Chapter 6

Applications of Positive Matrices

We have now accumulated enough material to pause for a while to discuss its consequences in concrete situations. We have revised linear algebra facts from a functional analytic perspective and obtained a construction to get functions of matrices in a coordinate-free manner, without the use of the Jordan normal form. This was useful when we considered positive matrices, and enabled us to see important and deep spectral consequences of positivity.

The applications of the developed theory are numerous and we have selected just a few representing our taste: graph matrices, the Google matrix, and age-structured population models.

6.1 Motivating Examples Revisited

We start by revisiting our motivating examples from Section 1.1.

Graphs

Let $G = (V, E)$ be a directed graph with n vertices $V = \{v_1, \ldots, v_n\}$ and a set of directed edges E. The graph G is called *strongly connected* if for every $v_i \in V$ and every $v_j \in V$ there is walk in G from v_i to v_j. This property of the graph can be read from its adjacency matrix.

Proposition 6.1. *A graph G is strongly connected if and only if its adjacency matrix is irreducible.*

Proof. Let $A = (a_{ij})$ be the adjacency matrix of G. By Lemma 5.10, A is reducible iff we can partition the sets of vertices $V = V_1 \cup V_2$ into two disjoint subsets such that after relabeling the vertices we obtain a block-triangular form for A,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \tag{6.1}$$

A. Bátkai et al., *Positive Operator Semigroups*, Operator Theory:
Advances and Applications 257, DOI 10.1007/978-3-319-42813-0_6

where the block $A_{k\ell}$ for each $k, \ell \in \{1, 2\}$ corresponds to connections from the set of vertices V_k to the set V_ℓ. Note that $A_{21} = 0$ is equivalent to the fact that there are no direct edges from a vertex in V_2 to a vertex in V_1.

Let $v_i \in V_2$ and $v_j \in V_1$ and assume there exists a walk in G from v_i to v_j. Then

$$a_{ii_1} a_{i_1 i_2} \cdots a_{i_s j} \neq 0$$

for some $i_1, \ldots, i_s \in \{1, \ldots, n\}$. Observe that in this product there must be a nonzero entry with "mixed" indices, i.e., $a_{i_k i_\ell} \neq 0$ with $v_{i_k} \in V_2$ and $v_{i_\ell} \in V_1$, which contradicts (6.1). So, if G is strongly connected, A must be irreducible.

For the converse assume that G is not strongly connected. Hence there exist vertices $v_i, v_j \in V$ such that there is no walk starting in v_i and ending in v_j. Let V_1 be the set of all initial vertices of walks which end in v_j, and let $V_2 = V \backslash V_1$. The sets V_1 and V_2 are disjoint and nonempty. According to the partition $V = V_1 \cup V_2$ the adjacency matrix has block-triangular form given in (6.1), so A is reducible. □

As a corollary we obtain a combinatorial characterization of positive irreducible matrices. Note that every positive matrix can be seen as the adjacency matrix of a graph.

Corollary 6.2. *A positive $n \times n$ matrix A, $n \geq 2$, is irreducible if and only if for every $i, j \in \{1, \ldots, n\}$ there exists an $s \in \mathbb{N}$ such that $(A^s)_{ij} > 0$.*

We will illustrate another property of the adjacency matrix A in terms of the structure of the graph G. Recall from Theorem 5.19 that any imprimitive matrix with index of imprimitivity h can be written in Frobenius form as follows:

$$PAP^{-1} = \begin{pmatrix} 0 & A_{12} & 0 & \ldots & 0 \\ 0 & 0 & A_{23} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & A_{h-1,h} \\ A_{h1} & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{6.2}$$

with square blocks on the main diagonal.

Lemma 6.3. *Let A be an imprimitive matrix with index of imprimitivity h and Frobenius form* (6.2)*. Then $A_{12} A_{23} \cdots A_{h1}$ is a primitive matrix.*

Proof. First introduce the matrices

$$\tilde{A}_1 := A_{12} A_{23} \cdots A_{h1}, \quad \tilde{A}_2 := A_{23} A_{34} \cdots A_{12}, \quad \ldots \quad \tilde{A}_h := A_{h1} A_{12} \cdots A_{h-1,h}.$$

Observe that all of them are positive matrices and their spectra coincide.

Using the Frobenius form (6.2), one sees that

$$\left(PAP^{-1}\right)^{sh} = \begin{pmatrix} \tilde{A}_1^s & 0 & \cdots & 0 \\ 0 & \tilde{A}_2^s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A}_h^s \end{pmatrix} \qquad \text{for all } s \in \mathbb{N}.$$

Since A is irreducible, so is PAP^{-1}. Combining Corollary 6.2 and the above block diagonal form yields the irreducibility of $\tilde{A}_1$.

By Theorem 5.19, the boundary spectrum of A equals

$$\sigma_{\mathrm{b}}(A) = \left\{r, r\omega, r\omega^2, \ldots, r\omega^{h-1}\right\},$$

where $r = \mathrm{r}(A)$ and $\omega = \mathrm{e}^{2\pi\mathrm{i}/h}$. Hence,

$$\{r^h\} = \sigma_{\mathrm{b}}(A^h) = \sigma_{\mathrm{b}}\left(PA^hP^{-1}\right) = \sigma_{\mathrm{b}}(\tilde{A}_1),$$

and the matrix $\tilde{A}_1$ is indeed primitive. □

Proposition 6.4. *Let G be a strongly connected graph whose adjacency matrix A is imprimitive with index of imprimitivity h. Then h equals the greatest common divisor*

- *d_i of lengths of all closed walks through a vertex v_i in G,*
- *d_{W} of lengths of all closed walks in G, and*
- *d_{C} of lengths of all cycles in G.*

Proof. Let us first show that $d_{\mathrm{C}} = d_{\mathrm{W}}$. Clearly, $d_{\mathrm{W}} | d_{\mathrm{C}}$, as every cycle is also a closed walk. Now observe that every closed walk can be partitioned into cycles and the length of the closed walk is the sum of the lengths of these cycles, hence divisible by d_{C}.

Now fix a vertex v_i of G. By definition, $d := d_{\mathrm{C}} = d_{\mathrm{W}}$ divides d_i. Choose an arbitrary closed walk C in G. If it contains v_i, then its length $\ell(C)$ is divisible by d_i. Otherwise, take a vertex $v_j \in C$. Since G is strongly connected, there exist a walk W_{ij} from v_i to v_j and a walk W_{ji} from v_j to v_i. Now $W_{ij}CW_{ji}$ is a closed walk in G that contains v_i, hence its length

$$\ell(W_{ij}CW_{ji}) = \ell(W_{ij}) + \ell(C) + \ell(W_{ji})$$

is divisible by d_i. But also $W_{ij}W_{ji}$ is a closed walk in G that contains v_i and thus also $\ell(W_{ij}W_{ji}) = \ell(W_{ij}) + \ell(W_{ji})$ is divisible by d. Therefore d_i divides $\ell(C)$ and since W was arbitrary, it divides d. We conclude that $d_i = d$.

Again take a vertex v_i of G. It remains to show that $d_i = h$. The existence of a closed walk in G of length ℓ through a vertex v_i is equivalent to the condition

$(A^\ell)_{ii} > 0$, see Proposition 1.1. Therefore $(A^{kd_i})_{ii} > 0$ for all sufficiently large $k \in \mathbb{N}$ and $(A^s)_{ii} = 0$ if s is not a multiple of d_i.

On the other hand, we may assume that A is in Frobenius form (6.2). Then only powers of A^h can have nonzero diagonal elements. Note that, by Lemma 6.3, the square diagonal blocks of A^h consist of primitive matrices, hence $A^{mh} \gg 0$ for some $m \in \mathbb{N}$, see Proposition 5.21. Therefore $(A^{mh})_{ii} > 0$ for all sufficiently large $m \in \mathbb{N}$ and $(A^s)_{ii} = 0$ if s is not a multiple of h.

Altogether we thus have that $h = d_i = d$. □

Remark 6.5. Observe that in the case when A is a primitive matrix the same proof yields $d_i = d_{\mathrm{W}} = d_{\mathrm{C}} = 1$.

Markov chains

Now let a positive stochastic $n \times n$ matrix $P = (p_{ij})$ be the transition matrix of a discrete finite homogeneous Markov chain with the state space $V = \{v_1, \dots, v_n\}$. The *kth step probability distribution vector* $p(k) = (p_1(k), p_2(k), \dots, p_n(k))^\top$ is defined as a positive stochastic vector, i.e.,

$$0 \leq p_i(k) \leq 1, \quad \sum_{i=1}^{n} p_i(k) = 1,$$

where $p_i(k)$ is the probability of Markov process being in the state v_i after k steps. By the Markov property and Remark 1.2, the kth step distribution is determined from the initial distribution $p(0)$ by means of the transition matrix:

$$p(k) = (P^k)^\top p(0), \quad k \in \mathbb{N}.$$

Therefore the long-run (or limiting) probability distribution depends on the behavior of P^k for $k \to \infty$. Using our results from Chapters 3 and 5 we can describe it in terms of spectral properties of P.

Let us first state some spectral properties of P.

Lemma 6.6. *For the transition matrix P the following holds.*

a) $\mathrm{r}(P) = 1$ *is an eigenvalue of P with corresponding eigenvector*

$$\mathbb{1} = (1, 1, \dots, 1)^\top.$$

b) *All eigenvalues of P with modulus 1 are simple poles of the resolvent.*

Proof. a) Since P is row-stochastic, $P^k \mathbb{1} = \mathbb{1}$ holds for all $k \geq 1$. Hence, by Gelfand's formula, $\mathrm{r}(P) = 1$ and 1 is an eigenvalue with eigenvector $\mathbb{1}$.

b) Since $\|P^k\|_\infty = 1$ for all $k \in \mathbb{N}$, the sequence (P^k) is bounded, and by Theorem 3.13, all eigenvalues with modulus 1 are simple poles of the resolvent. □

As a consequence, P is always Cesàro summable with Cesàro means converging to the spectral projection of P belonging to 1 (cf. Theorem 3.13). The sequence P^k, however, does not converge as $k \to \infty$ unless 1 is radially dominant (see Theorem 3.7).

The Cesàro means of $p(k)$ have an illustrative interpretation in the context of Markov chains. Pick a state v_j and define a sequence of random variables $(X_i)_{i=0}^{\infty}$ by

$$X_i = \begin{cases} 1 & \text{if the chain is in the state } v_j \text{ after } i \text{ steps,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\frac{1}{k}\sum_{i=0}^{k-1} X_i$ represents the fraction of time that the state v_j is visited in $k-1$ steps. Since the expected value of each X_i is $E(X_i) = p_j(i)$, we have

$$E\left(\frac{1}{k}\sum_{i=0}^{k-1} X_i\right) = \left(\frac{1}{k}\sum_{i=0}^{k-1} p(i)\right)_j .$$

This means that the jth component of the Cesàro limit vector represents the fraction of time that the chain spends in the state v_j in the long-run.

Assume now, that the matrix P is irreducible (i.e., all states v_i are reachable from each other in a finite number of steps). In this case we have two possibilities.

- If P is a primitive matrix, then

$$\lim_{k\to\infty} P^k = P_1 \text{ with } P_1 x = \langle x, y\rangle \, \mathbb{1} \quad \text{and} \quad \lim_{k\to\infty} p(k) = y, \tag{6.3}$$

  where y is the stochastic Perron vector for $P^\top$, see Proposition 5.21.
- If P is an imprimitive matrix, then the above limits do not exist. However, for the corresponding Cesàro means,

$$\lim_{k\to\infty} P^{(k)} = P_1 \text{ with } P_1 x = \langle x, y\rangle \, \mathbb{1} \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} p(i) = y, \tag{6.4}$$

  where again y is the stochastic Perron vector for $P^\top$, see Theorem 3.13.

A Markov chain with an irreducible and imprimitive transition matrix is called *periodic*. In such a chain all states are visited periodically, with the period equal to the index of imprimitivity of P, see Corollary 5.23.

Note that the value of the (Cesàro) limit is independent of the initial distribution $p(0)$. The vector y in equations (6.3) and (6.4) is called the *stationary distribution vector* for the Markov chain. It is the unique stochastic vector satisfying $P^\top y = y$. Its components represent the long-run fraction of time that the chain spends in the corresponding state.

6.2 The Google Matrix

We shall demonstrate now that we encounter positive matrices and their Perron vectors on an everyday basis. We will look at the mathematics behind Google[7], currently the world biggest web search engine.

Every web search engine must build its web-page repository and index the pages stored there in the best possible way. For this purpose they use crawler software that creates virtual robots, called spiders, that constantly travel the web. The spiders number each page, collect important data from it (such as title, key words, link names, anchors, etc.) and create an index of all visited pages. Now the pages have to be ranked according to their importance. When the user does an internet search it is desired that more relevant pages are placed at the beginning of the produced list. This is actually the most important and delicate step for a search engine. It is because of intelligent ranking that Google got ahead its competitors when it appeared on the market. The core of Google is the ranking algorithm *PageRank*, developed in 1998 by Larry Page and Sergey Brin, then PhD students at Stanford University, California.

PageRank

Assume we have n web pages $W = \{W_k \mid k = 1, \ldots, n\}$. For a page W_k we denote by $I_k := \{i \mid W_i \to W_k\}$ the set of indices of all *inlinks* to W_k, by $O_k := \{j \mid W_k \to W_j\}$ the set of indices of all *outlinks* of W_k, and by $x_k \geq 0$ the *rank of the page* W_k. Now the question is, how to define x_k properly?

The answer of Page and Brin is: *A page is important if it is pointed to by other important pages.* Their formula for the rank is thus recursive and it is not clear at this point whether it admits a solution:

$$x_k := \sum_{i \in I_k} \frac{x_i}{|O_i|}, \quad k = 1, \ldots, n. \tag{6.5}$$

Here it is assumed that a link from a page to itself does not count.

The internet can be viewed as a huge directed graph with n vertices (= web pages) whose edges are hyperlinks. Let H be the transposed adjacency matrix of this graph, called also the *hyperlink matrix*, with entries

$$H_{ij} = 1/|O_j| \quad \text{iff } W_j \to W_i \quad \text{and} \quad H_{ij} = 0 \text{ otherwise.}$$

We can interpret the values H_{ij} as probabilities of accessing page W_i from page W_j. Collecting single ranks into a *ranking vector* $x := (x_1, \ldots, x_n)^\top$, we can now write the recursive relation (6.5) as a matrix equation

$$x = Hx. \tag{6.6}$$

[7]The name comes from the misspelled number googol $= 10^{100}$.

The solution vector, if it exists, is thus the fixed vector of the hyperlink matrix H. To assure uniqueness, we impose from now on that the ranking vector x is stochastic, i.e., $\|x\|_1 = 1$.

Note that H is a positive matrix, thus by Perron's theorem (see Theorem 5.6), its spectral radius $\mathrm{r}(H)$ is an eigenvalue of H with positive eigenvector. Matrix H is also substochastic, i.e., $\sum_{i=1}^{n} H_{ij} \le 1$ for all j, hence $\mathrm{r}(H) \le 1$. Having equation (6.6) in mind, we would like that $\mathrm{r}(H) = 1$. Observe that the sum of non-zero columns actually equals 1, but H might have some zero columns which represent the so-called *dangling nodes*, that is, pages without outlinks. Brin and Page therefore suggested to adjust the matrix H: replace all zero columns with $(1/n, \dots, 1/n)^\top$. The adjusted matrix becomes stochastic and thus equation (6.6) with the modified matrix H has a solution. We can also interpret this adjustment. Imagine a random surfer traveling the web using hyperlinks, which he chooses randomly. At some point he might find himself at a dangling node. His way out is to randomly type an url and thus jump to any page with probability $1/n$.

In order to assure the uniqueness of the solution to equation (6.6), we would like H to be irreducible. By Proposition 6.1, H is irreducible if and only if the web is strongly connected, which is clearly a nonrealistic assumption. However, Brin and Page overcame also this problem with a new adjustment: they replaced the matrix H by the *Google matrix*

$$G := \alpha H + (1 - \alpha) S, \tag{6.7}$$

where $S = (1/n)_{n \times n}$ and $\alpha \in [0, 1]$ is some fixed number. The interpretation of this adjustment is a continuation of the one above: a random surfer sometimes decides to jump to some other page directly by typing an url instead of following some hyperlink, even if he is not at the dangling node. The role of the parameter α is to balance between the original web structure given by H and a fully connected web represented by S. We would of course like to weight the original hyperlink structure heavily and take α close to 1.

For any $\alpha \in [0, 1)$, the Google matrix G is positive, irreducible, and column stochastic, hence Frobenius Theorem 5.13 guarantees that the equation $Gx = x$ has a unique strictly positive stochastic solution. Thus the desired ranking vector is nothing but the Perron vector for G!

Computation of the Perron vector

To compute the Perron vector for G we can use a very simple numerical method called the *power method* that was already mentioned at the end of Chapter 3. It is an iterative method defined by

$$x^{(k+1)} = G x^{(k)}.$$

From this we infer that $x^{(k+1)} = G^k x^{(0)}$, thus convergence of this process is assured by Corollary 5.16, independent of the choice of the initial vector $x^{(0)} \neq 0$. Here it is important that 1 is a strictly dominant eigenvalue of the positive irreducible matrix G.

It is well known that the rate of convergence of the power method is governed by the magnitude of the second eigenvalue $|\lambda_2|$ of the matrix. For the Google matrix it can be shown that $|\lambda_2| \leq \alpha$. This means that the convergence is faster for smaller α. Since we argued above that α should be close to 1, one has to accept a compromise here. It is reported that Google uses $\alpha = 0.85$, the value set already by Brin and Page in 1998.

6.3 Age-structured Population Models

Plant, animal, and human population models are typical examples for positive dynamical systems in which the state variables represent biomass, density, or the number of individuals in the population. Many of these models, in particular those describing predation, competition, and symbiosis among species, are nonlinear and therefore deemed to investigation by other means. An important and still widely used exception is the well-known *Leslie model*, which describes the time evolution of a population in which fertility and survival rates of individuals strongly depend on their age. For this reason, such populations are called age-structured populations. In the Leslie model, the time is discrete and represents the reproduction season (typically the year in case of mammals), while the variables $x_1(t), x_2(t), \ldots, x_n(t)$ represent the number of females (or individuals, or couples) of age $1, 2, \ldots, n$ at the beginning of year t.

In the simplest possible case one can describe the aging process by means of the equations

$$x_{i+1}(t+1) = s_i x_i(t), \quad i = 1, 2, \ldots, n-1,$$

where $s_i > 0$ is the survival coefficient at age i, that is, the fraction of females of age i that survive at least for 1 year. The first state equation takes into account the reproduction process, and is

$$x_1(t+1) = s_0(f_1 x_1(t) + f_2 x_2(t) + \cdots + f_n x_n(t)),$$

where $s_0 > 0$ is the survival coefficient during the first year of life and $f_i \geq 0$ is the fertility rate of females of age i, that is, the mean number of females born from each female of age i. These equations, originally proposed by Leslie, lead to a positive linear autonomous model

$$x(t+1) = Ax(t),$$

where the matrix A, called the *Leslie matrix*, is given as

$$A = \begin{pmatrix} s_0 f_1 & s_0 f_2 & \dots & s_0 f_{n-1} & s_0 f_n \\ s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & s_{n-1} & 0 \end{pmatrix}. \tag{6.8}$$

Though Leslie models appear to be quite coarse at first sight, they are extensively used for making demographic projections, i.e., forecasting

$$x(k) = A^k x(0)$$

given $x(0)$.

Let us comment on the usefulness of these models first. In Leslie models, survival and fertility rates depend exclusively on age. In reality, this is more or less true provided the individuals in each age class are not too many. In fact, as soon as the density of the individuals increases, some phenomena show up, which may reduce fertility and/or survival rates. For example, finding appropriate niches for reproduction becomes more difficult if the number of fertile individuals increases; the spreading of epidemics is favoured by high population densities; the search for food becomes more and more difficult as a population increases, and so on. This means that Leslie models are well suited for describing the dynamics of populations doomed to extinction, that is, characterized by small densities $x_i(t)$ for which we can suppose that survival and fertility rates are constant as time evolves. Leslie models are also extremely effective yielding short term forecasts in growing populations.

Investigating the properties of the Leslie matrix, we see that it is positive and, if $f_n > 0$, it is also irreducible. Looking at the directed weighted graph whose adjacency matrix is given by equation (6.8) and using Proposition 6.4 one easily obtains that the index of imprimitivity of the Leslie matrix equals

$$h = \gcd \{k \in \{1, \dots, n\} \colon f_k > 0\}.$$

Hence, if there are two consecutive ages with strictly positive fertility age, then the Leslie matrix is primitive.

The (normalized) Perron eigenvector of the Leslie matrix is called the *stable age structure*, which is roughly the asymptotic age distribution as time evolves. More precisely, we have the following result as a consequence of Proposition 5.21.

Proposition 6.7. *Consider the Leslie matrix A given in* (6.8) *with $f_n > 0$ and assume that A is a primitive matrix. Denote the Perron eigenvalue by $\lambda_1 = \mathrm{r}(A)$ and the corresponding eigenvector by $x_1 \gg 0$. Then*

$$\lambda_1^{-k} A^k - P_1 \longrightarrow 0$$

as $k \to \infty$, where P_1 is the projection to the one-dimensional subspace spanned by x_1.

Age Class	Average Reprod./ Year	Low Reprod./ Year	High Reprod./ Year	Average Annual Survival	Low Annual Survival	High Annual Survival
Cub	0.00	0.00	0.00	0.80	0.41	0.99
1-year-old	0.00	0.00	0.00	0.75	0.41	0.99
2-year-old	0.00	0.00	0.00	0.71	0.41	0.90
3-year-old	0.28	0.00	0.50	0.84	0.69	0.93
Adult	0.58	0.23	0.82	0.84	0.69	0.93

Table 6.1: Input parameters for Leslie Matrix population model (based on females only) of Virginias hunted black bear populations as estimated between 1994–1999.

Let us note that in many applications it is better to structure the population not in age groups, but in so-called *stage groups*. As an example, we consider Virginias hunted black bear populations. A statistical analysis, the details of which we omit, leads to the following table, which is only reproduced here to show the complexity of such problems.

In this case, as we see, it is better to investigate the so-called stage-based Leslie model. Stage-based models are frequently used for long-lived species because data on specific ages are not available, demographic variables within age classes are not different, and individual age classes for a species that lives, for example, up to 30 years (like black bear), would result in matrices of sizes up to 30×30. Analysis of this table can lead to the following Leslie matrix, where various other effects have been taken into account, and which was used successfully in analysis done by biologists:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0.275 & 0.575 \\ 0.80 & 0 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 & 0 \\ 0 & 0 & 0.71 & 0 & 0 \\ 0 & 0 & 0 & 0.84 & 0.84 \end{pmatrix}. \tag{6.9}$$

Here the last row stands for the whole adult stage, the element in the lower right corner of the matrix representing the rate of the adult population remaining alive after the year.

We now consider a second model, which is famous in the literature. The eastern wild turkey (Meleagris gallopavo silvestris) inhabits more or less the eastern part of the United States. Turkey hunting has a substantial economic effect in many rural communities. It is not only important because of the actual turkey hunting, but it also takes part in the development of the related industries of turkey-hunting clothes and equipment. Improvement of the knowledge of turkey population dynamics is important for formulating hunting regulations and other turkey management practices. A Leslie matrix model can be developed for the population dynamics of eastern wild turkeys in Iowa based on local studies. Here a three-stage model is chosen in order to simplify the modeling procedure. The first category is "poults", aged from 0 to 1, the second category is "yearlings", aged from 1 to 2, and the last category is "adults", aged 2 and older. Reproduction occurs from yearlings onwards. The time unit is one year. The Leslie matrix obtained is

$$A = \begin{pmatrix} 0 & 0.880 & 1.860 \\ 0.445 & 0 & 0 \\ 0 & 0.616 & 0.610 \end{pmatrix}. \tag{6.10}$$

This grouping makes sense for example if there are regulations allowing only the adult population to be hunted, see Exercise 6.

6.4 Notes and Remarks

For further reading on search engines and the PageRank algorithm we recommend the excellent monograph by Langville and Meyer [82]. The modeling and investigation of age-structured populations was initiated by Leslie in 1945 [87], and extended to stage structured populations by Lefkovitch [86]. Virginia's hunted black bear populations is discussed in the PhD dissertation by Klenzendorf [75]. Much research about the rates of reproduction, mortality, and survival, and the movement of wild turkeys has been done by Dickson [30].

6.5 Exercises

1. Verify that the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

   is irreducible and imprimitive using graph-theoretical interpretations. Compute also its index of imprimitivity.

2. Explain the statements given in (6.3) and (6.4). Why is the limiting distribution independent of $p(0)$?

3. Find the limiting distribution for the Markov chain given by the transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}.$$

4. Translate the PageRank algorithm into the language of Markov chains.
5. Compute the ranking vector for the web depicted in Figure 6.1. Choose sev-

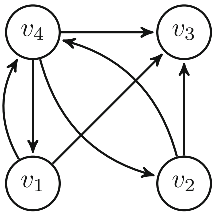

Figure 6.1: The web graph for Exercise 5.

eral values for α and observe how this choice does affect the ranking and the computation time.
6. Consider the Leslie matrix in (6.10) corresponding to the turkey population in Iowa. Use an appropriate computer software if necessary.
   a) Calculate the Perron eigenvalue and the corresponding stable age structure. Is the population growing?
   b) Assume we can change the survival rate of the adult population. How should we change the survival rate of the adult population to ensure that the Perron eigenvalue equals 1, meaning that the population remains balanced?
   c) Using a 1977 survey, the age structure in a region in Iowa was estimated as $x_1(0) = 580$, $x_2(0) = 123$, $x_3(0) = 156$. How many adults should be hunted down at the end of the first year to ensure this decrease in the survival rate of adults?
7. What is the Perron eigenvalue and the corresponding stable age distribution of the Leslie matrix in (6.9) corresponding to the bear population? Is the population growing, balanced, or dying out? Use an appropriate computer software if necessary.
8. To connect two topics of this chapter, google further Leslie matrix models, for example for the annual bluegrass (poa annua) or the brown rat (rattus norvegicus) populations.