved in a recent case study.

According to these premises, in the following section I will try to give some suggestions for answering the question "How can we didactically intervene in effective way?"

6 How the Functionalities of AlNuSeT Allow Designing Tasks to Construct Algebraic Meanings

I discuss here if and how this new approach to the meaning of algebraic notions in AlNuSet, can be effective for students with dyscalculia. In particular, I will present how a student with DD was able to make sense of the notions of *unknown* of an equation, and of *variable* of an algebraic expression, exploiting the functionalities of the Algebraic Line.

I refer to task as "what students are asked to do" (Mason and Johnston-Wilder 2006), and I expect the activity to be carried out in AlNuSet, so I am speaking of tasks expressed verbally that are designed to be effective with respect to specific didactical objectives. In this sense, the tasks are designed considering AlNuSet as a tool that allows and favors a multimodal approach to algebra.

The subject of this case study is Eleonora, a student with severe dyscalculia. The case is particularly interesting because, although she had taken algebra in high school (she is 27 years old), she had not been able to construct any (apparent) meaning for the various algebraic notions she had encountered and she was not able to use algebra when solving problems. I will show how the perceptive and dynamic approach, together with the visual non-verbal representations, offered in the Algebraic Line of AlNuSet, were effective in helping Eleonora grasp the desired concepts.

6.1 Methodology

I met Eleonora as a working university student, in 2014, when she was 27 years old. She had obtained her first diagnosis of dyscalculia the year before. Eleonora attended the fourth year of a 5-year undergraduate degree for becoming a primary school teacher. She claimed to have always had a bad relationship with mathematics. Indeed the word "mathematics" immediately created in her a state of anxiety and low self-esteem. She had difficulties in calculating the results of simple arithmetic operations; she found it hard to construct algebraic models and to recall algebraic processes for solving equations (even linear equations) or algebraic facts (for example that $a \cdot a = a^2$).

The experimental activities I proposed to Eleonora were structured in two moments: a pre-test to explore the meanings Eleonora initially attributed to the algebraic notions that would be treated, and, then, the sequences of tasks in AlNuSet. These activities were carried out outside of the customary university lectures, as additional hours in a quiet setting where I was alone with Eleonora, who willingly took part in the study.

During the AlNuSet tasks the researcher (the author) would guide Eleonora in using the software through additional questions.

Each session lasted about one hour and a half and it was video recorded. We worked through three sessions: the pre-test session, and two subsequent sessions using the Algebraic Line of Alnuset.

In the following sections I will analyse Eleonora's performance throughout main moments of each session.

6.2 Pre-test

A pre-test was presented to Eleonora in order to investigate the meaning she attributed to the notions of variable and of expression containing that variable, of unknown and of equation. The tasks (*Tn*) and Eleonora's written answers are in italics; they are followed by a brief analysis in regular font.

(T1) What does the letter "a" represent in the expression " $2 \cdot a$*"?*
E: "a" denotes any number which is, here, in relation, through an arithmetic operation, with 2. It [a] can take on any value.

Fig. 4 Eleonora's attempt at solving T2

Note that "*a"* denotes "any number" but no reference is given to the numerical set of reference. Moreover, Eleonora does seem to attribute to the expression $2 \cdot a$ the meaning of a symbolic representation of the arithmetic operation of multiplication but it seems, again, unrelated to a number set (in this case, the set of even numbers): Eleonora seems to think that varying the value of a, the expression 2a could take on any value, not just those of even numbers. In other words, the algebraic expression 2a seems to be, for Eleonora, a relation between a fixed number (2) and a variable (a), but it does not seem to be a value that depends on a.

(T2) 3 times a certain natural number is equal to 11. Find the number.
E: No, it isn't possible.

[She draws three dots and, at the same time, she says: "This one, plus this one, plus this one…". See the right side of Fig. 4]

The multiplication is interpreted as repeated addition and the equality is solved using an arithmetic pattern.

E: No, it [12] isn't possible [she means that 12 is not the right number] *but, it is possible with a fraction …*

She writes: $\frac{11}{3}$; $3 \cdot \frac{11}{3}$; 366,666,666,667 (Fig. 4) using trial and error and a calculator to look for a number that multiplied by 3 can give 11. Notice that no equation [$3 \cdot x = 11$] is used to represent and solve the situation. She only refers to arithmetical expressions: $\frac{11}{3}$ is the number, and $3 \cdot \frac{11}{3}$ corresponds to the arithmetic verification of the fact that she found the number such that 3 times it needs to be 11. Thus, one dot corresponds to 366,666,666,667 [note that 366,666,666,667 is written on one of the circled dots and it refers to a dot] but 3 times 366,666,666,667 is not 11 [she uses the calculator]. She says: "It *is nearly 11…*". It seems that $\frac{11}{3}$ cannot be considered the number she was looking for. This might be a misconception Eleonora holds relative to rational numbers, according to which they are not considered "numbers" unless they are transformed into decimal form (e.g. Fandiño Pinilla 2005).

(T3) When 3 is added to 3 times a certain number, the sum is 28; find the number.

As before, Eleonora is oriented towards computation and not towards the use of algebra (Fig. 5), so she subtracts 3 from 28 (obtaining 25) and then she divides by 3—"undoing" the operations stated in the text of the problem. She writes that the number 25/3 plus 3 is equal to 28. Once again, she uses a trial and error strategy

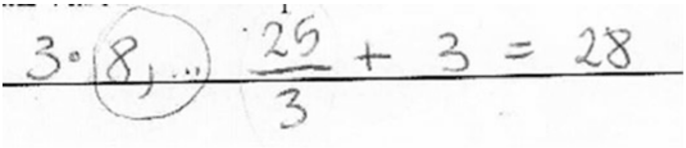

Fig. 5 Eleonora's answer to T3

looking for the number (8, …) that, multiplied by 3 and added to 3, gives 28. Once again, she does not seem to accept $\frac{25}{5}$ as a number. She does not seem to feel the need to use algebra; she does not see algebra as a tool for representing and solving a situation and this appears quite clearly from Eleonora's use of the equal sign (Fig. 5).

This paragraph should give a general idea of the meaning attributed by Eleonora to algebra, to the notion of variable, of unknown, of equation and of solution of an equation. Through this preliminary assessment I was able to determine that Eleonora, very likely, had not constructed proper meanings for the algebraic notions involved, even though she had constructed some meaning for the notion of variable, but one that was not useful for working with algebraic expressions.

In the next paragraph, I will describe the interventions and tasks in the working sessions that were based on the "little" Eleonora seemed to know. I will analyse how the meanings of expression depending on a variable, unknown, equation and solution of equation, are constructed by Eleonora working with the Algebraic Line of AlNuSet. Therefore, I will include excerpts from the dialogue between the researcher and Eleonora.

6.3 *Tasks to Construct the Meaning of the Notion of Variable and of Algebraic Expression Depending on that Variable*

As described above, in these working sessions Eleonora used the Algebraic Line in AlNuSet. The researcher designed tasks favouring a perceptive approach, which seems to be one of the most effective approaches to helping students with DD build mathematical notions. I did this exploiting the dynamic functionalities of the Algebraic Line in AlNuSet. I started by asking Eleonora to insert, in the Algebraic line of AlNuSet, the letter "a" and to drag the corresponding point along the line.

E: At this moment, we can see that "a" changes value,… it changes value if I drag it. We can see that, when I drag "a", when I drag the corresponding point to "a" along the line, it takes on all values of the numerical set. As we can see, the values can be positive or negative [she drags the point along the positive and along the negative parts of the number line].

The yellow square shows the value that the letter takes on. It is very useful to not "get lost" along the line.

The point corresponding to "a" is dragged along the line. Eleonora observes, in a very spontaneous way, that the point can take on different values in the instantiated numerical set. She states that the yellow square (post-it) plays a role in supporting her memory and in orienting her dragging of the point corresponding to *a* along the number line. We can see this as a new sign that allows Eleonora, by means of visual perception, to build an image for the equivalence between the letter "a" (label), the point on the line, and the values associated to that point on it. The post-it and the mobile point are new available signs in AlNuSet that help Eleonora construct meaning for the notion of variable as a symbol representing *any* quantity in the instantiated numerical domain. In other words, the notion of variable seems to get mapped within the sensory-motor system mediated by the task that exploits the functionalities of AlNuSet. In this sense, understanding of the mathematical concept of variable seems to be fostered, rather than through a definition, through perceptuo-motor activities, implemented within AlNuSet.

The most significant feature of AlNuSet in this activity seemed to be the dragging of the point: indeed the dynamicity allowed Eleonora to carry out actions producing images on the screen linked to visual verbal (the label "2a"), visual non-verbal (the mobile point, the post-it…) and symbolic representations, that were useful for constructing the meaning of variable. The multimodal interactions between the different registers within a unique integrated system made up of different modalities is clear: the gesture (point dragging along the line) and the symbols that make explicit the symbolic function of the signs (post-it, point on the line).

The researcher's aim is now to introduce the dependence of the variable on the numerical set in which it is instantiated. What the researcher asks to do in AlNuSet is still tied to the perception, to the dynamic images and to the manipulation through the mouse.

R: Now, select the Set of Natural numbers. What happens? Why?
E: we can see that now we are able to visualize only numbers on the right side of the number line [Fig. 6], *because…because the negative numbers are not present in the Natural numbers! If I drag "a" along the line, I observe that I cannot go on the left side of the line. I observe that "a" takes on integer values; but now it cannot take on values between two natural numbers.[…] The point jumps from one natural value to another. Moreover, we can observe that, even if I forget that I'm working in the set of the Natural numbers, AlNuSet's interface reminds me of it.*

Two main considerations can be drawn. Firstly, Eleonora seems to be associating an image to the fact that the set of Natural numbers is not "dense". As a matter of fact, she observes that the point corresponding to "a" "jumps" along the line. This is perceptually evident comparing the movement of the point in the Natural numbers with that in the Full Domain. This visual perception of the movement of the point along the line (Fig. 6), allows Eleonora to construct a new meaning for the structure of the set of Natural numbers. Secondly, AlNuSet seems to be perceived

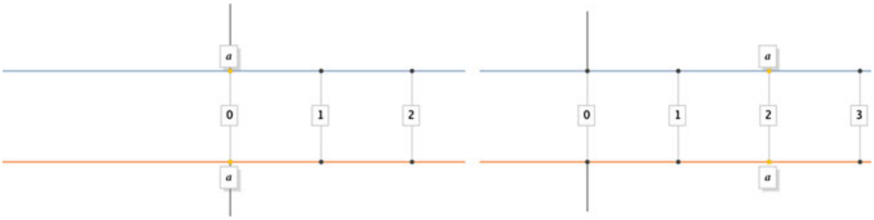

Fig. 6 Different positions of the point labelled “a”; one shows where it takes on the value 0, and the other where it takes on the value 2

as a compensatory tool for Eleonora’s memory: “*even if I forget that I’m working in the set of the Natural numbers, AlNuSet’s interface reminds me of it”*. This leads to a more “relaxed” approach to the mathematical task and it allows Eleonora to focus on the algebraic task rather than on recalling information that she finds difficult to retrieve.

R: In maths, this “a” is called “variable”. So, what is, in your opinion, a variable? In other words, how could you explain to a student the meaning of variable?
E: Variable indicates any value in the numerical set considered.

For Eleonora “variable” seems to denote the mathematical object “number”, in a specific numerical set. So, to grasp the meaning of variable, Eleonora mainly uses a visual and perceptual approach (visual-non verbal and kinaesthetic channels to access and elaborate the information) but, in order to explain the meaning of variable, she does not refer explicitly to visual signs such as “mobile point” or “post-it”. Once the notion of variable is introduced, the researcher introduces the expression depending on that variable.

R: Now, edit $2 \cdot a$ *on the editing bar and press the “enter” button. What happens?*
E: It appears on the line. So… what value can we give? [she drags the point corresponding to “a” along the line]. *If I give to “a” the value −1… 2a will be… will be −2, yes, of course!*

We can observe that, in addition to the answer “*it* [$2 \cdot a$] *appears on the line”*, Eleonora performs actions on the expression “$2 \cdot a$” dragging “a” along the line. This is a realization of the meaning of expression depending on a variable: Eleonora shows that, when the “a” *varies* (that is to say, when the corresponding point is dragged along the line), then the value of the expression “$2 \cdot a$” *varies* accordingly.

R: Ok, so, what does the letter “a” represent in the expression “ $2 \cdot a$*”?*
E: a … variable? Yes, “a” is a variable! And… $2 \cdot a$*,… takes on values, takes on values depending on “a”.*

Note that Eleonora uses the algebraic term “variable” to refer to “a”. She explains the dependence of the expression “$2 \cdot a$” on “a” stressing that the values taken on by “$2 \cdot a$” are dependent on those taken on by “a”. So here we see the expression not interpreted exclusively as a symbolic representation of the arithmetic operation (multiplication), as in the pre-test. This answer suggests that Eleonora has constructed a new meaning of variable and she uses it to construct the meaning for expression depending on that variable.

6.4 Tasks to Construct the Meaning of the Unknown Involved in Equation

The researcher's aim is now to construct meaning for the notion of *identity* and *conditioned equality* (equation). To this aim, it is necessary to first construct the meaning of unknown involved in equation.

R: For which value of "a" is the expression $2 \cdot a$ *equal to 8?*
E: The expression is equal to 8… that is 2a is equal to 8… If I move "a" along the line, I am looking for the right value to match to the letter. For example, I discovered that if I place "a" on 3 …if I give to "a" the value 3… $2 \cdot a$ *is [*equal to*] 6; Instead, if I put "a" on 4,* $2 \cdot a$ *is 8…*

Dragging the corresponding point to "a" along the line, Eleonora observes that there is only one value of "a" for which the point corresponding to the expression takes on the value 8 (Fig. 7a, b). This dynamic representation contributes to building new meaning for the equal sign between the expressions as *conditioned equality*. Indeed, when Eleonora tries to verify the equality between 2a and 8, the dragging of "a" is performed with a specific aim: to ensure that the expression is associated to the point 8 on the line and it belongs to the same post-it as 8. If dragging is accomplished with this aim, then the meaning of variable can be that of *unknown*, and the action associated to it is searching for a value to be assigned to "a" *so that* the equality is true.

Notice that the expression "If I place "a" on 3…", which refers to the perceptive approach to solving the equation is reformulated as "if I give to "a" the value 3…", which, instead, refers to the mathematical meaning of solving the equation (finding the value of the unknown that makes the equation true). This awareness is also present in the following exchanges. This shows how the sensory-motor system,

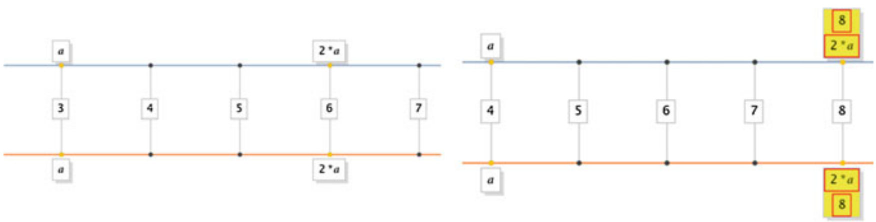

Fig. 7 **a** "a" takes on value 3, so that $2 \cdot a$ takes on value 6. **b** "a" takes on value 4, so that $2 \cdot a$ takes on value 8

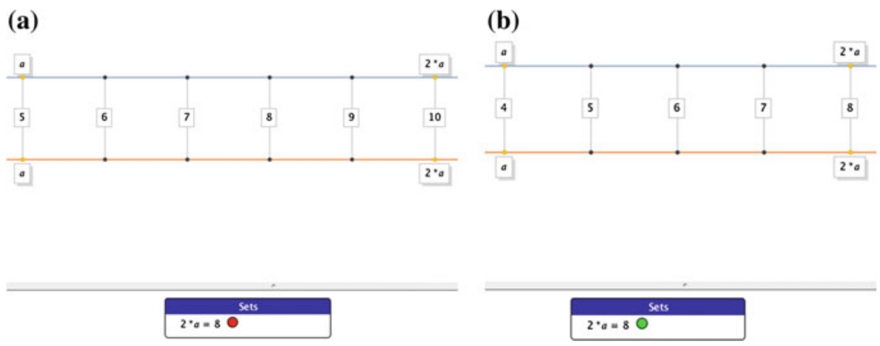

Fig. 8 ***a*** The *red dot* associated to the equation means that the value of "a" is not a truth-value. **b** The *green dot* associated to the equation means that the value of "a" is a truth-value

which acts through the perceptive approach, can contribute to the construction of mathematical knowledge also in presence of dyscalculia.

R: Ok, now insert $2 \cdot a = 8$ *into the edit bar and then, push "enter". What do you get?*
E: A verification …It's a check, if I drag "a", the red dot shows that I'm making a mistake. Because, in this moment, 2a equals 8 is not true [see Fig. 8a]. *There isn't equality. Because it's 2a equal to 10, if I give to "a" the value 5. But, if I give to "a"* [the value] *4, the green dot shows that it is right* [Fig. 8b], *because 2a is 8. So, I found the value of "a" which allows me to say that this equality is true. Yes, because I'm multiplying.*

The coloured dot next to the equation is interpreted as a visual aid to validate the truth of the equation. Moreover, we can observe how the colour of the dot supports reaching the appropriated mathematical interpretations: verbal expressions referring to the truth of the equation evolve from perceptual (linked to signs) to formal expressions (referred to maths object). For example, "..the red dot shows that I make a mistake" becomes "…the red dot shows that there isn't equality", or "the green dot shows that it is right" becomes "the green dot shows that this equality is true".

Here it is evident that for Eleonora the equality is conditioned by the values of "a" but the stronger meaning for the equal sign is still "give a numerical result" rather than showing a "relation" (probably due to the fact that the second term of the equality is a number). Indeed Eleonora uses the term "multiplying".

This is why I chose to now propose a task addressing this misconception. The task fosters both the construction of the idea that the equality between two expressions is conditioned by the value of unknown and that the equal sign denotes a relation between the expressions.

R: Now, edit the expressions: $2 \cdot a + 3$; $2 \cdot a + 3 \cdot a$; $5 \cdot a$. *Insert them into the Algebraic Line and drag "a" along the line. What happens?*

E: [after editing, without dragging] $5 \cdot a$ *automatically goes on 5* [the last position of "a" was on the point 1] *and this is written in the yellow square:* $2 \cdot a + 3$; $2 \cdot a + 3 \cdot a$; *and* $5 \cdot a$ *are all together* [they belong to the same post-it] *and they refer to the same value* 5. *But, for example, if "a" is 2* [she drags the mobile point along the line] *then* $2 \cdot a + 3$ *is 7 and the others are 10…*

Dragging "a" along the line Eleonora explores what happens to the expressions $2 \cdot a + 3$; $2 \cdot a + 3 \cdot a$; $5 \cdot a$. She knows that the expressions depend on the values of "a" (that is, they move because of the movement of "a"), but this exploration allows her to make sense of the *existential quantifier* and the *universal quantifier*. Indeed, dragging point "a" along the line, Eleonora observes that there is only one value of the "a" for which the points of the expressions $2 \cdot a + 3$ and $5 \cdot a$ take on the same value, that is to say, they correspond to the same value on the line. This dynamic representation contributes to building meaning for the equal sign between the expressions, guiding its interpretation as a conditioned equality. Indeed, she tries to verify the equality between the two expressions dragging "a" with a specific aim: to ensure that the two expressions take on equal values, that is, they are associated to the same point on the line and they belong to the same post-it. This fosters construction of a new meaning for the *existential quantifier* (∃).

In Fig. 9, two moments corresponding to dragging "a" along the line are represented (Fig. 9a, b). The expressions 2a + 3 and 5a refer to the same point, corresponding to the value 5, and they belong to the same post-it, only for a = 1. Since the expressions are equal, the equality is true and the dot associated to the equation is green.

A specific command allows the student to construct and visualize the truth set associated to the equality (Fig. 9). Moreover, a coloured dot is associated to that set: the red (Fig. 9a)/green (Fig. 9b) colour means that the current value of the unknown is/is not an element of the constructed set. So the fact that the colour of the dots

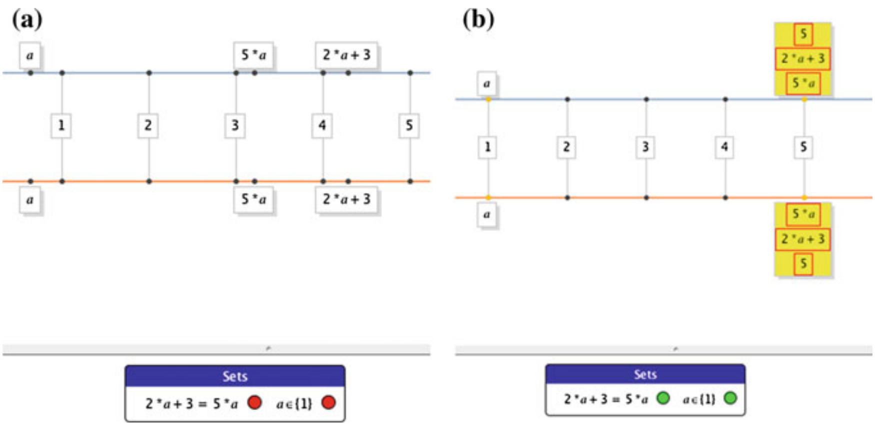

Fig. 9 **a** The expressions do not refer to the same point on the line. Thus, the equation is not true. **b** The expressions refer to the same point on the line. Thus, the equation is true for a = 1

matches/does not match during the dragging of “a” along the line is a visual representation that allows Eleonora to:

– construct the meaning for truth set for equality, as the set of the values making the equality true,
– check the correctness of the truth set.

Note that usually teachers replace truth-values in the equality with letters in order to make the equivalence explicit (passing through a calculation that forces the equal sign to be seen as a “result” rather than a “relation”). This kind of approach does not seem to be very effective for Eleonora. On the contrary, the dynamic representations of the Algebraic Line of AlNuSet allow Eleonora to elaborate the meaning of “=” found between the expressions as a conditioned equality. This suggests that the visuo-spatial approach rather than a computational or verbal one, can be very effective in helping some students with DD.

6.5 *Tasks to Construct the Meaning of the Identity in AlNuSet*

The following figure (Fig. 10) shows two moments when a is in different positions.

By dragging “a” along the line it is possible to verify that the expressions 2a + 3a and 5a refer, for all values of “a”, to the same point on the line and they belong to the same post-it (Fig. 10). The equality is verified for all values of “a”, hence it is an *identity*. This is highlighted by the green colour of the dot associated to the equation 2a + 3a = 5a and by the matching of the colour between

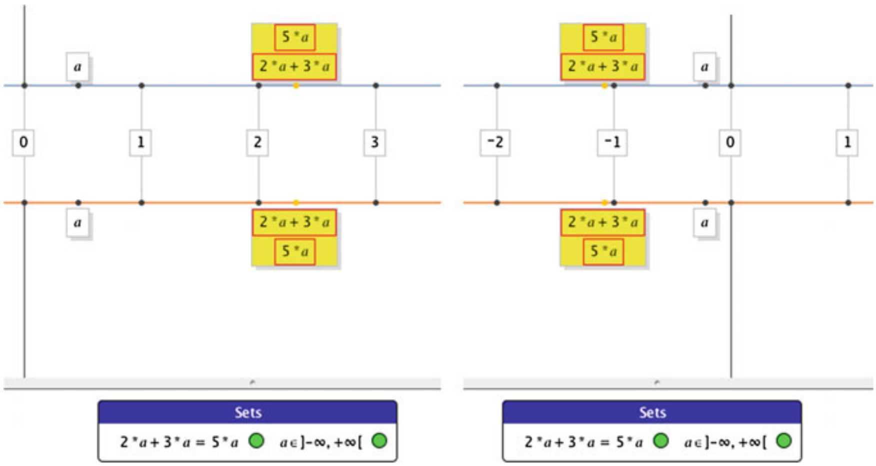

Fig. 10 By dragging *a* along the line, it is possible to verify the equivalence between 2a + 3a and 5a because they belong to the same post-it for all values of *a*

the dot associated to the equation and that associated to the truth set of the equation. Thus, a new meaning for *universal quantifier* (∀) can be constructed through the dragging of “a”.

E: $5 \cdot a$ *is inside* $2 \cdot a + 3 \cdot a$ [referring to the same point] *and they belong to the same post-it. This means that they can have the same value.*
R: Sometimes, always?
E: Always! If I drag “a”, they move correspondingly! But…Why don’t we have the window with the dot?
R: Because we haven’t typed an…
E: yes! We haven’t typed the equality!!

Eleonora edits $5 \cdot a = 2 \cdot a + 3 \cdot a$ on the edit bar of AlNuSet and presses the “enter” button.

E: et voilà, so, we can see that, dragging “a” along the line, the expressions belong to the same post-it… they belong always to the same post-it!

The meaning of *universal quantifier* is perceived by means of the “post-it” sign. This sign belongs to the new set of signs introduced by AlNuSet. The post-it can be related to the mathematical meaning expressed by different signs (e.g. the verbal sign “*for every*” associated to a visual sign, ∀). This last connection, between mathematical meanings and different signs, is only partially accomplished because the researcher did not introduce the sign “∀”. The same applies to the *existential quantifier*. Here, the “post-it” sign is related to the meaning of “equivalence” between the expressions belonging to it. So the equal sign (“=”) can be associated to the meaning of “relation” rather than of “result” of the arithmetical operation. Note that here two different signs (the “post-it” and the “=” sign), referring to different semiotic sets of signs, are related by means of dragging “a” along the line. This allows to construct a new meaning of universal quantifier (and existential quantifier in the previous task). Once again, the sensory-motor system allowed Eleonora to construct meanings for the mathematical objects. In other words, the meaning of *universal* and *existential* quantifiers are built exploiting the perceptive and visuo-spatial approach available in the Algebraic Line.

7 Conclusion

The standard teaching approach to algebra leads to finding, within the algebraic formalism, the meanings of algebraic notions: for instance, the manipulation of an equation allows to finding values that, replaced in the initial equality, make it true; thus, the meaning of the solutions of equation is found in the algebraic manipulation itself. As I tried to show, this does not seem to be an affective approach to algebra for Eleonora, a student with dyscalculia, whose case I discussed as I have found it to be representative of many students with DD that I have worked with or read about in other studies. Referring to studies in mathematics education and neuroscience, I discussed how a perceptual and dynamic approach can be used to effectively construct mathematical knowledge in the case of students with DD. This can be done with the software AlNuSet, designed for teaching dynamic algebra.

In this chapter I considered the Algebraic Line in AlNuSet and I discussed how its possibilities in terms of representation allowed me to design a sequence of verbal tasks that helped a student grasp new meanings of the algebraic notions involved in the solution of the equations and identities. AlNuSet seems to provide a context, in which it is possible to design mathematical tasks that activate particular perceptual-motor activities, as discussed in Nemirovsky, that foster understanding of the algebraic notions involved. Thus, in addition to the standard semiotic representations and registers, (e.g. written symbols, graphs, speech…), there are other important digital resources that can be exploited to design tasks that effectively promote algebraic learning. In particular, the dynamic representations available in AlnuSet (such as the mobile point dragged along the line), symbols (such as the post-it or the coloured dots), and, more in general, extra-linguistic modes of expressions, seem to be particularly appropriate for addressing certain algebraic notions, such as the one analysed.

In conclusion, the new dynamic representations available in the Algebraic Line of AlNuSet allow teachers to design tasks that support the construction of algebraic notions using the visual non-verbal and kinaesthetic channels of access to information. It seems that the dynamicity, expressed by the point dragging along the line, and the correlated representations (post-it, coloured dots, mobile point, truth set,…) are the key functions that allow the student to:

- construct algebraic meanings (for the notions of variable, unknown, solution of equation, truth set of an equation) and algebraic relations among expressions (for example, the equal sign as a relation rather than as a result indicator);
- support fact retrieval from memory when solving algebraic tasks (for example, recalling numerical sets of reference or the rules and the axioms to manipulate algebraic expressions).

These are core aspects of some ongoing and future studies.

References

Artigue, M., & Perrin-Glorian, M. J. (1991). Didactic engineering, research and development tool: some theoretical problems linked to this duality. *For the learning of Mathematics*, *11*(1), 13–17.

Arzarello, F. (2006). Semiosis as a multimodal process. *Relime, Revista latinoamericana de investigación en matemática educativa*, *9*(1), 267–299.

Arzarello, F., & Edwards, L. (2005). Gesture and the construction of mathematical meaning (research forum 2). In *Proceedings of 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 122–145). Melbourne, AU: PME.

Arzarello, F., & Robutti, O. (2001). From body motion to algebra through graphing. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference* (Vol. 1, pp. 33–40). Australia: The University of Melbourne.

Arzarello, F., Bazzini, L., & Chiappini, G. P. (1994). Intensional semantics as a tool to analyse algebraic thinking. *Rendiconti del Seminario Matematico dell'Università di Torino, 52*(1), 105–125.

Baccaglini-Frank, A., & Robotti, E. (2013). Gestire gli studenti con DSA in classe: alcuni elementi di un quadro commune. In *Convegno Grimed 18 "Per piacere voglio contare-difficoltà, disturbi di apprendimento e didattica della matematica*", *Padova*, (pp. 75–86).

Butterworth, B. (2003). *Dyscalculia screening*. London, Uk: nferNelson Publishing Company Limited. (ISBN: 0 7087 0366 6).

Butterworth, B. (2005). Developmental Dyscalculia. In *Handbook of mathematical cognition* (pp. 455–467). Hove, UK: Psychology Press.

Chiappini, G., Robotti, E., & Trgalova, J. (2010). Role of an artifact of dynamic algebra in the conceptualization of the algebraic equality. In *Proceedings of CERME 6*, *Lyon, France* (pp. 619–628). www.inrp.fr/editions/cerme6.

Chaachoua, H., Chiappini, G., Croset, M. C., Pedemonte, B., & Robotti, E., (2012). Introduction de nouvelles rerpésentations dans deux environnements pour l'apprentissage de l'algèbre. *Recherche en Didactique des mathématiques*, 253–281.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. New York, Oxford Unicersity Press.

DeThorne, L. S., & Schaefer, B. A. (2004). A guide to child nonverbal IQ measures. *American Journal of Speech-Language Pathology*, *13*(4), 275–290.

Fandiño Pinilla, M. I. (2005). *Le frazioni, aspetti concettuali e didattici*. Bologna: Pitagora Editrice.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology, 22*(3/4), 455–479.

Hittmair-Delazer, M., Sailer, U., & Benke, T. (1995). Impaired Arithmetic Facts But Intact Conceptual Knowledge a Single—Case Study of Dyscalculia. *Cortex*, *31*(1), 139–147.

Kieran, C. (2006). Research on the learning and teaching of algebra. In G. Gutierrez, & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education. Past, Present and Future* (pp. 11–49). Rotterdam, Taipei: Sense Publishers.

Landy, D., & Goldstone, R. L. (2010). Proximity and precedence in arithmetic. *The Quarterly Journal of Experimental Psychology (Colchester), 63*, 1953–1968.

Leung, A., & Bolite-Frant, J. (2015). Designing mathematics tasks: The role of tools. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: The 22nd ICMI study (new ICMI study series)* (pp. 191–225). New York: Springer.

Mammarella, I. C., Giofrè, D., Ferrara, R., & Cornoldi, C. (2013). Intuitive geometry and visuospatial working memory in children showing symptoms of nonverbal learning disabilities. *Child Neuropsychology*, *19*(3), 235–249.

Mammarella, I. C., Lucangeli, D., & Carnoldi, C. (2010). Spatial working memory and arithmetic deficits in children with non verbal learning difficulties. *Journal of Learning Disability*, *43*, 455–468.

Mariani, L. (1996). Investigating learning styles, perspectives. *Journal of TESOL-Italy, XXI, 2/ XXII, 1*, (pp. 35–49). Spring.

Mason, J., & Johnston-Wilder, S. (2006). *Designing and using mathematical tasks*. Tarquin.

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 103–135). Honolulu, Hawaii: PME. C. K. Ogden, & I. A. Richards (1923).

Núñez, R. (2000). Mathematical idea analysis: What embodied cognitive science can say about the human nature of mathematics. In XXX (Eds.), *Proceedings of the 24 PME Conference* (Vol. 1, pp. 3–22). Japan: Hiroshima University.

Radford, L. (2005). Body, tool, and symbol: semiotic reflections on cognition. In E. Simmt, & B. Davis (Eds.), *Proceedings of the 2004 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 111–117). Toronto, Canada.

Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and individual differences*, *20*(2), 110–122.

Robotti, E. (2013). Dynamic representations for algebraic objects available in AlNuSet: How develop meanings of the notions involved in the equation solution. In C. Margolinas (Ed.), *Task design in mathematics education: Proceedings of ICMI study 22* (pp. 99–108). UK: Oxford.

Robotti, E., & Baccaglini-Frank, A. (2016). Using digital environments to address students' mathematical learning difficulties. In E. Faggiano, F. Ferrara, & A. Montone (Eds.), *Innovation and technology enhancing mathematics education. Perspectives in the digital era.* Springer Publisher (accepted).

Robotti, E., & Ferrando, E. (2013). Difficulties in algebra: new educational approach by AlNuSet. In E. Faggiano, & A. Montone (Eds.), *Proceedings of ICTMT11*, 250–25. Italy: ICTMT.

Robutti, O. (2005). Hearing gestures in modelling activities with the use of technology. In F. Olivero, & R. Sutherland (Eds.), *Proceedings of the 7th international conference on technology in mathematics teaching* (pp. 252–261). University of Bristol.

Rourke, B. P., & Conway, J. A. (1997). Disabilities of arithmetic and mathematical reasoning perspectives from neurology and neuropsychology. *Journal of Learning disabilities*, *30*(1), 34–46.

Sfard, A., & Linchevsky, L. (1992). Equations and inequalities: Processes without objects? In W. Goeslin, K. Graham (Ed.), *Proceedings PME XVI, Durham, NH* (Vol. 3, p. 136).

Stella, G., & Grandi, L. (2011). *Conoscere la dislessia e i DSA*. Giunti Editore.

Szucs, D., Devine, A., Soltesz, F., Nobes, A., & Gabriel, F. (2013). Developmental dyscalculia is related to visuo-spatial memory and inhibition impairment. *Cortex*, *49*(10), 2674–2688.

Temple, C. M. (1991). Procedural dyscalculia and number fact dyscalculia: Double dissociation in developmental dyscalculia. *Cognitive Neuropsychology, 8*, 155–176.

Thomas, M. O. J., & Tall, D. O. (2001). The long-term cognitive development of symbolic algebra. In H. Chick, K. Stacey, J. Vincent & J. T. Zilliox (Ed.), *International Congress of Mathematical Instruction (ICMI) Working Group Proceedings—the future of the teaching and learning of algebra, Melbourne (*Vol. 2, pp. 590–597).

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. *Addition and subtraction: A cognitive perspective*, 39–59.