Chapter 4

The Matrix Exponential Function

We continue our investigation of the asymptotic behavior of dynamical systems described by matrices, which was started in last chapter, now moving to the continuous time case. This means that we investigate the asymptotic properties of the matrix exponential function.

The importance of the topic should be clear for everyone reading this: the matrix exponential function always solves a corresponding system of ordinary differential equations, hence the asymptotic properties of matrix exponential functions provide information on the long-time behavior of solutions of ODEs. This subject has more than 100 years of history, with the famous Lyapunov stability theorem as its starting point.

Topics we cover include boundedness, convergence to zero, convergence, mean convergence (or Cesàro convergence), periodicity, hyperbolic decomposition, and are presented in analogy to the results achieved in the previous chapter.

4.1 Main Properties

Let X be a n-dimensional vector space. The *exponential function of a complex matrix* $A \in \mathcal{L}(X)$ is the mapping

$$\exp : \mathbb{R} \longrightarrow \mathcal{L}(X), \quad t \longmapsto \exp(tA) = \mathrm{e}^{tA}.$$

Here, as explained in Section 2.2, $\exp(tA) = \mathrm{e}^{tA}$ stands for the matrix $f_t(A)$ with $f_t(\lambda) := \mathrm{e}^{t\lambda}$. Therefore, Theorem 2.11, formula (2.9), says that e^{tA} can be written as

$$\mathrm{e}^{tA} = \sum_{i=1}^{m} \sum_{\nu=0}^{\nu_i - 1} \frac{\mathrm{e}^{t\lambda_i} t^{\nu}}{\nu!} (A - \lambda_i)^{\nu} P_i, \tag{4.1}$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of A with corresponding multiplicities ν_1, …, ν_m (as roots of the minimal polynomial) and spectral projections $P_1, \ldots, P_m$.

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Alternatively, according to Corollary 2.14, the matrix e^{tA} is represented by the exponential series

$$\mathrm{e}^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}. \tag{4.2}$$

Formula (4.1) gives an easy access to the following properties of e^{tA}. Firstly, since $f_0(\lambda) = 1$, we have

$$f_0(A) = \mathrm{e}^{0A} = I.$$

By the multiplicativity of the functional calculus and the fact that

$$f_{s+t}(\lambda) = f_s(\lambda) \cdot f_t(\lambda), \qquad \lambda \in \mathbb{C},$$

we infer that

$$\mathrm{e}^{(s+t)A} = \mathrm{e}^{sA} \cdot \mathrm{e}^{tA} \tag{4.3}$$

for $s, t \in \mathbb{R}$. Hence, $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$ is a subgroup of the multiplicative semigroup $\mathcal{L}(X)$, and the mapping $t \mapsto \mathrm{e}^{tA}$ is a homomorphism of the additive group $(\mathbb{R}, +)$ into $\mathcal{L}(X)$.

Remark 4.1. It is usual to refer to these properties of $t \mapsto \mathrm{e}^{tA}$ by saying that $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$ is the *matrix group generated by* A. If we consider only $t \geq 0$, we call $(\mathrm{e}^{tA})_{t\geq 0}$ the *matrix semigroup* generated by A

Furthermore, the function $t \mapsto \mathrm{e}^{tA}$ has nice analytic properties.

Theorem 4.2. *The matrix exponential function $t \mapsto \mathrm{e}^{tA}$ is differentiable on $\mathbb{R}$ with derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA} = \mathrm{e}^{tA}A, \quad t \in \mathbb{R}. \tag{4.4}$$

Proof. Let $f(\lambda) := \lambda \mathrm{e}^{t\lambda}$ and observe that

$$\left(\lambda \mathrm{e}^{t\lambda}\right)^{(\nu)}(\lambda) = \left(\lambda t^{\nu} + \nu t^{\nu-1}\right)\mathrm{e}^{t\lambda}.$$

Hence applying Theorem 2.11 for $f(\lambda) = \lambda \mathrm{e}^{t\lambda}$ we obtain

$$\begin{aligned}
A\mathrm{e}^{tA} &= \sum_{i=1}^{m} \sum_{\nu=0}^{\nu_i-1} \left(\lambda_i t^{\nu} \mathrm{e}^{t\lambda_i} + \nu t^{\nu-1}\mathrm{e}^{t\lambda_i}\right) \frac{(A-\lambda_i)^{\nu}}{\nu!} P_i \\
&= \sum_{i=1}^{m} \sum_{\nu=0}^{\nu_i-1} \left[\frac{\mathrm{d}}{\mathrm{d}t}(t^{\nu}\mathrm{e}^{t\lambda_i})\right] \frac{(A-\lambda_i)^{\nu}}{\nu!} P_i \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{m} \sum_{\nu=0}^{\nu_i-1} t^{\nu}\mathrm{e}^{t\lambda_i} \frac{(A-\lambda_i)^{\nu}}{\nu!} P_i\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA}.
\end{aligned}$$

Notice that since $\lambda \mathrm{e}^{t\lambda} = \mathrm{e}^{t\lambda}\lambda$, by the properties of the functional calculus we see that $A\mathrm{e}^{tA} = \mathrm{e}^{tA}A$. □

The following consequence of formula (4.4) motivates our interest in the behavior of the function $t \mapsto \mathrm{e}^{tA}$ as $t \to \infty$.

Corollary 4.3. *Let $A = (a_{ij}) \in \mathcal{L}(X)$. Then for each $x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ the function*

$$t \longmapsto \mathrm{e}^{tA} x =: (x_1(t), x_2(t), \ldots, x_n(t))^\top$$

is the unique solution of the system of differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} x_1(t) &= a_{11} x_1(t) + \cdots + a_{1n} x_n(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} x_2(t) &= a_{21} x_1(t) + \cdots + a_{2n} x_n(t) \\ &\vdots \\ \frac{\mathrm{d}}{\mathrm{d}t} x_n(t) &= a_{n1} x_1(t) + \cdots + a_{nn} x_n(t), \end{aligned}$$

with the initial condition

$$(x_1(0), x_2(0), \ldots, x_n(0)) = (x_1, x_2, \ldots, x_n).$$

Proof. A look at the differential quotient defining the derivative $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{tA} x)$ and Theorem 4.2 convinces us that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{tA} x) = \left(\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{tA} \right) x = (A \mathrm{e}^{tA}) x = A(\mathrm{e}^{tA} x)$$

for all $t \in \mathbb{R}$. Since $\mathrm{e}^{0A} = I$, we infer that $\mathrm{e}^{tA} x$ is a solution of the above initial value problem. Now, let $x(t)$ be any solution and define $y(t) := \mathrm{e}^{-tA} x(t)$. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} y(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-tA} \right) x(t) + \mathrm{e}^{-tA} \frac{\mathrm{d}}{\mathrm{d}t} x(t) \\ &= -A \mathrm{e}^{-tA} x(t) + \mathrm{e}^{-tA} A x(t) \\ &= 0. \end{aligned}$$

Therefore, $t \mapsto y(t) = \mathrm{e}^{-tA} x(t)$ is constant. Since for $t = 0$ we have $y(0) = x$, we conclude that $x(t) = \mathrm{e}^{tA} x$ for all $t \in \mathbb{R}$. ☐

Remark 4.4. In short, Corollary 4.3 tells us that the matrix semigroup generated by $A = (a_{ij})$ solves the initial value problem

$$\begin{cases} \dot{x}(t) = A x(t), & t \geq 0, \\ x(0) = x_0, \end{cases}$$

in the sense that the orbit $\{\mathrm{e}^{tA} x_0 : t \in \mathbb{R}\} \geq 0$ of the initial value $x_0 \in \mathbb{C}^n$ is the unique solution of the problem. At present, we note that Theorem 4.2 remains true if t is allowed to run through $\mathbb{C}$, hence

$$z \longmapsto \mathrm{e}^{zA}$$

is a holomorphic function on $\mathbb{C}$.

4.2 Coordinate Functions

We intend to study the behavior of the function $t \mapsto \mathrm{e}^{tA}$ (or of $t \mapsto \mathrm{e}^{tA}x$ for a given $x \in X$), as $t \to \infty$, following the same pattern as in the previous Chapter 3 for the matrix powers. Nevertheless, a few comments seem to be appropriate.

While in Chapter 3 we studied the sequence (T^k) and based our considerations on the characterization of convergence of the coordinate sequences given in Section 3.1, we will now have to deal with a function $t \mapsto \mathrm{e}^{tA}$ of the real variable t. We will formulate, without going into a detailed discussion, the following versions of the convergence properties discussed in Section 1.2.

If $t \mapsto y(t)$ is a real function with values in a finite-dimensional vector space X with a basis $\{y_1, \ldots, y_n\}$, then

$$y(t) = \sum_{i=1}^{n} \eta_i(t) y_i$$

with uniquely determined values of $\eta_i(t)$ for each $t \in \mathbb{R}$. We call the functions $t \mapsto \eta_i(t)$ the *coordinate functions* of $y(t)$ with respect to $\{y_1, \ldots, y_n\}$. Convergence of $y(t)$ as $t \to \infty$ in X is equivalent to the convergence of all coordinate functions $\eta_i(t)$, no matter what basis is employed, the coordinates of the limit being the limits of the respective coordinate functions.

In order to discuss the function $t \mapsto \mathrm{e}^{tA}$, in analogy to Lemma 2.16, we use a basis $\mathcal{B}_A$ of $\mathcal{L}(X)$ containing the set

$$B_A := \left\{ \frac{(A - \lambda_i)^{\nu}}{\nu!} P_i : i = 1, \ldots, m;\ \nu = 0, \ldots, \nu_i - 1 \right\}.$$

By (4.1), the non-zero coordinate functions with respect to this basis are

$$g_{\nu,\lambda_i}(t) := t^{\nu} \mathrm{e}^{t\lambda_i} \tag{4.5}$$

for $i = 1, \ldots, m$ and $\nu = 0, \ldots, \nu_i - 1$. Likewise, if we wish to study $\mathrm{e}^{tA}x$ for a given $x \in X$, we use a basis $\mathcal{B}_{A,x}$ of X containing the non-zero elements of

$$B_{A,x} := \left\{ \frac{(A - \lambda_i)^{\nu} x}{\nu!} : i = 1, \ldots, m;\ \nu = 0, \ldots, \nu_i - 1 \right\}.$$

Again, the coordinate functions of $\mathrm{e}^{tA}x$ with respect to this basis are among the functions $g_{\nu,\lambda_i}(t)$ defined in (4.5).

The behavior of a function $g_{\nu,\lambda}(t) := t^{\nu} \mathrm{e}^{t\lambda}$ is easy to understand and essentially depends on the real part of λ. The following cases are possible.

- $\underline{\operatorname{Re} \lambda < 0}$. Then, for each fixed value of ν, $\mathrm{e}^{t\lambda} t^{\nu} \to 0$ as $t \to \infty$, where the decay is exponential in the following sense:
  for any $0 < \delta < -\operatorname{Re} \lambda$ there is $M_\delta \geq 1$ such that

$$\left| \mathrm{e}^{\lambda t} t^{\nu} \right| = t^{\nu} \mathrm{e}^{t \operatorname{Re} \lambda} \leq M_\delta \mathrm{e}^{-\delta t} \quad \text{for all } t \geq 0.$$

- $\underline{\operatorname{Re}\lambda > 0}$. Then, for each fixed value of ν, $\left|e^{t\lambda}t^{\nu}\right| \to \infty$ as $t \to \infty$, but it remains exponentially bounded in the following sense:
  for each $w > \operatorname{Re}\lambda > \delta > 0$ there is $M_w \geq 1$ such that
  $$\left|e^{\lambda t}t^{\nu}\right| \leq M_w e^{wt} \quad \text{for all } t \geq 0.$$
- $\underline{\operatorname{Re}\lambda = 0 \text{ and } \nu = 0}$. Then $e^{t\lambda}$ is constant (for $\lambda = 0$) or periodic of period $\frac{2\pi i}{\lambda}$ (for $\lambda \neq 0$).
- $\underline{\operatorname{Re}\lambda = 0 \text{ and } \nu \geq 1}$. Then $\left|e^{\lambda t}t^{\nu}\right| = t^{\nu} \to \infty$ as $t \to \infty$.

After these preparations, we now look at the behavior of $e^{tA}x$ on the spectral subspaces $X_i = \operatorname{im} P_i$ of X.

Theorem 4.5. *Let $A \in \mathcal{L}(X)$, let $\|\cdot\|$ be a norm on X, and fix an $i \in \{1, \ldots, m\}$. Then the following assertions hold.*

a) *For every $\rho < \operatorname{Re}\lambda_i < \omega$ there exist $M \geq 1$ and $N > 0$ such that*
$$N e^{\rho t}\|x\| \leq \left\|e^{tA}x\right\| \leq M e^{\omega t}\|x\|$$
for all $t \geq 0$ and all $x \in X_i$.

b) *If $\operatorname{Re}\lambda_i = 0$, then*
$$\left\{e^{tA}x : t \geq 0\right\}$$
is bounded for every $x \in X_i$ if and only if λ_i is a simple pole of $R(\cdot, A)$, i.e., if $\nu_i = 1$. In this case, $e^{tA}x = e^{t\lambda_i}x$ for every $x \in X_i$ and $t \geq 0$.

Proof. a) By formula (4.1), we have for $x \in X_i$ that
$$\left\|e^{tA}x\right\| = \left\|\sum_{\nu=0}^{\nu_i-1} e^{t\lambda_i}t^{\nu}\frac{(A-\lambda_i)^{\nu}}{\nu!}x\right\| \leq \sum_{\nu=0}^{\nu_i-1}\left\|\frac{(A-\lambda_i)^{\nu}}{\nu!}\right\|\left|e^{t\lambda_i}t^{\nu}\right| \cdot \|x\|$$
$$\leq M e^{\omega t}\|x\|,$$
for all $\omega > \operatorname{Re}\lambda_i$ and some $M \geq 1$.

Now, we apply the above estimate to $-A$ which has $-\lambda_i$ as an eigenvalue with the same spectral projection P_i and spectral subspace X_i as before. Hence,
$$\left\|e^{-tA}y\right\| \leq M e^{-\rho t}\|y\|$$
for all $y \in X_i$, $t \geq 0$, and some $M \geq 1$. Since $e^{-tA}e^{tA} = I$, we find for every $x \in X_i$ an element $y \in X_i$ such that $x = e^{-tA}y$. This implies
$$\left\|e^{tA}x\right\| = \|y\| \geq \frac{1}{M}e^{\rho t}\|x\|,$$
for all $t \geq 0$.

b) If $\nu_i = 1$, then $e^{tA}x = e^{t\lambda_i}x$ for all $x \in X_i$. On the other hand, if $\nu_i \geq 2$, then $\ker(A-\lambda_i) \subsetneq X_i$, hence there is an $x \in X_i$ with $(A-\lambda_i)x \neq 0$. The coordinate function of $(A-\lambda_i)x$ with respect to the basis element $x \in \mathcal{B}_{A,x}$ equals $te^{t\lambda_i}$, which is unbounded as $t \to \infty$. ☐

4.3 The Spectral Bound

Now we introduce the following constant which plays the same role for the exponential function $t \mapsto \mathrm{e}^{tA}$ as the spectral radius $\mathrm{r}(T)$ does for the powers $k \mapsto T^k$ (see Section 3.2).

Definition 4.6. For $A \in \mathcal{L}(X)$ the number

$$\mathrm{s}(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$$

is called the *spectral bound* of A.

We note that the spectral bound of A can be determined from $\|\mathrm{e}^{tA}\|$ in the following way (compare with Proposition 3.3).

Proposition 4.7. *If* $\|\cdot\|$ *is any norm on* $\mathcal{L}(X)$*, then*

$$\mathrm{s}(A) = \lim_{t\to\infty} \frac{1}{t} \log \|\mathrm{e}^{tA}\|. \tag{4.6}$$

If $\|\cdot\|$ *is an operator norm, then*

$$\mathrm{s}(A) = \inf_{t>0} \frac{1}{t} \log \|\mathrm{e}^{tA}\|. \tag{4.7}$$

Proof. By the equivalence of norms on $\mathcal{L}(X)$, the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \|\mathrm{e}^{tA}\|,$$

if it exists, does not depend on the specific norm. Hence, we can use the supremum norm $|\!|\!|\cdot|\!|\!|$ with respect to the basis $\mathcal{B}_A$ above. Then we have

$$|\!|\!|\mathrm{e}^{tA}|\!|\!| = \left|t^\nu \mathrm{e}^{t\lambda_i}\right| = t^\nu \mathrm{e}^{t\operatorname{Re}\lambda_i}$$

for some $i \in \{1,\dots,m\}$ and some $0 \le \nu \le n-1$. Hence,

$$\frac{1}{t} \log |\!|\!|\mathrm{e}^{tA}|\!|\!| = \frac{\nu}{t}\log t + \operatorname{Re}\lambda_i$$

for all $t > 0$ and i and ν as before. Since the function $t \mapsto \frac{\nu}{t}\log t$ tends to zero as $t \to \infty$, we obtain

$$\frac{\nu}{t}\log t + \operatorname{Re}\lambda_i \longrightarrow \operatorname{Re}\lambda_i$$

as $t \to \infty$, yielding Formula (4.6).

Now let $\|\cdot\|$ be an operator norm. The Spectral Mapping Theorem 2.20 and Corollary 1.10 imply

$$\log \|\mathrm{e}^{tA}\| \ge t|\lambda_i| \ge t\operatorname{Re}\lambda_i,$$

and (4.7) follows. □

A repeated application of Theorem 4.5 yields the following.

Corollary 4.8. *Let* $A \in \mathcal{L}(X)$ *and let* $\|\cdot\|$ *be any norm on* X. *Then for every* $w > \mathrm{s}(A)$ *there is a constant* $M \geq 1$ *such that*

$$\left\|\mathrm{e}^{tA}x\right\| \leq M\mathrm{e}^{wt}\|x\|$$

for all $t \geq 0$ *and* $x \in X$. *Furthermore,*

$$\mathrm{s}(A) = \omega_0(T)$$

where

$$\omega_0(T) := \inf\bigl\{w \in \mathbb{R} : \exists\, M \geq 1 \text{ such that } \|\mathrm{e}^{tA}\| \leq M\mathrm{e}^{wt} \text{ for } t \geq 0\bigr\}. \tag{4.8}$$

Remark 4.9. The number $\omega_0(T)$ defined in (4.8) is known as the *growth bound* of the matrix semigroup $T(t) := \mathrm{e}^{tA}$. Note that if X is an infinite-dimensional vector space, the equality $\mathrm{s}(A) = \omega_0(T)$ need no longer hold in general.

4.4 Asymptotics

Now we put all the information together to describe the action of e^{tA} on all of X. As in Section 3.3, we first define different types of long-time behavior of e^{tA}.

Definition 4.10. For $A \in \mathcal{L}(X)$ and any norm $\|\cdot\|$ on X we say that the semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ is

- *bounded*[4] if $\sup_{t\geq 0} \|\mathrm{e}^{tA}\| < \infty$;
- *stable* if $\lim_{t\to\infty} \|\mathrm{e}^{tA}\| = 0$;
- *exponentially stable* if there exist $M \geq 1$ and $\varepsilon > 0$ such that $\left\|\mathrm{e}^{tA}\right\| \leq M\mathrm{e}^{-\varepsilon t}$ for all $t \geq 0$;
- *convergent* if $\lim_{t\to\infty} \mathrm{e}^{tA} = P$ for some $P \in \mathcal{L}(X)$;
- *periodic* if $\mathrm{e}^{t_0 A} = I$ for some $t_0 > 0$; in this case the smallest such t_0 is called the *period* of e^{tA};
- *hyperbolic* if there exist A-invariant subspaces X_s and X_u, such that $X = X_\mathrm{s} \oplus X_\mathrm{u}$ and

$$\left\|\mathrm{e}^{tA}x\right\| \leq M\mathrm{e}^{-\varepsilon t}\|x\| \quad \text{for } x \in X_\mathrm{s}, \tag{4.9}$$

$$\left\|\mathrm{e}^{tA}x\right\| \geq \frac{1}{M}\mathrm{e}^{\varepsilon t}\|x\| \quad \text{for } x \in X_\mathrm{u}, \tag{4.10}$$

  for all $t \geq 0$ and some constants $M \geq 1$, $\varepsilon > 0$; X_s and X_u are called the *stable* and *unstable* subspaces, respectively.

[4] Note that working with ODEs or dynamical systems, a different terminology is also widely accepted: what we call "bounded" is often called "stable", what we call "stable" is often called "asymptotically stable", and what we call "hyperbolic" is often called "exponential dichotomy".

Remarks 4.11.

a) Since pointwise and norm convergence on $\mathcal{L}(X)$ coincide, a statement about the long-time behavior of $\|e^{tA}x\|$ for all $x \in X$ is equivalent to the same statement regarding $\|e^{tA}\|$ for the appropriate operator norm.

b) Note that stability of a matrix semigroup is equivalent to exponential stability, see Exercise 3. We will prove this in a more general form later (cf. Proposition 12.4).

c) Using (4.8), we see that

$$(e^{tA})_{t\geq 0} \text{ is (exponentially) stable} \iff \omega_0(T) < 0. \tag{4.11}$$

We now classify the asymptotic behavior of e^{tA} in terms of spectral properties of the matrix A.

Theorem 4.12. *Let $A \in \mathcal{L}(X)$ and take any norm $\|\cdot\|$ on X.*

a) $(e^{tA})_{t\geq 0}$ *is (exponentially) stable if and only if* $s(A) < 0$.

b) $(e^{tA})_{t\geq 0}$ *is bounded if and only if* $s(A) \leq 0$ *and all eigenvalues of A with real part equal to* 0 *are simple poles of the resolvent $R(\cdot, A)$.*

c) $(e^{tA})_{t\geq 0}$ *is periodic with period t_0 if and only if it is bounded and $\sigma(A) \subset \frac{2\pi i}{t_0}\mathbb{Z}$ for some $t_0 > 0$.*

d) $\lim_{t\to\infty} e^{tA} = P_1$ *(P_1 denotes the spectral projection of A belonging to the eigenvalue* 0*) if and only if* $s(A) = 0$ *is a simple pole of the resolvent $R(\cdot, A)$ and $\sigma(A) \cap i\mathbb{R} = \{0\}$.*

e) $(e^{tA})_{t\geq 0}$ *is hyperbolic if and only if $\sigma(A) \cap i\mathbb{R} = \emptyset$.*

Proof. a) This is a consequence of relation (4.11) and Corollary 4.8.

b) $(e^{tA})_{t\geq 0}$ is bounded iff the same is true for the coordinate functions in formula (4.5), which holds iff for each i, either $\operatorname{Re}\lambda_i < 0$ or $\operatorname{Re}\lambda_i = 0$ and $\nu_i = 1$, that is, if and only if $s(A) \leq 0$ and every eigenvalue λ_i with $\operatorname{Re}\lambda_i = 0$ is a simple pole of the resolvent.

c) Again we use coordinate functions and observe that $e^{t_0 A} = I$ for some $t_0 > 0$ iff $\lambda_i \in \frac{2\pi i}{t_0}\mathbb{Z}$, with $\nu_i = 1$, which by b) holds iff $(e^{tA})_{t\geq 0}$ is bounded and $\sigma(A) \subset \frac{2\pi i}{t_0}\mathbb{Z}$.

d) $\lim_{t\to\infty} e^{tA} = P_1$ iff all coordinate functions converge, that is, iff either $\operatorname{Re}\lambda_i < 0$, or $\lambda_i = 0$ and $\nu_i = 1$. This is true iff $s(A) \leq 0$, which is a simple pole of the resolvent and the only eigenvalue on the imaginary axis. Moreover, the semigroup converges to the corresponding spectral projection.

e) $(e^{tA})_{t\geq 0}$ is hyperbolic iff there exist A-invariant subspaces X_s and X_u such that $X = X_s \oplus X_u$ and inequality (4.9) holds, that is, iff $e^{tA}|_{X_s}$ and $e^{-tA}|_{X_u}$ are both exponentially stable. By a), this is equivalent to

$$s(A|_{X_s}) < 0 \text{ and } s(-A|_{X_u}) < 0 \iff \sigma(A) \cap i\mathbb{R} = \emptyset. \qquad \square$$

Thus, in complete analogy to the situation in Section 3, convergence of e^{tA} as $t \to \infty$ is restricted to one of the following situations.

- $\lim_{t\to\infty} e^{tA} = 0$: this is the case if and only if $s(A) < 0$;
- $\lim_{t\to\infty} e^{tA} = P_1$, where P_1 is the spectral projection belonging to $\lambda_1 = 0$: this is the case if and only if $s(A) = 0$, $\sigma(A) \cap i\mathbb{R} = \{0\}$, and 0 is a simple pole of the resolvent $R(\cdot, A)$.

Example 4.13. Analyzing the spectral properties of the matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$

one obtains that the semigroup $\left(e^{tA_1}\right)_{t\geq 0}$ is periodic with period 2π, $\left(e^{tA_2}\right)_{t\geq 0}$ is hyperbolic, $\left(e^{tA_3}\right)_{t\geq 0}$ is unbounded, $\left(e^{tA_4}\right)_{t\geq 0}$ converges to $P_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $\left(e^{tA_5}\right)_{t\geq 0}$ is exponentially stable.

Decomposing the space we can study stability concepts more in detail. One example of this approach is the definition of hyperbolicity of a matrix semigroup. Let us use this approach to obtain another asymptotic property.

Definition 4.14. For $A \in \mathcal{L}(X)$ we call the semigroup $(e^{tA})_{t\geq 0}$ *asymptotically periodic* if there is a direct sum decomposition

$$X = X_0 \oplus X_1$$

into A-invariant subspaces X_0 and X_1 such that

a) $e^{tA}|_{X_0}$ is stable, i.e., $\lim_{t\to\infty} e^{tA}x = 0$ for all $x \in X_0$, and
b) $e^{tA}|_{X_1}$ is periodic, i.e., there exists $t_0 > 0$ such that $e^{t_0 A}y = y$ for all $y \in X_1$.

Again, this property can be described by spectral properties of A.

Theorem 4.15. *For $A \in \mathcal{L}(X)$ the following assertions are equivalent.*

(i) $(e^{tA})_{t\geq 0}$ *is asymptotically periodic.*
(ii) $(e^{tA})_{t\geq 0}$ *is bounded and $\sigma(A) \cap i\mathbb{R} \subset 2\pi i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.*
(iii) $s(A) \leq 0$, *the set $\sigma(A) \cap i\mathbb{R}$ consists of simple poles of the resolvent $R(\cdot, A)$ and is contained in $2\pi i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.*

Proof. (i) $\Longrightarrow$ (ii): The boundedness of $(e^{tA})_{t\geq 0}$ follows directly. Let $X = X_0 \oplus X_1$ be the corresponding decomposition. Then

$$\sigma(A) = \sigma(A|_{X_0}) \cup \sigma(A|_{X_1}),$$

where $\sigma(A|_{X_0}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ by Theorem 4.12.a) and $\sigma(A|_{X_1}) \subset 2\pi i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$ by Theorem 4.12.c).

(ii) $\Longrightarrow$ (iii): This follows by Theorem 4.12.b).

(iii) $\Longrightarrow$ (i): Define

$$X_0 := \bigoplus_{\operatorname{Re}\lambda_i<0} X_i \qquad \text{and} \qquad X_1 := \bigoplus_{\operatorname{Re}\lambda_i=0} X_i$$

and apply Theorem 4.12.a) and c). □

Finally, we ask under which conditions a subsequence or the Cesàro means of $(\mathrm{e}^{tA})_{t\geq 0}$ converge. First recall the concept of a spectral contraction in Definition 3.11.

Theorem 4.16. *The following assertions are equivalent for* $A \in \mathcal{L}(X)$.

(i) e^{tA} *is spectral contraction for one/all* $t > 0$.

(ii) $\mathrm{s}(A) = 0$ *and all eigenvalues of* A *with real part equal to 0 are simple poles of the resolvent* $R(\lambda, A)$.

(iii) *There is an operator norm* $|||\cdot|||$ *on* $\mathcal{L}(X)$ *such that* $|||\mathrm{e}^{ktA}||| = 1$ *for all* $k \in \mathbb{N}$ *and one/all* $t > 0$.

(iv) *There exists a sequence* (t_m) *of the form* $t_m := tk_m$, *where* (k_m) *is a subsequence of* (k), *such that* $(\mathrm{e}^{t_m A})$ *converges to some limit* $P \neq 0$ *for one/all* $t > 0$.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) follows by Theorem 3.10 combined with the Spectral Mapping Theorem 2.20 and Theorem 2.28, the equivalence (i) $\Longleftrightarrow$ (iv) again by Theorem 3.10, while (i) $\Longleftrightarrow$ (iii) holds by Theorem 3.12. □

Definition 4.17. We say that $(\mathrm{e}^{tA})_{t\geq 0}$ is a *spectral contraction semigroup*, if any of the equivalent assertions of Theorem 4.16 is true.

The following is the continuous-time analogue of the Cesàro means introduced in Chapter 3.

Definition 4.18. Let $A \in \mathcal{L}(X)$. The matrices

$$C(r) := \frac{1}{r} \int_0^r \mathrm{e}^{sA}\,\mathrm{d}s, \qquad r > 0,$$

are called the *Cesàro means* of the semigroup $(\mathrm{e}^{tA})_{t\geq 0}$. The semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ is *mean ergodic* (or *Cesàro summable*), if $\lim_{r\to\infty} C(r)$ exists.

Theorem 4.19. *For* $A \in \mathcal{L}(X)$ *the semigroup* $(\mathrm{e}^{tA})_{t\geq 0}$ *is mean ergodic if and only if either* $\mathrm{s}(A) < 0$, *or* $(\mathrm{e}^{tA})_{t\geq 0}$ *is a spectral contraction semigroup.*

In the case $0 \in \sigma(A)$, *the Cesàro means* $C(r)$ *converge to the spectral projection of* A *belonging to* 0, *in all other cases of mean ergodic semigroups* $C(r)$ *converge to* 0.

Proof. First note that the coordinate functions of $C(r)$ with respect to $\mathcal{B}_A$ are of the form

$$g_{\nu,\lambda_i}^{(r)} := \frac{1}{r}\int_0^r g_{\nu,\lambda_i}(s)\,\mathrm{d}s = \frac{1}{r}\int_0^r \mathrm{e}^{s\lambda_i} s^\nu\,\mathrm{d}s. \tag{4.12}$$

Following the discussion on page 46–47, we see that $g_{\nu,\lambda}^{(r)}$ converges only in two cases: either for $\operatorname{Re}\lambda < 0$, or for $\operatorname{Re}\lambda = 0 = \nu$. This proves the first assertion of the theorem.

Since for $\operatorname{Re}\lambda < 0$ we have $g_{\nu,\lambda}^{(r)} \to 0$ as $r \to \infty$, in the case when $\mathrm{s}(A) < 0$ we obtain $C(r) \to 0$. On the other hand, in the case of a spectral contraction semigroup, the only nonzero limit of the coordinate functions as $r \to \infty$ is $g_{0,0}^{(r)} = 1$. By (4.1), the Cesàro means $C(r)$ then converge towards the spectral projection of A belonging to $\lambda = 0$. □

4.5 Notes and Remarks

There are many ways how to compute the exponential function of a matrix numerically and we refer here to an excellent survey paper by Moler and van Loan [99].

Theorem 4.12.a) is Lyapunov's Stability Theorem proved in 1892 (see [88]). The results of this chapter are presented in many books on ordinary differential equations, like for example Amann [3] or Teschl [139, Ch. 3].

4.6 Exercises

1. Show that if $A, B \in \mathcal{L}(X)$ commute, then $\mathrm{e}^{t(A+B)} = \mathrm{e}^{tA}\mathrm{e}^{tB}$. Find an example to show that the commutativity assumption is necessary.
2. Let $B \in \mathcal{L}(X)$. Under which conditions is there an $A \in \mathcal{L}(X)$ such that $\mathrm{e}^{kA} = B^k$ for all $k \in \mathbb{N}$?
3. Prove that for $A \in \mathcal{L}(X)$ the matrix semigroup e^{tA} is stable if and only if it is exponentially stable.
4. Show that $\{\mathrm{e}^{tA}x : t \in \mathbb{R}\}$ is bounded for every $x \in X$ if and only if $\sigma(A) \subset \mathrm{i}\mathbb{R}$ and all eigenvalues are simple poles of the resolvent $R(\cdot, A)$.
5. Show that the semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ is hyperbolic if and only if $\sigma\left(\mathrm{e}^{tA}\right) \cap \Gamma = \emptyset$ for some/all $t > 0$, where Γ denotes the unit circle in $\mathbb{C}$.
6. Compute the matrix exponential e^{tA} for
$$A = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix} \text{ with } a + b \neq 0.$$
7. For every one of the matrices in Example 4.13 compute the spectrum and the corresponding semigroup, and then describe its asymptotic behavior.