Chapter 2
On Iteration and Asymptotic Solutions of Differential Equations by Jacques Hadamard

Boris Hasselblatt (Translator)

The works of Mr. Poincaré have shown how a periodic solution of an arbitrary system of differential equations is, in general, accompanied by a family of asymptotic solutions. In certain cases every solution sufficiently near the periodic solution is asymptotic to it. In others, this asymptotic behavior only takes place for the trajectories that lie on certain surfaces through the closed curve in question. To generate the equations of these surfaces Mr. Poincaré uses series expansions, established under the assumption that the given equations are analytic.

It seemed interesting to me to treat the same question from an exclusively real point of view and without invoking analyticity of the given quantities. Moreover, I restrict myself to the simplest cases and the most immediate results.

It is known that the problem reduces to that of iteration in several variables. Therefore, let

$$x_1 = f(x,y) = ax + by + \dots,$$
$$y_1 = \varphi(x,y) = cx + dy + \dots,$$

be a point transformation of the plane that fixes the origin. We assume that the equation

$$\begin{vmatrix} a-s & b \\ c & d-s \end{vmatrix} = 0$$

Translation of Jacques Hadamard: *Sur l'itération et les solutions asymptotiques des équations différentielles*, Bulletin de la Société Mathématique de France **29** (1901), 224–228.

B. Hasselblatt (✉)
Department of Mathematics, Medford, MA 02155, USA
e-mail: Boris.Hasselblatt@tufts.edu

© Springer International Publishing Switzerland 2017

B. Hasselblatt (ed.), *Ergodic Theory and Negative Curvature*,
Lecture Notes in Mathematics 2164, DOI 10.1007/978-3-319-43059-1_2

has distinct real roots s, s'. Then, by means of a coordinate change the transformation can be written as

$$\begin{cases} x_1 = f(x,y) = sx + F(x,y), \\ y_1 = \varphi(x,y) = s'y + \Phi(x,y), \end{cases}$$

F and Φ being functions whose (formal) power series begin with terms of second order. We assume that F and Φ have partial derivatives in a certain neighborhood of the origin, these derivatives being continuous and, in particular, tending to zero with x and y. Finally, s, assumed larger than s' in absolute value (equality being excluded), should be positive and greater than 1.

The point (x_1, y_1) is called the *image* of (x, y), and the latter the *preimage* of (x_1, y_1); the point (x_2, y_2), the image of (x_1, y_1), is called the *second image* of (x, y), etc.

Due to the assumptions about s and s' a circle of sufficiently small radius centered at the origin has as its image a sort of ellipse elongated along the x-axis; this, in turn, has as its image a curve of analogous shape, but yet more elongated, and so on. Therefore, one should presume that the entire portion of the plane around the origin will become infinitely thin and will tend to a unique curve. This is what we will verify.

To this end, let us trace, starting from the origin, a curve segment C that is not tangent to the y-axis, i.e., an arc along which y is uniquely determined as a function of x and $\frac{dy}{dx}$ is always less in absolute value than a certain number α. It is easy to see, given the assumptions on the functions f and φ, that the image of the curve C will enjoy the same property, at least if one restricts oneself (as we shall do) to a certain closed neighborhood D of the origin. All subsequent images will therefore intersect an arbitrary line parallel to the y-axis in a unique point, and the slopes of their tangents and therefore also of any of their chords, will all be less than α.

Let C' be a second curve satisfying the same conditions as C. If we intersect these two curves with the same line $x = \text{const.}$, then the difference $y - y'$ will be zero with x, and the ratio $\dfrac{|y - y'|}{x}$ will have, in the domain D, a certain maximum μ.

Now let C_1, C'_1 be the images of C and C'; μ_1 the quantity analogous to μ determined from C_1 and C'_1. The ratio $\frac{\mu_1}{\mu}$ is *bounded above by a fixed number σ less than 1*, namely a number that exceeds $\dfrac{|s'|}{s}$ by arbitrarily little, if the neighborhood D is taken sufficiently small.

Let us denote, then, by Y, Y' the ordinates of C and of C' corresponding to the same value X of the abscissa (X assumed positive to fix ideas); y_1, y'_1 the analogous ordinates of C_1 and of C'_1. The points (X, y_1) and (X, y'_1) will be the images of two points (x, y) and (x', y') located on C and C_1, respectively.

The assumption on C gives

$$|y| < \alpha x, \tag{1}$$

and the assumptions on F and Φ,

$$|X - sx| < \eta(x + |y|),$$

where η can be taken as small as desired by taking the size of D small enough.

These two inequalities show that x, and likewise x', are both less than $\dfrac{X}{s - \eta(1 + \alpha)}$.

If, finally, y_0' is the point of C_1' that corresponds to the abscissa x, one has

$$|y_0' - y| < \mu x < \frac{\mu X}{s - \eta(1 + \alpha)}, \tag{2}$$

$$|y' - y_0'| < \alpha |x' - x|. \tag{3}$$

On the other hand, since the derivatives of F and Φ are in absolute value all smaller than η, one can write

$$|x - x'| < \frac{\eta}{s - \eta}|y - y'|,$$

and, therefore, by virtue of (2) and of (3),

$$|y - y'| < \frac{|y_0' - y|}{1 - \frac{\alpha\eta}{s - \eta}} < \frac{\mu X (s - \eta)}{[s - \eta(1 + \alpha)]^2},$$

then

$$|y_1 - y_1' - s'(y - y')| < \eta(|x - x'| + |y - y'|). \tag{4}$$

The inequalities (1) through (4) indeed give, as we announced,

$$|y_1 - y_1'| < \mu \left| \frac{s'}{s} + \epsilon \right| X,$$

ϵ being bounded as a function of η and α and tending to zero with η.

It is easy to deduce from this that *the successive images of C tend to a limit curve $\mathscr{C}$*. Indeed, suppose that C' coincides with C_1: then C_1' is the second image C_2 of C. The vertical segment between C and C_1 with abscissa X being smaller than μX, that between C_1 and C_2 will be smaller than $\mu\sigma X$; that between C_2 and C_3, smaller than $\mu\sigma^2 X, \ldots$. These segments will thus form a convergent series.

Furthermore, *the curve $\mathscr{C}$ we obtained is independent of the choice of the curve C*, because if C' is any other curve (through the origin and not tangent to the y-axis) and $C_1', C_2', \ldots, C_n', \ldots$ its successive images, then any vertical segment connecting C_n and C_n' tends to zero as n grows indefinitely.

The curve $\mathscr{C}$ is evidently invariant (that is to say, coincides with its image), and one sees furthermore that it is the only invariant curve through the origin that is regular and is not tangent to the y-axis.

Note that *these various results do not at all assume* $|s'| < 1$. If one adds this new assumption, it follows that the *preimages* of any curve through the origin (and not tangent to the x-axis) also tend to a definite limit position $\mathscr{C}'$, which will be tangent to the y-axis.

The two curves $\mathscr{C}$ and $\mathscr{C}'$ will then be geometric loci, one that of points whose successive preimages tend to the origin, the other of points whose successive images tend to the origin. They are those that occur in the theory of asymptotic solutions.