On Necessary and Sufficient Conditions for Eigenstrain-Type Control of Stresses in the Dynamics of Force-Loaded Elastic Bodies

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Abstract In the present contribution, the possibility of controlling dynamic stresses in force-loaded bodies by means of actuating eigenstrain fields is addressed. The action of eigenstrains, such as thermal or piezoelectric actuating strains, is subsequently gathered under the notion of actuating stresses. Our study is performed in the framework of the theory of small incremental dynamic deformations superimposed upon a state of possibly large static pre-deformation of a hyperelastic body. Particularly, we present a solution for the general problem of producing certain incremental stress trajectories by means of specifically tailored actuation stresses that are superimposed onto the force-loaded body. This we shortly call the stress tracking problem. The problem of suppressing incremental stresses is contained as a special case. Subsequently, particular emphasis is given to the systematic derivation of necessary and sufficient conditions that must be satisfied in order to solve the stress tracking problem. Necessary conditions are presented that must be satisfied by the intermediate configuration and by the desired incremental stress field that shall be tracked, and sufficient conditions are derived that must be satisfied by the incremental actuating stresses. As an illustrative example, our three-dimensional formulation is eventually applied to the one-dimensional dynamic case of a straight homogeneous rod with a support excitation at one end and a single point-mass at the other end.

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1 Fundamental Relations

1.1 Local Balance of Linear Momentum and Boundary Conditions in the Actual Configuration

In the following, we use a natural (unstressed) state of the body as common reference configuration. The local relation of balance of linear momentum reads:

$$\operatorname{Div} P + b_0 = \rho_0\, \ddot{u} \tag{1}$$

The first Piola–Kirchhoff stress tensor is denoted as P, and Div stands for the divergence operator with respect to the place in the reference configuration. The imposed body force per unit volume in the reference configuration is b_0, and ρ_0 is the mass density in the reference configuration. The total displacement vector from the reference configuration is written as u, and a superimposed dot denotes the material time derivative. On some portions ∂B_1 of the boundary $\partial B = \partial B_1 \cup \partial B_2$, kinematic boundary conditions are prescribed:

$$\partial B_1\colon \quad u = u^* \tag{2}$$

The imposed surface displacements at ∂B_1 are denoted as u^*. At ∂B_2, dynamic boundary conditions (boundary conditions of traction) are given:

$$\partial B_2 : \quad P\, n_0 = t_0^* \tag{3}$$

Here, n_0 stands for the unit outer normal vector at the surface in the reference configuration, and t_0^* is the imposed Lagrange surface traction.

1.2 Introduction of a Static Intermediate Configuration

Incremental dynamic displacements and stresses are considered relative to an intermediate configuration, which in general may be a (possibly) large static pre-deformation from the reference configuration. The use of a common reference configuration allows applying the following additive decomposition:

$$b_0 = b_{0i} + b_{0+}, \quad t_0^* = t_{0i}^* + t_{0+}^*, \quad u = u_i + u_+, \quad P = P_i + P_+ \tag{4}$$

The subscript (i) refers to the static intermediate configuration, and the subscript $(+)$ indicates the dynamic increments from the intermediate configuration. Substituting into Eqs. (1)–(4) and subtracting the relations for the intermediate state gives:

$$\operatorname{Div} P_+ + b_{0+} = \rho_0\, \ddot{u}_+ \tag{5}$$

$$\partial B_1: \quad u_+ = u_+^* \tag{6}$$

$$\partial B_2: \quad P_+ n_0 = t_{0+}^* \tag{7}$$

Since the intermediate state is assumed to be static, i.e., it is at rest, trivial initial conditions for the incremental motion are obtained:

$$t = 0: \quad u_+ = 0, \quad \dot{u}_+ = 0 \tag{8}$$

1.3 Linearization About the Incremental State

From now on, we consider infinitesimally small incremental dynamic deformations superimposed upon the intermediate state. The latter may represent a large static pre-deformation from the common reference configuration. Considering a hyperelastic body, the linearized constitutive relations read

$$P_+ = A_i\,[\mathrm{Grad}\,u_+] + P_{a+} \tag{9}$$

The fourth order tensor of elastic constants in the intermediate configuration is abbreviated by A_i, and Grad denotes the gradient operator with respect to the place in the reference configuration. For the square bracket notation, which indicates the linear mapping of a second order tensor onto a second order tensor via a fourth order tensor, see Gurtin [1]. The incremental actuation stress tensor, a second order tensor, is denoted as P_{a+}. It represents a linear mapping of the actuating incremental eigenstrains. In case of thermal eigenstrains, it relates the stress to the temperature via the second order stress–temperature tensor, see Carlson [2] for the linear theory of thermoelasticity, i.e., when intermediate and natural reference configuration do coincide. When using eigenstrains for the purpose of controlling deformations, one also talks about a smart actuation in the literature. Note that in case of an intermediate state with a large deformation from the reference configuration, A_i as well as P_{a+} depend on the intermediate state and thus do vary across the body, even if it is homogeneous in the natural reference configuration.

1.4 Stress-Based Reformulation

Since we deal with stress tracking, a stress-based reformulation of the above incremental relations is desirable. In the framework of the linear theory of elasticity, i.e., when the intermediate configuration coincides with the natural reference configuration, this strategy dates back to Ignaczak [3] and [4]. This reformulation requires that A_i is invertible:

$$\mathrm{Grad}\, u_+ = K_i\,[P_+ - P_{a+}] \quad \text{with} \quad K_i = {A_i}^{-1} \tag{10}$$

The existence of the compliance tensor K_i in any point of the body represents a first necessary condition in order that our subsequent solutions for the stress tracking problem make sense. Substituting Eq. (10) into Eqs. (5)–(8), we obtain the following stress-based reformulation of the incremental initial boundary value problem under consideration:

$$\mathrm{Grad}\left({\rho_0}^{-1}\,(\mathrm{Div}\, P_+ + b_{0+})\right) = K_i\left[\ddot{P}_+ - \ddot{P}_{a+}\right] \tag{11}$$

$$\partial B_1\!: \quad \mathrm{Div}\, P_+ + b_{0+} = \rho_0\, \ddot{u}^*_+ \tag{12}$$

$$\partial B_2: \quad P_+\, n_0 = t^*_{0+} \tag{13}$$

$$t = 0: \quad P_+ - P_{a+} = 0, \quad \dot{P}_+ - \dot{P}_{a+} = 0 \tag{14}$$

2 The Dynamic Stress Tracking Problem

2.1 Formulation of the Stress Tracking Problem

We now formulate the stress tracking problem as follows: derive a space- and time-wise distribution of an incremental actuation stress field P_{a+}, such that the above initial boundary value problem, Eqs. (5)–(9), results in a desired incremental stress field Z everywhere in the body under consideration and for all times:

$$P_+ \equiv Z \tag{15}$$

Note that the desired incremental stress field Z may be both, space- and time-dependent. For a convenient solution strategy, we introduce an error stress field:

$$P_e = P_+ - Z \tag{16}$$

Our goal in the following will be enforcing that the error stress field does vanish everywhere and for all times, $P_e = 0$.

2.2 Direct Solution of the Stress Tracking Problem

In a first step, we derive a direct solution strategy for the stress tracking problem by replacing the incremental stress P_+ by the entities Z and P_e, see Eq. (16), in the stress-based formulation in Eqs. (11)–(14). Putting the expressions that contain P_e

onto the left-hand side of the corresponding relations yields:

$$\begin{aligned} &\text{Grad}\left(\rho_0{}^{-1}\text{Div}\,P_e\right) - K_i\left[\ddot{P}_e\right] \\ &\quad = -\,\text{Grad}\left(\rho_0{}^{-1}\left(\text{Div}\,Z + b_{0+}\right)\right) + K_i\left[\ddot{Z} - \ddot{P}_{a+}\right] \end{aligned} \tag{17}$$

$$\partial B_1\colon \quad \text{Div}\,P_e = -\text{Div}\,Z - b_{0+} + \rho_0\,\ddot{u}^*_+ \tag{18}$$

$$\partial B_2: \quad P_e\,n_0 = -Z\,n_0 + t^*_{0+} \tag{19}$$

$$t = 0: \quad P_e = P_{a+} - Z, \quad \dot{P}_e = \dot{P}_{a+} - \dot{Z} \tag{20}$$

The desired goal, $P_e \equiv 0$, then may be reached, when the right-hand sides of Eqs. (17)–(20) do vanish. First, note that Eqs. (18) and (19) result into two necessary conditions that must be satisfied by the desired incremental stress field Z:

$$\partial B_1\colon \quad \text{Div}\,Z = -b_{0+} + \rho_0\,\ddot{u}^*_+ \tag{21}$$

$$\partial B_2: \quad Z\,n_0 = t^*_{0+} \tag{22}$$

In other words, at the boundary of the body B, the desired incremental stress field Z cannot be chosen independently from the imposed incremental body forces and boundary data. Moreover, two sufficient conditions for the incremental actuation stress follow from Eqs. (17) and (20):

$$\ddot{P}_{a+} = -A_i\left[\text{Grad}\left(\rho_0{}^{-1}\left(\text{Div}\,Z + b_{0+}\right)\right] + \ddot{Z} \tag{23}$$

$$t = 0: \quad P_{a+} = Z, \quad \dot{P}_{a+} = \dot{Z} \tag{24}$$

In the present context of (infinitesimally) small incremental deformations superimposed upon the large pre-deformation of the intermediate state, it is required that the latter is stable in some sense, such that a further necessary condition must be formulated.

2.3 Stability Issues

In order to derive a condition for the necessary stability of the intermediate configuration, we now utilize a strategy originally suggested by Ignaczak [3] for studying the completeness of a stress-based formulation in the framework of the linear case. By analogy, we introduce the following scalar error integral over the volume in the reference configuration, but referring to the incremental error stress:

$$I_e\left(t\right) = \int_{B_0} \left(\rho_0^{-1}\,\text{Div}\,P_e \cdot \text{Div}\,P_e + K_i\left[\dot{P}_e\right] \cdot \dot{P}_e\right)\,dV_0 \tag{25}$$

Now assume that initially no errors are present:

$$P_e\,(t=0)=0,\quad \dot{P}_e\,(t=0)=0\;\Rightarrow\quad I_e\,(t=0)=0 \tag{26}$$

Considering the major symmetry property of the fourth order tensor of compliance

$$K_i\,[B]\cdot D = K_i\,[D]\cdot B \tag{27}$$

see, e.g., Knops and Wilkes [5], the time derivative of Eq. (25) follows to:

$$\frac{1}{2}\frac{d}{dt}I_e = \int_{B_0}\left(\rho_0^{-1}\,\mathrm{Div}\,P_e\cdot\mathrm{Div}\,\dot{P}_e + \ddot{P}_e\cdot K_i\left[\dot{P}_e\right]\right)\,dV_0 \tag{28}$$

Using some results from tensor algebra and analysis, it can be shown after some reformulations, using Eqs. (21)–(24) and (26), that the necessary and sufficient conditions for $P_e\equiv 0$ yield that

$$\begin{aligned}&\frac{d}{dt}I_e = 0 \Rightarrow I_e\,(t) = \text{const.} = I_e\,(t=0)\\ &\quad\Rightarrow I_e\,(t) = \int_{B_0}\left(\rho_0^{-1}\,\mathrm{Div}\,P_e\cdot\mathrm{Div}\,P_e + K_i\left[\dot{P}_e\right]\cdot\dot{P}_e\right)\,dV_0\equiv 0\end{aligned} \tag{29}$$

Now, the first part of the integral in Eq. (29) is positive semi-definite:

$$\int_{B_0}\left(\rho_0^{-1}\,\mathrm{Div}\,P_e\cdot\mathrm{Div}\,P_e\right)\,dV_0\begin{cases}=0 & \text{for}\quad P_e=0\\ >0 & \text{for}\quad P_e\neq 0\end{cases} \tag{30}$$

However, the second part of the integral is generally indefinite

$$\int_{B_0}\left(K_i\left[\dot{P}_e\right]\cdot\dot{P}_e\right)\,dV_0\begin{cases}=0 & \text{for}\quad \dot{P}_e=0\\ <0 & \\ >0 & \text{for}\quad \dot{P}_e\neq 0\end{cases} \tag{31}$$

Thus, vanishing of the error integral implies that the error stress vanishes only if:

$$P_e\equiv 0\quad\text{if}\quad\int_{B_0}K_i\left[\dot{P}_e\right]\cdot\dot{P}_e\,dV_0>0\quad\text{for}\quad\dot{P}_e\neq 0 \tag{32}$$

This necessary condition is analogous to the Hadamard stability condition, see Knops and Wilkes [5]

$$\int_{B_0} K_i\left[\dot{P}_e\right] \cdot \dot{P}_e \, dV_0 > 0 \quad \text{for} \quad \dot{P}_e \neq 0 \tag{33}$$

This necessary condition is also known as infinitesimal superstability of the intermediate configuration under consideration. When the intermediate configuration and the natural reference configuration do coincide, i.e., in the linear theory of infinitesimally small deformations superimposed upon an undeformed configuration, stability is pre-assumed. In the present case of a possibly large deformation of the intermediate configuration from the reference configuration, however, Eq. (33) represents a practically important requirement.

2.4 Recalling the Three-Dimensional Solution

The above results for solving the stress tracking problem are shortly summarized. If the following two necessary conditions hold at the boundary of the body:

$$\partial B_1: \quad \text{Div}\, Z = -b_{0+} + \rho_0 \, \ddot{u}_+^*, \tag{34}$$

$$\partial B_2: \quad Z\, n_0 = t_{0+}^*, \tag{35}$$

and moreover if the compliance tensor does exist in every point in the intermediate configuration, and if the Hadamard stability condition stated in Eq. (33) does hold, then, in order that the goal of stress tracking is reached,

$$P_+ \equiv Z, \tag{36}$$

it is sufficient to use an eigenstrain actuation satisfying the following two relations:

$$\ddot{P}_{a+} = -A_i\left[\text{Grad}\left(\rho_0{}^{-1}\left(\text{Div}\, Z + b_{0+}\right)\right)\right] + \ddot{Z}, \tag{37}$$

$$t = 0: \quad P_{a+} = Z, \quad \dot{P}_{a+} = \dot{Z} \tag{38}$$

For preliminary formulations concerning the linear case of infinitesimally small deformations superimposed upon the natural reference configuration, see Irschik, Gusenbauer and Pichler [6] and Irschik [7]. The solution strategy gathered in Eqs. (34)–(38) will be subsequently exemplified.

3 Illustrative Example: Straight Rod

3.1 One-Dimensional Boundary Value Problem

In the one-dimensional case, the relation of balance of incremental linear momentum, Eq. (5), becomes

$$\frac{\partial}{\partial X} P_+(X,\ t) + b_{0+}(X,\ t) = \rho_0(X)\,\frac{\partial^2}{\partial t^2}\,u_+(X,\ t) \tag{39}$$

The axial coordinate in the reference configuration is denoted as $0 \leq X \leq L$. The incremental boundary conditions of place and traction, Eqs. (6) and (7), and the trivial initial conditions for the incremental motion from the static intermediate configuration, see Eq. (8), read

$$X = 0\colon \quad u_+(X = 0,\ t) = u_+^*(t) \tag{40}$$

$$X = L: \quad P_+(X = L,\ t) = t_{0+}^*(t) \tag{41}$$

$$t = 0: \quad u_+(X,\ t = 0) = 0, \quad \dot{u}_+(X,\ t = 0) = 0 \tag{42}$$

The one-dimensional form of the linearized constitutive relation in Eq. (10) is

$$\frac{\partial}{\partial X}\,u_+(X,\ t) = K_i(X)\,(P_+(X,\ t) - P_{a+}(X,\ t)) \tag{43}$$

The Hadamard stability condition for the intermediate configuration, Eq. (33), locally reduces to

$$K_i = 1/A_i > 0 \tag{44}$$

3.2 Conditions for Stress Tracking in the One-Dimensional Case

We assume a support excitation $u_+^*(t)$ at $X = 0$ and boundary conditions of traction at $X = L$:

$$X = 0\colon \quad \frac{\partial}{\partial X} Z(0,\ t) = -b_{0+}(0,\ t) + \rho_0(0)\,\frac{\partial^2}{\partial t^2} u_+^*(t) \tag{45}$$

$$X = L: \quad Z(L,\ t) = t_{0+}^*(t) \tag{46}$$

Sufficient conditions for stress tracking, $P_+(X,\ t) \equiv Z(X,\ t)$, then become, see Eqs. (37) and (38):

$$\frac{\partial^2}{\partial t^2} P_{a+}(X,\ t) = -A_i(X) \left[\frac{\partial}{\partial X} \left({\rho_0}^{-1}(X) \, (\frac{\partial}{\partial X} Z(X,\ t) + b_{0+}(X,\ t)) \right) \right] + \frac{\partial^2}{\partial t^2} Z(X,\ t) \tag{47}$$

$$t = 0: \quad P_{a+}(X,\ 0) = Z(X,\ 0), \quad \dot{P}_{a+}(X,\ 0) = \dot{Z}(X,\ 0) \tag{48}$$

3.3 Linear Elastic Rod with End-Mass and Support Excitation

For simplicity sake, from now on we restrict to the case of an intermediate configuration, which coincides with the natural reference configuration, such that we deal with the linear theory of elastic bodies in the presence of eigenstrains. We take body forces to be absent, $b_{0+}(X,\ t) = 0$, and assume that the mass density ρ_0, the cross-section a_0 and the elastic constant A_0 (the effective Young's modulus) of the rod are constant. As complicating aspects, the rod, however, is assumed to be firmly connected to a single point-mass M at the free end $X = L$. The boundary condition of traction at this end of the rod thus becomes, see Eqs. (39) and (41):

$$X = L: \quad P_+(L,\ t) = t^*_{0+}(t) = -\frac{M}{a_0} \frac{\partial^2}{\partial t^2} u_+(L,\ t) = -\frac{M}{\rho_0 \, a_0} \frac{\partial}{\partial X} P_+(L,\ t) \tag{49}$$

Now, let the desired stress be separable in space and time $Z(X, t) = z(X) \ \ \rho_0 \frac{\partial^2}{\partial t^2} u^*_+(t)$, then the necessary conditions stated in Eqs. (45) and (46) yield that

$$X = 0: \quad \frac{\partial}{\partial X} z(0) = 1 \tag{50}$$

$$X = L: \quad z(L) + \frac{M}{\rho_0 \, a_0} \frac{\partial}{\partial X} z(L) = 0 \tag{51}$$

A suitable function $z(X)$ is chosen, which satisfies the necessary conditions, Eqs. (50) and (51), and the sufficient conditions, Eqs. (47) and (48), are eventually solved for a given $u^*_+(t)$, resulting in an actuation stress field that satisfies $P_+(X,\ t) \equiv Z(X,\ t)$. This strategy is subsequently demonstrated in an example. The following peak-type functional dependence for the support excitation is taken into consideration in this example, see also Fig. 1, where C and α are constants:

$$u^*_+(t) = C(\alpha \, t)^3 \exp{(-\alpha \, t)} \, L \tag{52}$$

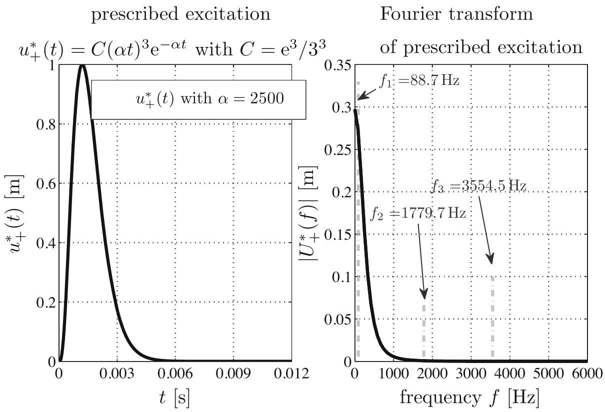

Fig. 1 Support excitation at $X = 0$ and Fourier transform

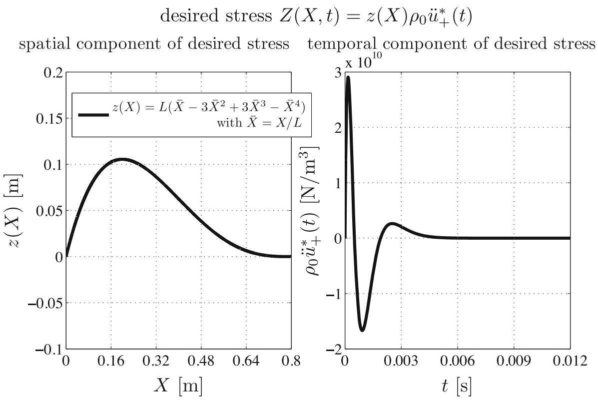

Fig. 2 Spatial and temporal distribution of the desired stress $Z(X,\ t)$

In Fig. 1, the first three natural frequencies of the rod are marked as f_1, f_2 and f_3. The following values (in SI-units) have been chosen in the numerical computations: $C = (e/3)^3$, $\alpha = 2500$, modulus of elasticity $A_0 = 6.29 \times 10^9$, cross-section $a_0 = 4 \times 10^{-5}$, mass density $\rho_0 = 7750$, single mass $M = 10$, length of rod $L = 0.8$. From Eqs. (48) and (52), we find that:

$$P_{a+}(X,\ 0) = 0, \quad \frac{\partial}{\partial t} P_{a+}(X,\ 0) = 0 \tag{53}$$

For the desired stress $Z(X,\ t)$, we use, see also Fig. 2:

$$z(X) = L\left(\bar{X} - 3\bar{X}^2 + 3\bar{X}^3 - \bar{X}^4\right) \quad \text{with} \quad \bar{X} = X/L \tag{54}$$

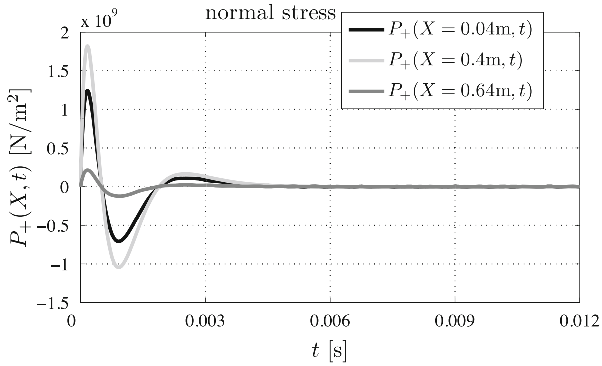

Fig. 3 Time-wise distribution of stress in three locations; controlled case

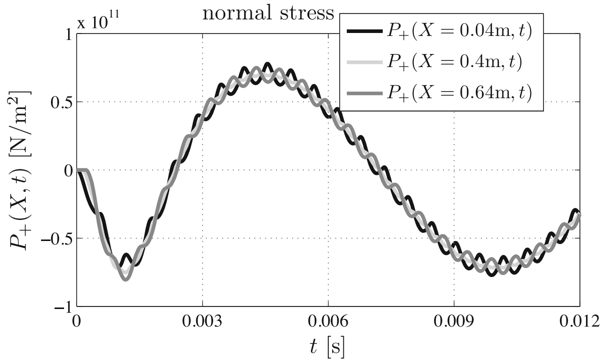

Fig. 4 Time-wise distribution of stress in three locations; uncontrolled case ($P_{a+}(X,\ t) = 0$)

Note that the selected spatial distribution $z(X)$ in Eq. (54) satisfies the necessary conditions stated in Eqs. (50) and (51). The required actuation stress $P_{a+}(X,\ t)$ is eventually found by integration of Eq. (47), using the trivial initial conditions stated in Eq. (53). It is found that the desired stress $Z(X, t)$ is indeed obtained by our method. Results are depicted in the following figures for the controlled (Fig. 3) and the uncontrolled case (Fig. 4). Note that the stress maxima in the uncontrolled case are substantially higher than in the controlled case. A more detailed discussion and further examples will be given in a forthcoming contribution, Schoeftner and Irschik [8].

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