Chapter 16

Koopman Semigroups

We present here a class of positive operator semigroups that arise in studying dynamical systems. The main idea is to linearize a given (nonlinear) system by considering another state space. The linear operator which acts on this new space is called the Koopman operator. It is named after B. O. Koopman, who used this in the 1930s together with G. D. Birkhoff and J. von Neumann to prove the so-called ergodic theorems.

We start with a nonlinear system of ordinary differential equations, associate a semiflow to it, and then derive the corresponding Koopman semigroup. Subsequently we present the main properties of this semigroup and its generator. At the end we show some properties of the semiflow that can be deduced from the appropriate properties of the associated Koopman semigroup or its generator.

In this chapter we assume some general knowledge of measure theory.

16.1 Ordinary Differential Equations and Semiflows

Consider the ordinary differential equation

$$\begin{cases} \dot{x}(t) = F(x(t)), \quad t \geq 0, \\ x(0) = x_0 \in \Omega, \end{cases} \tag{16.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set. We make the following standing assumptions.

Assumptions 16.1.

a) $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable.

b) Equation (16.1) has global solutions for all $x_0 \in \Omega$.

c) $\overline{\Omega} \subset \mathbb{R}^n$ is *positively invariant* for the solution of equation (16.1), i.e.,

$$x_0 \in \overline{\Omega} \implies x(t) \in \overline{\Omega} \text{ for } t \geq 0.$$

A. Bátkai et al., *Positive Operator Semigroups*, Operator Theory: Advances and Applications 257, DOI 10.1007/978-3-319-42813-0_16

We comment on these assumptions based on standard theorems from ordinary differential equations. First recall that Assumption 16.1.a) implies the existence and uniqueness of local solutions to equation (16.1). Assumption 16.1.b) is satisfied whenever F grows at most linearly, i.e., if there are constants $c, d > 0$ such that

$$\|F(x)\| \leq c\|x\| + d.$$

Finally, the set $\bar{\Omega}$ is positively invariant if the subtangent condition

$$\liminf_{h\downarrow 0} \frac{1}{h}\mathrm{d}(x + hF(x), \Omega) = \liminf_{h\downarrow 0} \frac{1}{h} \inf_{z\in\Omega} \|x + hF(x) - z\| = 0$$

holds for every $x \in \partial\Omega$. If Ω is convex, then this is equivalent to the angle condition

$$(F(x)|y) \leq 0$$

for $x \in \partial\Omega$ and y being an outer normal vector to Ω at x (see Figure 16.1).

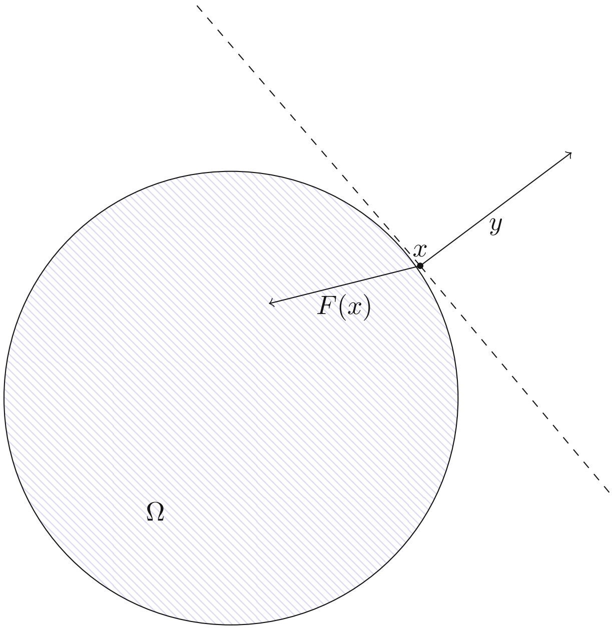

Figure 16.1: The angle condition $\sphericalangle(y, F(x)) \geq \frac{\pi}{2}$.

The assumptions imply that there exists a continuous mapping solving the differential equation in (16.1). More precisely, there exists a function $\varphi : \mathbb{R}^+ \times \overline{\Omega} \to \overline{\Omega}$, which is continuously differentiable in its first variable, satisfying

$$\begin{aligned} \varphi(0, x) &= x && \text{for all } x \in \overline{\Omega} \text{ and} \\ \varphi(t, \varphi(s, x)) &= \varphi(t + s, x) && \text{for all } t, s \geq 0,\ x \in \overline{\Omega}, \end{aligned} \tag{16.2}$$

such that the solutions of equation (16.1) are given by

$$x(t) = \varphi(t, x_0).$$

Such a mapping φ is called a continuous *semiflow*.

It was the fundamental observation of Koopman and von Neumann that such nonlinear dynamical systems give rise to linear ones. One motivation for this construction is that in many situations we do not see the state space Ω and the dynamics on it, but we only observe some quantity (heat, concentration, density, etc.). Hence, an *observable* is simply a (scalar) function defined on Ω. The nonlinear action of φ induces a linear action on observables.

To make these considerations precise, we suppose that Ω is a bounded open set, choose the Banach function space $E = \mathrm{C}(\overline{\Omega})$, and define the operators

$$T(t)f(x) := f(\varphi(t, x)) \tag{16.3}$$

for $f \in E$, $t \geq 0$ and $x \in \overline{\Omega}$. Recall that the linear operator T on the Banach lattice E is called a *lattice homomorphism* if $|Tf| = T|f|$ for every $f \in E$ (see Definition 10.19).

Proposition 16.2. *The family $(T(t))_{t\geq 0}$, defined by formula* (16.3)*, is a C_0-semigroup of positive contractions on the Banach lattice E. Its generator A is given as the closure of the operator*

$$(A_0 f)(x) = (\nabla f(x)|F(x))$$

with $f \in D(A_0) = \mathrm{C}^1(\overline{\Omega})$ and $x \in \overline{\Omega}$. Further, $T(t)$ is a lattice homomorphism for each $t \geq 0$.

Here

$$\mathrm{C}^1(\overline{\Omega}) := \left\{ f \in \mathrm{C}(\overline{\Omega}) \,:\, \exists U \text{ open, } \Omega \subset U, \text{ and } g \in \mathrm{C}^1(U) \text{ such that } f = g|_{\overline{\Omega}} \right\}.$$

The operator semigroup $(T(t))_{t\geq 0}$ from Proposition 16.2 is usually called the *Koopman semigroup* associated to the semiflow φ.

Proof. Clearly, each operator $T(t)$ is linear, positive, even a lattice homomorphism, and $T(t)\mathbb{1} = \mathbb{1}$ holds. By Lemma 10.27, the operators $T(t)$ are all contractions. For $t, s \geq 0$ and $x \in \overline{\Omega}$ we see that, by (16.2),

$$\begin{aligned}(T(t)T(s)f)(x) &= (T(t)f)(\varphi(s, x)) = f(\varphi(t, \varphi(s, x))) \\ &= f(\varphi(t + s, x)) = (T(t + s)f)(x),\end{aligned}$$

hence the semigroup property holds. To show strong continuity of the map $T(\cdot)$, note that $\varphi : [0, 1] \times \overline{\Omega} \to \overline{\Omega}$ is uniformly Lipschitz continuous in its first variable. We will denote its Lipschitz constant by L. Take $f \in E = \mathrm{C}(\overline{\Omega})$. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in \overline{\Omega}$ with $\|x - y\| < \delta$ we have

$$\|f(x) - f(y)\| < \varepsilon.$$

The Lipschitz continuity of the semiflow implies that if $0 \le t < \delta/L$ holds, then $\|\varphi(t,x) - x\| < \delta$. Hence, for $0 \le t < \delta/L$ we have

$$\|T(t)f - f\| = \sup_{x \in \overline{\Omega}} \|f(\varphi(t,x)) - f(x)\| \le \varepsilon.$$

Now we turn our attention to the characterization of the generator A of $(T(t))_{t\ge0}$. Taking $f \in D(A_0)$, $x \in \overline{\Omega}$, we note that the function $t \mapsto (T(t)f)(x) = f(\varphi(t,x))$ is also continuously differentiable, hence $T(t)D(A_0) \subset D(A_0)$. Since $D(A_0)$ is dense in E, we see that it is a core for the generator[16]. It only remains to show that for $f \in D(A_0)$ we have $Af = A_0 f$. To verify this, take $f \in \mathrm{C}^1(\overline{\Omega})$ and consider

$$\begin{aligned} I &:= \left\| \frac{T(t)f - f}{t} - A_0 f \right\| = \sup_{x\in\overline{\Omega}} \left| \frac{(T(t)f)(x) - f(x)}{t} - (\nabla f(x)|F(x)) \right| \\ &= \sup_{x\in\overline{\Omega}} \left| \frac{f(\varphi(t,x)) - f(\varphi(0,x))}{t} - (\nabla f(\varphi(0,x))|F(\varphi(0,x)) \right|. \end{aligned}$$

By the mean value theorem, for every $t > 0$ and $x \in \overline{\Omega}$, there is $0 < \xi = \xi(t,x) < t$ such that

$$I = \sup_{x\in\overline{\Omega}} \left|(\nabla f(\varphi(\xi(t,x),x))|F(\varphi(\xi(t,x),x))) - (\nabla f(\varphi(0,x))|F(\varphi(0,x))\right|.$$

If $t \downarrow 0$, then $\xi(t,x) \to 0$ uniformly in x. Since $f \in \mathrm{C}^1(\overline{\Omega})$, and φ and F are also continuously differentiable in their first variable, it follows that $I \to 0$ as $t \downarrow 0$. This shows that $f \in D(A)$ and $A_0 f = Af$. □

Remark 16.3. If $\overline{\Omega}$ is not invariant, then we have to require boundary conditions on the domain of the generator A of the Koopman semigroup. Typical here are the so-called *Wentzell boundary conditions*. As an illustration, we present the simplest situation without a proof.

Suppose that Ω is a convex domain with C^1-boundary. In this case we have at each boundary point $x \in \partial\Omega$ a unique outer normal vector $\nu(x)$. Our main assumption is now that

$$(F(x)|\nu(x)) > 0$$

for all $x \in \partial\Omega$. We consider the operator $A_0 f := (\nabla f|F)$ with domain

$$D(A_0) = \{f \in \mathrm{C}^1(\overline{\Omega}) \,:\, A_0 f(x) = 0 \text{ for } x \in \partial\Omega\}.$$

Then the closure of this operator generates a positive semigroup of lattice homomorphisms.

[16] A subspace $D \subset D(A)$ is called a *core* of A if it is dense in the graph norm of A. This implies that if we know the generator on a core, the we can determine it uniquely. An important result in semigroup theory states that if a dense set $D \subset A$ is invariant under the semigroup, then it is a core.

16.2 Koopman Semigroups

In this section we collect some properties characterizing Koopman semigroups and their generators. We start with some technical lemmata.

Lemma 16.4. *Let K be a compact Hausdorff space and $f^* \in \mathrm{C}(K)^*$ a continuous linear functional. Then the following statements are equivalent.*

(i) *f^* is a lattice homomorphism.*

(ii) *There are a $c \geq 0$ and $x \in K$ such that $f^* = c\delta_x$, i.e., $f^*(f) = \langle f, f^* \rangle = cf(x)$ holds for all $f \in \mathrm{C}(K)$.*

Proof. The implication (ii) $\Longrightarrow$ (i) is straightforward, hence we only treat the other one. Let $f^* \in \mathrm{C}(K)^*$ be a lattice homomorphism. By the Riesz representation theorem (see Theorem A.29), there is a Borel measure μ such that

$$\langle f, f^* \rangle = \int_K f \, \mathrm{d}\mu$$

for all $f \in \mathrm{C}(K)$.

Assume by contradiction that $x, y \in K$ are two distinct points in the support of μ, and take two disjoint open sets $U, V \subset K$ such that $x \in U$ and $y \in V$. Applying Urysohn's lemma (see Lemma A.19) to these sets, we obtain functions $f, g \in \mathrm{C}(K)$ such that

$$f(x) = 1 \text{ and } f(s) = 0 \text{ for } s \in K \setminus U,$$
$$g(y) = 1 \text{ and } g(s) = 0 \text{ for } s \in K \setminus V.$$

Then $f \wedge g = 0$ holds, but $\langle f, f^* \rangle \wedge \langle g, f^* \rangle > 0$, a contradiction. □

This allows us to characterize lattice homomorphisms between $\mathrm{C}(K)$ spaces.

Lemma 16.5. *Let K, L be compact Hausdorff spaces and consider $T : \mathrm{C}(K) \to \mathrm{C}(L)$. Then the following are equivalent.*

(i) *T is a lattice homomorphism.*

(ii) *There exist a unique positive function $g \in \mathrm{C}(L)$ and a unique function $\psi : L \to K$ which is continuous on the set $\{y \in L : g(y) > 0\}$, such that*

$$(Tf)(s) = g(s)f(\psi(s))$$

holds for all $f \in \mathrm{C}(K)$, $s \in L$.

Proof. Again, as in the proof of the previous lemma, we only have to prove the implication (i) $\Longrightarrow$ (ii). Suppose that T is a lattice homomorphism, and let $y \in L$. Since

$$(Tf)(y) = \delta_y(Tf) = (\delta_y \circ T)(f),$$

we see that $\delta_y \circ T \in \mathrm{C}(L)^*$ is a lattice homomorphism. By Lemma 16.4, there exist a scalar $g(y) \geq 0$ and a point $\psi(y) \in K$ satisfying

$$(Tf)(y) = g(y)f(\psi(y)).$$

Clearly, $g = T\mathbb{1}_K \in \mathrm{C}(L)$. It remains to show that ψ is continuous. To this end, take a net $(y_\alpha) \subset L$ such that $y_\alpha \to y$ in L, $g(y) > 0$ and $g(y_\alpha) > 0$ for all α. Then $g(y_\alpha) \to g(y)$, and for all $f \in \mathrm{C}(K)$ we have

$$g(y_\alpha)f(\psi(y_\alpha)) = (Tf)(y_\alpha) \longrightarrow (Tf)(y) = g(y)f(\psi(y)).$$

Since $g(y_\alpha), g(y) \neq 0$, this implies that

$$f(\psi(y_\alpha)) \longrightarrow f(\psi(y))$$

for all $f \in \mathrm{C}(K)$. This establishes the continuity of ψ. □

Recall that a positive linear operator $T : C(K) \to C(L)$ is called a *Markov operator*, if the identity $T\mathbb{1}_K = \mathbb{1}_L$ holds, see Definition 10.28. Further, a semiflow on K is defined analogously to the previous section: it is a continuous mapping $\varphi : \mathbb{R}^+ \times K \to K$ satisfying

$$\begin{aligned} \varphi(0, x) &= x && \text{for all } x \in K, \text{ and} \\ \varphi(t, \varphi(s, x)) &= \varphi(t + s, x) && \text{for all } t, s \geq 0,\ x \in K. \end{aligned} \tag{16.4}$$

This leads to the following characterization of Koopman semigroups on $\mathrm{C}(K)$ spaces.

Theorem 16.6. *Let K be a compact Hausdorff space, $E = \mathrm{C}(K)$, and $(T(t))_{t\geq 0}$ a strongly continuous semigroup on E. Then the following are equivalent.*

(i) *Each $T(t)$, $t \geq 0$, is a Markov lattice homomorphism.*
(ii) *$(T(t))_{t\geq 0}$ is induced by a continuous semiflow $\varphi : \mathbb{R}^+ \times K \to K$, i.e.,*

$$(T(t)f)(x) = f(\varphi(t, x)), \quad f \in E,\ x \in K,\ t \geq 0.$$

Proof. (ii) $\Longrightarrow$ (i): This implication was the content of Proposition 16.2.

(i) $\Longrightarrow$ (ii): From the proof of Lemma 16.5 we see that if the operator T is Markov, then we have $g(s) = 1$ for each $s \in L$. Hence, Lemma 16.5 implies the existence of a function $\varphi : \mathbb{R}^+ \times K \to K$ such that $(T(t)f)(x) = f(\varphi(t, x))$. The semiflow property defined in (16.4) follows directly from the semigroup property of the operators $T(t)$, since

$$\begin{aligned} f(\varphi(t+s,x)) &= (T(t+s)f)(x) = (T(t)T(s)f)(x) \\ &= (T(t)(f(\varphi(s,x))) = f(\varphi(t,\varphi(s,x))) \end{aligned}$$

holds for each $f \in \mathrm{C}(K)$, implying that $\varphi(t+s,x) = \varphi(t,\varphi(s,x))$. It thus remains to show that φ is continuous in t. Take a sequence $(t_n) \subset \mathbb{R}^+$ such that $t_n \to t$. Then, by the strong continuity of the semigroup,

$$f(\varphi(t_n,x)) = (T(t_n)f)(x) \longrightarrow (T(t)f)(x) = f(\varphi(t,x))$$

for all $f \in \mathrm{C}(K)$ and $x \in K$. Hence, $\varphi(t_n,x) \to \varphi(t,x)$, implying continuity of φ in t. □

Remark 16.7. Note that $\mathrm{C}(K)$ is not only a Banach lattice, but also a *C^*-algebra*. If the conditions of Lemma 16.5 are satisfied, then T is also a C^*-algebra homomorphism. This leads us to the following additional equivalences in Theorem 16.6.

(iii) Each operator $T(t)$ is a C^*-algebra homomorphism.

(iv) The generator A is a *derivation*, i.e., the domain $D(A)$ is a *-subalgebra and the identities $A(f\cdot g) = f\cdot Ag + Af\cdot g$ and $\overline{Af} = A\bar{f}$ hold for all $f,g \in D(A)$.

It seems desirable to have not only an algebraic, but also an order theoretic characterization of generators of Koopman semigroups. Before deriving one such characterization, let us make the following informal comment. If $(T(t))_{t\geq 0}$ is a semigroup of lattice homomorphisms, then

$$T(t)|f| = |T(t)f|$$

for each $f \in E$. Accepting that the real function $\mathrm{abs}(s) = |s|$ has derivative $\mathrm{abs}'(s) = \operatorname{sgn} s$, we obtain formally that

$$A|f| = \frac{\mathrm{d}}{\mathrm{d}t}T(t)|f|\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}|T(t)f|\Big|_{t=0} = (\operatorname{sgn} f)Af.$$

To give this calculation a meaning and to see how to interpret it correctly, we need the following preparations.

Definition 16.8. Let X be a Banach space, $\eta : X \to X$ a mapping, $f,u \in X$. The mapping η is called *right Gâteaux differentiable in f in direction u*, if

$$\partial_u\eta(f) := \lim_{t\downarrow 0} \frac{\eta(f+tu)-\eta(f)}{t} \tag{16.5}$$

exists. The function η is called *right Gâteaux differentiable in f* if $\partial_u\eta(f)$ exists for all directions $u \in X$.

Example 16.9. Let $X = \mathbb{C}$ and consider the function $\eta(z) = |z|$. Using the notation $\operatorname{sgn} z = \frac{z}{|z|}$ for $z \neq 0$ and $\operatorname{sgn} 0 = 0$, we can show that η is right Gâteaux differentiable and

$$\partial_u \eta(z) = \begin{cases} \operatorname{Re}((\operatorname{sgn} \bar{z})u) & \text{if } z \neq 0, \\ |u| & \text{if } z = 0. \end{cases}$$

This can be obtained by direct calculations which are left as Exercise 3.

We shall use the following chain rule for the right Gâteaux derivative.

Lemma 16.10. *Let $\psi : \mathbb{R} \to X$ be right differentiable at $a \in \mathbb{R}$ with the right derivative $\psi'(a)$, and suppose that $\eta : E \to E$ is Lipschitz continuous. If η is right Gâteaux differentiable at the point $\psi(a)$ in the direction $\psi'(a)$, then $\eta \circ \psi$ is right differentiable at a and its right derivative in a equals*

$$(\eta \circ \psi)'(a) = \partial_{\psi'(a)} \eta(\psi(a)).$$

Proof. Take $L > 0$ such that $\|\eta(f) - \eta(g)\| \leq L\|f - g\|$ holds for all $f, g \in X$. Then

$$\begin{aligned} &\lim_{t \downarrow 0} \left\| \frac{1}{t} \big(\eta(\psi(a+t)) - \eta(\psi(a))\big) - \partial_{\psi'(a)} \eta(\psi(a)) \right\| \\ &\leq \limsup_{t \downarrow 0} \left\| \frac{1}{t} \big(\eta(\psi(a+t)) - \eta(\psi(a) + t\psi'(a))\big) \right\| \\ &\quad + \limsup_{t \downarrow 0} \left\| \frac{1}{t} \big(\eta(\psi(a) + t\psi'(a)) - \eta(\psi(a))\big) - \partial_{\psi'(a)} \eta(\psi(a)) \right\| \\ &\leq L \limsup_{t \downarrow 0} \left\| \frac{1}{t} (\psi(a+t) - \psi(a)) - \psi'(a) \right\| + 0 = 0, \end{aligned}$$

by the right Gâteaux differentiability of η at $\psi(a)$ in the direction of $\psi'(a)$ and by the right differentiability of ψ at $a \in \mathbb{R}$. □

Let us introduce some further notation. For $f, g \in \mathrm{C}(K)$ we write[17]

$$((\operatorname{sgn} f)(g))(x) := (\operatorname{sgn} f(x)) \cdot g(x)$$

and

$$((\widehat{\operatorname{sgn}} f)(g))(x) := \begin{cases} (\operatorname{sgn} f)(g)(x) & \text{if } f(x) \neq 0, \\ |g(x)| & \text{if } f(x) = 0. \end{cases}$$

Note that, though this may not be a continuous map, we can extend the duality map to it by the following straightforward construction. For $\mu \in \mathrm{C}(K)^*$ we define

$$\langle (\widehat{\operatorname{sgn}} f)(g), \mu \rangle = \int_K (\widehat{\operatorname{sgn}} f)(g) \mathrm{d}\mu.$$

[17] Compare with (14.17).

Lemma 16.11. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on $E = \mathrm{C}(K)$ with generator A. Then for every $f \in D(A)$ and $\mu \in E^*$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle |T(t)f|, \mu\rangle\Big|_{t=0} = \langle \mathrm{Re}((\widehat{\mathrm{sgn}}\bar{f})(Af)), \mu\rangle.$$

Proof. For $f \in D(A)$ and $x \in K$, define the function

$$\eta(t) = (T(t)f)(x).$$

It is right differentiable in 0 with $\eta'(0) = (Af)(x)$. The chain rule, Lemma 16.10, and Example 16.9 imply that

$$|\eta(0)|' = \mathrm{Re}((\widehat{\mathrm{sgn}}\bar{f})(Af))(x).$$

Moreover,

$$\frac{1}{t}\big||T(t)f| - |f|\big| \leq \frac{1}{t}|T(t)f - f|$$

implies that

$$\sup_{0<t\leq 1} \frac{1}{t}\||T(t)f| - |f|\| < \infty.$$

Hence, the functions

$$k_t(x) = \frac{1}{t}(|T(t)f(x)| - |f(x)|)$$

are uniformly bounded on K. Lebesgue's dominated convergence theorem (see Theorem A.23) then implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle |T(t)f|, \mu\rangle\Big|_{t=0} = \lim_{t\downarrow 0}\langle k_t, \mu\rangle = \langle \mathrm{Re}((\widehat{\mathrm{sgn}}\bar{f})(Af)), \mu\rangle. \qquad \square$$

We are now able to characterize a Koopman semigroup by its generator in the following way.

Theorem 16.12. *A strongly continuous semigroup $(T(t))_{t\geq 0}$ with the generator A on the Banach lattice $E = \mathrm{C}(K)$ is a semigroup of lattice homomorphisms if and only if the* Kato equality

$$\left\langle \mathrm{Re}\left((\widehat{\mathrm{sgn}}\bar{f})(Af)\right), \mu\right\rangle = \langle |f|, A^*\mu\rangle \tag{16.6}$$

holds for all $f \in D(A)$ and $\mu \in D(A^*)$.

Proof. Suppose that $(T(t))_{t\geq 0}$ is a semigroup of lattice homomorphisms and let $f \in D(A)$ and $\mu \in D(A^*)$. Lemma 16.11 implies that

$$\langle \mathrm{Re}((\widehat{\mathrm{sgn}}\bar{f})(Af)), \mu\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle |T(t)f|, \mu\rangle\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\langle T(t)|f|, \mu\rangle\Big|_{t=0} = \langle |f|, A^*\mu\rangle.$$

Conversely, suppose that (16.6) holds. We have to show that

$$T(t)|f| = |T(t)f|$$

holds for all $t > 0$ and $f \in E$. Since $D(A)$ is dense, it is sufficient to show this equality for $f \in D(A)$. Moreover, it suffices to show

$$\langle |T(t)f|, \mu\rangle = \langle T(t)|f|, \mu\rangle \tag{16.7}$$

for all $\mu \in D(A^*)$.

Let $\mu \in D(A^*)$, $t > 0$, and define the function

$$\xi(s) = \langle T(t-s)|T(s)f|, \mu\rangle$$

for $s \in [0, t]$. If we show that ξ is constant, then $\xi(0) = \xi(t)$, which is exactly relation (16.7).

Since $\mu \in D(A^*)$,

$$\lim_{h\downarrow 0} \frac{1}{h}\langle g, (T(t-(s+h)) - T(t-s))^*\mu\rangle = -\langle g, A^*T(t-s)^*\mu\rangle$$

holds for all $g \in E$. Consequently, by the Uniform Boundedness Principle (see Theorem A.15), we see that

$$\limsup_{h\downarrow 0} \frac{1}{h}\|(T(t-(s+h)) - T(t-s))^*\mu\| < \infty.$$

Hence,

$$\lim_{h\downarrow 0} \frac{1}{h}\langle |T(s+h)f|, (T(t-(s+h)) - T(t-s))^*\mu\rangle = -\langle |T(s)f|, A^*T(t-s)^*\mu\rangle.$$

Using this equality, we obtain

$$\begin{aligned}
&\lim_{h\downarrow 0} \frac{1}{h}(\xi(s+h) - \xi(s)) \\
&\quad = \lim_{h\downarrow 0} \frac{1}{h}\left(\langle T(t-(s+h))|T(s+h)f|, \mu\rangle - \langle T(t-s)|T(s+h)f|, \mu\rangle\right. \\
&\qquad \left.+\langle T(t-s)|T(s+h)f| - T(t-s)|T(s)f|, \mu\rangle\right) \\
&\quad = -\langle |T(s)f|, A^*T(t-s)^*\mu\rangle + \lim_{h\downarrow 0} \frac{1}{h}\langle T(t-s)|T(s+h)f| - T(t-s)|T(s)f|, \mu\rangle.
\end{aligned}$$

By Lemma 16.11,

$$\begin{aligned}
&\lim_{h\downarrow 0} \frac{1}{h}(\xi(s+h) - \xi(s)) \\
&\qquad = -\langle |T(s)f|, A^*T(t-s)^*\mu\rangle + \langle \mathrm{Re}((\widehat{\mathrm{sgn}}\overline{T(s)f})(AT(s)f)), T(t-s)^*\mu\rangle.
\end{aligned}$$

By the Kato Equality (16.6), this last expression equals zero, proving that ξ is constant. □

16.3 Applications of Koopman Semigroups

We show by some simple examples how one can translate properties of the semiflow into appropriate properties of the Koopman semigroup. As before, let K be a compact Hausdorff space, $\varphi : \mathbb{R}^+ \times K \to K$ a continuous semiflow on it, and $(T(t))_{t\geq 0}$ the associated Koopman semigroup.

Lemma 16.13. *A closed subset $L \subset K$ is invariant under φ if and only if the ideal J_L generated by L is invariant under the Koopman semigroup $(T(t))_{t\geq 0}$.*

Proof. Recall from Proposition 10.13 the characterization of closed ideals in $\mathrm{C}(K)$. Suppose now that L is invariant under φ, that is, $\varphi(t, L) \subset L$. Then, by definition, taking a function $f \in J_L$, we see that $f \circ \varphi(t, \cdot) \in J_L$, showing that

$$T(t)J_L \subset J_L.$$

For the other direction, let us assume that $T(t)J_L \subset J_L$ and $x \in K \setminus L$. By Urysohn's lemma (see Lemma A.19), there is $f \in J_L$ such that $f(x) = 1$. On the other hand, since $T(t)J_L \subset J_L$, we see that

$$f(\varphi(t, y)) = 0 \quad \text{for all } y \in L.$$

This implies that $x \notin \varphi(t, L)$. Since $x \in K \setminus L$ was arbitrary, the invariance $\varphi(t, L) \subset L$ follows. □

We need some standard notions from topological dynamical systems.

A semiflow is called *minimal* if it has no nontrivial closed invariant sets. In view of the characterization of ideals in $\mathrm{C}(K)$ in Proposition 10.13, the following is a straightforward consequence of Lemma 16.13.

Corollary 16.14. *The semiflow φ is minimal if and only if the Koopman semigroup $(T(t))_{t\geq 0}$ is irreducible.*

The semiflow is called *topologically* (*forward*) *transitive* if there is a point $x \in K$ such that its orbit $\mathrm{orb}\,(x) := \{\varphi(t, x) : t \geq 0\}$ is dense in K.

Proposition 16.15. *If the semiflow is topologically transitive, then the generator A of the Koopman semigroup satisfies*

$$\dim \ker(A) = 1.$$

Proof. If $f \in \mathrm{C}(K)$ is a constant function, then $T(t)f = f$, hence $f \in \ker(A)$, meaning that $\dim \ker(A) \geq 1$ is true for every Koopman semigroup.

Further, if $T(t)f = f$ holds for $f \in \mathrm{C}(K)$, then $f(\varphi(t, x)) = T(t)f(x) = f(x)$ implies that f is constant along orbits. Hence, if there is a point $x \in K$ with dense orbit, then f has to be constant, implying that $\ker(A)$ consists of the constant functions. □

Definition 16.16. We say that a positive measure $\mu \in \mathrm{C}(K)^*$ is an *invariant Borel measure* for the semiflow φ if

$$\mu(\varphi^{-1}(t, H)) = \mu(H)$$

holds for each $t \geq 0$ and for each Borel measurable set $H \subset K$.

Interesting and important is the connection between invariant measures and the adjoint of the Koopman semigroup.

Lemma 16.17. *A measure $\mu \in \mathrm{C}(K)^*$ is an invariant probability measure for the continuous semiflow φ if and only if it is an eigenvector associated to the eigenvalue* 1 *for the adjoint of the corresponding Koopman semigroup, i.e., if and only if*

$$T^*(t)\mu = \mu$$

holds for all $t \geq 0$.

Proof. Note that, by the definition of the Koopman semigroup, a measure μ is invariant if and only if

$$\begin{aligned}\langle f, T^*(t)\mu\rangle = \langle T(t)f, \mu\rangle &= \int_K f(\varphi(t,x))\mathrm{d}\mu(x)\\ &= \int_K f(x)\mathrm{d}\mu(\varphi^{-1}(t,x)) = \int_K f(x)\mathrm{d}\mu(x) = \langle f, \mu\rangle\end{aligned}$$

holds for all $f \in \mathrm{C}(K)$ and $t \geq 0$, and hence

$$T^*(t)\mu = \mu$$

for all $t \geq 0$. ☐

It turns out that continuous flows always have an invariant measure.

Theorem 16.18 (Krylov–Bogoliubov)**.** *Let K be a compact Hausdorff space and $\varphi : \mathbb{R}^+ \times K \to K$ a continuous semiflow. Then there is at least one invariant probability measure for the semiflow φ.*

Proof. Fix $y \in K$ and let $\mu_0 := \delta_y$ be the Dirac measure supported at y. Define the probability measures μ_t for which

$$\int_K f(x)\mathrm{d}\mu_t(x) = \frac{1}{t}\int_0^t \int_K T(t)f(x)\mathrm{d}\mu_0(x)\mathrm{d}t$$

holds for all $f \in \mathrm{C}(K)$. This means that

$$\mu_t = \frac{1}{t}\int_0^t T^*(t)\mu_0 \mathrm{d}t,$$

where the integral is defined in the weak*-topology. By the Banach–Alaoglu theorem, Theorem A.31, there is a weak*-accumulation point μ of μ_t as $t \to \infty$. Since μ_t are positive probability measures, so is μ. By Exercise 5, this measure satisfies

$$T^*(t)\mu = \mu,$$

hence it is an invariant measure. □

We can apply the theory of irreducible semigroups here.

Proposition 16.19. *If the semiflow φ is minimal, then all invariant probability measures are strictly positive.*

Proof. This is a straightforward consequence of Proposition 14.12. □

Suppose again that K is a compact Hausdorff space and $\varphi : \mathbb{R}^+ \times K \to K$ a continuous semiflow, and let μ be an invariant measure for φ.

For each $t > 0$ and $f \in \mathrm{L}^1(K, \mu)$, we define the measure

$$\nu_t(H) := \int_{\varphi^{-1}(t,H)} f \mathrm{d}\mu.$$

Notice that if for $B \subset K$ we have $\mu(B) = 0$, then $\nu_t(B) = 0$. Hence, by the Radon–Nikodým theorem (see Theorem A.25), there is a unique $P(t)f \in \mathrm{L}^1(K, \mu)$ such that

$$\int_H P(t)f \mathrm{d}\mu = \int_{\varphi^{-1}(t,H)} f \mathrm{d}\mu.$$

Definition 16.20. The operator family $(P(t))_{t\geq 0}$ defined above is called the *Perron–Frobenius semigroup* associated with the semiflow φ.

We summarize the main properties of the Perron–Frobenius semigroup.

Proposition 16.21. *The Perron–Frobenius semigroup is a strongly continuous positive contraction semigroup on $\mathrm{L}^1(K, \mu)$. It can be identified with the restriction of the adjoint of the Koopman semigroup when $\mathrm{L}^1(K, \mu)$ is identified with the set of absolutely continuous measures with respect to μ.*

Note that, since the Perron–Frobenius semigroup is a restriction of the adjoint of the Koopman semigroup to an invariant subspace, the only thing to prove is its strong continuity. Then one shows directly that the Perron–Frobenius semigroup is weakly continuous. The proof is finished by applying the fact that weakly continuous semigroups are already strongly continuous.

16.4 Notes and Remarks

A standard reference for ordinary differential equations is Amann [3]. The generator A is sometimes referred to as the "Lie generator" of the semiflow φ, see for example Neuberger [104]. The contents of Proposition 16.2 and the boundary conditions appearing in Remark 16.3 are due to Ulmet [144]. For the fact that a dense and invariant subset of the domain of the generator is a core in the proof of Proposition 16.2, see Engel and Nagel [43, Proposition II.1.7].

The operator theoretic characterization of Koopman semigroups as Markov lattice homomorphism and the Kato equality are due to Nagel, Arendt, and the research group in Tübingen, and is documented in Nagel (ed.) [101, Sections B-II.2,3]. Relation (16.6) has its origins in Kato's investigations on the positivity of Schrödinger semigroups in L^2, see [72].

For the proof of Proposition 16.21 the main technical tool is the fact that weakly continuous semigroups are already strongly continuous, see Engel and Nagel, [43, Theorem I.5.8] and Exercise 6.

Basic properties of Perron-Frobenius and Koopman semigroups can be also found in the book by Lasota and Mackey [83]. A characterization of generators of Koopman semigroups in L^p spaces can be found in Edeko and Kühner [38].

Applications of Koopman semigroups to ergodic theory are numerous. The connection of nonlinear dynamical systems and their "linearization" using the Koopman operator has a long history, and in these notes here we only scratched the surface. For a comprehensive treatment of the time discrete case we refer to the monograph by Eisner, Farkas, Haase and Nagel [40].

16.5 Exercises

1. Let $F : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $\sup_{x\in\mathbb{R}} |F'(x)| < \infty$. Define the flow $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as the solution of the nonlinear ODE

$$\begin{cases} \dot{y}(t) = F(y(t)), \\ y(0) = s, \end{cases}$$

i.e., $\varphi(t, s) := y(t)$. Take $E := \mathrm{C}_0(\mathbb{R})$ and define

$$\big(T(t)f\big)(s) := f\big(\varphi(t, s)\big)$$

for $t \geq 0$, $s \in \mathbb{R}$.

   a) Show that $(T(t))_{t\geq 0}$ is a positive contraction semigroup (i.e., of type $(1, 0)$) and identify its generator.
   b) What is the corresponding abstract Cauchy problem? Which partial differential equation can we associate with it? Relate the semigroup $(T(t))_{t\geq 0}$ to the method of characteristics.

2. Consider the ordinary differential equation

$$\begin{cases} \dot{y}(t) = y(t)(y(t) - 1), \\ y(0) = s \in (0, 1), \end{cases}$$

Write down the explicit formula for the corresponding Koopman semigroup and its generator. Identify all invariant ideals of this semigroup.

3. Show that $z \mapsto |z|$ is right Gâteaux differentiable as stated in Example 16.9.

4. Let $\Omega \subset \mathbb{R}^n$ be an open set, φ a continuous semiflow on it, and define the Koopman semigroup on $\mathrm{L}^2(\Omega)$ by the rule (16.3).
   a) Show that it consists of unitary operators.
   b) Show that it is strongly continuous. [Hint: Use that it is continuous for $f \in \mathrm{C}_0(\Omega)$.]

5. Consider the adjoint $(T^*(t))_{t\geq 0}$ of the Koopman semigroup $(T(t))_{t\geq 0}$ and show that

$$(I - T^*(t))\frac{1}{r}\int_0^r T^*(s)\nu \mathrm{d}s = (I - T^*(r))\frac{1}{r}\int_0^t T^*(s)\nu \mathrm{d}s$$

for $r > t$. Use this identity to show that each weak*-accumulation point of the Cesàro-means in the proof of Theorem 16.18 is a fixed point of the adjoint semigroup.

6. Show that the Perron–Frobenius semigroup is weakly continuous.