Chapter 8

Positive Linear Systems

We present one important large field of applications to the theory developed so far: control theory. More specifically, we present an elementary introduction to positive linear systems.

We cover some very special aspects of linear time-invariant systems, like controllability or stabilizability. Many of these problems can be naturally posed with additional positivity assumptions: we have a positive system, we would like to apply positive controls, or we would like to steer our system into positive states.

We discuss only continuous-time systems, but the definitions and most results can be modified for the discrete-time case in a straightforward way.

8.1 Externally and Internally Positive Systems

First, we set the stage and present the relevant notation and terminology. For the sake of simplicity, we only refer to the case of time-invariant, finite-dimensional input-output systems, which are described by state equations of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0, \\ y(t) = Cx(t), \end{cases} \tag{8.1}$$

where the objects involved are the following:

- $X = \mathbb{C}^n$ is the *state space*, $Y = \mathbb{C}^q$ is the *observability space*, and $U = \mathbb{C}^p$ is the *control space*.
- The function $x : \mathbb{R}^+ \to X$ is the *state vector*, the operator $A \in \mathcal{L}(X)$ is the *state (or system) operator*.
- The function $u : \mathbb{R}^+ \to U$ is the *control*, the operator $B \in \mathcal{L}(U, X)$ is the *input (or control) operator*.
- The function $y : \mathbb{R}^+ \to Y$ is the *output (or observation)*, the operator $C \in \mathcal{L}(X, Y)$ is the *output (or observation) operator*.
- The vector $x_0 \in X$ is the *initial value.*

A. Bátkai et al., *Positive Operator Semigroups*, Operator Theory: Advances and Applications 257, DOI 10.1007/978-3-319-42813-0_8

System (8.1) is often referred to as $\Sigma(A, B, C)$. The interpretation of this set of equations is the following. There is a system described by a set of n equations and governed by the operator A. This is also referred to as the "free system", the system without intervention. The function u is the control we apply from the outside, and the operator B represents the action of u on the system. Finally, the function y is the set of parameters we are able to measure, and the measurement process is described by the observation operator C.

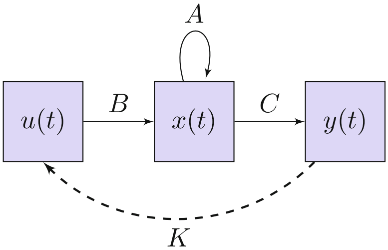

Figure 8.1: An input-output system (with feedback).

If we need to stress the dependence of the solution x on the initial value x_0, then we shall write $x(t) = x(t; x_0)$.

Before turning our attention to controllability concepts, let us make the following crucial observation and present a representation formula. Suppose that the control u is locally integrable and set

$$z(t) = \mathrm{e}^{tA}x_0 + \int_0^t \mathrm{e}^{(t-s)A}Bu(s)\,\mathrm{d}s.$$

Then, $\dot{z}(t) = Az(t) + Bu(t)$ and $z(0) = x_0$. Since x is the solution of $\Sigma(A, B, C)$ in (8.1), we infer that $\dot{z}(t) - \dot{x}(t) = A(z(t) - x(t))$ and $z(0) - x(0) = 0$. Hence, by uniqueness,

$$x(t) = z(t) = \mathrm{e}^{tA}x_0 + \int_0^t \mathrm{e}^{(t-s)A}Bu(s)\,\mathrm{d}s. \tag{8.2}$$

This yields the formula

$$y(t) = C\mathrm{e}^{tA}x_0 + \int_0^t C\mathrm{e}^{(t-s)A}Bu(s)\,\mathrm{d}s. \tag{8.3}$$

The function $h(t) = C\mathrm{e}^{tA}B$ is sometimes called the *impulse response*.

Often the control u is designed depending on the observation, and such systems are called *feedback systems*. If $u(t) = Ky(t)$, then the operator $K \in \mathcal{L}(Y, U)$ is called the feedback operator. Note that in this case we have the representation formula

$$x(t) = \mathrm{e}^{t(A+BKC)}x_0. \tag{8.4}$$

We list now a few properties that a time-invariant linear system can have, and which are important in view of applications.

Definition 8.1. The linear system $\Sigma(A, B, C)$ in (8.1) is said to be *externally positive*, if the output corresponding to the zero initial state is positive for every positive input function. In other words, $u(t) \geq 0$ implies $y(t) \geq 0$ if $x_0 = 0$.

In the following we characterize externally positive linear systems.

Proposition 8.2. *A linear system is externally positive if and only if its impulse response is positive.*

Proof. By the representation formula (8.3), the sufficiency is clear. Suppose now that there is $t_0 > 0$ such that $h(t_0) = C\mathrm{e}^{t_0 A}B$ is not positive. Then, by continuity, at least one entry of $h(t)$ would be negative on a whole nondegenerate interval $[t_1, t_2]$. Thus, the appropriate entry of the output would be negative for every input function which is strictly positive in $[t-t_2, t-t_1]$ and zero elsewhere. Hence, the system cannot be externally positive. □

Let us give a simple example of an externally positive linear system.

Example 8.3. Here we suppose that $U = Y = \mathbb{C}$ and $X = \mathbb{C}^2$. Let us consider

$$A = \begin{pmatrix} -a & -a \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

for a parameter $a > 0$. For which values of a will the system $\Sigma(A, B, C)$ in (8.1) be externally positive?

Using the above proposition one has to check for which values of a the corresponding impulse response $h(t) = C\mathrm{e}^{tA}B$ is positive. The special forms of B and C imply that $h(t) = a(\mathrm{e}^{tA})_{2,1}$, where $(\mathrm{e}^{tA})_{2,1}$ is the $(2,1)$th entry of e^{tA}.
The eigenvalues λ_i, $i = 1, 2$, of A are the roots of $\lambda^2 + (a+1)\lambda + 2a$. So, the following holds:

$$\lambda_{1,2} = \begin{cases} \frac{-1-a\pm\sqrt{a^2-6a+1}}{2} & \text{if } 0 < a < 3-2\sqrt{2} \text{ or } a > 3+2\sqrt{2}, \\ -\frac{1+a}{2} =: \lambda_0 & \text{if } a = 3-2\sqrt{2} \text{ or } a = 3+2\sqrt{2}, \\ \in \mathbb{C}\setminus\mathbb{R} & \text{if } 3-2\sqrt{2} < a < 3+2\sqrt{2}. \end{cases}$$

We have to investigate only the first two cases, since the third one corresponds to an oscillating e^{tA}, which can have negative values. From the Theorem 2.11 one has $\mathrm{e}^{tA} = \alpha_0 I + \alpha_1 A$, where α_0 and α_1 satisfy the system of equations

$$\begin{aligned} \mathrm{e}^{t\lambda_1} &= \alpha_0 + \alpha_1\lambda_1, \\ \mathrm{e}^{t\lambda_2} &= \alpha_0 + \alpha_1\lambda_2 \end{aligned}$$

in the first case above, and

$$\begin{aligned} \mathrm{e}^{t\lambda_0} &= \alpha_0 + \alpha_1\lambda_0, \\ t\mathrm{e}^{t\lambda_0} &= \alpha_1 \end{aligned}$$

in the second one. Thus, by a simple computation, we obtain

$$h(t) = \begin{cases} \frac{a}{\sqrt{a^2-6a+1}} \mathrm{e}^{t\lambda_2} \left(e^{2t\sqrt{a^2-6a+1}} - 1 \right) & \text{if } 0 < a < 3-2\sqrt{2} \text{ or } a > 3+2\sqrt{2}, \\ at\mathrm{e}^{t\lambda_0} & \text{if } a = 3-2\sqrt{2} \text{ or } a = 3+2\sqrt{2}. \end{cases}$$

Therefore, for $a > 0$, the system $\Sigma(A, B, C)$ is externally positive, iff $0 < a \leq 3 - 2\sqrt{2}$ or $a \geq 3 + 2\sqrt{2}$.

The concept of (internal) positivity and irreducibility is defined as follows.

Definition 8.4. The linear system $\Sigma(A, B, C)$ in (8.1) is said to be *positive* (or internally positive), if the state and the output corresponding to a positive initial state are positive for every positive input function. In other words, $u(t) \geq 0$ and $x_0 \geq 0$ implies $x(t) \geq 0$ and $y(t) \geq 0$. The system is said to be *reducible*, if the matrix A is reducible, and *irreducible* otherwise.

The positivity of linear systems can be characterized in terms of the positivity of B, C, and e^{tA}.

Proposition 8.5. *The linear system $\Sigma(A, B, C)$ is positive if and only if $B \geq 0$, $C \geq 0$, and A generates a positive matrix semigroup.*

Proof. Assume that the system is positive. Then, letting $x_0 = 0$ and $u(t) = u$, a nonnegative constant, we see that

$$0 \leq \frac{x(t)}{t} = \frac{1}{t} \int_0^t \mathrm{e}^{(t-s)A} Bu \;\mathrm{d}s = \left(\frac{1}{t} \int_0^t \mathrm{e}^{sA} \;\mathrm{d}s \right) Bu.$$

So, by letting $t \to 0$, we obtain the positivity of B.

Since $Cx_0 = Cx(0) = y(0) \geq 0$ for every $x_0 \geq 0$, the operator C has to be positive too. Finally, applying the zero control, we see that $x(t) = \mathrm{e}^{tA} x_0 \geq 0$ for every $x_0 \geq 0$.

To prove the converse implication, suppose that $B \geq 0$, $C \geq 0$, and that A generates a positive matrix semigroup. Taking $x_0 \geq 0$ and $u \geq 0$, we see that

$$\mathrm{e}^{tA} x_0 \geq 0$$

and that $Bu(s) \geq 0$ for each $s \in [0, t]$, hence $\mathrm{e}^{(t-s)A} Bu(s) \geq 0$, implying

$$\int_0^t \mathrm{e}^{(t-s)A} Bu(s) \;\mathrm{d}s \geq 0,$$

which in view of (8.2) and (8.3) proves the statement. □

Let us define now excitable positive linear systems.

Definition 8.6. A positive system is said to be *excitable*, if each state variable can be made strictly positive by applying an appropriate positive input to the system initially at rest. In other words, for each $i = 1, 2, \ldots, n$ there are a control $u_i \geq 0$ and a time t_i such that $x_i(t_i) > 0$ if $x_0 = 0$.

Excitable systems enjoy some remarkable properties. To be able to present some of them, we introduce some new concepts. To keep the presentation as simple as possible, we restrict ourselves for the rest of this section to the case $Y = U = \mathbb{C}$, i.e., we only consider one-dimensional control and observation spaces.

The *influence graph* of the system $\Sigma(A, B, C)$ in (8.1) is a directed graph $G = (V, E)$ with $n+2$ vertices $V = \{v_0, v_1, \ldots, v_{n+1}\}$. Vertex v_0 is associated with the input u and vertex v_{n+1} with the output y. The remaining vertices $v_1, \ldots, v_n$, correspond to the state variables $x_1, \ldots, x_n$. The edges represent the influence relations among the variables and are constructed as follows.

- $(v_0, v_j) \in E$ if and only if $b_j \neq 0$, $j = 1, \ldots, n$;
- for $i, j = 1, \ldots, n$, $i \neq j$, $(v_i, v_j) \in E$ if and only if $a_{ji} \neq 0$;
- $(v_i, v_{n+1}) \in E$ if and only if $c_i \neq 0$, $i = 1, \ldots, n$.

No other edges are present in the graph.

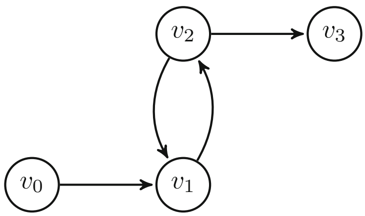

Figure 8.2: The influence graph of the system in Example 8.3.

The corresponding graph matrices are constructed as follows: $\hat{A} = (\hat{a}_{ij})$, where $\hat{a}_{ij} = 1$ if and only if $i \neq j$ and $a_{ji} \neq 0$, otherwise $\hat{a}_{ij} = 0$. The row- and column- matrix $\hat{B}$ and $\hat{C}$, respectively, are constructed in a similar manner. The following $(n+2) \times (n+2)$ matrix

$$\mathbb{A} := \begin{pmatrix} 0 & \hat{B} & 0 \\ 0 & \hat{A} & \hat{C} \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.5}$$

is thus the $0-1$ adjacency matrix of the unweighted influence graph.

Many properties of a positive linear system can be described in terms of its influence graph. Observe, for example, that by Proposition 6.1 the following holds.

Corollary 8.7. *A system is irreducible if and only if the subgraph of its influence graph, consisting only of vertices $v_1, \ldots, v_n$ and edges between them, is strongly connected.*

We can also express excitability of the system in terms of the above graph matrices.

Proposition 8.8. *The positive linear system* $\Sigma(A,B,C)$ *in* (8.1) *is excitable if and only if there exists at least one walk from the input vertex* v_0 *to each vertex* v_i*,* $i=1,\dots,n$*, in the influence graph* G*, or, equivalently, if and only if*

$$\hat{B}+\hat{B}\hat{A}+\cdots+\hat{B}\hat{A}^{n-1}\gg 0.$$

Proof. Excitability means that each state variable x_i can be influenced by the input u. This implies that there has to be at least one walk from the vertex v_0 to the vertex v_i in the influence graph G.

Note that the powers of the adjacency matrix $\mathbb{A}$ in (8.5) have the same block form:

$$\mathbb{A}^k=\begin{pmatrix}0 & \hat{B}\hat{A}^{k-1} & \hat{B}\hat{A}^{k-2}\hat{C}\\ 0 & \hat{A}^k & \hat{A}^{k-1}\hat{C}\\ 0 & 0 & 0\end{pmatrix},\quad k\in\mathbb{N}.$$

Recall from Proposition 1.1 that the ith component of the row vector $\hat{B}\hat{A}^{k-1}$ represents the number of walks of length k from vertex v_0 to vertex v_i, $i=1,\dots,n$. Hence, there is a walk to every vertex, if and only if

$$\hat{B}+\hat{B}\hat{A}+\cdots+\hat{B}\hat{A}^{n-1}\gg 0.$$

Assume that the positive system $\Sigma(A,B,C)$ is not excitable. Then, there exists $i\in\{1,\dots,n\}$ such for all t and all controls $u\geq 0$,

$$x_i(t)=\left(\int_0^t \mathrm{e}^{(t-s)A}Bu(s)\ \mathrm{d}s\right)_i=0.$$

By taking $u(t)=1$, we obtain

$$b_i=\lim_{t\to 0}\left(\frac{1}{t}\int_0^t \mathrm{e}^{sA}\ \mathrm{d}sB\right)_i=0.$$

On the other hand,

$$\dot{x_i}(t)=\left(Bu(t)+A\int_0^t \mathrm{e}^{(t-s)A}Bu(s)\ \mathrm{d}s\right)_i=0.$$

This implies that

$$\left(A\int_0^t \mathrm{e}^{(t-s)A}Bu(s)\ \mathrm{d}s\right)_i=0.$$

As above, by taking $u(t)=1$, we obtain $(AB)_i=\sum_{j=1}^n a_{ij}b_j=0$. Using the positivity, we deduce that $a_{ij}b_j=0$ for all j (note that, as we have seen above, $b_i=0$, so also $a_{ii}b_i=0$). Hence for the graph matrices we have $(\hat{B}\hat{A})_i=0$. By repeating the same arguments we obtain $(\hat{B}\hat{A}^k)_i=0$ for all $k=0,\dots,n-1$, and this ends the proof of the proposition. □

We consider now rather special constant inputs, $u(t) = \bar{u} > 0$.

Theorem 8.9. *An excitable and asymptotically stable positive linear system has a strictly positive equilibrium state.*

Proof. Since by asymptotic stability all the eigenvalues of A have negative real part, A is invertible and $\bar{x} := -A^{-1}B\bar{u}$ is the unique equilibrium of the system, which is asymptotically stable. We only have to show that it is strictly positive.

Suppose that there are indices i such that $\bar{x}_i = 0$, and collect these indices in the set $I := \{i \in \{1, \ldots, n\} : \bar{x}_i = 0\}$. Then, since

$$A\bar{x} + B\bar{u} = 0,$$

we see that

$$\sum_{j \notin I} a_{ij}\bar{x}_j + b_i\bar{u} = 0 \quad \text{ for } i \in I.$$

This implies that $b_i = 0$ and that $a_{ij} = 0$ for $i \in I$ and $j \notin I$, since the system is positive. Hence, there is no walk from the input vertex 0 to vertices $i \in I$ and the system is not excitable. □

8.2 Controllability

For simplicity, we consider here systems without observation, i.e., where $Y = X$ and $C = I$. We denote by $\Sigma(A, B)$ the system (8.1) simplified this way.

Definition 8.10. The system $\Sigma(A, B)$ is called *controllable in time* τ if for every initial value $x_0 \in X$ and every state $x_1 \in X$ there is a control u such that for the solution x we have $x(\tau; x_0) = x_1$.

We will briefly call a system *controllable*, if there exists a $\tau > 0$ such that it is controllable in time τ.

Lemma 8.11. *The system $\Sigma(A, B)$ is controllable in time τ if and only if every state $x_1 \in X$ can be reached from $x_0 = 0$ in time τ.*

Proof. We only have to prove the converse. Let us take $x_1 \in X$ and set $x_2 = x_1 - \mathrm{e}^{\tau A}x_0$. Then, by assumption, there is a control u such that $x_2 = x(\tau; 0)$. Then

$$x(\tau; 0) = \int_0^\tau \mathrm{e}^{(\tau-s)A}Bu(s)\,\mathrm{d}s = x_2,$$

hence

$$x(\tau; x_0) = \mathrm{e}^{\tau A}x_0 + \int_0^\tau \mathrm{e}^{(\tau-s)A}Bu(s)\,\mathrm{d}s = \mathrm{e}^{\tau A}x_0 + x_2 = x_1,$$

and we are done. □

To investigate the possible reachable states, we take a functional analytic point of view and introduce an operator which maps control functions to states which are reached from the origin by using this control.

Definition 8.12. Fix $\tau > 0$. The *controllability operator* $\mathcal{B}_\tau : \mathrm{L}^1([0,\tau],U) \to X$ is defined by

$$\mathcal{B}_\tau(u) := \int_0^\tau \mathrm{e}^{(\tau-s)A} Bu(s)\,\mathrm{d}s.$$

Hence, the system is controllable in time τ if and only if $\mathcal{B}_\tau$ is surjective.

Let us begin with the following simple properties. The proof is left to the reader.

Lemma 8.13. *The operator $\mathcal{B}_\tau$ has the following properties.*

a) *The operator $\mathcal{B}_\tau$ is linear.*
b) *The operator $\mathcal{B}_\tau : \mathrm{L}^1([0,\tau],U) \to X$ is bounded, i.e.,*

$$\sup_{\|u\|\le 1} \|\mathcal{B}_\tau(u)\| < \infty.$$

Fortunately, there is an important characterization of the range of $\mathcal{B}_\tau$.

Theorem 8.14. *For every $\tau > 0$ we have*

$$\operatorname{im}(\mathcal{B}_\tau) = \operatorname{span}\left\{x,\ Ax,\ A^2x, \dots, A^{n-1}x \,:\, x \in \operatorname{im}(B)\right\}.$$

Proof. Let us introduce first some shorthand notation for this proof and introduce

$$X_1 := \operatorname{span}\left\{x,\ Ax,\ A^2x, \dots, A^{n-1}x \,:\, x \in \operatorname{im}(B)\right\},$$

as well as two further spaces,

$$X_2^\tau := \operatorname{span}\left\{\mathrm{e}^{tA}y \,:\, 0 \le t \le \tau,\ y \in \operatorname{im}(B)\right\},$$
$$X_3^\tau := \operatorname{span}\left\{\int_0^t \mathrm{e}^{sA}y\,\mathrm{d}s \,:\, 0 \le t \le \tau,\ y \in \operatorname{im}(B)\right\}.$$

Since step functions are dense in L^1, and since in X every subspace is automatically closed, we conclude by the continuity of $\mathcal{B}_\tau$ that

$$\operatorname{im}(\mathcal{B}_\tau) = X_3^\tau.$$

Observe also, using Corollary 2.14 and Theorem 2.12, that e^{tA} is a polynomial in A of degree at most $n-1$, and thus, $X_2^\tau = X_1$.

Now let us take $y \in \operatorname{im}(B)$. Then $x(t) = \int_0^t \mathrm{e}^{sA}y\,\mathrm{d}s \in X_3^\tau$ for $t \le \tau$. Clearly, all the derivatives of x lie in X_3^τ, hence

$$x(0) = 0 \in X_3^\tau,$$
$$\dot{x}(0) = y \in X_3^\tau,$$
$$\ddot{x}(0) = Ay \in X_3^\tau,$$
$$\text{etc.}$$

Thus, $X_1 \subset X_3^\tau$. On the other hand, if $y \in \operatorname{im}(B)$, then $\mathrm{e}^{sA}y \in X_1$, implying that

$$\int_0^t \mathrm{e}^{sA} y \,\mathrm{d}s \in X_1,$$

i.e., $X_3^\tau \subset X_1$. □

Corollary 8.15 (Kàlmàn criterion)**.** *For a control system $\Sigma(A,B)$ the following are equivalent.*

(i) *The system is controllable in time τ for all $\tau > 0$.*
(ii) *The controllability operator $\mathcal{B}_\tau$ is surjective for every $\tau > 0$.*
(iii) *The rank condition* $\operatorname{rank}(B, AB, A^2B, \dots, A^{n-1}B) = n$ *is satisfied.*
(iv) *The system is controllable.*

In many applications it is natural to consider only positive initial values, positive controls, and expect the states of the system to remain positive for all times. Hence, we restrict our investigations here to this case.

By $X_+ := \{x \in X : x \geq 0\}$ we denote the *positive cone* of X. The *reachability set* $X_{\tau,+}$ of a positive system $\Sigma(A,B)$ is defined as the set of points that can be reached from the origin in time τ by applying a positive control. In other words,

$$X_{\tau,+} := \left\{ \int_0^\tau \mathrm{e}^{(\tau-s)A} Bu(s)\,\mathrm{d}s \;:\; u \geq 0 \right\}.$$

By linearity and positivity of the operators, the set $X_{\tau,+} \subseteq X_+$ is a *convex cone* (i.e., for every $x, y \in X_{\tau,+}$ and $\alpha, \beta \geq 0$, $\alpha x + \beta y \in X_{\tau,+}$). Actually, much more can be said.

Theorem 8.16. *The set $X_{\tau,+}$ is a convex cone which is non-degenerate (i.e., it contains an open ball) if and only if the positive system $\Sigma(A,B)$ is controllable, i.e., the Kàlmàn rank condition is satisfied.*

Proof. It can be shown directly that $X_{\tau,+}$ is a convex cone. Assume now that the Kàlmàn rank is less than n. This means that there exists a nonzero $y \in X$ such that $y^\top A^i B = 0$ for $i = 0, \dots, n$. Hence, $y^\top \mathrm{e}^{tA} B = 0$ for all $t \geq 0$, since the degree of the minimal polynomial of A is less than n. So, for all $x \in X_{\tau,+}$ we have

$$\begin{aligned}
(y \,|\, x) &= \left(y \,\middle|\, \int_0^\tau \mathrm{e}^{A(\tau-s)} Bu(s) \right) \mathrm{d}s \\
&= \int_0^\tau \left((\mathrm{e}^{A(\tau-s)} B)^\top y \,\middle|\, u(s) \right) \mathrm{d}s \\
&= 0.
\end{aligned}$$

Therefore, $X_{\tau,+}$ lies in an $(n-1)$-dimensional subspace of X, which means that $X_{\tau,+}$ is degenerate.

Conversely, if $X_{\tau,+}$ is degenerate, it lies in an $(n-1)$-dimensional subspace of X, since $X_{\tau,+}$ is convex. Thus, there exists $y \in X$ such that $(y|x) = 0$ for all $x \in X_{\tau,+}$ and so

$$\int_0^\tau \left((\mathrm{e}^{A(\tau-s)}B)^\top y \,\Big|\, u(s) \right) \, \mathrm{d}s = 0$$

for all $u \geq 0$ (and hence for all $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, U)$). Taking now a constant function $u(s) = v \in U$ and differentiate the above equation with respect to τ, one obtains

$$\left(B^\top y \,\middle|\, v \right) = 0,$$
$$\int_0^\tau \left((A\mathrm{e}^{A(\tau-s)}B)^\top y \,\Big|\, v \right) \, \mathrm{d}s = 0$$

for all $v \in U$. Thus, $y^\top B = 0$. Differentiating again the above equation with respect to τ one has

$$\left((AB)^\top y \,\middle|\, v \right) = 0,$$
$$\int_0^\tau \left((A^2\mathrm{e}^{A(\tau-s)}B)^\top y \,\Big|\, v \right) \, \mathrm{d}s = 0$$

for all $v \in U$. Repeating the above process $(n-1)$-times one gets $y^\top BA^i = 0$ for $i = 0, 1, \ldots, n-1$, which implies that the Kàlmàn rank is less than n. □

An important case is when $X_{\tau,+}$ is actually the whole positive cone X_+, i.e., when each positive state can be reached by applying a positive control from the origin.

Definition 8.17. A positive system $\Sigma(A, B)$ is called

(i) (*exactly*) *positive controllable in time* τ, if

$$X_{\tau,+} = X_+$$

(ii) (*exactly*) *positive controllable*, if

$$\bigcup_{\tau \geq 0} X_{\tau,+} = X_+,$$

(iii) *approximately positive controllable in time* τ, if

$$\overline{X_{\tau,+}} = X_+.$$

(iv) *approximately positive controllable*, if

$$\overline{\bigcup_{\tau \geq 0} X_{\tau,+}} = X_+.$$

Positive controllability is a much more delicate question then usual controllability and the different notions in the definition above do not coincide (as is the case with the usual controllability). In the proof of Theorem 8.14, the range of the controllability operator, that is, the reachability set for the usual case, was characterized via three linear subspaces. Unfortunately $X_{\tau,+}$ is in general not a closed linear subspace and we can only show the following characterization.

By $\operatorname{co} M$ we denote he smallest convex set containing M and by cocone M the smallest convex cone containing M and 0.

Proposition 8.18. *Let $u_1, \dots, u_p$ be the standard basis vectors in $U = \mathbb{C}^p$. For a positive system $\Sigma(A,B)$,*

$$\begin{aligned}\overline{X_{\tau,+}} &= \overline{\operatorname{co}} \left\{ \mathrm{e}^{tA} Bu \colon 0 \le t \le \tau, u \in U_+ \right\} \\ &= \overline{\operatorname{cocone}} \left\{ \mathrm{e}^{tA} Bu_j \colon 0 \le t \le \tau, 1 \le j \le p \right\}.\end{aligned}$$

Proof. By the definition of the integral,

$$\overline{X_{\tau,+}} \subseteq \overline{\operatorname{co}} \left\{ \mathrm{e}^{tA} Bu \colon 0 \le t \le \tau, u \in U_+ \right\}.$$

Now choose any $u \in U_+$. Since the equations are autonomous, it is enough to show that $\mathrm{e}^{tA} Bu \in \overline{X_{t,+}}$ for all $0 \le t \le \tau$. To this aim take

$$u_m(s) := \begin{cases} mu, & \text{for } 0 \le s \le \frac{1}{m}, \\ 0, & \text{for } \frac{1}{m} < s \le \tau, \end{cases}$$

and compute

$$\left\| \int_0^t \mathrm{e}^{(t-s)A} Bu_m(s)\, \mathrm{d}s - \mathrm{e}^{tA} Bu \right\| \le m \int_0^{1/m} \left\| \mathrm{e}^{(t-s)A} Bu - \mathrm{e}^{tA} Bu \right\| \, \mathrm{d}s$$

which converges to 0 as $m \to \infty$. Hence, the first equality is proved. The second one now also follows, since $BU_+ = \operatorname{cocone}\{Bu_1, \dots, Bu_p\}$. □

8.3 Stabilization

We restrict ourselves again to systems without observation and where the control is given by a suitable feedback K.

Definition 8.19. A system $\Sigma(A,B)$ is called *stabilizable* if there is a feedback K such that the state converges to zero for every initial value, i.e.,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \mathrm{e}^{t(A+BK)} x_0 = 0 \tag{8.6}$$

for every $x_0 \in X$.

Note that by Theorem 4.12 we have the following characterization of stabilizable systems.

Corollary 8.20. *System* $\Sigma(A, B)$ *is stabilizable if and only if there is a feedback* K *such that* $\mathrm{s}(A + BK) < 0$.

A positive system $\Sigma(A, B)$ is called *positively stabilizable*, if there is a positive feedback operator K such that (8.6) holds for every $x_0 \geq 0$.

Proposition 8.21. *A positive system is positively stabilizable if and only if it is stabilizable with a positive feedback.*

Proof. Note that for every element in X its real part and imaginary part can be represented as the difference of two positive elements in the positive cone of X. Since

$$\lim_{t\to\infty} \mathrm{e}^{t(A+BK)} x_0 = 0$$

for every x_0 is equivalent to

$$\lim_{t\to\infty} \mathrm{e}^{t(A+BK)} (x_1 - x_2) = 0$$

for every $x_1, x_2 \geq 0$, the statement follows. □

8.4 Notes and Remarks

There are many excellent introductions to systems and control theory. We based our presentation on the monograph by Jacob and Zwart [69], on the work by Mehrmann [93], and on the monograph by Zabczyk [158].

Positivity aspects of control problems are discussed by Schanbacher in [127] and in the monograph by Farina and Rinaldi [45]. Many further interesting topics could be studied here, and in case we succeeded to make you curious, you can look them up in the above-mentioned sources.

8.5 Exercises

1. Prove the basic properties of the controllability operator $\mathcal{B}_\tau$ as stated in Lemma 8.13.
2. Show that a system $\Sigma(A, B)$ is controllable if and only if for every eigenvector v of $A^\top$ we have $vB \neq 0$.
3. Show that a system $\Sigma(A, B)$ is controllable if and only if $\operatorname{rank}(\lambda - A, B) = n$ for all $\lambda \in \mathbb{C}$.
4. Let $U = \mathbb{C}$ and $X = \mathbb{C}^2$, and consider

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with $a > 0$. What can you say about the reachability set $X_{\tau,+}$ of this positive linear system? In other words, which states can be reached from the origin by applying a positive control u in time τ?

5. Let $U = \mathbb{C}$ and $X = \mathbb{C}^2$, and consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Show that $\Sigma(A, B)$ is controllable, but not approximately positive controllable.

6. Let $U = \mathbb{C}$ and $X = \mathbb{C}^2$, and consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ a \end{pmatrix},$$

with $a > 0$. Is the system $\Sigma(A, B)$ stabilizable? Is it positively stabilizable?