Chapter 5
Ergodicity of Geodesic Flows on Incomplete Negatively Curved Manifolds

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5.1 Introduction

This chapter arose from notes taken by the third author during a series of lectures by the first author in the workshop *Young mathematicians in dynamical systems*. Its goal is to explain an *ergodicity criterion* (for geodesic flows on incomplete negatively curved manifolds) used by Burns–Masur–Wilkinson in their proof [BMW] of the ergodicity of the so-called Weil–Petersson (WP) flow.

5.1.1 Ergodicity Criterion for a Certain Class of Geodesic Flows

Consider the quotient $N = M/\Gamma$ where M is a contractible, negatively curved, Riemannian manifold and Γ is a subgroup of isometries of M acting freely and

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properly discontinuously. The metrics on N and M induced by the Riemannian metric of M are both denoted by d.

Let $\overline{N}$ be the *metric completion* (N, d), i.e., the metric space formed by equivalence classes of Cauchy sequences $\{x_n\} \subset N$ under the relation

$$\{x_n\} \sim \{y_n\} \iff \lim_{n\to\infty} d(x_n, y_n) = 0$$

equipped with the metric $d(\{x_n\}, \{z_n\}) = \lim_{n\to\infty} d(x_n, z_n)$. In this context, the set $\partial N := \overline{N} - N$ is called (Cauchy) *boundary* of N.

In [BMW], Burns–Masur–Wilkinson developed the following criterion for the ergodicity of *certain* geodesic flows.

Theorem 5.1.1 (Ergodicity criterion for singular hyperbolic geodesic flows) *Let $N = M/\Gamma$ be a manifold as above. Suppose that:*

(I) the universal cover M of N is geodesically convex*, i.e., for every $p, q \in M$, there exists an unique geodesic segment* in *M connecting p and q.*
(II) the metric completion $\overline{N}$ of (N, d) *is* compact*.*
(III) the boundary ∂N is volumetrically cusplike*, i.e., for some constants $C > 1$ and $\nu > 0$, the volume of a ρ-neighborhood of the boundary satisfies*

$$Vol(\{x \in N : d(x, \partial N) < \rho\}) \leq C\rho^{2+\nu}$$

for every $\rho > 0$.
(IV) N has polynomially controlled curvature*, i.e., there are constants $C > 1$ and $\beta > 0$ such that the curvature tensor R of N and its first two derivatives satisfy the following polynomial bound*

$$\max\{\|R(x)\|, \|\nabla R(x)\|, \|\nabla^2 R(x)\|\} \leq Cd(x, \partial N)^{-\beta}$$

for every $x \in N$.
(V) N has polynomially controlled injectivity radius*, i.e., there are constants $C > 1$ and $\beta > 0$ such that*

$$inj(x) \geq (1/C)d(x, \partial N)^{\beta}$$

for every $x \in N$ (where $inj(x)$ denotes the injectivity radius at x).
(VI) The first derivative of the geodesic flow *φ_t is* polynomially controlled*, i.e., there are constants $C > 1$ and $\beta > 0$ such that, for every infinite geodesic γ on N and every $t \in [0, 1]$:*

$$\|D_{\dot\gamma(0)}\varphi_t\| \leq Cd(\gamma([-t, t]), \partial N)^{\beta}$$

Then, the Liouville (volume) measure m of N is finite, the geodesic flow φ_t on the unit cotangent bundle T^1N of N is defined at m-almost every point for all time t, and the geodesic flow φ_t is nonuniformly hyperbolic *(in the sense of Pesin's theory) and* ergodic.

Actually, the geodesic flow φ_t is Bernoulli and, furthermore, its measure-theoretic entropy $h(\varphi_t)$ is positive, finite and $h(\varphi_t)$ is given by Pesin's entropy formula (i.e., $h(\varphi_t)$ is the sum of positive Lyapunov exponents of φ_t counted with multiplicities).

This statement appears as Theorem 3.1 in [BMW]. Its main application was Theorem 1 in [BMW] ensuring the ergodicity of the so-called *Weil–Petersson geodesic flow* on *moduli spaces of Riemann surfaces*:

Theorem 5.1.2 (Ergodicity of the Weil–Petersson geodesic flow) *The WP flow on the unit cotangent bundle $T^1\mathscr{M}_{g,n}$ of the moduli space $\mathscr{M}_{g,n}$ of Riemann surfaces of genus g with n marked points is ergodic with respect to the Liouville measure μ_{WP} of the WP metric whenever $3g - 3 + n \geq 1$. Actually, it is Bernoulli (i.e., it is measurably isomorphic to a Bernoulli shift) and, a fortiori, mixing. Furthermore, its measure-theoretic entropy $h(\mu_{WP})$ is positive and finite.*

The Teichmüller-theoretical aspects involved the reduction of Theorem 5.1.2 to Theorem 5.1.1 were the subject of Matheus' lectures (see [Ma]) and, for this reason, from now on we shall focus exclusively on the discussion of the proof of Theorem 5.1.1.

5.1.2 Outline of Proof of Theorem 5.1.1

The starting point of the proof of Burns–Masur–Wilkinson Theorem is the so-called *Hopf's argument* for the ergodicity of dynamical systems with some hyperbolicity.

5.1.2.1 Hopf's Argument for Anosov Systems

Let (X, d) be a compact Riemannian manifold and denote by μ the corresponding volume measure.

Given a smooth flow $(\psi^t)_{t\in\mathbb{R}}: X \to X$ preserving μ and a continuous observable $f: X \to \mathbb{R}$, let

$$f^+(x) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(\psi^s(x))\,ds \quad \text{and} \quad f^-(x) := \lim_{T\to-\infty} \frac{1}{T}\int_0^T f(\psi^s(x))\,ds$$

be the future and past *Birkhoff averages*. This nomenclature is motivated by the *Birkhoff Ergodic Theorem* (cf. Sect. 6.3 of [Ha]) stating that, for μ-almost every $x \in X$, the Birkhoff averages $f^+(x)$ and $f^-(x)$ exist and, actually, they coincide $f^+(x) = f^-(x) := \tilde{f}(x)$.

Remark 5.1.3 A point x such that $f^+(x), f^-(x)$ exist and $f^+(x) = f^-(x) = \tilde{f}(x)$ is called a μ-*Birkhoff generic* point.

Recall that (ψ^t, μ) is ergodic if and only if the Birkhoff averages f^+ and f^- are *constant* at μ-almost every point.

In this direction, Hopf [Ho] notices that the function f^+ is constant along stable sets

$$W^s(x) := \{y : \lim_{t\to+\infty} d(\psi^t(y), \psi^t(x)) = 0\},$$

i.e., $f^+(x) = f^+(y)$ whenever $y \in W^s(x)$. We leave the verification of this elementary fact as an exercise to the reader.

If ψ^t is an *Anosov flow* on X, then the stable and unstable sets are immersed *submanifolds*: see, e.g., Sect. 5.5 of Hasselblatt's notes [Ha]. Moreover, if one forgets about the flow direction, the stable and unstable manifolds are transverse. Thus, we can connected any $p, q \in X$ (in distinct ψ^t-orbits) with pieces of stable and unstable manifolds.

This *suggests* that volume-preserving Anosov flows are ergodic because the future and past Birkhoff averages are constant along stable and unstable manifolds, they coincide almost everywhere and any pair of points can be connected via pieces of stable and unstable manifolds.

Nevertheless, there is an important ingredient missing in the ergodicity argument above: in fact, one needs to know that the intersection points $z_1, \ldots, z_n$ between the pieces of stable and unstable manifolds connecting p and q are *Birkhoff generic* in order to conclude that $\tilde{f}(p) = \tilde{f}(z_1) = \cdots = \tilde{f}(z_n) = \tilde{f}(q)$.

In his original paper, Hopf [Ho] dealt with geodesic flows on compact surfaces of *constant* negative curvature. In this setting, the stable and unstable manifolds form C^1 foliations and this permits to take Birkhoff generic intersection points $z_1, \ldots, z_n$. Indeed, the C^1-regularity allows to apply the *Fubini Theorem* to the full μ-volume set $\mathscr{B}$ of Birkhoff generic points, so that almost all stable and unstable manifolds $W^s(x)$ and $W^u(x)$ intersect $\mathscr{B}$ in a subset of total length measure of $W^s(x)$ and $W^u(x)$ (see the proof of Proposition 4.10 of [Ha]).

After this quick review of Hopf's argument, let us explain some of the difficulties in extending this idea to the setting of Theorem 5.1.1.

5.1.2.2 Hopf's Argument in the Context of Singular Hyperbolic Geodesic Flows

The celebrated *Pesin Stable-Manifold Theorem* [Pe2] asserts that, if ψ^t is a volume-preserving *nonuniformly hyperbolic system*, then the stable and unstable sets of almost all points are immersed submanifolds. Moreover, modulo the flow direction, the stable and unstable manifolds have complementary dimensions. Furthermore, the stable and unstable manifolds are part of absolutely continuous laminations.

This gives some hope that Hopf's argument *could be* extended to show the ergodicity of volume-preserving nonuniformly hyperbolic systems.

However, an inspection of Hopf's argument reveals that the uniform hyperbolicity of Anosov flows was strongly used. For example, if we want to connect two points at *large* distances using pieces of stable and unstable manifolds, then we might need some "uniformity" of stable and unstable manifolds, e.g., they are graphs of *definite size* and *bounded curvature* intersecting with an angle *uniformly bounded away from zero*.

In other terms, it is not easy to run Hopf's argument if the stable and unstable manifolds get shorter and/or have huge curvature and/or intersect with very small angle at certain points, then one might not be able to connect certain pairs of points with pieces of stable and unstable manifolds.

As it turns out, there are some concrete examples where the presence of these "nonuniformities" actually *prevent* the utilization of Hopf's argument: for instance, Dolgopyat–Hu-Pesin [DHP] constructed volume-preserving nonuniformly hyperbolic systems with countably many ergodic components consisting of invariant sets of positive volumes that are essentially open.

In summary, the ergodicity of a nonuniformly hyperbolic system *depends* on the particular dynamical features of the given system.

For the proof of Theorem 5.1.1, the inspiration comes from Katok–Strelcyn [KS] extension of Pesin's theory [Pe2] to *singular hyperbolic systems*.

Very roughly speaking, the basic philosophy behind Katok–Strelcyn's work is the following. Given a nonuniformly hyperbolic system with some nontrivial singular set, all statements predicted by Pesin theory in virtue of the *exponential* contraction/expansion are *not* affected *if* the loss of control of (the first two derivatives of) the system is at most *polynomial* as one approaches the singular set. In particular, this *indicates* that Hopf's argument can be extended to singular hyperbolic systems with "polynomially bad" singular sets and this is exactly the basic idea behind the proof of Theorem 5.1.1.

5.1.3 Organization of the Text

We organize these lectures notes as follows. Section 5.2 contains introductory material on the geometry of the Sasaki metric on the tangent bundle of a Riemannian manifold. Section 5.3 discusses the description of the first derivative of a geodesic flow in terms of Jacobi fields. Section 5.4 contains the proof of the nonuniform hyperbolicity statement in Theorem 5.1.1. Section 5.5 is dedicated to the application of the results of Katok–Strelcyn in order to derive the absolute continuity of the stable and unstable foliations of certain geodesic flows. Finally, Sect. 5.6 contains the proof of Theorem 5.1.1 along the lines of Hopf's argument.

5.2 Geometry of Tangent Bundles

5.2.1 Riemannian Metrics and Curvature Tensors

Let M be a Riemannian manifold and denote by $\langle\cdot,\cdot\rangle$ its Riemannian metric of M.

Let ∇ be the associated *Levi-Cività connection*, i.e., the unique connection[1] that is symmetric and compatible with the Riemannian metric $\langle\cdot,\cdot\rangle$. Given a curve $c\colon t \mapsto c(t)$ on M, the *covariant derivative* along c is

$$D_c := \nabla_{\dot{c}(t)} =: \frac{D}{dt}$$

(and it should *not* be confused with $\dot{c}(s) = \frac{dc}{dt}(s)$). Sometimes we will also denote the covariant derivative simply by $'$ when the curve c is implicitly specified: for example, given a vector field $V(t)$ along a curve c (of footprints), we write $V'(t) = D_c\widehat{V}$ where $\widehat{V}$ is an extension of V to M.

In this setting, recall that a curve c is *geodesic* if and only if $D_c\dot{c}(t) = 0\ \forall\, t$.

Since the equation $D_c\dot{c}(t) = 0$ is a *first order ODE* (in the variables $(c, \dot{c})$), geodesics are determined by the initial vector $(c(0), \dot{c}(0))$. Furthermore, any geodesic has constant speed, i.e., the quantity $\langle\dot{c}(t), \dot{c}(t)\rangle$ measuring the square of size (norm) of the tangent vector $\dot{c}(t)$ is constant along c: in fact, using the compatibility between ∇ and $\langle\cdot,\cdot\rangle$, one gets

$$\frac{d}{dt}\langle\dot{c}(t), \dot{c}(t)\rangle = 2\langle D_c\dot{c}(t), \dot{c}(t)\rangle = 0$$

for all t.

The lack of commutativity of the Levi-Cività connection is measured by the *Riemannian curvature tensor*

$$R(A, B)C := \nabla_A\nabla_B C - \nabla_B\nabla_A C - \nabla_{[A,B]}C.$$

In terms of the Riemannian curvature tensor R, the sectional curvature $K(A, B)$ of a 2-plane spanned by two vectors A and B is

$$K(A, B) := \frac{\langle R(A, B)B, A\rangle}{\|A \wedge B\|^2}$$

[1] Notion of parallel transport.

5.2.2 The Tangent Bundle to a Tangent Bundle

The tangent bundle *TTM* of the tangent bundle *TM* of *M* is a bundle over *M* in three natural ways:

(a) $TTM \overset{D\pi_M}{\to} TM \overset{\pi_M}{\to} M$ where $\pi_M\colon TM \to M$ is the natural projection;
(b) $TTM \overset{\pi_{TM}}{\to} TM \overset{\pi_M}{\to} M$ where $\pi_{TM}\colon TTM \to TM$ is the natural projection;
(c) $TTM \overset{\kappa}{\to} TM \overset{\pi_M}{\to} M$ where κ is defined as follows: given $\xi \in TTM$ tangent at $t = 0$ to a curve $t \mapsto V(t) \in TM$, we set $\kappa(\xi) := D_c V(0)$ where $c(t) = \pi_M(V(t))$ is the curve of footprints of the vectors $V(t)$;

In this context, the *vertical*, resp. *horizontal*, subbundle of *TTM* is $\ker(D\pi_M)$, resp. $\ker(\kappa)$. The vertical, resp. horizontal, subbundle is naturally identified with *TM* via κ, resp. $D\pi_M$. The vertical subbundle is transverse to the horizontal subbundle and the fiber T_vTM of *TTM* at $v \in T_pM$ can be identified $T_pM \times T_pM$ via the map $D\pi_M \times \kappa\colon TTM \to TM \times TM$.

Geometrically, the roles of the vertical and horizontal subbundles are easier to understand in the following way. Given an element of $\xi \in T_vTM$ tangent to a curve $V\colon t \mapsto V(t) \in TM$ with $V(0) = v$, let $c(t) = \pi_M(V(t))$ be the curve of footprints of $V(t)$ in M. In this setting, the identification of ξ with a pair of vectors $(v_1, v_2) \in T_pM \times T_pM$ via the horizontal and vertical subbundles simply amounts to take

$$\xi = (v_1, v_2) = (\dot{c}(0), D_c V(0))$$

In other terms, the component $v_1 = \dot{c}(0)$ of ξ in the horizontal subbundle measures how fast $V(t)$ is moving in M while the component $v_2 = D_c V(0)$ of ξ in the vertical subbundle measure how fast $V(t)$ is moving in the fibers of *TM*.

This way of thinking *TTM* as a bundle over *M* leads to the following natural Riemannian metric on *TM*: given $(v_1, v_2), (w_1, w_2) \in T_vTM$, we define

$$\langle (v_1, v_2), (w_1, w_2)\rangle_{Sas} := \langle v_1, w_1\rangle + \langle v_2, w_2\rangle$$

This metric is called *Sasaki metric* and the geometry of *TM* with respect to this Riemannian metric will be useful in our study of geodesic flows.

Remark 5.2.1 Sasaki metric is induced by the symplectic form

$$\omega((v_1, v_2), (w_1, w_2)) := \langle v_1, w_2\rangle - \langle v_2, w_1\rangle$$

in the sense that

$$\langle (v_1, v_2), (w_1, w_2)\rangle_{Sas} = \omega((v_1, v_2), J(w_1, w_2))$$

where $J(w_1, w_2) := (-w_2, w_1)$. The symplectic form ω is the pullback of the *canonical symplectic form* on the cotangent bundle T^*M by the map $TM \to T^*M$ associating to $v \in TM$ the linear functional $\langle v, . \rangle \in T^*M$.

For the reader's convenience, let us mention the following three useful facts about Sasaki metric:

- Sasaki [Sas] showed that the fibers of the tangent bundle TM are totally geodesic submanifolds of TM equipped with Sasaki metric;
- A parallel vector field on M viewed as a curve on TM is a geodesic for Sasaki metric that is always orthogonal to the fibers of TM;
- by *Topogonov Comparison Theorem*, for $v' \in T_{p'}M$ close to $v \in T_pM$, one has

$$\frac{1}{2}(d(p,p') + \|w - v'\|) \le d_{Sas}(v, v') \le 2(d(p,p') + \|w - v'\|)$$

  where $w \in T_{p'}M$ is the vector obtained by parallel transporting v along the geodesic connecting p to p' and d_{Sas} is the distance associated to Sasaki metric; here, how close v' must be from v depends only on the sectional curvatures of Sasaki metric in a neighborhood of v;

5.3 First Derivative of Geodesic Flows and Jacobi Fields

5.3.1 Computation of the First Derivative of Geodesic Flows

Let φ_t be the geodesic flow associated to a Riemannian manifold M. By definition, given a tangent vector $v \in TM$, we define $\varphi_t(v) := \dot{\gamma}_v(t)$ where $\gamma_v(s)$ is the unique geodesic of M with $\dot{\gamma}_v(0) = v$. Here, it is worth to point out that the geodesic flow is always locally well-defined but it might be globally ill-defined. Moreover, the geodesic flow φ_t preserves the Liouville measure (i.e., the volume form on TM induced by Riemannian metric of M).

We want to describe $D\varphi_t$ and, from the definition of first derivative, this amounts to study (1-parameter) *variations of geodesics*.

More precisely, let $\alpha: (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to M$ be a (smooth) map such that, for each s, $\alpha(s, .)$ is a geodesic of M. Intuitively, α is a one-parameter variation of the geodesic $\gamma(t) := \alpha(0, t)$.

Define the vector field $J(t) := \frac{\partial \alpha}{\partial s}(0, t)$ along the geodesic $\gamma(t) = \alpha(0, t)$. It is well-known that J satisfies the *Jacobi equation*

$$J'' + R(J, \dot{\gamma})\dot{\gamma} = 0$$

where $'$ is the covariant derivative (along γ) and R is the Riemannian curvature tensor. In other terms, $\frac{\partial \alpha}{\partial s}(0, t)$ is a *Jacobi field*, i.e., a vector field satisfying Jacobi's equation (Fig. 5.1).

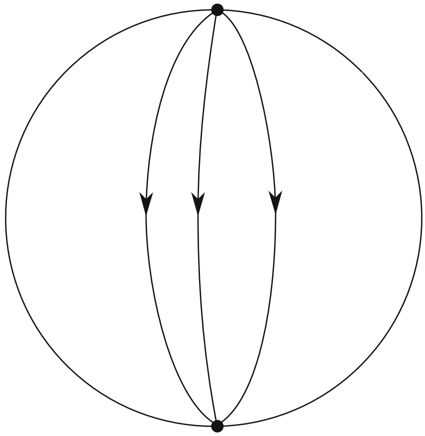

Fig. 5.1 A Jacobi field associated to a variation of geodesics connecting the north and south poles of a round sphere

Observe that Jacobi's equation is a second order linear ODE. In particular, a Jacobi field J is determined by the initial data $(J(0), J'(0)) \in T_{\gamma(0)}M \times T_{\gamma(0)}M$.

The pair $(J(0), J'(0)) \in T_{\gamma(0)}M \times T_{\gamma(0)}M$ corresponds to the tangent vector $\dot{V}(0)$ at $s = 0$ to the curve $V(s) = \frac{\partial \alpha}{\partial t}(s, 0)$ in TM (under the identification $T_{\dot{\gamma}(0)}TM \simeq T_{\gamma(0)}M \times T_{\gamma(0)}M$ described above [in terms vertical and horizontal subbundles]). Indeed, the curve $c(s)$ of footprints of $V(s)$ is $c(s) = \alpha(s, 0)$, so that the tangent vector at $s = 0$ of $V(s)$ is represented by

$$(\dot{c}(0), D_c \frac{\partial \alpha}{\partial t}(0, 0)) := (J(0), \frac{D}{\partial s}\frac{\partial \alpha}{\partial t}(0, 0)) = (J(0), \frac{D}{\partial t}\frac{\partial \alpha}{\partial s}(0, 0)) =: (J(0), J'(0))$$

Here, the symmetry $\frac{D}{\partial s}\frac{\partial \alpha}{\partial t}(s, t) = \frac{D}{\partial t}\frac{\partial \alpha}{\partial s}(s, t)$ of the Levi-Cività connection was used.

Similarly, the pair $(J(t), J'(t))$ represents the tangent vector $\dot{V}(t)$ at $s = 0$ to the curve $s \mapsto \frac{\partial \alpha}{\partial t}(s, t) = \varphi_t \circ V(s)$. Therefore, $(J(t), J'(t))$ represents

$$\dot{V}(t) = D\varphi_t(\dot{V}(0)) = D\varphi_t(\dot{\gamma}(0)).$$

In summary, Jacobi fields are intimately related to the first derivatives of geodesic flows:

Proposition 5.3.1 *The image of the tangent vector* $(v_1, v_2) \in T_v TM$ *under the derivative* $D_v\varphi_t$ *of the geodesic flow is the tangent vector* $(J(t), J'(t)) \in T_{\varphi_t(v)}TM$ *where* J *is the (unique) Jacobi field with initial data* $(J(0), J'(0)) = (v_1, v_2)$ *along the (unique) geodesic* γ *with* $\dot{\gamma}(0) := v$.

5.3.2 Perpendicular Jacobi Fields and Invariant Subbundles

A concrete example of Jacobi field along a geodesic γ is $J(t) = (a + bt)\dot{\gamma}(t)$: indeed, in this context, $R(J, \dot{\gamma})\dot{\gamma} = 0$ and $J'' = 0$, so that Jacobi's equation is trivially verified. Geometrically, this Jacobi field correspond to a trivial variation α of the geodesic γ where the initial point $\gamma(0)$ moves along γ and/or the speed of the parametrization of γ changes, i.e., $\alpha(s, t) = \gamma(as + bt)$.

In general, a Jacobi field J along a geodesic γ that is tangent to γ has the form $J(t) = (a + bt)\dot{\gamma}$ for some $a, b \in \mathbb{R}$: in fact, for $J(t) = j(t)\dot{\gamma}(t)$ with $j(t) \in \mathbb{R}$, one has $R(J, \dot{\gamma})\dot{\gamma} = 0$, so that Jacobi's equation reduces to $J'' = 0$, i.e., $j''(t) = 0$ for all t.

Hence, a Jacobi field J along a geodesic is interesting only when it is not completely tangent to the geodesic, or, equivalently, when it has some nontrivial component in the perpendicular direction to the geodesic.

A Jacobi field J along a geodesic γ has the following geometrical properties:

- the component J' of (J, J') makes constant angle with γ, i.e., the quantity $\langle J', \dot{\gamma}\rangle$ is constant;
- if both components of (J, J') are orthogonal to γ at some point, then they stay orthogonal all along γ, i.e., if $J(t_0) \perp \dot{\gamma}(t_0)$ and $J'(t_0) \perp \dot{\gamma}(t_0)$ for some t_0, then $J(t) \perp \dot{\gamma}(t)$ for all t;

We say that a Jacobi field J along a geodesic γ is a *perpendicular Jacobi field* whenever both components of (J, J') are orthogonal to γ.

From the properties of Jacobi fields discussed above, we see that any Jacobi field J along a geodesic γ has a decomposition

$$J = J_{\parallel} + J_{\perp}$$

where $J_{\perp}$ is a perpendicular Jacobi field and $J_{\parallel}$ is a Jacobi field tangent to γ.

After this little digression on Jacobi fields, let us use them to introduce relevant invariant subbundles under the first derivative of a geodesic flow φ_t.

We begin by recalling that the norm of a tangent vector $v \in TM$ stays constant along its φ_t-orbit, i.e., φ_t preserves the energy hypersurfaces $\{v \in TM : \|v\| = E\}$ (for each $E \geq 0$). In particular, the first derivative $D\varphi_t$ of the geodesic flow φ_t preserves the tangent bundle TT^1M (to the unit tangent bundle T^1M of M).

We affirm that, under the identification $T_vTM \simeq T_pM \times T_pM$ for $v \in T_pM$, the fiber T_vT^1M (of the subbundle TT^1M of TTM) corresponds to the set of pairs (w_0, w_1) with $w_1 \perp v$.

In fact, note that an element (w_0, w_1) of T_vT^1M is tangent at $s = 0$ to a variation of geodesics $\alpha(s, t)$ parametrized by arc-length, i.e.,

$$\left\| \frac{\partial \alpha}{\partial t}(s, t) \right\| = 1$$

for all s, t, such that the geodesic $\gamma(t) = \alpha(0, t)$ satisfies $\dot{\gamma}(0) = v$ and the Jacobi field J corresponding to $\alpha(s, t)$ verifies $(J(0), J'(0)) = (w_0, w_1)$.

The desired property $J'(0) = w_1 \perp v = \dot{\gamma}(0)$ now follows from the following calculation:

$$\begin{aligned} 0 &= \frac{\partial D}{\partial s} \left\| \frac{\partial \alpha}{\partial t} \right\|^2 (0,0) = 2\langle \frac{D^2\alpha}{\partial s \partial t}, \frac{\partial \alpha}{\partial t} \rangle (0,0) \\ &= 2\langle \frac{D^2\alpha}{\partial t \partial s}, \frac{\partial \alpha}{\partial t} \rangle (0,0) = 2\langle J'(0), \dot{\gamma}(0) \rangle \end{aligned}$$

The invariant subbundle TT^1M itself admits a decomposition into two invariant subbundles, namely,

$$TT^1M = \mathbb{R}\dot{\varphi} \oplus \dot{\varphi}^{\perp}$$

where $\dot{\varphi}$ is the vector field generating the geodesic flow φ_t and $\dot{\varphi}^{\perp}$ is the orthogonal complement of $\dot{\varphi}$. In fact, under the identification $T_vTM \simeq T_pM \times T_pM$ for $v \in T_pM$, the vector $\dot{\varphi}(v)$ is $(v, 0)$ and the elements of $\dot{\varphi}^{\perp}$ have the form (w_0, w_1) with $w_0 \perp v$, $w_1 \perp v$. In particular, the $D\varphi_t$-invariance of $\dot{\varphi}^{\perp}$ follows from the fact (mentioned above) that a Jacobi field J satisfying $J(t_0) \perp v$ and $J'(t_0) \perp v$ for some t_0 is a perpendicular Jacobi field (i.e., $J(t) \perp v$ and $J'(t) \perp v$ for all t).

In summary, the action of $D\varphi_t$ on TT^1M has two complementary invariant subbundles, namely, the span of the vector field $\dot{\varphi}$ generating the geodesic flow and its orthogonal $\dot{\varphi}^{\perp}$ consisting of perpendicular Jacobi fields. Since $D\varphi_t$ acts isometrically in the direction of $\dot{\varphi}$, our task is reduced to study the action of $D\varphi_t$ on perpendicular Jacobi fields.

5.3.3 Matrix Jacobi and Ricatti Equations

We want to describe the matrix of $D\varphi_t$ acting on the vector space of perpendicular Jacobi fields. For this sake, let $e_1 = \dot{\gamma}(0), e_2, \ldots, e_n$ be an orthonormal basis for the tangent space of $\gamma(0)$, and denote by $e_1(t) = \dot{\gamma}(t), e_2(t), \ldots, e_n(t)$ the parallel transport of this orthonormal basis along the geodesic $\gamma(t)$.

Define the matrix $\mathscr{R}(t)$ whose entries are

$$\mathscr{R}_{jk}(t) = \langle R(e_j(t), e_1(t))e_1(t), e_k(t) \rangle$$

where R is the Riemannian curvature tensor.

Note that any Jacobi field J along γ can be written as $J(t) = \sum_{k=1}^{n} y^k(t)e_k(t)$. In this setting, Jacobi equation becomes

$$\frac{d^2y^k}{dt} + \sum_{j=1}^{n} y^j(t)\mathscr{R}_{jk}(t) = 0$$

and, as usual, a solution is determined by the values $y^k(0)$ and $\frac{dy^k}{dt}(0)$.

We can write solutions of the Jacobi equation above in a practical way by considering a matrix solution $\mathscr{J}$ of the matrix Jacobi equation:

$$\mathscr{J}''(t) + \mathscr{R}(t)\mathscr{J}(t) = 0$$

If $\mathscr{J}$ is nonsingular, the matrix

$$U := \mathscr{J}'\mathscr{J}^{-1}$$

satisfies the matrix Ricatti equation

$$U' + U^2 + \mathscr{R} = 0$$

Remark 5.3.2 The matrix $\mathscr{U}$ is symmetric if and only if one has

$$\omega_{\mathbb{R}^{2n}}((J_i, J_i'), (J_j, J_j')) = 0$$

for any two columns J_i and J_j of $\mathscr{J}$. Here, $\omega_{\mathbb{R}^{2n}}$ is the standard symplectic form $\omega_{\mathbb{R}^{2n}}((x, y), (z, w)) = \sum_{i=1}^{n}(x_i w_i - y_i z_i)$ of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

5.3.4 An Estimate for the First Derivative of a Geodesic Flow

After these preliminaries on the geometry of tangent bundles, geodesic flows and Jacobi fields, we are ready to prove the following result:

Theorem 5.3.3 *Let M be a negatively curved manifold. Let $0 \leq \tau \leq 1$ and consider $\gamma: [-\tau, \tau] \to M$ a geodesic. Suppose that $\kappa: [-\tau, \tau] \to \mathbb{R}_+$ is a Lipschitz function such that, for each $-\tau \leq t \leq \tau$, the sectional curvature of any plane containing $\dot{\gamma}(t)$ is greater than or equal to $-\kappa(t)^2$, and denote by $u: [-\tau, \tau] \to \mathbb{R}_+$ the solution of Ricatti's equation*

$$u' + u^2 = \kappa^2$$

with initial data $u(-\tau) = 0$. Then, the first derivative of the geodesic flow φ_t at time τ satisfies the estimate

$$\|D\varphi_\tau(\dot{\gamma}(0))\| \leq 1 + 2(1 + u(0))^2 \left(1 + \sqrt{1 + u(\tau)^2}\right) \exp\left(\int_0^\tau u(s)\, ds\right)$$

From our discussion so far, the task of estimating the norm $\|D\varphi_\tau(\dot{\gamma}(0))\|$ is equivalent to provide bounds for $\|(J(\tau), J'(\tau))\|_{Sas}$ in terms of $\|(J(0), J'(0))\|_{Sas}$ where $J(t)$ is a perpendicular Jacobi field along γ (cf. Proposition 5.3.1 and Sect. 5.3.2).

We begin by estimating these quantities for two special subclasses of perpendicular Jacobi fields defined as follows. Let $\mathscr{X}(t)$ and $\mathscr{Y}(t)$ be the (fundamental) solutions of the matrix Jacobi equation

$$\mathscr{J}''(t) + \mathscr{R}(t)\mathscr{J}(t) = 0$$

with initial data $\mathscr{X}(0) = \text{Id} = \mathscr{Y}(0)$ and $\mathscr{X}'(-\tau) = 0 = \mathscr{Y}(\tau)$. Note that, by definition, all Jacobi fields $X(t)$ with $X'(-\tau) = 0$, resp. all Jacobi fields $Y(t)$ with $Y(\tau) = 0$, have the form $X(t) = \mathscr{X}(t) \cdot X(0)$, resp. $Y(t) = \mathscr{Y}(t) \cdot Y(0)$, i.e., they are obtained by applying the matrices $\mathscr{X}(t)$, resp. $\mathscr{Y}(t)$, to a vector $X(0)$, resp. $Y(0)$. In this setting, the "other" component $X'(t)$, resp. $Y'(t)$ (of the Jacobi field $X(t)$, resp. $Y(t)$, viewed as a tangent vector to TM) can be recovered by applying the matrix $\mathscr{U}(t) := \mathscr{X}'(t)\mathscr{X}(t)^{-1}$, resp. $\mathscr{V}(t) := \mathscr{Y}'(t)\mathscr{Y}(t)^{-1}$, to $X(t)$, resp. $Y(t)$.

Remark 5.3.4 Very roughly speaking, the idea behind the choice of the subclasses $X(t)$ and $Y(t)$ is that $Y(t)$ are Jacobi fields belonging to a certain "stable cone" and $X(t)$ are Jacobi fields belonging to a certain "unstable cone" (compare with the discussion in the next Section).

Our first lemma says that the tangent vectors $(Y(t), Y'(t))$ associated to Jacobi fields $Y(t)$ as above do *not* growth in forward time.

Lemma 5.3.5 *Let $Y(t)$ be a perpendicular Jacobi field along γ such that $Y(\tau) = 0$. Then,*

$$\|Y'(0)\| \geq \|Y(0)\|/\tau \geq \|Y'(\tau)\|$$

In particular,

$$\|(Y(\tau), Y'(\tau)\|_{Sas} = \|Y'(\tau)\| \leq \|Y'(0)\| \leq \|(Y(0), Y'(0))\|_{Sas}$$

Proof One of the consequences of negative sectional curvatures along γ is the fact that the functions $\|J(t)\|$ and $\|J(t)\|^2$ are strictly convex for any perpendicular Jacobi field $J(t)$ (see, e.g., Eberlein's survey [Ebs]).

In our context, this implies that $\|Y\|$ is a (strictly) convex function decreasing from $\|Y(0)\|$ to 0 in the interval $[0, \tau]$. Therefore,

$$-\|Y\|'(0) \geq \|Y(0)\|/\tau \geq -\|Y\|'(\tau-)$$

Since $\|Y\|' = \langle Y', Y/\|Y\|\rangle$ and $Y(t) = Y'(\tau)(t-\tau) + o(\tau - t)$ for t close to τ (because $Y(\tau) = 0$), we deduce that

$$\|Y'(0)\| \geq -\|Y\|'(0) \geq \|Y(0)\|/\tau \geq -\|Y\|'(\tau-) = \|Y'(\tau)\|$$

This completes the proof of the lemma.

Our second lemma says that the growth in forward time of tangent vectors $(X(t), X'(t))$ associated to Jacobi fields $X(t)$ as above is reasonably controlled in terms of the solution u of Ricatti's equation $u' + u^2 = \kappa^2$ with $u(-\tau) = 0$ (where κ is the Lipschitz function controlling some sectional curvatures of M).

Lemma 5.3.6 *Let X be a perpendicular Jacobi field along γ with $X'(-\tau) = 0$. Then,*

$$\|(X(\tau), X'(\tau))\|_{Sas} \leq \sqrt{1 + u(\tau)^2} \exp\left(\int_0^\tau u(s)\, ds\right) \|(X(0), X'(0))\|_{Sas}$$

Proof By definition, $X'(t) = \mathscr{U}(t) \cdot X(t)$. Thus, $\|X'(t)\| \leq \|\mathscr{U}(t)\| \cdot \|X(t)\|$ and, a fortiori,

$$\|(X(t), X'(t))\|_{Sas} = \sqrt{\|X(t)\|^2 + \|X'(t)\|^2} \leq \sqrt{1 + \|\mathscr{U}(t)\|^2}\|X(t)\|$$

On the other hand, since $\|X\|'(t) = \langle X'(t), X(t)/\|X(t)\|\rangle$, we see that $\|X\|'(t) \leq \|X'(t)\| \leq \|\mathscr{U}(t)\| \cdot \|X(t)\|$ and, hence,

$$\|X(\tau)\| \leq \exp\left(\int_0^\tau \|\mathscr{U}(s)\|\, ds\right) \|X(0)\|$$

These inequalities show that the proof of the lemma is complete once we can prove that $\|\mathscr{U}(t)\| \leq u(t)$ for all $t \in [0, \tau]$.

In this direction, let us observe that the matrix $\mathscr{U}$ is symmetric because it verifies (in a trivial way) the condition of Remark 5.3.2. Therefore, the norm of $\mathscr{U}$ is given by the expression

$$\|\mathscr{U}(t)\| = \sup_{\|e\|=1} |\langle \mathscr{U}(t)e, e\rangle|$$

where $e \in \mathbb{R}^{\dim(M)-1}$ ranges from all unit vectors. In particular, our task is reduced to show that

$$|u_e(t)| \leq u(t)$$

for all unit vectors e, where $u_e(t) := \langle \mathscr{U}(t)e, e\rangle$.

From the matrix Ricatti equation, we see that

$$u_e'(t) = \langle \mathscr{U}'(t)e, e\rangle = \langle \mathscr{R}(t)e, e\rangle - \langle \mathscr{U}(t)^2 e, e\rangle$$

Since the Lipschitz function κ controls the sectional curvatures (of planes containing $\dot{\gamma}$) along γ and the matrix $\mathscr{U}$ is symmetric, we can estimate the right-hand side of the previous inequality as

$$u_e'(t) = \langle \mathscr{R}(t)e, e\rangle - \langle \mathscr{U}(t)^2 e, e\rangle \leq \kappa(t)^2 - \langle \mathscr{U}(t)e, \mathscr{U}(t)e\rangle$$

On the other hand, since e is a unit vector, the Cauchy–Schwarz inequality implies that $u_e(t)^2 = \langle \mathscr{U}(t)e, e\rangle^2 \leq \langle \mathscr{U}(t)e, \mathscr{U}(t)e\rangle$. Therefore, the right-hand side of the previous inequality is bounded by

$$u_e'(t) \leq \kappa(t)^2 - \langle \mathscr{U}(t)e, \mathscr{U}(t)e\rangle \leq \kappa(t)^2 - u_e(t)^2$$

From this differential inequality and the facts that $u_e(-\tau) = 0 = u(-\tau)$ and $u' = \kappa^2 - u^2$, we can easily deduce that $u_e(t) \leq u(t)$ for all $|t| \leq \tau$ from the standard continuity argument.

Finally, we can complete the proof of the lemma by observing that the symmetric matrix $\mathscr{U}$ is positive-definite for $-\tau < t \leq \tau$: this follows from the facts that $\mathscr{U}(-\tau) = 0$ and $\mathscr{U}(t)$ satisfies matrix Ricatti equation associated to a negatively curved manifold M (cf. Eberlein's book [Ebb]). Therefore, $u_e(-\tau) = 0$ and $u_e(t) > 0$ for all $-\tau < t \leq \tau$ and all unit vector e, so that

$$|u_e(t)| = u_e(t) \leq u(t)$$

as desired.

Once we know how to control the growth of $(Y(t), Y'(t))$ and $(X(t), X'(t))$ for Jacobi fields $Y(t)$ and $X(t)$ as above, the idea to estimate the growth of $(J(t), J'(t))$ for an arbitrary perpendicular Jacobi field $J(t)$ (thus completing the proof of Theorem 5.3.3) is to produce a decomposition of the form

$$(J(t), J'(t)) = (X(t), X'(t)) + (Y(t), Y'(t))$$

where $X'(-\tau) = 0 = Y(\tau)$ and the norms of $(X(0), X'(0))$ and $(Y(0), Y'(0))$ are controlled in terms of the norm of $(J(0), J'(0))$.

For this sake, define

$$v := J(0) \quad \text{and} \quad w := (\mathscr{U}(0) - \mathscr{V}(0))^{-1}(J'(0) - \mathscr{U}(0)J(0)))$$

and we set $X(t) := \mathscr{X}(t)(v + w)$ and $Y(t) := \mathscr{Y}(t)(-w)$.

First, note that the vector w is well-defined, i.e., the matrix $\mathscr{U}(0) - \mathscr{V}(0)$ is invertible. Indeed, we already saw that the matrices $\mathscr{U}(0)$ and $\mathscr{V}(0)$ are symmetric (because they satisfy (in a trivial way) the condition of Remark 5.3.2) and that the matrix $\mathscr{U}(0)$ is positive-definite (because $\mathscr{U}(t)$ satisfies matrix Ricatti equation), the manifold M is negatively curved and $\mathscr{U}(-\tau) = 0$ imply that $\mathscr{U}(t)$ is positive-definite for $-\tau < t \leq \tau$ (cf. Eberlein's book [Ebb]). Furthermore, all eigenvalues of the matrix $\mathscr{V}(0)$ are ≤ -1: in fact, any eigenvalue of $\mathscr{V}(0)$ has the form

$\langle \mathscr{V}(0)Y(0), Y(0)\rangle = \langle Y'(0), Y(0)\rangle = \|Y\|'(0)$ for some unit vector $Y(0)$, and $\|Y\|'(0) \leq -1/\tau \leq -1$ because $\|Y\|$ is a convex function (see, e.g., Eberlein's survey [Ebs]) decreasing from $\|Y(0)\| = 1$ to 0 in the interval $[0, \tau]$ (with $0 < \tau \leq 1$). Therefore, the matrix $\mathscr{U}(0)-\mathscr{V}(0)$ is a symmetric matrix whose eigenvalues are ≥ 1 and, hence, $\mathscr{U}(0) - \mathscr{V}(0)$ is an invertible matrix satisfying

$$\|(\mathscr{U}(0) - \mathscr{V}(0))^{-1}\| \leq 1$$

Secondly, we claim that the Jacobi fields $X(t)$ and $Y(t)$ give the desired decomposition. In fact, since $J(t)$, $X(t)$ and $Y(t)$ are Jacobi fields, our claim follows from the facts that $J(0) = (v + w) - w$ and $J'(0) = \mathscr{U}(0)(v + w) + \mathscr{V}(0)(-w)$.

Finally, let us estimate the (Sasaki) norms of $(X(0), X'(0))$ and $(Y(0), Y'(0))$ in terms of $(J(0), J'(0))$. We begin by observing that it suffices to estimate the Sasaki norm of $(X(0), X'(0))$ because

$$\|(Y(0), Y'(0))\|_{Sas} \leq \|(J(0), J'(0))\|_{Sas} + \|(X(0), X'(0))\|_{Sas}$$

On the other hand, the (Sasaki) norm of $(X(0), X'(0))$ is not difficult to bound:

$$\|(X(0), X'(0)\|_{Sas} \leq (\|v\| + \|w\|)\sqrt{1 + \|\mathscr{U}(0)\|^2}$$

Since $\|\mathscr{U}(0)\| \leq u(0)$ (cf. the proof of Lemma 5.3.6) and $\|(\mathscr{U}(0) - \mathscr{V}(0))^{-1}\| \leq 1$, we can estimate the right-hand side of the previous inequality by

$$\begin{aligned}
\|(X(0), X'(0)\|_{Sas} &\leq (\|v\| + \|w\|)\sqrt{1 + \|\mathscr{U}(0)\|^2} \\
&\leq (\|J(0)\| + \|J'(0)\| + u(0)\|J(0\|))\sqrt{1 + u(0)^2} \\
&\leq \sqrt{2}(1 + u(0)^2)(\|J(0)\| + \|J'(0)\|) \\
&= \sqrt{2}(1 + u(0)^2)\|(J(0), J'(0))\|_{Sas}
\end{aligned}$$

By putting together these estimates of the Sasaki norms of $(X(0), X'(0))$ and $(Y(0), Y'(0))$ and Lemmas 5.3.5 and 5.3.6, we deduce that

$$\begin{aligned}
\|(J(\tau), J'(\tau))\|_{Sas} &\leq \|(X(\tau), X'(\tau))\|_{Sas} + \|(Y(\tau), Y'(\tau))\|_{Sas} \\
&\leq \sqrt{1 + u(\tau)^2}\, e^{\int_0^\tau u(s)\,ds}\, \|(X(0), X'(0))\|_{Sas} + \|(Y(0), Y'(0))\|_{Sas} \\
&\leq (1 + \sqrt{1 + u(\tau)^2}\, e^{\int_0^\tau u(s)\,ds})\, \|(X(0), X'(0))\|_{Sas} + \|(J(0), J'(0))\|_{Sas} \\
&= (1 + \sqrt{2}(1 + u(0)^2)(1 + \sqrt{1 + u(\tau)^2})\, e^{\int_0^\tau u(s)\,ds})\|(J(0), J'(0))\|_{Sas}
\end{aligned}$$

This completes the proof of Theorem 5.3.3.

5.4 Hyperbolicity of Certain Geodesic Flows

In this section, we show the nonuniform hyperbolicity of a geodesic flow φ_t satisfying properties (II), (III) and (VI) in the statement of Theorem 5.1.1.

We start by noticing that N has finite m-volume: this is an easy consequence of the compactness of $\overline{N}$ (assumption (II)) and the volumetrically cusp-like assumption (III) on ∂N.

Next, let us check that the geodesic flow φ_t in the statement of Burns–Masur–Wilkinson ergodicity criterion is defined for all time for almost every initial data $v \in T^1N$. For this sake, denote by $\pi : T^1N \to N$ the natural projection and set

$$U_\rho := \{v \in T^1N : d(\pi(v), \partial N) < \rho\}$$

and

$$S^+(\rho) := \{v \in T^1N : \varphi_t(v) \in U_\rho \text{ for some } 0 \leq t \leq 1\}$$

By definition,

$$\{v \in T^1N : \varphi_t(v) \text{ is not defined for some } 0 \leq t \leq 1\} \subset \bigcap_{\rho>0} S^+(\rho)$$

and, a fortiori,

$$\{v \in T^1N : \varphi_t(v) \text{ is not defined for some } t \in \mathbb{R}\} \subset \bigcup_{k\in\mathbb{Z}} \varphi_{-k}\left(\bigcap_{\rho>0} S^+(\rho)\right)$$

In particular, since the geodesic flow φ_t preserves the measure m, our task of showing that φ_t is defined for all time for almost every initial data is reduced to prove that $\bigcap_{\rho>0} S^+(\rho)$ has zero m-measure.

In order to compute the m-measure of $\bigcap_{\rho>0} S^+(\rho)$, let us estimate the m-measure of $S^+(\rho)$ for each $0 < \rho < 1$ along the following lines. Note that

$$S^+(\rho) \subset \bigcup_{k=0}^{\lfloor 1/\rho \rfloor} V_k(\rho)$$

where $V_k(\rho)$ consists of the unit tangent vectors $v \in T^1N$ flowing into U_ρ for some time between $k\rho$ and $(k+1)\rho$. By definition, $\varphi_{(k+1)\rho}(V_k(\rho)) \subset U_{2\rho}$, so that

$$m(V_k(\rho)) \leq m(U_{2\rho}) \leq C\rho^{2+\nu}$$

where $\nu > 0$. Here, we used the fact that m is φ_t-invariant (for the first inequality) and the assumption (III) (for the second inequality). It follows that

$$m(S^+(\rho)) \leq C\frac{1}{\rho}\rho^{2+\nu} = C\rho^{1+\nu}$$

for all $0 < \rho < 1$. Hence, $\bigcap_{\rho>0} S^+(\rho)$ has zero m-measure and φ_t is defined for all time for almost all initial data.

Remark 5.4.1 The reader certainly noticed that we do not the full strength of assumption (III) to deduce the long-term existence of φ_t at almost every point: in fact, the weaker condition $m(U_\rho) \leq C\rho^{1+\nu}$ works equally well. Nevertheless, we will see below that the full strength of assumption (III) is helpful to ensure the existence of Lyapunov exponents for the geodesic flow φ_t.

Now, let us show that the geodesic flow φ_t is nonuniformly hyperbolic in the sense of Pesin theory, i.e., all (transverse) Lyapunov exponents are nonzero.

We start by verifying that the Lyapunov exponents of φ_t are well-defined (at almost every point): by Oseledets Multiplicative Ergodic Theorem [Os], it suffices to check the log-integrability of the derivative cocycles $D\varphi_1$ and $D\varphi_{-1}$ associated to the time-1 and time-(-1)maps φ_1 and φ_{-1}, that is,

$$\int_{T^1N} \log^+ \|D\varphi_{\pm 1}\| dm < \infty$$

By symmetry (or reversibility of the geodesic flow), we have to consider only the log-integrability of $D\varphi_1$. We estimate the integral above for $D\varphi_1$ by noticing that

$$\int_{T^1N} \log^+ \|D\varphi_1\| dm \leq \int_{T^1N-U_1} \log^+ \|D\varphi_1\| dm + \sum_{n\in\mathbb{N}} \int_{S^+(1/n)-S^+(1/(n+1))} \log^+ \|D\varphi_1\| dm$$

Since $T^1N - U_1$ is compact (by assumption (II)), we need to show only that the series above is convergent and this is not hard to see: on one hand, we already saw that $m(S^+(1/n)) \leq C/n^{1+\nu}$ for some $\nu > 0$ (as a consequence of assumption (III), and, on the other hand, $\|D\varphi_1\| \leq \frac{C}{\beta}\log(n+1)$ on $S^+(1/n) - S^+(1/(n+1))$ by assumption (VI), so that

$$\sum_{n\in\mathbb{N}} \int_{S^+(1/n)-S^+(1/(n+1))} \log^+ \|D\varphi_1\| dm \leq \frac{C^2}{\beta} \sum_{n\in\mathbb{N}} \frac{\log(n+1)}{n^{1+\nu}} < \infty$$

By the Oseledets Theorem, once we know the log-integrability of the derivative cocycle, we have that, for almost every $v \in T^1N$, there are $k(v) \leq 2n - 1$ real numbers

$$\lambda_1(v) > \cdots > \lambda_{k(v)}(v)$$

called Lyapunov exponents and a $D\varphi_t$-invariant splitting

$$T_v T^1 N = \bigoplus_{i=1}^{k(v)} E_i(v)$$

into Lyapunov subspaces $E_i(v)$ such that, for every $\xi \in E_i(v) - \{0\}$,

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|D_v \varphi_t(\xi)\| = \lambda_i(v)$$

In the context of a geodesic flow φ_t, recall that the derivative cocycle $D\varphi_t$ preserves the decomposition $T_v T^1 N = \mathbb{R}\dot{\varphi} \oplus \dot{\varphi}^\perp$, and $D\varphi_t$ acts isometrically along $\mathbb{R}\dot{\varphi}$ and $D\varphi_t$ preserves $\dot{\varphi}^\perp$. This implies that the Lyapunov exponent of φ_t along $\mathbb{R}\dot{\varphi}$ is zero and the derivative cocycle $D\varphi_t$ has $2n-2$ Lyapunov exponents counted with multiplicity (i.e., we count $\dim(E_i(v))$-times the Lyapunov exponent $\lambda_i(v)$) along $\dot{\varphi}^\perp$.

Remark 5.4.2 In fact, the derivative cocycle $D\varphi_t$ preserves a natural symplectic form on $\dot{\varphi}^\perp$. In particular, the $2n - 2$ Lyapunov exponents are organized in a symmetric way around the origin in the sense that $-\lambda$ is a Lyapunov exponent whenever λ is a Lyapunov exponent.

By definition, φ_t is said to be *nonuniformly hyperbolic* whenever all Lyapunov exponents along $\dot{\varphi}^\perp$ (sometimes called transverse Lyapunov exponents) are non-zero.

In our context (of the statement of Burns–Masur–Wilkinson ergodicity criterion), we will prove the nonuniform hyperbolicity of φ_t by exploiting the negative curvature of N. More concretely, the negative curvature of N implies that:

- for any nontrivial perpendicular Jacobi field $J(t)$, the functions $\|J(t)\|$ and $\|J(t)\|^2$ are strictly convex (thanks to Jacobi's equation);
- for each geodesic ray $\gamma\colon (-\infty, 0] \to N$ and for each $w \in \dot{\gamma}(0) = v$, there exists an unique perpendicular Jacobi field $J_{w,+}$ along γ with $J_{w,+}(0) = w$ such that

$$\|J_{w,+}(t)\| \leq \|w\|$$

for all $t \leq 0$.

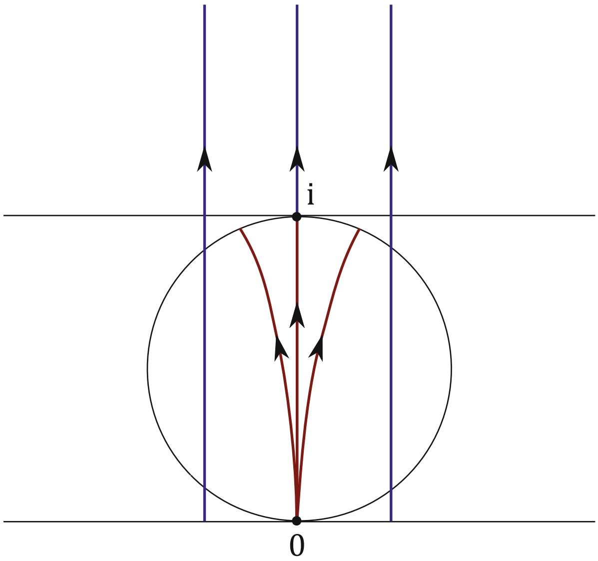

Fig. 5.2 Jacobi fields associated to two (*blue* and *red*) variations of the geodesic in hyperbolic plane "connecting" 0 to ∞

See, e.g., Eberlein's book [Ebb] for more explanations. In the literature, $J_{w,+}$ is called an *unstable* Jacobi field and it is usually constructed as the limit $J_{w,+} = \lim_{\tau\to-\infty} J_{w,+,\tau}$ where $J_{w,+,\tau}$ is the Jacobi field with $J_{w,+,\tau}(0) = w$ and $J_{w,+,\tau}(\tau) = 0$. Similarly, we can define *stable* Jacobi fields along geodesic rays $\gamma\colon [0,+\infty) \to N$ by reversing the time of the geodesic flow. Figure 5.2 illustrates stable ("blue") and unstable ("red") Jacobi fields along a vertical geodesic in the hyperbolic plane.

We will discuss stable and unstable Jacobi fields in more details in Sect. 5.5 (because they describe the stable and unstable manifolds of φ_t and Hopf's argument depend crucially on the features of stable and unstable manifolds). For now, we just need to know that, if N is negatively curved and $\varphi_t(v)$ is defined for all time at $v \in T^1N$, then

$$T_vT^1N = E^s(v) \oplus E^0(v) \oplus E^u(v)$$

where $E^0(v) := \mathbb{R}\dot{\varphi}(v)$,

$$E^s(v) = \{(J(0), J'(0)) : J(t) \text{ is a stable Jacobi field}\}$$

and

$$E^u(v) = \{(J(0), J'(0)) : J(t) \text{ is an unstable Jacobi field}\}.$$

In other terms, $\dot{\varphi}(v)^{\perp} = E^s(v) \oplus E^u(v)$ where $E^s(v)$ and $E^u(v)$ are $(n-1)$-dimensional subspaces related to stable and unstable Jacobi fields. See, e.g., Eberlein's book [Ebb] for a proof of this fact.

In this setting, the nonuniform hyperbolicity of φ_t is a direct consequence of the following lemma relating stable and unstable Jacobi fields to Lyapunov subspaces:

Lemma 5.4.3 *There exists a φ_t-invariant subset $\Lambda_0 \subset T^1N$ of full m-measure such that*

$$E^s(v) = \bigoplus_{\lambda_i(v)<0} E_i(v) \quad \textit{and} \quad E^u(v) = \bigoplus_{\lambda_j(v)>0} E_j(v)$$

Proof Denote by Λ_0 the set of unit vectors $v \in T^1N$ such that:

- $\varphi_t(v)$ is defined for all time $t \in \mathbb{R}$;
- the Lyapunov exponents $\lambda_i(v)$ and Lyapunov subspaces $E_i(v)$ are defined for $i = 1, \dots, k(v)$;
- v is *uniformly recurrent* under φ_t in the sense that, for any neighborhood U of v, there exists $\delta > 0$ such that the sets $R_{\pm}(T) = \{\pm t \in [0, T] : \varphi_t(v) \in U\}$ have Lebesgue measure $\geq \delta T$ for all T sufficiently large.

Note that Λ_0 is φ_t-invariant and it has full m-measure: our previous discussion showed that the first two conditions hold for almost every $v \in T^1N$ and the third condition holds in a full measure subset thanks to the Birkhoff Ergodic Theorem.

We affirm that Λ_0 satisfies the conclusions of the lemma. In fact, by the reversibility of the geodesic flow φ_t, it suffices to show that

$$E^u(v) = \bigoplus_{\lambda_j(v)>0} E_j(v)$$

for all $v \in \Lambda_0$.

For this sake, given $v \in \Lambda_0$, we fix a neighborhood U of v and a real number $\eta > 0$ such that if $J(t)$ is an unstable Jacobi field along a geodesic γ with $\dot{\gamma}(0) \in U$, then

$$\|J(1)\| \geq (1+\eta)\|J(0)\|$$

The choice of U and η is possible because N is negatively curved and $\|J(t)\|$ is an increasing strictly convex function whose second derivative is controlled by Jacobi's equation.

Since $v \in \Lambda_0$ is uniformly recurrent for φ_t, we have that

$$\|J(t+1)\| \geq (1+\eta)\|J(t)\|$$

for all $t \in R_+(T) := \{s \in [0, T] : \varphi_s(v) \in U\}$. Because $v \in \Lambda_0$, we know that $R_+(T)$ has Lebesgue measure $\geq \delta T$for some $\delta > 0$ and for all T sufficiently large. Therefore, for any unstable Jacobi field $J(t)$ along $\varphi_t(v)$, one has

$$\|J(T)\| \geq (1+\eta)^{\delta T - 1} \|J(0)\|$$

for all T sufficiently large. It follows from the definitions that

$$\lim_{T \to +\infty} \frac{1}{T} \log \|D_v \varphi_T(\xi)\| \geq \delta \log(1+\eta) > 0$$

for any $\xi \in E^u(v)$, and, hence,

$$E^u(v) \subset \bigoplus_{\lambda_j(v) > 0} E_j(v)$$

Similarly, $E^s(v) \subset \bigoplus_{\lambda_i(v)<0} E_i(v)$. Because $E^s(v) \oplus E^u(v) = \dot{\varphi}(v)^{\perp}$, these inclusions must be equalities and the proof of the lemma is complete.

For later reference, we summarize the results proved in this section in the following statement:

Theorem 5.4.4 *Under the assumptions (II), (III) and (VI) in Theorem 5.1.1 above, the geodesic flow φ_t is* nonuniformly hyperbolic*: more concretely, there exists a subset $\Lambda_0 \subset T^1N$ of full m-measure such that the $D\varphi_t$-invariant splitting*

$$T_v T^1 N = E^s(v) \oplus E^0(v) \oplus E^u(v)$$

into the flow direction $E^0(v) = \mathbb{R}\dot{\varphi}(v)$ and the spaces $E^s(v)$ and $E^u(v)$ of stable and unstable Jacobi fields *along $\gamma(t) = \varphi_t(v)$ have the property that*

$$0 < \lim_{t \to \infty} \frac{1}{t} \log \|D_v \varphi_t(\xi^u)\| < \infty \quad \textit{and} \quad -\infty < \lim_{t \to \infty} \frac{1}{t} \log \|D_v \varphi_t(\xi^s)\| < 0$$

for all $\xi^u \in E^u(v) - \{0\}$ and $\xi^s \in E^s(v) - \{0\}$.

5.5 Stable Manifolds of Certain Geodesic Flows

Our long-term goal is to exploit the nonuniform hyperbolicity of φ_t in order to deduce the ergodicity of φ_t via Hopf's argument.

For this sake, we take in this section an important preliminary step, namely, we will show that the stable and unstable manifolds of φ_t form *global* laminations with useful *absolute continuity* properties.

5.5.1 Local (Pesin) Stable Manifolds for Certain Geodesic Flows

We begin by noticing that a geodesic flow φ_t satisfying the assumptions (I) to (VI) of Theorem 5.1.1 has "nice" local (Pesin) stable and unstable manifolds through almost every point.

The reader with some experience with nonuniformly hyperbolic systems might think that this is an immediate consequence of the so-called *Pesin's theory*. However, this is *not* the case in our setting because the phase space T^1N of φ_t is *not* assumed to be compact. In other words, we are facing a dilemma: while the noncompactness of N is an important point for the applications of Theorem 5.1.1 (to Weil–Petersson geodesic flows), it forbids a naive utilization of Pesin's theory because of the competition between the dynamical behaviors of φ_t in compact regions of N and near "infinity" ∂N.

Fortunately, Katok and Strelcyn [KS] (with the aid of Ledrappier and Przytycki) developed a *generalization* of Pesin's theory where any "well-behaved" dynamics on noncompact phase space is allowed. Furthermore, Katok–Strelcyn successfully applied their version of Pesin's theory to the study of *dynamical billiards*.

Roughly speaking, Katok–Strelcyn say that if the nonuniformly hyperbolic system φ_t "blows up at most polinomially" at infinity ∂N, then the hyperbolic (exponential) behavior of φ_t is strong enough so that Pesin's theory can be applied (because N is "essentially compact" for practical purposes).

Evidently, this is much easier said than done, and, unfortunately, the discussion of the details of Katok–Strelcyn's generalization of Pesin's theory is out of the scope of these notes. In particular, we will content ourselves in just mentioning the conditions (I) to (VI) in Theorem 5.1.1 were set up in [BMW] in such a way that a geodesic flow φ_t satisfying (I) to (VI) also verifies all the requirements to apply Katok–Strelcyn's work. Here, even though this is philosophically natural, it is worth to point out that the deduction of the conditions to use Katok–Strelcyn's technology from (I) to (VI) is *far from trivial*: indeed, Burns–Masur–Wilkinson [BMW] do this after studying (in Appendices A and B of their paper) several C^3 properties of Sasaki metric and C^2 properties of φ_t.

In summary, the hypothesis (I) to (VI) in Theorem 5.1.1 ensure that Katok–Strelcyn's generalization of Pesin's theory applies in the setting of Theorem 5.1.1. As a by-product, they deduce the following statement about the existence and absolute continuity of local (Pesin) stable manifolds (cf. Proposition 3.10 in [BMW]).

Theorem 5.5.1 ("Pesin Stable-Manifold Theorem") *Let φ_t be the geodesic flow on the unit tangent bundle T^1N of a n-dimensional Riemannian manifold N satisfying the conditions (I) to (VI) of Theorem 5.1.1. Denote by $\Lambda_0 \subset T^1N$ the subset of full volume provided by Theorem 5.4.4 where φ_t is nonuniformly hyperbolic.*

Then, there exists a subset $\Lambda_1 \subset \Lambda_0$ of full volume, a measurable function $r\colon \Lambda_1 \to \mathbb{R}_+$, and a measurable family

$$W^s_{loc} = \{W^s_{loc}(v) : v \in \Lambda_1\}$$

of smooth (C^∞) embedded disks $W^s_{loc}(v)$ with the following properties. For all $v \in \Lambda_1$:

- *$T_v W^s_{loc}(v) = E^s(v)$, i.e., $W^s_{loc}(v)$ is tangent to $E^s(v)$;*
- *$\varphi_t(W^s_{loc}(v)) \subset W^s_{loc}(v)$ for all $t \geq 0$, i.e., $W^s_{loc}(v)$ is topologically contracted in forward time by φ_t;*
- *$w \in W^s_{loc}(v)$ if and only if $d_{Sas}(v, w) < r(v)$ and $\lim_{t\to+\infty} d_{Sas}(\varphi_t(v), \varphi_t(w)) = 0$, i.e., $W^s_{loc}(v)$ is local stable manifold (in the sense that it is dynamically characterized as the set of w close to v whose forward φ_t-orbit approaches the forward φ_t-orbit of w).*

  Moreover, the family W^s_{loc} is absolutely continuous in the sense that the following "Fubini-like statements" hold.
- *given $Z \subset T^1N$ a subset of zero volume, one has that the set $Z \cap W^s_{loc}(v)$ has zero measure in $W^s_{loc}(v)$ (with respect to the induced $(n-1)$-dimensional Lebesgue measure on $W^s_{loc}(v)$) for almost every $v \in \Lambda_1$;*
- *given a C^1-embedded n-dimensional open disk $D \subset T^1N$ and $B \subset D$ a subset of zero measure (for the induced Lebesgue measure of D), the set*

$$Sat^s_{loc}(B) := \bigcup_{\substack{v\in\Lambda_1,\\ W^s_{loc}(v)\cap B\neq\varnothing}} W^s_{loc}(v)$$

  (obtained by saturating B by the local stable manifolds $W^s_{loc}(v)$ passing through it) has zero volume in T^1N.

Finally, the analogous assertions about unstable manifolds are also true.

5.5.2 Global Stable Manifolds of Certain Geodesic Flows

The Pesin stable and unstable laminations provided by Theorem 5.5.1 are *not* sufficient to run Hopf's argument: as it was explained in Sect. 5.1, the local stable manifolds $W^s_{loc}(v)$ could be a priori very *short* (because their radii $r(v)$ vary only *measurably* with $v \in \Lambda_1$ and so one does not expect for uniform lower bounds on $r(v)$).

Hence, it is important (for our purposes of using Hopf's argument) to compare Pesin's local stable manifolds W^s_{loc} with *global* objects. Here, the key point is to observe that Theorem 5.4.4 says that the tangent space of $W^s_{loc}(v)$ at v is exactly the vector space of *stable Jacobi fields* along the geodesic $\varphi_t(v)$ and, as we will recall in a moment, stable Jacobi fields are naturally related to global objects called *stable horospheres*.

5.5.2.1 Stable Jacobi Fields and Stable Horospheres

Let N be a Riemannian manifold. Given an unit tangent $v \in T^1N$ generating a geodesic ray $\gamma\colon [0,\infty) \to N$ such that the sectional curvatures of N are negative along γ and $w \in \dot{\varphi}(v)^{\perp}$, let us denote by $J_{-,w}$ the *stable Jacobi field* associated to w: by definition, this is the Jacobi field

$$J_{-,w}(t) := \lim_{\tau\to+\infty} J_{-,w,\tau}(t)$$

where $J_{-,w,\tau}(t)$ is the Jacobi field satisfying $J_{-,w,\tau}(0) = w$ and $J_{-,w,\tau}(\tau) = 0$.

In terms of the description of Jacobi fields via variations of geodesics, the stable Jacobi fields along γ are obtained by varying γ through geodesics $\beta\colon [0,+\infty) \to N$ such that $d(\beta(t),\gamma(t)) \leq d(\beta(0),\gamma(0))$ for all $t \geq 0$ (that is, β stays always close to γ in *forward time*). These geodesics β are *orthogonal* to a family of immersed hypersurfaces of N whose lifts to the universal cover M of N are the so-called *stable horospheres*.

The stable horospheres can be constructed "by hands" with the aid of the so-called *Busemann functions* as follows.

Let N be the quotient $N = M/\Gamma$ of a contractible, negatively curved, Riemannian manifold M by a subgroup Γ of isometries of M acting freely and properly discontinuously and suppose that the universal cover M of N is geodesically convex (i.e., M satisfies item (I) of Theorem 5.1.1).

In this situation, it is possible to show (see, e.g., Proposition 3.5 in [BMW]) that given an unit vector $v \in T^1M$ generating an infinite geodesic ray $\gamma_v\colon [0,+\infty) \to M$, the functions $b^s_{v,t}\colon M \to \mathbb{R}$ given by

$$b^s_{v,t}(y) = d(y,\gamma_v(t)) - t$$

converge (uniformly on compact sets) as $t \to +\infty$ to a C^1 convex function

$$b^s_v\colon M \to \mathbb{R}$$

called *stable Busemann function* such that $\|\mathrm{grad}(b^s_v)\| = 1$ and, for every $y \in M$, the unit vector $w^s_v(y) := -\mathrm{grad}(b^s_v)(y)$ defines an infinite geodesic ray $\gamma_{w^s_v(y)}\colon [0,+\infty) \to M$ with

$$d(\gamma_{w^s_v(y)}(t),\gamma_v(t)) \leq d(\gamma_v(0),y)$$

for all $t \geq 0$. In particular, the geodesics $\gamma_{w^s_v(y)}(t)$ give variations of γ leading to stable Jacobi fields.

For each $t \in \mathbb{R}$, the level set $\mathscr{H}_v^s(t) = (b_v^s)^{-1}(t) \subset M$ is a connected, complete, codimension 1 submanifold of M called *stable horosphere of level t*. By definition, the geodesics $\gamma_{w_v^s(y)}$ are orthogonal to the 1-parameter family $\mathscr{H}_v^s(t)$ of stable horospheres (because stable horospheres are level sets of b_v^s and the geodesics $\gamma_{w_v^s(y)}$ point in the direction $w_v^s(y) := -\text{grad}(b_v^s)(y)$ of the gradient).

The submanifold

$$W^s(v) := \{w_v^s(y) : y \in \mathscr{H}_v^s(0)\}$$

of T^1M consisting of unit vectors that are orthogonal to the stable horosphere $\mathscr{H}_v^s(0)$ of level 0 is called the *(global) stable manifold* of $v \in T^1M$. This nomenclature is justified by the following property (corresponding to Proposition 3.6 in [BMW]). In the context of Theorem 5.1.1, suppose that the infinite geodesic ray $\gamma_v: [0, +\infty) \to M$ projecting to a *forward recurrent* geodesic on $N = M/\Gamma$ (i.e., *after* projection to N, the unit vector $\dot{\gamma}(0)$ becomes an accumulation point of the set $\{\dot{\gamma}(t) : t \geq 1\}$). Then, for any $y \in M$, the unit vector $w = w_v^s(y) \in T_y^1M$ is tangent to an infinite geodesic ray $\gamma_w: [0, +\infty) \to M$ such that

$$\lim_{t\to+\infty} d(\gamma_v(t), \gamma_w(t + b_v^s(y))) = 0$$

Furthermore, $d_{Sas}(\varphi_t(v), \varphi_{t+b_v^s(y)}(w)) \to 0$ as $t \to +\infty$. In particular, $\varphi_t(W^s(v)) = W^s(\varphi_t(v))$ (stable manifolds are φ_t-invariant) and $\lim_{t\to+\infty} d_{Sas}(\varphi_t(v), \varphi_t(w)) = 0$ for all $w \in W^s(v)$ (stable manifolds are dynamically characterized by future orbits getting close together).

Remark 5.5.2 As usual, by reversing the time (via the symmetry $\gamma_v(t) = \gamma_{-v}(-t)$), one can define unstable Jacobi fields, unstable Busemann functions and unstable horospheres.

Remark 5.5.3 We already met the stable and unstable horospheres associated to the vertical geodesic in the hyperbolic plane passing through i in Fig. 5.2.

5.5.2.2 Geometry of the Stable and Unstable Horospheres

In this subsection, we make a couple of comments on the geometry of stable and unstable horospheres. More precisely, besides explaining the computation of their second fundamental forms from matrix Riccati equations, we will see that the stable and unstable horospheres are mutually transverse in a quantitative way. Of course, this transversality property of horospheres is another important point in Hopf's argument (as it allows to control the angle between stable and unstable manifolds).

Let $\gamma: (-\infty, 0] \to M$ be a geodesic ray such that the sectional curvatures of M along γ are negative. For each $w \in \dot{\gamma}(0)^{\perp}$, let us denote by $J_{+,w}(t)$ the unstable Jacobi field along γ with $J_{+,w}(0) = w$ (as usual).

Consider the 1-parameter family of matrices (linear operators) $U_+(t)\colon \dot\gamma(t)^\perp \to \dot\gamma(t)^\perp$ defined by the formula

$$U_+(t)(J_{+,w}(t)) = J'_{+,w}(t)$$

As we mentioned in Sect. 5.3, $U_+(t)$ are symmetric, positive-definite operators satisfying the matrix Ricatti equation

$$U'_+ + U_+ + \mathscr{R} = 0,$$

(i.e., $\langle a, U_+(t)(a)\rangle = -\langle a, R(a, \dot\gamma(t))\dot\gamma(t)\rangle - \langle a, U_+(t)^2(a)\rangle$ for all $a \in \dot\gamma(t)^\perp$).

It is possible to show (cf. Eberlein's survey [Ebs]) that the operator $U_+(t)$ is precisely the *second fundamental form* at $\dot\gamma(t)$ of the unstable horosphere $\mathscr{H}_v^u(t)$ of level t.

By reversing the time, we have an analogous operator $U_-(t)$ related to stable horospheres.

Note that, by definition, the stable and unstable subspaces $E^s(v)$ and $E^u(v)$ at an unit vector $v = \dot\gamma(0)$ defining an *infinite* geodesic ray $\gamma\colon \mathbb{R} \to M$ are

$$E^u(v) = \{(a, U_+(0)a) : a \in v^\perp\} \text{ and } E^s(v) = \{(b, U_-(b)) : b \in v^\perp\}$$

In other terms, we have a $D\varphi_t$-invariant splitting

$$T_{\mathscr{D}}T^1M = E^s \oplus E^0 \oplus E^u$$

over the set

$$\mathscr{D} := \{v \in T^1M : v \text{ defines an infinite geodesic ray } \gamma\colon \mathbb{R} \to M\}$$

(where $E^0 = \mathbb{R}\dot\varphi$).

Let us now show that this splitting is locally uniform over $\mathscr{D}$.

Proposition 5.5.4 *There exists a continuous function $\delta\colon T^1M \to \mathbb{R}_+$ such that the continuous family of conefields*

$$\mathscr{C}^s(v) = \{(w, w') \in \dot\varphi(v)^\perp : \langle w, w'\rangle \leq -\delta(v)\|(w, w')\|_{Sas}\}$$

and

$$\mathscr{C}^u(v) = \{(w, w') \in \dot\varphi(v)^\perp : \langle w, w'\rangle \geq \delta(v)\|(w, w')\|_{Sas}\}$$

meeting only at the origin have the property that

$$E^s(v) \subset \mathscr{C}^s(v) \text{ and } E^u(v) \subset \mathscr{C}^u(v)$$

for all $v \in \mathscr{D}$.

Proof Our task consists of showing that the functions

$$\delta^u(v) := \inf_{(w,w')\in E^u(v)-\{0\}} \frac{\langle w, w'\rangle}{\|(w,w')\|_{Sas}^2} \text{ and } \delta^s(v) := \inf_{(w,w')\in E^s(v)-\{0\}} -\frac{\langle w, w'\rangle}{\|(w,w')\|_{Sas}^2}$$

of $v \in \mathscr{D}$ are locally uniformly bounded away from zero.

By symmetry, it suffices to prove that δ^s is locally uniformly bounded from below. For the sake of reaching a contradiction, suppose this is not the case. This means that there are sequences $v_n \in \mathscr{D}$, $\xi_n \in E^s(v_n) - \{0\}$ with $\|\xi_n\|_{Sas} = 1$ such that $v_n \to v \in \mathscr{D}$, $\xi_n \to \xi = (w, w')$ and $\langle w, w'\rangle = 0$.

For each $n \in \mathbb{N}$, let J_n be the stable Jacobi fields along γ_{v_n} induced by ξ_n, and denote by J the (limit) Jacobi field along γ_v induced by ξ.

On one hand, for each n, the square $\|J_n(t)\|^2$ of the norm of the stable Jacobi field $J_n(t)$ is a decreasing function of t. In particular, since $\xi_n \to \xi$, we deduce that $\|J(t)\|^2$ is a nonincreasing function of t.

On the other hand, $\|J(t)\|^2$ is a strictly convex function of t (because J is a perpendicular Jacobi field, cf. Eberlein's survey [Ebs]).

By putting these two facts together, we see that the function $t \mapsto \|J(t)\|^2$ has no critical points. However, $(\|J\|^2)'(0) = 2\langle w, w'\rangle = 0$. This contradiction proves the desired proposition.

5.5.2.3 Absolute Continuity of Global Stable Manifolds

Once we have related Pesin's stable and unstable manifolds W^s_{loc} (local objects) to stable and unstable horospheres (global objects), it is not entirely surprising that the absolute continuity properties of Pesin stable manifolds (described in Theorem 5.5.1 above) can be "transferred" to horospherical laminations:

Proposition 5.5.5 *Let φ_t be the geodesic flow on the unit tangent bundle $\pi: T^1N \to N$ of a n-dimensional Riemannian manifold N satisfying the conditions (I) to (VI) of Theorem 5.1.1. Denote by $\Omega_1 \subset T^1M$ the subset of the unit tangent bundle of the universal cover $p: M \to N$ of N consisting of unit vectors $v \in T^1M$ projecting into a forward and backward recurrent geodesic γ_v in T^1N.*

Then, there exists a subset $\Omega_2 \subset \Omega_1$ of full volume such that the stable Busemann functions $b^s_v: M \to \mathbb{R}$ are C^∞ for all $v \in \Omega_2$. Moreover, the leaves of the stable lamination $W^s = \{W^s(v) : v \in \Omega_2\}$ are C^∞-submanifolds of T^1M diffeomorphic to $\mathbb{R}^{n-1}$. Furthermore, the stable horospherical lamination

$$\{W^s(v,\delta) : v \in \Lambda_2, \delta < inj(\pi(v))\}$$

obtained by taking the family of manifolds $W^s(v,\delta) :=$ connected component of $W^s(v) \cap B_{T^1N}(v,\delta)$ containing v through the vectors $v \in \Lambda_2$ in the projection

$\Lambda_2 = Dp(\Omega_2)$ of Ω_2 to T^1N (via $Dp: T^1M \to T^1N$) has the following absolute continuity properties:

- *if $Z \subset T^1N$ has zero m-volume, then for m-almost every $v \in \Lambda_2$ and any $\delta < inj(\pi(v))$, the set $Z \cap W^s(v,\delta)$ has zero $(n-1)$-dimensional volume in $W^s(v,\delta)$;*
- *if $D \subset T^1N$ is a smooth, embedded, n-dimensional open disk and $B \subset D$ has zero n-dimensional volume in D, then for any $\delta < \frac{1}{2}\inf_{v\in D} inj(\pi(v))$ one has $m(Sat^s(B,\delta)) = 0$ where*

$$Sat^s(B,\delta) := \bigcup_{v\in\Lambda_2: W^s(v,\delta)\cap B\neq\varnothing} W^s(v,\delta)$$

  is the set obtained by saturating B with the leaves of the lamination $W^s(v,\delta)$.

Finally, a similar statement holds for the corresponding unstable lamination.

Logically, the statement of this proposition is close to Theorem 5.5.1 about the absolute continuity of Pesin stable manifolds, but the crucial point is that we have now an absolutely continuous stable lamination W^s whose leaves have radii essentially equal to $\text{inj}(\pi(v))/2$. In other words, the leaves of the stable lamination W^s have a size controlled by the injectivity radius of N, a global smooth function, instead of the a priori merely measurable function $r(v)$ giving the radii of leaves of Pesin's stable lamination W^S_{loc}.

The proof of Proposition 5.5.5 is not very difficult: it uses the absolute continuity properties of Pesin's lamination W^s_{loc} in Theorem 5.5.1 and the "contraction of stable horospheres" (i.e., the fact that the forward dynamics of φ_t eventually contracts $W^s(v,\delta)$ inside $W^s(\varphi_t(v), r(v))$), and it occupies two pages in Burns–Masur–Wilkinson paper [BMW] (cf. the proof of their Proposition 3.11). However, we will skip this point in favor of discussing Hopf's argument in the next section.

5.6 Proof of Theorem 5.1.1 via Hopf's Argument

Let φ_t be a geodesic flow satisfying the assumptions (I) to (VI) of Theorem 5.1.1. We want to show that φ_t is ergodic with respect to the volume measure m (with normalized total mass).

By the Birkhoff Ergodic Theorem, given a continuous function $f: T^1N \to \mathbb{R}$ with compact support, the Birkhoff ergodic averages

$$\frac{1}{T}\int_0^T f(\varphi_t(v))dt$$

converge as $T \to \pm\infty$ to the same limit $B(f)(v)$ for m-almost every $v \in T^1N$.

By definition of ergodicity, our task consists of showing that the function $B(f)(v)$ is constant m-almost everywhere.

For this sake, let us define the measurable functions

$$f^s(v) = \limsup_{T\to+\infty} \frac{1}{T}\int_0^T \varphi_t(v)dt$$

and

$$f^u(v) = \limsup_{T\to-\infty} \frac{1}{T}\int_0^T \varphi_s(v)ds$$

Note that, by the Birkhoff Ergodic Theorem, there exists a subset $G \subset T^1N$ of full m-measure such that

$$f^s(v) = f^u(v) = B(f)(v)$$

Moreover, from their definitions, note that the functions f^s, f^u and $B(f)$ are φ_t-invariant.

The initial observation in Hopf's argument is the fact that the function f^s, resp. f^u, is *constant* along the stable manifolds $W^s(v)$, resp. unstable manifolds $W^u(v)$. In fact, this follows easily from the uniform continuity of the (compactly supported, continuous) function f and the fact that $d(\varphi_t(v), \varphi_t(w)) \to 0$ as $t \to +\infty$ (resp. $t \to -\infty$) whenever $w \in W^s(v)$ (resp. $w \in W^u(v)$).

The basic strategy of Hopf's argument can be summarized as follows. We want to combine this initial observation with the absolute continuity properties of the stable and unstable horospherical laminations to deduce that φ_t is "*locally ergodic*" in the sense that *every* $v \in T^1N$ possesses a neighborhood U_v such that the restriction $B(f)$ to U_v is m-almost everywhere constant.

Of course, since T^1N is connected, this local ergodicity property implies that the function $B(f)$ is constant m-almost everywhere, and, a fortiori, φ_t is ergodic with respect to m. In other terms, our task is reduced to prove the local ergodicity property stated in the previous paragraph.

In this direction, we fix once and for all $v \in T^1N$, we set

$$\delta = \delta(v) := \frac{1}{4}\min\{\text{inj}(\pi(v)), d(v, \partial N)\},$$

and we denote by V the δ-neighborhood of $v \in T^1N$.

Let $\Lambda_2 \subset T^1N$ be the full m-volume subset constructed in Proposition 5.5.5. For each $w \in \Lambda_2 \cap V$, we consider the stable leaf $W^s(w,\delta)$, we take its iterates under φ_t for $|t| < \delta$, and we saturate the resulting subset $\varphi_{(-\delta,\delta)}(W^s(w,\delta)) = \bigcup_{|t|<\delta} \varphi_t(W^s(w,\delta))$ with the leaves of the unstable horospherical lamination $W^u = \{W^u(.,\delta)\}$ to obtain the subset

$$N_\delta(w) := Sat^u(\varphi_{(-\delta,\delta)}(W^s(w,\delta)), \delta)$$

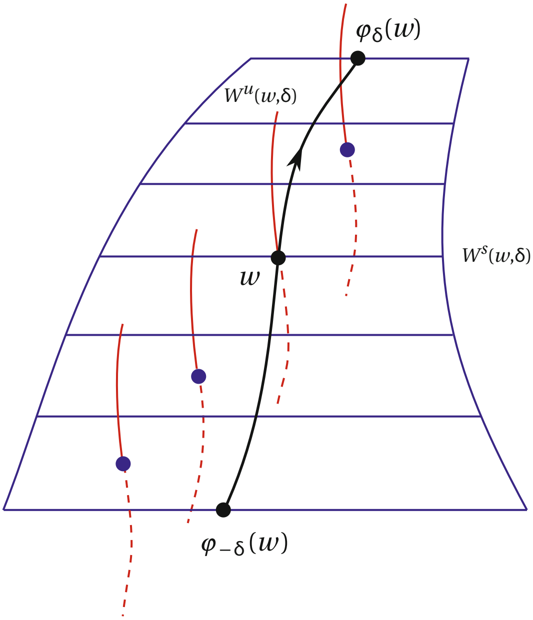

Fig. 5.3 Geometry of $N_\delta(w)$

The construction of $N_\delta(w)$ is illustrated in Fig. 5.3: the subset $\varphi_{(-\delta,\delta)}(W^s(w,\delta))$ is marked in blue and some leaves of W^u passing through points of $\varphi_{(-\delta,\delta)}(W^s(w,\delta))$ are marked in red.

The local ergodicity property stated above is an immediate consequence of the following two claims:

(a) the restriction of the function $B(f)$ to $N_\delta(w)$ is almost everywhere constant for almost every choice of $w \in \Lambda_2 \cap V$;
(b) $N_\delta(w)$ is essentially open for almost every w near v in the sense that there exists a neighborhood U_v of v such that $N_\delta(w) \cap U_v$ has full volume in U_v for almost every choice of $w \in U_v$.

We establish the first claim (a) by exploiting the initial observation that Birkhoff averages are constant along stable and unstable manifolds and the absolute continuity properties of the stable and unstable horospherical laminations.

More precisely, let us consider again the full volume subset G of T^1N where $f^s(v) = f^u(v) = B(f)(v)$ (provided by the Birkhoff Ergodic Theorem).

By absolute continuity property of W^s (cf. the first item of conclusion of Proposition 5.5.5), for almost every $w \in \Lambda_2 \cap V$, the intersection $G \cap W^s(w,\delta)$ has full volume in $W^s(w,\delta)$. We affirm that $B(f)|_{N_\delta(w)}$ is almost everywhere constant for any such w.

In fact, f^s takes a constant value $a := f^s(w)$ on $W^s(w,\delta)$. Moreover, since $f^s = f^u$ on G, we also have that f^u takes the constant value a on $G \cap W^s(w,\delta)$. By combining this fact with the φ_t-invariance of f^u, we deduce that f^u takes the constant value a

on $G' := \varphi_{(-\delta,\delta)}(G \cap W^s(w,\delta))$. Furthermore, by putting together this fact with the initial observation that f^u is constant along unstable manifolds $W^u(.,\delta)$, we obtain that f^u takes the constant value a on $Sat^u(G',\delta)$.

Note that, by assumption, $G \cap W^s(w,\delta)$ is a full volume subset of $W^s(w,\delta)$. Since φ_t is a C^∞-flow, it follows that G' is a full volume subset of the n-dimensional smooth submanifold $D = \varphi_{(-\delta,\delta)}(W^s(w,\delta))$. Therefore, from the absolute continuity property of W^u (cf. the second item of conclusion of Proposition 5.5.5), we conclude that $Sat^u(G',\delta)$ is a full volume subset of $Sat^u(D,\delta) := N_\delta(w)$. In particular, f^u takes the constant value a on the full volume subset $Sat^u(G',\delta)$ of $N_\delta(w)$. Because $f^u = B(f)$ on G, we get that $B(f)$ takes the constant value a on the full volume subset $G \cap Sat^u(G',\delta)$ of $N_\delta(w)$, i.e., $B(f)|_{N_\delta(w)}$ is almost everywhere constant. This completes the proof of the claim (a).

Remark 5.6.1 The reader is encouraged to interpret this argument in the light of Fig. 5.3 in order to get a clear picture of the roles of the subsets G', D and $N_\delta(w)$.

We establish now the second claim (b) from the absolute continuity properties of the horospherical laminations and the local uniform transversality of the stable and unstable manifolds.

More concretely, from the absolute continuity property in the first item of the conclusion of Proposition 5.5.5, the stable disk $W^s(w,\delta)$, resp. unstable disk $W^u(w,\delta)$, is almost everywhere tangent to the stable direction E^s, resp. unstable direction E^u, for almost every $w \in \Lambda_2 \cap V$. Since the stable and unstable directions E^s and E^u are contained in the *continuous* families of cones $\mathscr{C}^s$ and $\mathscr{C}^u$ from Proposition 5.5.4, we have that $W^s(w,\delta)$, resp. $W^u(w,\delta)$, is *everywhere* tangent to $\mathscr{C}^s$, resp. $\mathscr{C}^u$ for almost every $w \in \Lambda_2 \cap V$.

In particular, from the φ_t-invariance of the stable lamination W^s, we see that the n-dimensional disk $D = D(w) := \varphi_{(-\delta,\delta)}(W^s(w,\delta))$ is everywhere tangent to $\mathscr{C}^s \oplus E^0$ for almost every $w \in \Lambda_2 \cap V$. Since the continuous conefields $\mathscr{C}^s$ and $\mathscr{C}^u$ meet only at the origin (cf. Proposition 5.5.4), that is, they are locally uniformly transverse, we conclude that there exists a neighborhood U_v of v such that

$$W^u(w',\delta) \cap D(w) \neq \varnothing$$

for almost any $w, w' \in \Lambda_2 \cap U_v$. In other words, $Sat^u(D(w)) := N_\delta(w)$ intersects $\Lambda_2 \cap U_v$ in a full volume subset. This completes the proof of claim (b).

This concludes our discussion of Hopf's argument (namely, the derivation of claims (a) and (b)) for the ergodicity of φ_t.

Closing these notes, let us say a few words about the mixing and Bernoulli properties in the statement of Theorem 5.1.1. In [BMW], these properties are deduced from general results of Katok [K] saying that if a *contact* flow is nonuniformly hyperbolic and ergodic, then it is Bernoulli (and, in particular, mixing).

Nevertheless, as it was brought to our attention by B. Hasselblatt and Y. Coudène, the Hopf argument above can be slightly adapted in certain contexts to give mixing and/or mixing of all orders. For example, concerning the mixing property, Y.

Coudène, B. Hasselblatt and S. Troubetzkoy showed (in Theorem 3.3 of their paper [CHT]) that if any L^2-function f saturated by stable and unstable sets (in the sense that there is a full measure subset G such that $f(x) = f(y)$ whenever $x, y \in G$ and $y \in W^s(x)$ or $y \in W^u(x)$) is almost everywhere constant, then the dynamical system is mixing. Also, they have a similar criterion for multiple mixing, and, furthermore, they discuss a couple of nontrivial examples of applications of their criteria.

In the context of Theorem 5.1.1, we can deduce the mixing property for φ_t from the result of Coudène–Hasselblatt–Troubetzkoy. Indeed, the argument used in the proof of the claim (a) above (during the discussion of Hopf's argument) also shows that any L^2-function saturated by stable and unstable sets (such as $B(f) = f^s = f^u$) is almost everywhere constant, so that Coudène–Hasselblatt–Troubetzkoy mixing criterion "à la Hopf" applies in this setting.

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