Chapter 4
Ergodicity of the Weil–Petersson Geodesic Flow

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4.1 The Proof of Ergodicity

4.1.1 The Ergodicity Theorem

These notes describe some of the dynamical properties of the Weil–Petersson geodesic flow for the moduli space of Riemann surfaces, notably that this flow is ergodic. Ergodicity implies that a randomly chosen, unit-speed Weil–Petersson geodesic in moduli space becomes equidistributed over time. What is more, the tangent vectors to such a geodesic also become equidistributed in the space of all unit tangent vectors to moduli space. To state this more precisely and to put it in context, we first review the basic setting of Teichmüller theory. Let S be a surface of genus $g \geq 0$ with $n \geq 0$ punctures, and let $M(S)$ be the moduli space of conformal structures on S, up to conformal equivalence. Assume that $3g+n \geq 4$, which implies that in each conformal class there is a complete hyperbolic metric. Then $M(S)$ has the alternate description of the moduli space of hyperbolic structures on S, up to isometry. The orbifold universal cover of $M(S)$ is the *Teichmüller space* Teich(S) of marked conformal structures on S. It is a classical result due to Fricke and Klein that Teich(S) is homeomorphic to a ball of dimension $6g - 6 + 2n$. Teichmüller space carries a natural complex structure via a special embedding of Teich(S) into a complex representation variety QF(S), called *quasi-Fuchsian space*. Under this

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map, called the Bers embedding, the image of Teich(S) sits as a complex subvariety (indeed there is a biholomorphic equivalence QF(S) $\simeq$ Teich(S) $\times$ Teich(S)). The orbifold fundamental group of $M(S)$ is the mapping-class group MCG(S) of orientation-preserving homeomorphisms of S modulo isotopy. The mapping-class group acts holomorphically on Teich(S). The stabilizer of each point is finite, which gives $M(S)$ the structure of a complex orbifold.

A naturally defined and well-studied metric on Teichmüller space, and the focus of this course, is the Weil–Petersson metric g_{WP}, which is the Kähler metric induced by the Weil–Petersson symplectic form ω_{WP} and the almost complex structure J on Teich(S):

$$g_{\mathrm{WP}}(v, w) = \omega_{\mathrm{WP}}(v, Jw).$$

The Weil–Petersson metric is invariant under MCG(S) and so descends to a metric on $M(S)$. It has finite volume determined by the volume form $\left|\omega_{\mathrm{WP}}^{\wedge 3g-3+n}\right|$. A striking feature of the Weil–Petersson metric is its intimate connection with hyperbolic geometry:

- The hyperbolic length of a closed geodesic (for a fixed free homotopy class on S) is a convex function along Weil–Petersson geodesics in Teich(S) [Wo08];
- Fenchel–Nielsen coordinates $(\ell_i, \tau_i)_{i=1}^{3g-3+n}$ on Teich(S) are Darboux coordinates: the Weil–Petersson symplectic form has the simple expression $\omega_{\mathrm{WP}} = \frac{1}{2}\sum_{i=1}^{3g-3+n} \mathrm{d}\ell_i \wedge \mathrm{d}\tau_i$ [Wo82];
- the growth of the hyperbolic lengths of simple closed curves on S is related to the Weil–Petersson volume of $M(S)$ [Mi08]; and
- the Weil–Petersson metric has a formulation in terms of dynamical invariants of the geodesic flow on hyperbolic surfaces [Br10, Mc08].

The Weil–Petersson metric has several notable features that make it an interesting geometric object of study in its own right. It is negatively curved, but incomplete. The sectional curvatures are neither bounded away from 0 (except in the simplest cases of $(g, n) = (1, 1)$ and $(g, n) = (0, 4)$), nor bounded away from $-\infty$.

The Weil–Petersson geodesic flow thus presents a naturally-occurring example of a hyperbolic dynamical system with singularities, for which one might hope to reproduce the known properties of the geodesic flow for a compact, negatively curved manifold, such as: ergodicity, equidistribution of closed orbits, exponentially fast mixing and decay of correlations, and Central Limit Theorem.

We summarize the pertinent literature prior to [BMW12]. Wolpert [Wo03] showed that the geodesic flow is defined for all time on a full-volume subset of the unit tangent bundle T^1Teich(S) and thus descends to a volume-preserving flow on the finite-volume quotient $M^1(S) := T^1\mathrm{Teich}(S)/\mathrm{MCG}(S)$. Pollicott et al. proved in the case $(g, n) = (1, 1)$ that the geodesic flow is transitive on $M^1(S)$ and that periodic orbits are dense in $M^1(S)$ [PWW10]. Brock et al. [BMM10] proved transitivity and denseness of periodic orbits for arbitrary (g, n) and also showed that the topological entropy of the geodesic flow is infinite (that is, unbounded on compact invariant sets). Hamenstädt [Ha10] proved a measure-theoretic version of density of closed orbits: the set of invariant Borel probability measures for the Weil–Petersson geodesic flow that are supported on a closed orbit is dense in the space of all ergodic invariant probability measures.

The focus of the CIRM workshop was:

Theorem 4.1.1 *Let S be a Riemann surface of genus $g \geq 0$, with $n \geq 0$ punctures. Assume that $3g + n \geq 4$. The Weil–Petersson geodesic flow on $M^1(S)$ is ergodic (and in fact Bernoulli) with respect to Weil–Petersson volume and has finite, positive measure-theoretic entropy.*

The Bernoulli property means that the time-1 map of the geodesic flow is abstractly isomorphic (as a measure-preserving system) to a Bernoulli process on a finite alphabet. In particular, it is mixing of all orders. An interesting open question is to determine the rate of mixing of this flow.

4.1.2 Hyperbolic Dynamics

Our basic approach to proving Theorem 4.1.1 is as follows. The Weil–Petersson geodesic flow φ^t preserves a finite volume m on $M^1(S)$ (which we may then assume normalized to a probability measure), and one can show using properties of the Weil–Petersson metric that $\log \|D\varphi^1\|$ is integrable with respect to the measure m. The Oseledec Multiplicative Ergodic Theorem (cf. [KH95, Theorem S.2.9]) then implies that there is a full-volume subset $\Omega \subset M^1(S)$ such that for every $v \in \Omega$ and every nonzero tangent vector $\xi \in T_v M^1(S)$, the limit

$$\lambda(\xi) := \lim_{t\to\infty} \frac{1}{t} \log \|D_v \varphi^t(\xi)\|$$

exists and is finite. This real number $\lambda(\xi)$ is called the (forward) Lyapunov exponent of φ^t at ξ. Observe that if ξ is in the line bundle $\mathbb{R}\dot{\varphi}(v)$ tangent to the orbits of the flow, then $\lambda(\xi) = 0$. We say that φ^t is *nonuniformly hyperbolic* if the converse holds a.e.: for almost every $v \in \Omega$ and every $\xi \in T_v M^1(S) \smallsetminus \mathbb{R}\dot{\varphi}(v)$, the Lyapunov exponent $\lambda(\xi)$ is nonzero.

Using that the Weil–Petersson sectional curvatures are negative, we establish that the Weil–Petersson geodesic flow is nonuniformly hyperbolic. Nonuniform hyperbolicity is the starting point for a rich ergodic theory of volume-preserving diffeomorphisms and flows, developed first by Pesin for closed manifolds and expanded by Sinai, Katok–Strelcyn, Chernov, and others to systems with singularities, such as the Weil–Petersson geodesic flow.

4.1.3 The Hopf Argument

The main achievement of Burns et al. [BMW12] is the step from nonuniform hyperbolicity to ergodicity. The basic argument for establishing ergodicity of such systems originates with Eberhard Hopf and his proof of ergodicity for geodesic flows for closed, negatively curved surfaces [Ho39, Ha17]. His method was to study the Birkhoff averages of continuous functions along leaves of the stable and unstable foliations of the flow. This type of argument has been used since then in increasingly general contexts and has come to be known as the Hopf Argument (see also [Ha17]).

The core of the Hopf Argument is simple. Suppose that ψ^t is a C^∞ flow defined on a full-measure subset Ω of a Riemannian manifold V, preserving a finite volume on V. For any $x \in \Omega$, one defines the stable and unstable sets

$$W^s(x) := \{x' \in \Omega : \lim_{t\to+\infty} \mathrm{d}(\psi^t(x), \psi^t(x')) = 0\}$$

$$W^u(x) := \{x' \in \Omega : \lim_{t\to-\infty} \mathrm{d}(\psi^t(x), \psi^t(x')) = 0\}.$$

The stable (respectively unstable) sets partition Ω into measurable subsets.

The first step in the Hopf Argument is to observe that for any continuous function $f\colon V \to \mathbb{R}$ with compact support, the forward and backward upper Birkhoff averages

$$f^s := \limsup_{T\to+\infty} \frac{1}{T}\int_0^T f\circ\psi^t \,\mathrm{d}t \text{ and } f^u := \limsup_{T\to-\infty} \frac{1}{T}\int_0^T f\circ\psi^t \,\mathrm{d}t$$

have the property that f^s is constant on any stable set $W^s(x)$ and f^u is constant on any unstable set $W^u(x)$. Both functions f^s and f^u are evidently invariant under the flow ψ^t, and the Birkhoff and von Neumann Ergodic Theorems (see, e.g., [Ha17, KH95]) imply that $f^s = f^u$ almost everywhere. To show that ψ^t is ergodic it suffices to show that f^s is constant almost everywhere for every continuous f with compact support. The fundamental idea is to use the properties of the equivalence relation generated by the stable sets, the unstable sets, and the flow to conclude that $f^s = f^u$ must be constant.

In the next step in the Hopf Argument, one assumes some form of hyperbolicity of the flow, which will imply that the stable and unstable sets are in fact smooth manifolds. In the original context of Hopf's argument, $V = \Omega = T^1S$ is the unit tangent bundle of a compact, negatively curved surface S and ψ^t is the geodesic flow. In this setting, the stable and unstable sets have a particularly nice description. For almost every unit vector v, the stable and unstable Busemann functions $\mathscr{B}^s_v$ and $\mathscr{B}^u_v$ are globally defined C^∞ functions. The stable and unstable sets are the orthogonal vectors to the level sets of these functions or equivalently the gradients of these functions on the level sets. They are C^∞, globally defined, and for $* \in s, u$, the collection $W^* := \{W^*(v) : v \in T^1S\}$ defines a C^1 foliation of T^1S. At each point $v \in T^1S$, the tangent space T_vT^1S is spanned by the tangents to $W^s(v)$, $W^u(v)$ and the direction $\dot\psi(v)$ of the flow. A local argument in C^1 charts using the Fubini Theorem shows that any ψ^t-invariant function that is almost everywhere constant along leaves of W^s and W^u must be locally almost everywhere constant, and hence globally almost everywhere constant, since T^1S is connected. In particular, the function f^s is constant for any continuous, compactly supported f, and so ψ^t is ergodic.

Hopf's original argument does not generalize immediately to geodesic flows for higher-dimensional compact, negatively curved manifolds. In this higher-dimensional setting, the stable and unstable foliations W^s and W^u exist, again arise from the level sets of Busemann functions, and have C^∞ leaves. In general, however, they fail to be C^1 foliations (except when the curvature is 1/4-pinched) and so the argument using the Fubini Theorem in local C^1 charts fails.

In the late 1960s Anosov [An67] overcame this obstacle by proving that for any compact, negatively curved manifold, the foliations W^s and W^u are absolutely continuous. Absolute continuity, a strictly weaker property than C^1, is sufficient to carry out a Fubini-type argument to show that any ψ^t-invariant function almost everywhere constant along leaves of W^s and W^u is locally constant. See Sect. 4.3 for a more detailed discussion of absolute continuity. Anosov thereby proved that the geodesic flow for any compact manifold of negative sectional curvatures is ergodic.

4.1.4 Nonuniform Hyperbolicity

There is an extensive literature devoted to extending the Hopf Argument beyond the uniformly hyperbolic setting of geodesic flows on compact negatively curved manifolds. For smooth flows defined everywhere on compact manifolds, Pesin [Pe76] introduced an ergodic theory of nonuniformly hyperbolic systems. In short, Pesin theory shows that if $\psi^t: V \to V$ preserves a finite volume and is nonuniformly hyperbolic, then almost everywhere the stable and unstable sets are smooth manifolds. The family of stable manifolds is measurable and absolutely continuous in a suitable sense.

From Pesin theory, one deduces that a nonuniformly hyperbolic diffeomorphism of a compact manifold has countably many ergodic components of positive measure. More information about the flow can be used in some contexts to deduce ergodicity. The obstruction to using the full Hopf Argument in this setting is that stable manifolds are defined only almost everywhere, and they may be arbitrarily small in diameter, with poorly controlled curvatures, etc.

In a somewhat different direction than Pesin theory, Sinai [Si70] introduced methods for proving ergodicity of hyperbolic flows with singularities and applied them in his study of the n-body problem of celestial mechanics. Here the flow ψ^t locally resembles the geodesic flow for a compact, negatively curved manifold, but globally encounters discontinuities and places where the norms of the derivatives $\|D\psi^t\|$ and $\|D^2\psi^t\|$ become unbounded.

Introducing new techniques in the Hopf argument, Sinai was able to show that for several important classes of systems, including some billiards and flows connected to the n-body system, ergodicity holds. These arguments have since been generalized to much larger classes of singular hyperbolic systems and singular, nonuniformly hyperbolic systems.

4.1.5 Addressing Singularities: The Katok–Strelcyn Criteria

In the singular, nonuniformly hyperbolic setting, all aspects of Hopf's argument require careful revisiting. The mere existence of local stable manifolds is a delicate matter and depends in a strong way on the growth of the derivative of ψ^t near the singularities. To give a sense of how delicate these issues can be, we remark that

- for compact surfaces of nonpositive curvature and genus $g \geq 2$, it is unknown whether the geodesic flow is always ergodic (even though it is always transitive);
- there exist complete, finite-volume surfaces of pinched negative curvature (but unbounded derivative of curvature) whose stable foliations are not even Hölder-continuous [BBB87];
- for C^1 nonuniformly hyperbolic systems that are not C^2, stable sets can fail to be manifolds [Pu84];
- nonuniformly hyperbolic systems on compact manifolds can fail to be ergodic and can even have infinitely many ergodic components with positive measure [DHP01].

A general result providing for the existence and absolute continuity of local stable and unstable manifolds for singular, nonuniformly hyperbolic systems was proved by Katok–Strelcyn [KSLP86]. We use this work in an important way.

Returning to the present context, the Weil–Petersson geodesic flow is a singular, nonuniformly hyperbolic system. To prove that it is ergodic, the first step is to verify the Katok–Strelcyn conditions to establish existence and absolute continuity of local stable and unstable manifolds. In particular, one needs to control the norm of the first two derivatives of the geodesic flow in a neighborhood of the boundary of $M^1(S)$.

To control the first derivative, we use the asymptotic expansions of Wolpert for the Weil–Petersson curvature and covariant derivative found in [Wo03, Wo09, Wo11], combined with a careful analysis of the solutions to the Weil–Petersson Jacobi equations. The precise estimates obtained by Wolpert appear to be essential for these calculations.

Since Wolpert's expansions of the Weil–Petersson metric are only to second order, and we need third order control to estimate the second derivative of the flow, we borrow ideas of McMullen in [Mc00]. There is a nonholomorphic (in fact totally real) embedding of Teich(S) into quasi-Fuchsian space QF(S), under which the Weil–Petersson symplectic form has a holomorphic extension. This holomorphic form is the derivative of a one-form that is bounded in the Teichmüller metric. Using the Cauchy Integral Formula and a comparison formula between Teichmüller and Weil–Petersson metrics, one can then obtain bounds on all derivatives of the

Weil–Petersson metric. These bounds are adequate to control the second derivative of the geodesic flow, using the bounds on the first derivative already obtained.

Once the conditions of [KSLP86] have been verified, we are guaranteed the almost everywhere existence of absolutely continuous families W^s and W^u of local stable and unstable manifolds. Nonetheless these stable and unstable manifolds may not have uniform size. At this point, we use negative curvature and another key property of the Weil–Petersson metric called geodesic convexity to show that in fact W^s and W^u have well-controlled uniform size.

As a by-product of our arguments, we obtain that the Weil–Petersson Busemann function is C^∞ for almost every tangent direction to Teich(S). The local geometry of W^s and W^u is sufficiently nice that Hopf's original argument can be used with small modifications. In particular, none of the more complicated local ergodicity arguments, such as the "Hopf chains" developed by Sinai, are necessary. We also obtain positive, finite entropy of the Weil–Petersson flow using results of Katok–Strelcyn and Ledrappier–Strelcyn in [KSLP86].

The paper [BMW12] does not quite follow the structure of this outline. Rather than restricting to the special case of the Weil–Petersson metric, we instead develop an abstract criterion, Theorem 4.3.1 below and in [BMW12], for ergodicity of the geodesic flow for an incomplete, negatively curved manifold. This has the advantage of clarifying the issues involved and also might allow for further applications.

4.1.6 The Case of the Punctured Torus

Several interesting features of the Weil–Petersson metric are already present in the simplest cases $(g,n) = (1,1)$ and $(g,n) = (0,4)$, where S is the once-punctured torus or the four-times punctured sphere, respectively.[1]

Here, Teich(S) is the upper half-space H and $M(S)$ is the classical moduli space of elliptic curves $H/\mathrm{PSL}(2,\mathbb{Z})$, which is a sphere with one puncture and two cone singularities of order 2 and 3. The mapping-class group MCG(S) is the modular group $\mathrm{SL}(2,\mathbb{Z})$. Due to the presence of torsion elements in $\mathrm{PSL}(2,\mathbb{Z})$, the space $M(S)$ is not a manifold, but the finite branched cover $H/\Gamma[k]$ for $k \geq 3$ is a manifold [Se60], where $\Gamma[k] = \{A \in \mathrm{PSL}(2,\mathbb{Z}) | A \equiv I \mod k\}$ is the level-k congruence subgroup. The tangent bundle to Teich(S) is canonically identified with $\mathrm{PGL}(2,R)$.

There are global coordinates (ℓ,τ) in Teich(S), the so-called Fenchel–Nielsen coordinates, which have the asymptotic (first-order) expansions $\ell(z) \sim \frac{1}{\mathrm{Im}(z)}$ and $\tau(z) \sim \frac{\mathrm{Re}(z)}{\mathrm{Im}(z)}$ as $\mathrm{Im}(z) \to \infty$, and the Weil–Petersson form has the first-order

[1] Before our work appeared, Pollicott and Weiss [PW09] gave a fairly complete outline of how to prove ergodicity for the Weil–Petersson metric in these cases. They studied the model case of a negatively curved surface whose singularities coincide with a surface of revolution for a polynomial and proved ergodicity of the geodesic flow in this case. They say that the missing ingredients are the bounds on the first and second derivatives of the geodesic flow.

asymptotic expansion $\omega_{\mathrm{WP}} = \frac{1}{2}\mathrm{d}\ell \wedge \mathrm{d}\tau \sim \frac{1}{\mathrm{Im}(z)^3} dz \wedge \mathrm{d}\bar{z}$, as $\mathrm{Im}(z) \to \infty$. Since the complex structure on Teich(S) is the standard one on H, we obtain the expansion $g^2_{\mathrm{WP}} \sim \frac{|dz|^2}{\mathrm{Im}(z)^3}$. A neighborhood of the cusp in $M(S)$ is formed by taking the quotient of the points above the line $\mathrm{Im}(z) = \mathrm{Im}(z_0)$, for $\mathrm{Im}(z_0)$ sufficiently large, by the mapping class element $z \mapsto z + 1$. A model for this neighborhood is the surface of revolution for the curve $\{y = x^3 : x > 0\}$ about the x-axis.

From the form of the metric one can see the incompleteness: a vertical ray to the cusp at infinity starting at $\mathrm{Im}(z) = y_0$ has length $\sim 2y_0^{-1/2} \sim 2\ell^{1/2}$. Moreover the curvature K satisfies $K \sim -\frac{3}{2\ell} \to -\infty$ as $\mathrm{Im}(z) \to \infty$. These precise rates of divergence for the minimum sectional curvature hold as well in higher genus and are crucial to our investigations.

4.2 Geodesic Flows

This section is devoted to making explicit the connections between geometry and the dynamics of the geodesic flow, notably between negative (or nonpositive) curvature and hyperbolicity of the geodesic flow, as well as the connections between geometric structures then present and the structures in a hyperbolic flow, specifically, Jacobi fields on one hand and the invariant foliations of a hyperbolic flow on the other hand.

Let M be a Riemannian manifold. As usual, $\langle v, w\rangle$ denotes the inner product of two vectors and ∇ is the Levi-Cività connection defined by the Riemannian metric. It is the unique connection that is symmetric and compatible with the metric. The covariant derivative along a curve $t \mapsto c(t)$ in M is denoted by D_c, $\frac{\mathrm{D}}{\mathrm{d}t}$ or simply ' if it is not necessary to specify the curve. If $V(t)$ is a vector field along c that extends to a vector field on M, then $V'(t) = \nabla_{\dot{c}(t)} V$. Given a smooth map $(s, t) \mapsto \alpha(s, t)$, we let $\frac{\mathrm{D}}{\partial s}$ denote covariant differentiation along a curve of the form $s \mapsto \alpha(s, t)$ for a fixed t. Similarly, $\frac{\mathrm{D}}{\partial t}$ denotes covariant differentiation along a curve of the form $t \mapsto \alpha(s, t)$ for a fixed s. The symmetry of the Levi-Cività connection means that

$$\frac{\mathrm{D}}{\partial s}\frac{\partial \alpha}{\partial t}(s,t) = \frac{\mathrm{D}}{\partial t}\frac{\partial \alpha}{\partial s}(s,t)$$

for all s and t. The curve c is a geodesic if it satisfies the equation $\mathrm{D}_c\dot{c}(t) = 0$. Since this equation is a first order ordinary differential equation in the variables $(c, \dot{c})$, a geodesic is uniquely determined by its initial tangent vector. Geodesics have constant speed, since $\frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{c}(t), \dot{c}(t)\rangle = 2\langle \mathrm{D}_c\dot{c}(t), \dot{c}(t)\rangle = 0$ if c is a geodesic.

The Riemannian curvature tensor R is defined by

$$R(A, B)C := \big(\nabla_A \nabla_B - \nabla_B \nabla_A - \nabla_{[A,B]}\big)C,$$

and the sectional curvature of the 2-plane spanned by vectors A and B is

$$K(A,B) := \frac{\langle R(A,B)B, A\rangle}{\|A \wedge B\|^2}.$$

The action of the Levi-Cività connection extends to covectors and tensors in such a way that the product rule holds. In particular,

$$(\nabla_W R)(X,Y)Z = \nabla_W(R(X,Y)Z) - R(\nabla_W X, Y)Z - R(X, \nabla_W Y)Z - R(X,Y)\nabla_W Z.$$

Similarly, the second derivative $\nabla^2_{X,Y}T$ of a tensor T is defined by the product rule formula

$$\nabla_X(\nabla_Y T) = \nabla^2_{X,Y}T + \nabla_{\nabla_X Y}T.$$

We will use this later in the case $T = R$. If T is a vector field Z, a short calculation using the symmetry of the Levi-Cività connection yields

$$\nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z = R(X,Y)Z.$$

4.2.1 Vertical and Horizontal Subspaces and the Sasaki Metric

The tangent bundle TTM to TM may be viewed as a bundle over M in three natural ways shown in the following commutative diagram:

$$\begin{array}{ccc} TTM & \xrightarrow{D\pi_M} & TM \\ \Big\downarrow{\scriptstyle \kappa} & \searrow^{\pi_M \circ \pi_{TM}} & \Big\downarrow{\scriptstyle \pi_M} \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

The first is via the composition of the natural bundle projections $\pi_{TM}: TTM \to TM$ and $\pi_M: TM \to M$. The second is via the composition of the derivative map $D\pi_M: TTM \to TM$ with π_M. The third involves a map $\kappa: TTM \to TM$, often called the connector map, which is determined by the Levi-Cività connection. If $\xi \in TTM$ is tangent at $t = 0$ to a curve $t \mapsto V(t)$ in TM and $c(t) = \pi_M(V(t))$ is the curve of footpoints of the vectors $V(t)$, then

$$\kappa(\xi) = D_c V(0).$$

The *vertical subbundle* is the subbundle $\ker(D\pi_M)$. It is naturally identified with TM via the map κ. The horizontal subbundle is the subbundle $\ker(\kappa)$. It is naturally identified with TM via the map $D\pi_M$ and is transverse to the vertical subbundle. If

$v \in T_pM$, we may identify T_vTM with $T_pM \times T_pM$ via the map $D\pi_M \times \kappa: TTM \to TM \times TM$. Each element of T_vTM can thus be represented uniquely by a pair (v_1, v_2) with $v_1, v_2 \in T_pM$. Put another way, every element ξ of T_vTM is tangent to a curve $V: (-1, 1) \to TM$ with $V(0) = v$. Let $c = \pi_M \circ V: (-1, 1) \to M$ be the curve of base-points of V in M. Then ξ is represented by the pair

$$(\dot{c}(0), D_c V(0)) \in T_pM \times T_pM.$$

These coordinates on the fibers of TTM restrict to coordinates on TT^1M. Regarding TTM as a bundle over M in this way gives rise to a natural Riemannian metric on TM, called the Sasaki metric. In this metric, the inner product of two elements (v_1, w_1) and (v_2, w_2) of T_vTM is defined:

$$\langle (v_1, w_1), (v_2, w_2) \rangle_{\mathrm{Sas}} = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle.$$

This metric is induced by a symplectic form ω on TTM. For vectors (v_1, w_1) and (v_2, w_2) in T_vTM, we have

$$\omega((v_1, w_1), (v_2, w_2)) = \langle v_1, w_2 \rangle - \langle w_1, v_2 \rangle.$$

This symplectic form is the pullback of the canonical symplectic form on the cotangent bundle T^*M by the map from TM to T^*M induced by identifying a vector $v \in T_pM$ with the linear functional $\langle v, \cdot \rangle$ on T_pM. Sasaki [Sa58] showed that the fibers of the tangent bundle are totally geodesic submanifolds of TTM with the Sasaki metric. A parallel vector field along a geodesic of M (viewed as a curve in TM) is a geodesic of the Sasaki metric. Such a geodesic is orthogonal to the fibers of TM. If $v \in T_pM$ and $v' \in T_{p'}M$, we can join them by first parallel translating v along a geodesic from p to p' to obtain $w \in T_{p'}M$ and then moving from w to v' along a line in $T_{p'}M$. If v' is close to v, we can choose the geodesic so that its length is $d(p, p')$. It follows easily from Topogonov's comparison theorem [CE08, Theorem 2.2] that $d_{\mathrm{Sas}}(v, v') \asymp d(p, p') + \|w - v'\|$, as $v' \to v$, where the rate of convergence is controlled by the curvatures of the Sasaki metric in a neighborhood of v. The notation $a \asymp b$ means that the ratios a/b and b/a are bounded from above by a constant. In this case the constant is 2.

4.2.2 The Geodesic Flow and Jacobi Fields

For $v \in TM$, let γ_v denote the unique geodesic satisfying $\dot{\gamma}_v(0) = v$. The geodesic flow $\varphi^t: TM \to TM$ is defined by $\varphi^t(v) = \dot{\gamma}_v(t)$ wherever this is well-defined. The geodesic flow is always defined locally. Since the geodesic flow is Hamiltonian, it preserves a natural volume form on T^1M called the Liouville volume form. When the integral of this form is finite, it induces a unique probability measure on T^1M called the Liouville measure or Liouville volume.

Consider now a one-parameter family of geodesics, i.e., a map $\alpha\colon (-1,1)^2 \to M$ with the property that $\alpha(s,\cdot)$ is a geodesic for each $s \in (-1,1)$. Denote by $J(t)$ the vector field

$$J(t) = \frac{\partial\alpha}{\partial s}(0,t)$$

along the geodesic $\gamma(t) = \alpha(0,t)$. Then J satisfies the Jacobi equation

$$J'' + R(J,\dot\gamma)\dot\gamma = 0,$$

in which $'$ denotes covariant differentiation along γ. Since this is a second-order linear ordinary differential equation, the pair of vectors $(J(0),J'(0)) \in T_{\gamma(0)}M \times T_{\gamma(0)}M$ uniquely determines the vectors $J(t)$ and $J'(t)$ along $\gamma(t)$. A vector field J along a geodesic γ satisfying the Jacobi equation is called a Jacobi field. The pair $(J(0),J'(0))$ corresponds in the manner described above to the tangent vector at $s=0$ to the curve $V(s) = \frac{\partial\alpha}{\partial t}(s,0)$. To see this, note that $V(s)$ is a vector field along the curve $c(s) = \alpha(s,0)$, so $V'(0)$ corresponds to the pair

$$\Big(\dot c(0), \mathrm{D}_c\frac{\partial\alpha}{\partial t}(s,0)\Big) = \Big(J(0), \frac{\mathrm{D}}{\partial s}\frac{\partial\alpha}{\partial t}(s,0)\Big) = \Big(J(0), \frac{\mathrm{D}}{\partial t}\frac{\partial\alpha}{\partial s}(s,0)\Big) = (J(0),J'(0)).$$

In the same way one sees that $(J(t),J'(t))$ corresponds to the tangent vector at $s=0$ to the curve $s \mapsto \frac{\partial\alpha}{\partial t}(s,t) = \varphi^t \circ V(s)$, which is $\mathrm{D}\varphi^t(V'(0))$.

To summarize the preceding discussion, there is a one-to-one correspondence between elements of T_vTM and Jacobi fields along the geodesic γ with $\dot\gamma(0) = v$. Note that the pair $(J(t),J'(t))$ defines a section of TTM over $\gamma(t)$. We have the following key proposition.

Proposition 4.2.1 *The image of the tangent vector $(v_1,v_2) \in T_vTM$ under the derivative of the geodesic flow $\mathrm{D}_v\varphi^t$ is the tangent vector $(J(t),J'(t)) \in T_{\varphi^t(v)}TM$, where J is the unique Jacobi field along γ satisfying $J(0) = v_1$ and $J'(0) = v_2$.*

Any vector field of the form $J(t) = (a+bt)\dot\gamma(t)$ is a Jacobi field, since in that case $R(J,\dot\gamma) = 0$ and the Jacobi equation reduces to $J'' = 0$, which holds since $\dot\gamma' = 0$. Conversely, any Jacobi field that is always tangent to γ must have this form. Computing the Wronskian of the Jacobi field $\dot\gamma$ and an arbitrary Jacobi field J shows that $\langle J', \dot\gamma\rangle$ is constant. It follows that if $J'(t_0) \perp \dot\gamma(t_0)$ for some t_0, then $J'(t) \perp \dot\gamma(t)$ for all t. Similarly, if $J(t_0) \perp \dot\gamma(t_0)$ and $J'(t_0) \perp \dot\gamma(t_0)$ for some t_0, then $J(t) \perp \dot\gamma(t)$ and $J'(t) \perp \dot\gamma(t)$ for all t; in this case we call J a perpendicular Jacobi field. An easy consequence of the above discussion is that any Jacobi field J along a geodesic γ can be expressed uniquely as $J = J_\parallel + J_\perp$, where $J_\parallel$ is a Jacobi field tangent to γ and $J_\perp$ is a perpendicular Jacobi field.

4.2.3 Matrix Jacobi and Riccati Equations

Choose an orthonormal basis $e_1 = \dot{\gamma}(0), e_2, \ldots, e_n$ at 0 for the tangent space at $\gamma(0)$ and parallel-transport the basis along $\gamma(t)$. Let $R(t)$ be the matrix with entries $\mathscr{R}_{jk}(t) = \langle R(e_j(t), e_1(t))e_1(t), e_k(t)\rangle$. Any Jacobi field can be written in terms of the basis as $J(t) = \sum_{k=1}^{n} y^k e_k(t)$, and the Jacobi equation can be written as

$$\frac{\mathrm{d}^2 y^k}{\mathrm{d}t^2}(t) + \sum_j y^j(t)\mathscr{R}_{jk}(t) = 0.$$

A solution is determined by values and derivatives at 0 of the y^k. Let $\mathscr{J}(t)$ denote any matrix of solutions to the Jacobi equation. When the matrix $\mathscr{J}$ is nonsingular, we can define

$$U = \mathscr{J}' \mathscr{J}^{-1}.$$

Then U satisfies the *matrix Riccati equation*

$$U' + U^2 + \mathscr{R} = 0, \tag{4.1}$$

where $\mathscr{R}$ is the matrix above. A standard calculation using the Wronskian shows that the operator $U = \mathscr{J}' \mathscr{J}^{-1}$ is symmetric if and only if for any two columns J_i, J_j of $\mathscr{J}$, we have

$$\omega_{\mathbb{R}^{2n}}((J_i, J_i'), (J_j, J_j')) = 0,$$

where $\omega_{\mathbb{R}^{2n}}$ is the standard symplectic form on $\mathbb{R}^{2n}$.

4.2.4 Perpendicular Jacobi Fields and Invariant Subbundles

There are two natural subbundles of TTM that are invariant under the derivative $\mathrm{D}\varphi^t$ of the geodesic flow, the first containing the second. The first is the tangent bundle TT^1M to the unit tangent bundle of M. Under the natural identification $T_vTM \simeq T_xM \times T_xM$ for $v \in T_x^1M$, the subspace T_vT^1M is the set of all pairs (w_0, w_1) such that $\langle v, w_1\rangle = 0$. To see this, note that if $\alpha(s,t)$ is a variation of geodesics generating the Jacobi field J along the geodesic γ, with $\dot{\gamma}(0) = v$ and $\|\partial\alpha/\partial t(s,t)\| = 1$ for all s, t, then

$$\begin{aligned} 0 = \frac{\mathrm{D}}{\partial s}\left\|\frac{\partial\alpha}{\partial t}\right\|^2\Big|_{(0,0)} &= 2\left\langle\frac{\mathrm{D}^2}{\partial s\partial t}\alpha, \frac{\partial\alpha}{\partial t}\right\rangle\Big|_{(0,0)} \\ &= 2\left\langle\frac{\mathrm{D}^2}{\partial t\partial s}\alpha, \frac{\partial\alpha}{\partial t}\right\rangle\Big|_{(0,0)} = 2\langle J'(0), \dot{\gamma}(0)\rangle. \end{aligned}$$

The Dφ^t-invariance of TT^1M follows from the φ^t-invariance of T^1M. It is reflected in the fact, noted at the end of Sect. 4.2.2, that $\langle J'(t), \dot{\gamma}\rangle$ is constant for any Jacobi field J along a geodesic γ.

The second natural invariant subbundle is the orthogonal complement $\dot{\varphi}^{\perp}$ in TT^1M to the vector field $\dot{\varphi}$ generating the geodesic flow. Under the natural identification $T_vTM \simeq T_xM \times T_xM$, for $v \in T^1_xM$, the vector $\dot{\varphi}(v)$ is $(v, 0)$, and the subspace $\dot{\varphi}^{\perp}(v)$ is the set of all pairs (w_0, w_1) such that $\langle v, w_0\rangle = \langle v, w_1\rangle = 0$. The D$\varphi^t$-invariance of $\dot{\varphi}^{\perp}$ follows from the earlier observation (end of Sect. 4.2.2) that a Jacobi field J with $J(t_0) \perp \dot{\gamma}(t_0)$ and $J'(t_0) \perp \dot{\gamma}(t_0)$ for some t_0 is perpendicular to γ for all t. To summarize, the space of all perpendicular Jacobi fields along γ corresponds to the orthogonal complement to the direction of the geodesic flow $\dot{\varphi}(v)$ at the point $v = \dot{\gamma}(0) \in T^1M$. To estimate the norm of the derivative Dφ^t on TT^1M, it suffices to restrict attention to vectors in the invariant subspace $\dot{\varphi}^{\perp}$; that is, it suffices to estimate the growth of perpendicular Jacobi fields along geodesics.

4.2.5 Consequences of Negative Curvature and Unstable Jacobi Fields

If the sectional curvatures of the Riemannian metric are negative along γ, then it follows from the Jacobi equation that $\langle J'', J\rangle > 0$ for any Jacobi field with the property that $J(t)$ and $\dot{\gamma}(t)$ are linearly independent. This has the following consequence:

Lemma 4.2.2 ([Eb01]) *If the sectional curvatures are negative along γ, then the functions $\|J(t)\|$ and $\|J(t)\|^2$ are strictly convex for any nontrivial perpendicular Jacobi field J along γ.*

We also have the following results from [Eb96, §1.10]. Let $\gamma\colon (-\infty, a] \to M$ be a geodesic ray along which the sectional curvatures of the Riemannian metric are always negative. Then, for each $w \in \dot{\gamma}(a)^{\perp}$, there is a unique perpendicular Jacobi field $J_{+,w}$ along γ such that $J_{+,w}(a) = w$ and

$$\|J_{+,w}(t)\| \leq \|w\| \text{ for all } t \leq a.$$

Since $\|J_{+,w}(t)\|$ is a strictly convex function of t by Lemma 4.2.2, $\|J_{+,w}(t)\|$ is strictly increasing for $t \leq a$. In fact, $J_{+,w} = \lim_{\tau\to-\infty} J_{+,w,\tau}$, where $J_{+,w,\tau}$ is the Jacobi field such that $J_{+,w,\tau}(a) = v$ and $J_{+,w,\tau}(\tau) = 0$. We call $J_{+,w}$ an unstable Jacobi field. For each $t \leq a$, there is a linear map $U_+(t)\colon \dot{\gamma}(t)^{\perp} \to \dot{\gamma}(t)^{\perp}$ such that

$$J'_+(t) = U_+(t)(J_+(t))$$

for every unstable Jacobi field J_+. A Jacobi field along γ is unstable if and only if it satisfies $J' = U_+J$.

Proposition 4.2.3 *The operators $U_+(t)$ are symmetric and positive-definite. They satisfy the matrix Riccati equation (4.1). Thus*

$$U_+' + U_+^2 + \mathscr{R} = 0.$$

In other words, for any vector $w \in \dot{\gamma}(t)^{\perp}$, we have

$$\langle w, U_+'(w)\rangle = -\langle R(w,\dot{\gamma})\dot{\gamma}, w\rangle - \langle w, U_+^2(w)\rangle.$$

We call U_+ the unstable solution of the Riccati equation along the ray γ. If $v \in T^1M$ is a vector such that $\gamma_v(t)$ is defined for all $t < 0$, then we define $U_+(v)$ to be the operator $U_+(0)$ associated to the ray $\gamma_v\colon (-\infty, 0] \to M$. If γ is a geodesic in a complete Riemannian manifold with negative curvature, the unstable Jacobi fields along γ are obtained by varying γ through geodesics β such that $\mathrm{d}(\beta(t), \gamma(t)) \le \mathrm{d}(\beta(0), \gamma(0))$ for $t < 0$. These geodesics are orthogonal to a family of immersed hypersurfaces whose lifts to the universal cover of M are called horospheres. The operators $U_+(t)$ are the second fundamental forms of horospheres. There is an analogous definition of stable Jacobi fields and the stable solution of the Riccati equation along a ray $\gamma\colon [a,\infty) \to M$. If $\gamma\colon (-\infty,\infty) \to M$ is a complete geodesic, the unstable Jacobi fields along γ are the stable Jacobi fields along the geodesic $t \mapsto \gamma(-t)$. We define $U_-(v)$ analogously to $U_+(v)$; it is symmetric and negative definite. The norm of a stable Jacobi field $J(t)$ defined on a ray $\gamma\colon [a,\infty) \to M$ is strictly decreasing for $t \ge a$. Let

$$\mathscr{D} := \left\{v \in T^1M : \gamma_v(t) \text{ is defined for all } t\right\}.$$

If $v \in \mathscr{D}$, both $U_+(v)$ and $U_-(v)$ exist. This allows us to define a splitting of the $2n-1$-dimensional space T_vT^1M as the direct sum of a one-dimensional space $E^0(v)$ and two spaces $E^u(v)$ and $E^s(v)$, each of dimension $n-1$. The space $E^0(v)$ is $\mathbb{R}\dot{\varphi}(v)$, and $E^u(v) \oplus E^s(v) = \dot{\varphi}(v)^{\perp}$. In our usual coordinates, $E^0(v)$ is spanned by $(v,0)$ while

$$E^u(v) = \left\{(w, U_+(v)w) : w \in v^{\perp}\right\} \text{ and } E^s(v) = \left\{(w, U_-(v)w) : w \in v^{\perp}\right\}.$$

The splitting at v is mapped to the splitting at $\varphi^t(v)$ by $\mathrm{D}\varphi^t$.

The next proposition shows that while the splitting $T_{\mathscr{D}}T^1M = E^u \oplus E^0 \oplus E^s$ is defined only over the set $\mathscr{D}$, the geometry of this splitting is locally uniformly controlled.

Proposition 4.2.4 *There exists a continuous function $\delta\colon T^1M \to \mathbb{R}_{>0}$ such that for all $v \in \mathscr{D}$, if $(w,w') \in E^u(v)$, then*

$$\langle w, w'\rangle \ge \delta(v)\|(w,w')\|^2_{\mathrm{Sas}},$$

and if $(w, w') \in E^s(v)$*, then*

$$\langle w, w' \rangle \leq -\delta(v)\|(w, w')\|^2_{\mathrm{Sas}}.$$

Proof It suffices to show that the functions

$$\delta^u(v) = \inf_{(w,w')\in E^u(v)\smallsetminus\{0\}} \frac{\langle w, w' \rangle}{\|(w, w')\|^2_{\mathrm{Sas}}} \text{ and } \delta^s(v) = \inf_{(w,w')\in E^s(v)\smallsetminus\{0\}} -\frac{\langle w, w' \rangle}{\|(w, w')\|^2_{\mathrm{Sas}}}$$

are locally uniformly bounded away from 0 for $v \in \mathscr{D}$. We prove the statement for δ^s.

Suppose that δ^s is not locally bounded away from 0. Then there would be $v \in \mathscr{D}$, a sequence of vectors v_n in $\mathscr{D}$ with $\lim_{n\to\infty} v_n = v$, and a sequence $\xi_n \in E^s(v_n)$ such that ξ_n converges to a vector $\xi = (w, w')$ with $\langle w, w' \rangle = 0$. By renormalizing we may assume that $\|\xi_n\|_{\mathrm{Sas}} = \|\xi\|_{\mathrm{Sas}} = 1$ for each n.

Since $v \in \mathscr{D}$, there exists $\tau > 0$ such that $\gamma_v(t)$ is defined for $|t| < \tau$. Let J be the Jacobi field along the geodesic γ_v determined by ξ, and let J_n be the (stable) Jacobi field along γ_{v_n} defined by ξ_n. Then $(\|J\|^2)'(0) = 2\langle w, w' \rangle = 0$.

On the other hand, since $\xi_n \to \xi$ and $\|J_n(t)\|$ is a decreasing function of t for each n, we see that $\|J\|$ is nonincreasing on $(-\tau, \tau)$. It follows from this and the strict convexity of $\|J\|^2$ given by Lemma 4.2.2 that the function $\|J\|^2$ cannot have a critical point in the interval $(-\tau, \tau)$.

The Hopf argument uses the following corollary of Proposition 4.2.4.

Corollary 4.2.5 *Let* $\delta\colon T^1M \to \mathbb{R}_{>0}$ *be the function given by Proposition 4.2.4. The continuous cone fields*

$$C^u(v) := \left\{(w, w') \in \dot{\varphi}^{\perp}(v) : \langle w, w' \rangle \geq \delta(v)\|(w, w')\|^2_{\mathrm{Sas}}\right\}$$

and

$$C^s(v) := \left\{(w, w') \in \dot{\varphi}^{\perp}(v) : \langle w, w' \rangle \leq -\delta(v)\|(w, w')\|^2_{\mathrm{Sas}}\right\},$$

defined for $v \in T^1M$*, intersect only at the origin and satisfy*

$$E^u(v) \subset C^u(v) \text{ and } E^s(v) \subset C^s(v)$$

for all $v \in \mathscr{D}$.

4.3 An Ergodicity Criterion for Incomplete Geodesic Flows

In this section we present the general criterion from [BMW12] for ergodicity of the geodesic flow on a negatively curved manifold, not necessarily complete.

If R is the curvature tensor of a Riemannian metric on a manifold M, then for $x \in M$, we define

$$\|R_x\| := \sup_{v_1,v_2,v_3 \in T_x^1 N} \|R_x(v_1,v_2)v_3\|, \quad \|\nabla Rx\| := \sup_{v_1,v_2,v_3,v_4 \in T_x^1 N} \|\nabla_{v_1} R_x(v_2,v_3)v_4\|$$

and

$$\|\nabla^2 Rx\| := \sup_{v_1,v_2,v_3,v_4,v_5 \in T_x^1 N} \|\nabla^2_{v_1,v_2} R_x(v_3,v_4)v_5\|,$$

where $\nabla^2_{X,Y} R := \nabla_X \nabla_Y R - \nabla_{\nabla_X Y} R$.

Let M be a contractible Riemannian manifold, negatively curved, possibly incomplete. Let Γ be a group that acts freely and properly discontinuously on M by isometries, and denote by $N = M/\Gamma$ the quotient manifold. We denote by d both the path metric on M and the quotient metric on N, which is just the path metric for the induced Riemannian metric on N. The quotient map $p: M \to N$ is a covering map and a local isometry. Recall that the completion X of a metric space (X, d) is the set of all Cauchy sequences $\langle x_n \rangle$ in X modulo the equivalence relation $\langle x_n \rangle \sim \langle y_n \rangle \Leftrightarrow \lim_{n\to\infty} \mathrm{d}(x_n, y_n) = 0$, with the induced metric $\mathrm{d}(\langle x_n \rangle, \langle y_n \rangle) := \lim_{n\to\infty} \mathrm{d}(x_n, y_n)$. Let $\bar{M}$ be the metric completion of M, and let $\bar{N}$ be the completion of N. Let $\partial N = \bar{N} \smallsetminus N$. We will use d to denote the metric on all of these spaces. Consider the following additional six assumptions on M and N:

I. M is a geodesically convex: for every $p, p' \in M$, there is a unique geodesic segment in M connecting p to p'. (This implies that M and $\bar{M}$ are both CAT(0) spaces.)

II. N is compact.

III. ∂N is volumetrically cusplike: there are constants $C > 1$ and $\nu > 0$ with

$$\mathrm{Vol}(p \in N : \mathrm{d}(p, \partial N) < \rho) \le C\rho^{2+\nu} \text{ for every } \rho > 0.$$

If these last two assumptions hold, then N has finite volume, and we denote by m the Riemannian volume (measure) on N, normalized so that $m(T^1 N) = 1$. We note that in the case of the Weil–Petersson metric, these three assumptions were either already known or follow in a straightforward way from known results. If they hold, then the flow is a.e. defined for all time,[2] horospheres are contracted by it, and $\log \|\mathrm{D}\varphi^1\|$ is integrable. Thus, the Oseledec Theorem can be applied to $\mathrm{D}\varphi^1$, and φ^t is nonuniformly hyperbolic [BMW12, Proposition 3.9].

The final (and main) three assumptions are made in order to establish the existence and absolute continuity of families of local stable manifolds by showing

[2]For almost every $v \in T^1 M$, there exists an infinite geodesic (necessarily unique) tangent to v.

that the hypotheses of the main results of [KSLP86] are satisfied [BMW12, Proposition 3.10 and Appendix B]. These assumptions are that there exist constants $C > 1$ and $\beta > 0$ such that

IV. N has controlled curvature: for all $x \in N$, the curvature tensor R satisfies $\max \|R_x\|, \|\nabla R_x\|, \|\nabla^2 R_x\| \leq C\mathrm{d}(x, \partial N) - \beta$.
V. N has controlled injectivity radius: $\mathrm{inj}(x) \geq C^{-1}\mathrm{d}(x, \partial N)\beta$ for every $x \in N$.
VI. The derivative of the geodesic flow is controlled: for every infinite geodesic γ in N and every $t \in [0, 1]$, we have $\|\mathrm{D}_{\dot\gamma(0)}\varphi^t\| \leq C\mathrm{d}(\gamma([-t, t]), \partial N) - \beta$.

The Burns–Masur–Wilkinson ergodicity criterion is

Theorem 4.3.1 ([BMW12, Theorem 3.1]) *Under assumptions I–VI, the geodesic flow on T^1N is m-a.e. defined for all time. It is nonuniformly hyperbolic and ergodic (in fact, Bernoulli), and its entropy is positive and finite (in fact, equal to the sum of the positive Lyapunov exponents with respect to m, counted with multiplicity).*

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