Chapter 14

Advanced Spectral Theory and Asymptotics

In this chapter we continue our investigation of spectral properties of positive C_0-semigroups on Banach lattices and show how the Perron–Frobenius theory can be generalized to the infinite-dimensional setting. We also list some important properties of irreducible semigroups. We will see that many results valid for positive matrix semigroups continue to hold also in infinite dimensions.

Our main goal is to describe the asymptotic behavior of a semigroup (such as asymptotic periodicity or balanced exponential growth) via the spectral properties of its generator.

14.1 Spectral Decomposition

First we define and discuss spectral projections and spectral decompositions for an unbounded closed operator. Recall that in finite dimensions we have constructed a functional calculus using spectral projections corresponding to the eigenvalues (cf. Theorem 2.11). As already mentioned in Section 2.4, these projections can be obtained by Dunford's integral representation. We now start with such a representation in the case of bounded operators.

Let $T \in \mathcal{L}(X)$, where X is a Banach space. For a function f holomorphic on a neighborhood of $\overline{W}$ for some open neighborhood W of $\sigma(T)$ with a smooth, positively oriented boundary ∂W^+, we define

$$f(T) := \frac{1}{2\pi \mathrm{i}} \int_{\partial W^+} f(\lambda) R(\lambda, T) \, \mathrm{d}\lambda.$$

As in the finite-dimensional situation, the map $f \mapsto f(T)$ is linear and multiplicative, and for $g(z) := z^k$, $k \in \mathbb{N}$, one obtains $g(T) = T^k$.

Assume now that the spectrum $\sigma(T)$ can be decomposed as

$$\sigma(T) = \sigma_1 \cup \sigma_2, \tag{14.1}$$

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where σ_1, σ_2 are closed and disjoint sets. The *spectral projection* P_i of T belonging to σ_i is defined to be $\chi_i(T)$, where χ_i is the characteristic function of a neighborhood W_i of σ_i such that $\overline{W_i} \cap \sigma(T) = \sigma_i$ (compare with relation (2.5) in the finite-dimensional case). Hence, P_i can be written as

$$P_i := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_i} R(\lambda, T)\, \mathrm{d}\lambda, \tag{14.2}$$

where γ_i is a smooth curve in $\rho(T)$ enclosing σ_i. These projections commute with T and yield the *spectral decomposition*

$$X = X_1 \oplus X_2$$

with the T-invariant spaces $X_1 := \operatorname{im} P_1 = \ker P_2$, $X_2 := \operatorname{im} P_2 = \ker P_1$. The restrictions $T_i \in \mathcal{L}(X_i)$ of T to X_i satisfy

$$\sigma(T_i) = \sigma_i, \quad i = 1, 2,$$

a property that characterizes the above decomposition of X and T (again recall corresponding results in finite dimensions, e.g., Theorem 2.9).

For an unbounded operator A and an arbitrary decomposition of the spectrum $\sigma(A)$ into disjoint closed sets, it is not always possible to find an associated spectral decomposition. However, the spectral mapping theorem for the resolvent allows us to construct such decompositions if one of the subsets is compact.

Proposition 14.1. *Let $A : D(A) \subset X \to X$ be a closed operator such that its spectrum $\sigma(A)$ can be decomposed into the disjoint union of two closed subsets σ_{c} and σ_{u}, i.e.,*

$$\sigma(A) = \sigma_{\mathrm{c}} \cup \sigma_{\mathrm{u}}.$$

If σ_{c} is compact, then there exists a spectral decomposition $X = X_{\mathrm{c}} \oplus X_{\mathrm{u}}$ *for A in the following sense.*

a) *The restriction $A_{\mathrm{c}} := A|_{X_{\mathrm{c}}}$ is bounded on the Banach space X_{c}.*
b) *$D(A) = X_{\mathrm{c}} \oplus D(A_{\mathrm{u}})$, where A_{u} is the* part *of A in X_{u}, i.e., $A_{\mathrm{u}} := A|_{X_{\mathrm{u}}}$, $A_{\mathrm{u}} f := Af$ for $f \in D(A_{\mathrm{u}}) := \{g \in X_{\mathrm{u}} \cap D(A) : Ag \in X_{\mathrm{u}}\}$.*
c) *The operator A decomposes as $A = A_{\mathrm{c}} \oplus A_{\mathrm{u}}$.*
d) *$\sigma(A_{\mathrm{c}}) = \sigma_{\mathrm{c}}$ and $\sigma(A_{\mathrm{u}}) = \sigma_{\mathrm{u}}$.*

Proof. Supposing that A is unbounded and taking $\lambda_0 \in \rho(A)$, we see that $0 \in \sigma(R(\lambda_0, A))$ and, by Proposition 9.29, we obtain

$$\sigma\big(R(\lambda_0, A)\big) = \underbrace{\left\{\frac{1}{\lambda_0 - \mu} : \mu \in \sigma_c\right\}}_{\tau_{\mathrm{c}}} \cup \underbrace{\left\{\frac{1}{\lambda_0 - \mu} : \mu \in \sigma_u\right\} \cup \{0\}}_{\tau_{\mathrm{u}}}, \tag{14.3}$$

where τ_{c}, τ_{u} are compact and disjoint subsets of $\mathbb{C}$. (If σ_{c} is not compact, 0 is in the closure of τ_{c}.) Let now P be the spectral projection for $R(\lambda_0, A)$ associated to the decomposition in (14.3) and put $X_{\mathrm{c}} := \operatorname{im} P$ and $X_{\mathrm{u}} := \ker P$. Since $R(\lambda_0, A)$ and P commute, we have $R(\lambda_0, A)X_c \subseteq X_c$, hence

$$\lambda_0 \in \rho(A_{\mathrm{c}}) \qquad \text{and} \qquad R(\lambda_0, A_{\mathrm{c}}) = R(\lambda_0, A)|_{X_{\mathrm{c}}}. \tag{14.4}$$

Moreover, we know that $\sigma(R(\lambda_0, A_{\mathrm{c}})) = \tau_{\mathrm{c}} \not\ni 0$. Therefore, $A_{\mathrm{c}} = \lambda_0 - R(\lambda_0, A_{\mathrm{c}})^{-1}$ is bounded on X_{c}, and we obtain a).

To verify b), observe that by similar arguments as above we obtain

$$\lambda_0 \in \rho(A_{\mathrm{u}}) \quad \text{and} \quad \mathrm{R}(\lambda_0, A_{\mathrm{u}}) = R(\lambda_0, A)|_{X_{\mathrm{u}}}. \tag{14.5}$$

Combining this with (14.4) yields

$$\begin{aligned} X_{\mathrm{c}} + D(A_{\mathrm{u}}) &= R(\lambda_0, A_{\mathrm{c}})X_{\mathrm{c}} + R(\lambda_0, A_{\mathrm{u}})X_{\mathrm{u}} \\ &\subseteq D(A) = R(\lambda_0, A)(X_{\mathrm{c}} + X_{\mathrm{u}}) \\ &\subseteq R(\lambda_0, A_{\mathrm{c}})X_{\mathrm{c}} + R(\lambda_0, A_{\mathrm{u}})X_{\mathrm{u}} \\ &= X_{\mathrm{c}} + D(A_{\mathrm{u}}), \end{aligned}$$

implying that $D(A) = X_{\mathrm{c}} + D(A_{\mathrm{u}})$. This proves b), while c) follows from a) and b).

Finally, d) is a consequence of Proposition 9.29 and (14.3), (14.4), and (14.5). □

A particularly important case of the above decomposition occurs when $\sigma_c = \{\mu\}$ consists of a single point. This means that μ is isolated in $\sigma(A)$ and therefore the holomorphic function $\rho(A) \ni \lambda \mapsto R(\lambda, A) \in \mathcal{L}(X)$ can be expanded in a Laurent series

$$R(\lambda, A) = \sum_{k=-\infty}^{\infty} (\lambda - \mu)^k U_k$$

for $0 < |\lambda - \mu| < \delta$ and some sufficiently small $\delta > 0$. The coefficients U_k of this series are bounded operators given by the formulas

$$U_k = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{R(\lambda, A)}{(\lambda - \mu)^{k+1}} \,\mathrm{d}\lambda, \quad k \in \mathbb{Z}, \tag{14.6}$$

where γ is, for example, the positively oriented boundary of the disc with radius $\frac{\delta}{2}$ centered at μ. The coefficient U_{-1} is called the *residue* of $R(\cdot, A)$ at μ. From formula (14.6) one deduces

$$U_{k+1} = (A - \mu)^k U_{-1} \tag{14.7}$$

and the identity

$$U_{-(k+1)} \cdot U_{-(\ell+1)} = U_{-(k+\ell+1)} \tag{14.8}$$

for $k, \ell \geq 0$. Indeed,

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int_\gamma (\lambda-\mu)^k(\lambda-\mu)^\ell R(\lambda,A)\,\mathrm{d}\lambda\\ &\qquad = \left(\frac{1}{2\pi\mathrm{i}}\int_\gamma (\lambda-\mu)^k R(\lambda,A)\,\mathrm{d}\lambda\right)\cdot\left(\frac{1}{2\pi\mathrm{i}}\int_\gamma (\lambda-\mu)^\ell R(\lambda,A)\,\mathrm{d}\lambda\right)\end{aligned}$$

can be proved as in the case of a bounded operator A since the proof only uses the resolvent equation and the residue theorem.

If there exists $k > 0$ such that $U_{-k} \neq 0$, while $U_{-\ell} = 0$ for all $\ell > k$, then the spectral value μ is called a *pole of* $R(\cdot, A)$ *of order* k (compare with Remark 2.17). In view of (14.8), this is true if and only if $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. Moreover, we obtain U_{-k} as

$$U_{-k} = \lim_{\lambda\to\mu} (\lambda-\mu)^k R(\lambda, A). \tag{14.9}$$

The dimension of the spectral subspace $\operatorname{im} P$ is called the *algebraic multiplicity* m_{a} of μ, while $m_{\mathrm{g}} := \dim\ker(\mu - A)$ is its *geometric multiplicity*. One can show that the following relation holds:

$$m_{\mathrm{g}} + k - 1 \leq m_{\mathrm{a}} \leq m_{\mathrm{g}} \cdot k. \tag{14.10}$$

In the case $m_{\mathrm{a}} = 1$, we call μ an *algebraically simple pole*. We also denote by

$$\operatorname{Pol}(A) := \{\mu \in \mathbb{C}\colon \mu \text{ is a pole of } R(\cdot, A)\}. \tag{14.11}$$

The following result shows that, as in the case of a bounded operator, the spectral projection of A belonging to an isolated point $\mu \in \sigma(A)$ is the residue of $R(\cdot, A)$ at μ.

Proposition 14.2. *Let A be a closed linear operator having nonempty resolvent set $\rho(A)$ and take some $\lambda_0 \in \rho(A)$. Then $\mu \in \mathbb{C}$ is an isolated point of $\sigma(A)$ if and only if $(\lambda_0 - \mu)^{-1}$ is isolated in $\sigma(R(\lambda_0, A))$. In this case, the residues and the orders of the poles of $R(\cdot, A)$ at μ and of $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0 - \mu)^{-1}$ coincide.*

Proof. The first claim follows easily from Proposition 9.29 and the fact that the map $z \mapsto (\lambda_0 - z)^{-1}$ is homeomorphic between $\mathbb{C} \setminus \{\lambda_0\}$ and $\mathbb{C} \setminus \{0\}$.

In order to prove the assertion concerning the residues, we choose a positively oriented circle $\gamma \subset \rho(A)$ with center μ such that λ_0 lies in the exterior of γ. Then the residue P of $R(\cdot, A)$ at μ is given by

$$\begin{aligned}P &= \frac{1}{2\pi\mathrm{i}}\int_\gamma R(\lambda,A)\,\mathrm{d}\lambda\\ &= \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{R\big((\lambda_0-\lambda)^{-1}, R(\lambda_0,A)\big)}{(\lambda_0-\lambda)^2}\,\mathrm{d}\lambda - \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}\lambda}{(\lambda_0-\lambda)}\\ &= \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{R\big((\lambda_0-\lambda)^{-1}, R(\lambda_0,A)\big)}{(\lambda_0-\lambda)^2}\,\mathrm{d}\lambda,\end{aligned}$$

where we used the identity

$$R(\lambda, A) = \frac{R\big((\lambda_0-\lambda)^{-1}, R(\lambda_0, A)\big)}{(\lambda_0-\lambda)^2} - \frac{1}{(\lambda_0-\lambda)}$$

and Cauchy's integral theorem. The substitution $z := (\lambda_0-\lambda)^{-1}$ then yields a path $\tilde{\gamma}$ around $(\lambda_0-\mu)^{-1}$, and we obtain

$$P = \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\gamma}} R\big(z, R(\lambda_0, A)\big)\, \mathrm{d}z,$$

which is the residue of $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0-\mu)^{-1}$.

The final assertion concerning the pole orders is obtained as follows. By the same calculations as above we see that for $k \in \mathbb{N}$

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} (\lambda-\mu)^{k-1} R(\lambda, A)\, \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\gamma}} \left(\lambda_0 - \mu - \frac{1}{z}\right)^{k-1} R\big(z, R(\lambda_0, A)\big)\, \mathrm{d}z.$$

Since $\lambda_0 - \mu - \frac{1}{z} = \big(\frac{\lambda_0-\mu}{z}\big)\big(z - \frac{1}{\lambda_0-\mu}\big)$, by the multiplicativity of the functional calculus for $R(\lambda_0, A)$ the last integral can be interpreted as

$$\big((\lambda_0-\mu)(\lambda_0-A)\big)^{k-1} V_{-k},$$

where V_{-k} denotes the $-k$th coefficient in the Laurent expansion of $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0-\mu)^{-1}$. Hence, we obtain for the coefficients U_{-k} of the Laurent expansion of $R(\cdot, A)$ at μ

$$U_{-k} = \big((\lambda_0-\mu)(\lambda_0-A)\big)^{k-1} V_{-k}, \qquad k \in \mathbb{N},$$

and therefore

$$V_{-k} = \big((\lambda_0-\mu)^{-1} R(\lambda_0, A)\big)^{k-1} U_{-k}, \quad k \in \mathbb{N},$$

which proves the assertion. □

We continue by further refining the spectral decomposition. First recall that the *essential spectrum* of a bounded operator $T \in \mathcal{L}(X)$ is the spectrum of $T + \mathcal{K}(X)$ in the Calkin algebra $\mathcal{C}(X) := \mathcal{L}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the ideal of compact operators. Accordingly, the *essential spectral radius* is

$$\mathrm{r}_{\mathrm{ess}}(S) := \mathrm{r}(S + \mathcal{K}(X)),$$

see also Appendix A.9.

Analogously, we define the *essential growth bound* $\omega_{\mathrm{ess}}(T)$ of a C_0-semigroup $(T(t))_{t\geq 0}$ as the growth bound of the quotient semigroup $(T(t) + \mathcal{K}(X))_{t\geq 0}$ on $\mathcal{C}(X)$, i.e.,

$$\omega_{\mathrm{ess}}(T) := \inf\{\omega \in \mathbb{R} : \exists M > 0 \,\text{such that}\, \|T(t)\|_{\mathrm{ess}} \leq M\mathrm{e}^{\omega t} \text{ for all } t \geq 0\},$$

where $\|\cdot\|_{\mathrm{ess}}$ is the quotient norm in $\mathcal{C}(X)$. Then, as in Proposition 12.1, one can see that

$$\omega_{\mathrm{ess}}(T) = \frac{\log \mathrm{r}_{\mathrm{ess}}(T(t_0))}{t_0} = \lim_{t\to\infty} \frac{\log \|T(t)\|_{\mathrm{ess}}}{t} \tag{14.12}$$

holds for all $t_0 > 0$. The following result gives the relationship between $\omega_{\mathrm{ess}}(T)$ and $\omega_0(T)$.

Proposition 14.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator A on a Banach space X. Then*

$$\omega_0(T) = \max\{\mathrm{s}(A), \omega_{\mathrm{ess}}(T)\}.$$

Proof. If $\omega_{\mathrm{ess}}(T) < \omega_0(T)$, then $\mathrm{r}_{\mathrm{ess}}(T(1)) < \mathrm{r}(T(1))$. Let $\lambda \in \sigma(T(1))$ such that $|\lambda| = \mathrm{r}(T(1))$. Then by Proposition A.34, λ is an eigenvalue of $T(1)$ and by the spectral mapping theorem for the point spectrum, Theorem A.33, there is a $\lambda_1 \in \sigma_{\mathrm{p}}(A)$ with $\mathrm{e}^{\lambda_1} = \lambda$. Therefore, $\mathrm{Re}\,\lambda_1 = \omega_0(T)$, and thus $\omega_0(T) = \mathrm{s}(A)$. □

We are finally able to give an infinite-dimensional analogue of the formula for the matrix exponential function given in (2.9). As in finite dimensions, this will be an important tool to study the asymptotic behavior of the semigroup.

Theorem 14.4. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X such that $\omega_{\mathrm{ess}}(T) < 0$. Then the following assertions hold.*

a) *The set $\sigma_+ := \{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda \geq 0\}$ is finite (or empty) and consists of poles of $R(\cdot, A)$ of finite algebraic multiplicity.*
b) *Let $\sigma_+ := \{\lambda_1, \ldots, \lambda_m\}$ where λ_j is a pole of order k_j with the corresponding spectral projection P_j, $j = 1, \ldots, m$. Then $T(t) = T_1(t) + \cdots + T_m(t) + R(t)$, where*

$$T_j(t) := \mathrm{e}^{\lambda_j t} \sum_{k=0}^{k_j - 1} \frac{t^k}{k!}(A - \lambda_j)^k P_j, \quad j = 1, \ldots, m, \ \textit{and } t \geq 0,$$

and

$$\|R(t)\| \leq M\mathrm{e}^{-\varepsilon t}, \quad \textit{for some } \varepsilon > 0,\ M \geq 1,\ \textit{and all } t \geq 0.$$

Proof. a) Let $t_0 > 0$. Since $\omega_{\mathrm{ess}}(T) < 0$, (14.12) shows that $\mathrm{r}_{\mathrm{ess}}(T(t_0)) < 1$. So, by Proposition A.34, every $\lambda \in \sigma(T(t_0))$ with $|\lambda| \geq 1$ is an isolated point. The set

$$\sigma_{\mathrm{c}} := \sigma(T(t_0)) \cap \{z \in \mathbb{C} : |z| \geq 1\}$$

is thus finite and consists of the points $\{\lambda_1, \ldots, \lambda_m\}$.

Set $\sigma_u := \sigma(T(t_0)) \setminus \sigma_c$. Then $\sigma(T(t_0))$ is the disjoint union of the closed sets σ_{c} and σ_{u} with σ_{c} compact, and we can apply Proposition 14.1, yielding the spectral decomposition

$$X = \operatorname{im} P_{\mathrm{c}} \oplus \ker P_{\mathrm{c}} =: X_{\mathrm{c}} \oplus X_{\mathrm{u}}$$

with the associated spectral projection P_c. Since σ_c is finite and any of its elements is a pole of $R(\cdot, T(t_0))$, we deduce that X_c is finite-dimensional. To this decomposition we associate semigroups $T_c(\cdot) := T(\cdot)|_{X_c}$ and $T_u(\cdot) := T(\cdot)|_{X_u}$, and the corresponding generators are, respectively, $A_c := A|_{X_c} \in \mathcal{L}(X_c)$ and A_u the part of A in X_u. Moreover, $\sigma(A_c) = \sigma_c$ and $\sigma(A_u) = \sigma_u$.

Since X_c is finite-dimensional, $\sigma(A_c)$ is finite and $A = A_c \oplus A_u$. Moreover, every element of σ_c is a pole of $R(\cdot, A) = R(\cdot, A_c) \oplus R(\cdot, A_u)$. Thus, the spectral mapping theorem (see Theorems 2.20 and 2.28) yields

$$\sigma(A_c) = \{\lambda_1, \ldots, \lambda_m\} \quad \text{and} \quad \sigma(T_c(t)) = \{e^{\lambda_1 t}, \ldots, e^{\lambda_m t}\}.$$

In particular,

$$\sigma_c = \sigma(T_c(t_0)) \subset \{z \in \mathbb{C} : |z| \geq 1\},$$

and hence $\operatorname{Re} \lambda_j \geq 0$ for all $j = 1, \ldots, m$.

Next, we show that $(T_u(t))_{t \geq 0}$ is uniformly exponentially stable. By the spectral decomposition, we know that $\sigma(T_u(t_0)) = \sigma_u \subset \{z \in \mathbb{C} : |z| < 1\}$. So, $\mathrm{r}(T_u(t_0)) < 1$ and by Proposition 12.1 we obtain $\omega_0(T_u) < 0$, which also implies $\mathrm{s}(A_u) < 0$. This proves a).

b) In order to verify this, we define the spectral projection $P := \sum_{j=1}^m P_j$ of A corresponding to the spectral set $\{\lambda_1, \ldots, \lambda_m\}$, i.e., $P = P_c$. We decompose now the semigroup $(T(t))_{t \geq 0}$ as

$$T(t) = T(t)P_1 + \cdots + T(t)P_m + T(t)(I - P),$$

where each restricted semigroup $T(\cdot)P_j$ has generator $A|_{\operatorname{im} P_j}$. Since $\operatorname{im} P_j$ is finite-dimensional and $(A - \lambda_j)^{k_j} P_j = 0$, we can use Theorem 2.11 and get

$$T_j(t) := T(t)P_j = e^{\lambda_j t} \sum_{k=0}^{k_j - 1} \frac{t^k}{k!} (A - \lambda_j)^k P_j, \quad t \geq 0.$$

To show the last assertion, it suffices to note that

$$R(t) = T(t)(I - P) = T_u(t)(I - P_c) = T_u(t)$$

and $\omega_0(T_u) < 0$. This ends the proof of the theorem. □

Semigroups satisfying $\omega_{\mathrm{ess}}(T) < 0$ are also called *quasi-compact* semigroups (for an explanation of this name see Exercise 1). They include uniformly exponentially stable semigroups and eventually compact semigroups.

14.2 Periodic Semigroups

In this section we characterize periodic semigroups in terms of their spectrum. As we shall see later in this chapter, this class of semigroups plays an important role for the asymptotics of general semigroups.

Definition 14.5. A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called *periodic* if there is $t_0 > 0$ such that $T(t_0) = I$. In this case its *period* is defined as the smallest $\tau > 0$ such that $T(\tau) = I$.

Since, for every $k \in \mathbb{N}$ and $0 \leq t \leq k\tau$, $T(t)T(k\tau - t) = I$, we readily see that a periodic semigroup always extends to a group.

As in finite dimensions, we can characterize periodic semigroups in terms of their spectrum, compare with Theorem 4.12.c) in the finite-dimensional situation.

Theorem 14.6. *For a C_0-semigroup $(T(t))_{t\geq 0}$ with generator A on a Banach space X the following assertions are equivalent.*

(i) *$(T(t))_{t\geq 0}$ is a periodic semigroup.*

(ii) *$\sigma(A) = \sigma_{\mathrm{p}}(A) \subset 2\pi\mathrm{i}\alpha\mathbb{Z}$ for some $\alpha > 0$ and the corresponding eigenvectors span a dense subspace of X.*

Proof. (ii) $\Longrightarrow$ (i): First observe that for any $\lambda \in \sigma_{\mathrm{p}}(A)$ and a corresponding eigenvector $f \in D(A)$, Corollary 9.32 yields

$$T(t)f = \mathrm{e}^{\lambda t} f, \quad t \geq 0. \tag{14.13}$$

Thus, taking $\lambda = 2\pi\mathrm{i}k\alpha \in \sigma_{\mathrm{p}}(A)$ we deduce that $T(t)f = \mathrm{e}^{2\pi\mathrm{i}k\alpha t} f$ for all $t \geq 0$. Since these eigenvectors span a dense subspace of X, we obtain that $(T(t))_{t\geq 0}$ is periodic with period $\tau \leq \frac{1}{\alpha}$.

(i) $\Longrightarrow$ (ii): Let τ be the period of $(T(t))_{t\geq 0}$ and $\lambda \neq \frac{2\pi k\mathrm{i}}{\tau}$, $k \in \mathbb{Z}$. From Lemma 9.31 we infer that $\lambda \in \rho(A)$ and

$$R(\lambda, A) = \frac{1}{1 - \mathrm{e}^{-\lambda\tau}} \int_0^\tau \mathrm{e}^{-\lambda s} T(s)\ \mathrm{d}s. \tag{14.14}$$

So the resolvent is a meromorphic function having poles only at (some) $\mu_k = \frac{2\pi k\mathrm{i}}{\tau}$, $k \in \mathbb{Z}$, of order less than or equal to one. Using formula (14.14) and the residue theorem one obtains the residues in μ_k as

$$P_k = \frac{1}{\tau} \int_0^\tau \mathrm{e}^{-\mu_k s} T(s)\ \mathrm{d}s, \quad k \in \mathbb{Z}. \tag{14.15}$$

Now we prove that $\overline{\mathrm{span}} \bigcup_{k\in\mathbb{Z}} P_k X = X$. More precisely, we prove that

$$f = \sum_{k=-\infty}^{+\infty} P_k f \quad \text{for all } f \in D(A), \tag{14.16}$$

which clearly implies the assertion, since $D(A)$ is dense in X.

Setting $g = Af$, we have $P_k g = P_k A f = \frac{2\pi k\mathrm{i}}{\tau} P_k f$. This implies $P_k f = \frac{\tau}{2\pi k\mathrm{i}} P_k g$. Hence, by applying the Cauchy–Schwarz inequality (A.3), we obtain

$$\begin{aligned}
\left|\sum_{k\in F}\langle P_k f, f^*\rangle\right| &= \left|\sum_{k\in F}\frac{\tau}{2\pi k\mathrm{i}}\langle P_k g, f^*\rangle\right| \\
&\le \frac{\tau}{2\pi}\left(\sum_{k\in F}\frac{1}{k^2}\right)^{1/2}\left(\sum_{k\in F}|\langle P_k g, f^*\rangle|^2\right)^{1/2} \\
&\le \frac{\tau}{2\pi}\left(\sum_{k\in F}\frac{1}{k^2}\right)^{1/2}\left(\frac{1}{\tau}\int_0^\tau |\langle T(s)g, f^*\rangle|^2 \,\mathrm{d}s\right)^{1/2} \\
&\le \frac{\tau}{2\pi}\left(\sum_{k\in F}\frac{1}{k^2}\right)^{1/2}\|f^*\|\underbrace{\left(\frac{1}{\tau}\int_0^\tau \|T(s)g\|^2 \,\mathrm{d}s\right)^{1/2}}_{C} \\
&= \frac{C\tau}{2\pi}\left(\sum_{k\in F}\frac{1}{k^2}\right)^{1/2}\|f^*\|
\end{aligned}$$

for any finite subset $F \subset \mathbb{Z}$. Thus,

$$\left\|\sum_{k\in F} P_k f\right\| \le \frac{C\tau}{2\pi}\left(\sum_{k\in F}\frac{1}{k^2}\right)^{1/2}$$

for any finite subset $F \subset \mathbb{Z}$. This gives the convergence of $\sum_{k\in\mathbb{Z}} P_k f$ for all $f \in D(A)$.

On the other hand, using the relation (14.13) with $\lambda = \mu_m$ and the corresponding eigenvector $P_m f$, we obtain

$$P_k P_m f = \frac{1}{\tau}\int_0^\tau \mathrm{e}^{-\mu_k s} T(s) P_m f \,\mathrm{d}s = \frac{1}{\tau}\int_0^\tau \mathrm{e}^{(\mu_m-\mu_k)s} P_m f \,\mathrm{d}s = 0,$$

if $k \neq m$. From this we see that, for any $f^* \in X^*$, the Fourier coefficients of the functions $s \mapsto \langle T(s)(\sum_{k\in\mathbb{Z}} P_k f), f^*\rangle$ and $s \mapsto \langle T(s)f, f^*\rangle$ coincide. So, the two functions are equal and, in particular,

$$\left\langle \sum_{k\in\mathbb{Z}} P_k f, f^* \right\rangle = \left\langle T(0)\left(\sum_{k\in\mathbb{Z}} P_k f\right), f^* \right\rangle = \langle T(0)f, f^*\rangle = \langle f, f^*\rangle .$$

This proves that $f = \sum_{k\in\mathbb{Z}} P_k f$. □

The calculations in the proof above yield the following expansion formula for a periodic semigroup and its generator.

Corollary 14.7. *Let $(T(t))_{t\geq 0}$ be a periodic C_0-semigroup with period τ and generator A on a Banach space X. Then*

$$T(t)f = \sum_{-\infty}^{+\infty} \mathrm{e}^{\mu_k t} P_k f \quad \textit{for } f \in D(A) \quad \textit{and}$$

$$Af = \sum_{-\infty}^{+\infty} \mu_k P_k f \quad \textit{for } f \in D(A^2),$$

where P_k are the residues of $R(\cdot, A)$ at $\mu_k := \frac{2\pi \mathrm{i} k}{\tau}$ given in (14.15).

Proof. One has to apply expansion (14.16) to $T(t)f$ and Af instead of f, respectively, and use the identities $AP_k = \mu_k P_k$ and $T(t)P_k = \mathrm{e}^{\mu_k t} P_k$, see (14.13). □

Example 14.8. Let $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle and $\tau > 0$. On $X := \mathrm{L}^p(\Gamma)$, $1 \leq p < \infty$, we define

$$R_\tau(t)f(z) := f\left(z\mathrm{e}^{(2\pi \mathrm{i}/\tau)t}\right), \quad z \in \Gamma, \, t \in \mathbb{R}.$$

Then $R_\tau(\cdot)$ defines a periodic C_0-group with period τ. Moreover, one can prove that its generator is given by

$$D(A) = \{f \in X : f \text{ absolutely continuous }, \, f' \in X\}$$

$$Af(z) = \frac{2\pi \mathrm{i}}{\tau} z f'(z), \quad f \in D(A),$$

with

$$\sigma(A) = \frac{2\pi \mathrm{i}}{\tau}\mathbb{Z} \quad \text{and} \quad P_k f(z) = \frac{z^k}{2\pi \mathrm{i}} \int_\Gamma f(u) u^{-(k+1)} \, \mathrm{d}u.$$

14.3 Irreducible Semigroups

We now return to positive semigroups. The concept of irreducibility of bounded operators on a Banach lattice was already introduced in Definition 10.26. Let us restate it for operators forming a C_0-semigroup.

Definition 14.9. A positive C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E is called *irreducible* if $\{0\}$ and E are the only closed ideals that are invariant under all the operators $T(t)$, $t \geq 0$.

The following result gives two properties equivalent to irreducibility that are sometimes easier to verify.

Proposition 14.10. *Let $(T(t))_{t\geq 0}$ be a positive C_0-semigroup on a Banach lattice E with generator A. The following assertions are equivalent.*

(i) *$(T(t))_{t\geq 0}$ is irreducible.*

(ii) *For some (and then for every)* $\lambda > \mathrm{s}(A)$, *there is no* $R(\lambda, A)$*-invariant closed ideal except* $\{0\}$ *and* E.

(iii) *For some (and then for every)* $\mu > \mathrm{s}(A)$ *and for every* $f > 0$, $R(\mu, A)f$ *is a quasi-interior point of* E_+.

Proof. We prove first that if for some $\mu > \mathrm{s}(A)$ there is no $R(\mu, A)$-invariant closed ideal except $\{0\}$ and E, then this holds for every $\lambda > \mathrm{s}(A)$. Let I be a closed ideal of E such that $R(\mu, A)I \subset I$ for $\mu > \mathrm{s}(A)$. The inequality

$$0 \leq R(\lambda, A) \leq R(\mu, A),$$

holding for all $\lambda \geq \mu$, and the definition of ideals imply that $R(\lambda, A)I \subset I$.

On the other hand, for $\mu - \frac{1}{\mathrm{r}(R(\mu,A))} < \lambda < \mu$, we have $R(\lambda, A)I \subset I$, since

$$R(\lambda, A) = R(\mu, A) \sum_{k=0}^{\infty} ((\mu - \lambda)R(\mu, A))^k$$

(see (9.10)) and $R(\mu, A)I \subset I$.

Iteration of the argument establishes that $R(\lambda, A)I \subset I$ for every $\lambda > \mathrm{s}(A)$. This proves the above claim.

(i) $\Longrightarrow$ (ii): Let $I \neq \{0\}$ be a closed ideal of E such that $R(\lambda, A)I \subset I$ for some (and then for every) $\lambda > \mathrm{s}(A)$. By the approximation formula (see the proof of Theorem 11.1),

$$T(t)f = \lim_{k \to \infty} \mathrm{e}^{tA_k} f, \quad f \in E,$$

where $A_k = kAR(k, A) \in \mathcal{L}(E)$ are the Yosida approximants, we obtain that $T(t)I \subset I$ for all $t > 0$, hence $I = E$.

(ii) $\Longrightarrow$ (i): This follows from Theorem 12.7.

(ii) $\Longrightarrow$ (iii): Let $\lambda > \mathrm{s}(A)$, $0 \neq f \in E_+$ and consider the ideal generated by $R(\lambda, A)f$, i.e.,

$$E_{R(\lambda,A)f} := \bigcup_{k \in \mathbb{N}} [-kR(\lambda, A)f, kR(\lambda, A)f].$$

We infer from the resolvent equation that for $g \in E_{R(\lambda,A)f}$,

$$|R(\mu, A)g| \leq R(\mu, A)|g| \leq kR(\mu, A)R(\lambda, A)f \leq \frac{k}{\mu - \lambda} R(\lambda, A)f$$

for $\mu > \lambda$. Hence, $R(\mu, A)g \in E_{R(\lambda,A)f}$ for any $g \in E_{R(\lambda,A)f}$ and $\mu > \lambda$. Thus, we see that $\overline{E_{R(\lambda,A)f}}$ is a nontrivial $R(\mu, A)$-invariant closed ideal and hence equals E. This means that $R(\lambda, A)f$ is a quasi-interior point of E_+.

(iii) $\Longrightarrow$ (ii): Let $I \neq \{0\}$ be an $R(\mu, A)$-invariant closed ideal for some $\mu > \mathrm{s}(A)$, and let $0 \neq f \in E_+ \cap I$. It follows that for any $g \in E_{R(\mu,A)f}$ we have $|g| \leq nR(\mu, A)f$ for some $n \in \mathbb{N}$ and hence $g \in I$. This implies that $E_{R(\mu,A)f} \subset I$ and, furthermore, $E = \overline{E_{R(\mu,A)f}} = I$. □

Example 14.11. From the characterization of closed ideals given in Propositions 10.13, 10.14, and 10.15 (see also Examples 10.16) we obtain the following characterization of irreducible semigroups in certain function spaces.

a) Let $E := \mathrm{L}^p(\Omega, \mu)$, $1 \leq p < \infty$, and let $(T(t))_{t\geq 0}$ be a positive C_0-semigroup on E with generator A. Then, $(T(t))_{t\geq 0}$ is irreducible if and only if

$$0 \lneqq f \in E \Longrightarrow (R(\lambda, A)f)(s) > 0 \text{ for a.e. } s \in \Omega \text{ and some } \lambda > \mathrm{s}(A).$$

b) Let $E := C_0(\Omega)$, where Ω is locally compact Hausdorff space, and let $(T(t))_{t\geq 0}$ be a positive C_0-semigroup on E with generator A. Then $(T(t))_{t\geq 0}$ is irreducible if and only if

$$0 \lneqq f \in E \Longrightarrow (R(\lambda, A)f)(s) > 0 \text{ for all } s \in \Omega \text{ and some } \lambda > \mathrm{s}(A).$$

We collect here some properties of irreducible C_0-semigroups. Many of them resemble properties already observed in finite dimensions. The most import one is a generalization of the Perron–Frobenius theorem, Theorem 5.13 (see also the same result for matrix semigroups given in Theorem 7.6).

Proposition 14.12. *Assume that A is the generator of an irreducible C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E. Then the following assertions hold.*

a) *Every positive eigenvector of A is a quasi-interior point.*
b) *Every positive eigenvector of A^* is strictly positive.*
c) *If* $\ker(\mathrm{s}(A) - A^*)$ *contains a positive element, then* $\dim \ker(\mathrm{s}(A) - A) \leq 1$.
d) *If* $\mathrm{s}(A)$ *is a pole of the resolvent, then it has algebraic (and geometric) multiplicity equal to* 1. *The corresponding residue has the form*

$$P_{\mathrm{s}(A)} = u^* \otimes f,$$

where $f \in E$ is a strictly positive eigenvector of A, $u^* \in E^*$ is a strictly positive eigenvector of A^*, and $\langle f, u^* \rangle = 1$.

Proof. a) Let f be a positive eigenvector of A and λ its corresponding eigenvalue. Since $\lambda f = Af = \lim_{t\to 0^+} \frac{1}{t}(T(t)f - f)$, we have $\lambda \in \mathbb{R}$. We also have

$$f = (\mu - \lambda)R(\mu, A)f \quad \text{ for } \mu > \mathrm{s}(A) > \lambda.$$

Thus a) follows from Proposition 14.10.

b) Let f^* be a positive eigenvector of A^* and λ its corresponding eigenvalue. By the same argument as above, $\lambda \in \mathbb{R}$ and, by Corollary 9.32, $T(t)^* f^* = \mathrm{e}^{\lambda t} f^*$ for $t \geq 0$. Hence,

$$\langle |T(t)u|, f^* \rangle \leq \langle T(t)|u|, f^* \rangle = \langle |u|, \mathrm{e}^{\lambda t} f^* \rangle, \quad u \in E,\, t \geq 0.$$

Thus $I := \{u \in E : \langle |u|, f^* \rangle = 0\}$ is a $(T(t))_{t\geq 0}$ invariant closed ideal. Since $f^* \neq 0$, we have $I \subsetneq E$, and so by irreducibility we obtain $I = \{0\}$. Therefore $f^* > 0$.

c) For $0 \lneqq f^* \in \ker(\mathrm{s}(A) - A^*)$ we see from b) that f^* is strictly positive. Assume that $\ker(\mathrm{s}(A) - A) \neq \{0\}$ and define the rescaled positive semigroup as

$$T_{-\mathrm{s}(A)}(t)g := \mathrm{e}^{-\mathrm{s}(A)t}T(t)g,$$

see also Exercise 9.10.4. Then for $f \in \ker(\mathrm{s}(A) - A)$ we have by Corollary 9.32 that $T_{-\mathrm{s}(A)}(t)f = f$ and hence, by Lemma 10.18,

$$|f| = |T_{-\mathrm{s}(A)}(t)f| \leq T_{-\mathrm{s}(A)}(t)|f|, \quad t \geq 0.$$

Thus, for $t \geq 0$,

$$\begin{aligned} \langle |f|, f^* \rangle &\leq \langle T_{-\mathrm{s}(A)}(t)|f|, f^* \rangle \\ &= \langle |f|, f^* \rangle. \end{aligned}$$

This implies that $\langle T_{-\mathrm{s}(A)}(t)|f| - |f|, f^* \rangle = 0$, and since $f^* > 0$, we obtain $T_{-\mathrm{s}(A)}(t)|f| = |f|$ for $t \geq 0$. Therefore,

$$|f| \in \ker(\mathrm{s}(A) - A).$$

By Lemma 10.18, we also have $(T_{-\mathrm{s}(A)}(t)f)^+ \leq T_{-\mathrm{s}(A)}(t)f^+$ and $(T_{-\mathrm{s}(A)}(t)f)^- \leq T_{-\mathrm{s}(A)}(t)f^-$. By the same arguments as above, we obtain $f^+ \in \ker(\mathrm{s}(A) - A)$ and $f^- \in \ker(\mathrm{s}(A) - A)$. This implies that $F := E_{\mathbb{R}} \cap \ker(\mathrm{s}(A) - A)$ is a real sublattice of E. For $f \in F$ we consider the ideal E_{f^+} (resp. E_{f^-}) generated by f^+ (resp. f^-). Then E_{f^+} and E_{f^-} are $T_{-\mathrm{s}(A)}(t)$-invariant for all $t \geq 0$. Since E_{f^+} and E_{f^-} are orthogonal, see Proposition 10.4, the irreducibility of $(T_{-\mathrm{s}(A)}(t))_{t\geq 0}$ implies that either $f^+ = 0$ or $f^- = 0$. Consequently, F is totally ordered and, by Lemma 10.10, we have

$$\dim F = \dim \ker(\mathrm{s}(A) - A) = 1.$$

d) We claim first that, if $\mathrm{s}(A)$ is a pole of the resolvent, then there is an eigenvector $0 \lneqq f \in E$ of A corresponding to $\mathrm{s}(A)$. Indeed, let k be the pole order of $\mathrm{s}(A)$ and

$$U_{-k} = \lim_{\lambda \to \mathrm{s}(A)^+} (\lambda - \mathrm{s}(A))^k R(\lambda, A),$$

see (14.9). Then $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. Moreover, by Corollary 12.10, we have $U_{-k} \geq 0$. Hence, there is $0 \leq g \in E$ with $f := U_{-k}g \gneqq 0$. By the relation $U_{-(k+1)} = (A - \mathrm{s}(A))U_{-k} = 0$, we obtain $(A - \mathrm{s}(A))f = 0$. This proves the claim.

We can now use a) to obtain $\overline{E_f} = E$. By taking the adjoint $U^*_{-(k+1)}$ of $U_{-(k+1)}$ and by the same computation as before, one deduces that there is $0 \lneqq f^* \in \ker(\mathrm{s}(A) - A^*)$. So by c) we have $\dim \ker(\mathrm{s}(A) - A) = 1$.

Assume now that $k \geq 2$. Then we have

$$\begin{aligned}\langle f, f^* \rangle &= \langle U_{-k} g, f^* \rangle \\ &= \langle g, U_{-k}^* f^* \rangle \\ &= \langle g, U_{-(k-1)}^* (A^* - \mathrm{s}(A)) f^* \rangle = 0.\end{aligned}$$

Since $\overline{E_f} = E$, we infer that $\langle g, f^* \rangle = 0$ for all $g \in E_+$. This contradicts the assertion b), hence $k = 1$. From the inequality $m_{\mathrm{g}} + k - 1 \leq m_{\mathrm{a}} \leq m_{\mathrm{g}} k$, see (14.10), we further obtain

$$m_{\mathrm{a}} = m_{\mathrm{g}} = \dim P_{\mathrm{s}(A)} E = \dim \ker(\mathrm{s}(A) - A) = 1,$$

and

$$P_{\mathrm{s}(A)} E = \ker(\mathrm{s}(A) - A),$$

where we recall that $P_{\mathrm{s}(A)} = U_{-1}$.

We now show the last part of assertion d). To this end, let

$$0 \lneqq f \in \ker(\mathrm{s}(A) - A).$$

Without loss of generality we suppose that $\|f\| = 1$. Then $P_{\mathrm{s}(A)} E = \mathrm{span}\{f\}$, i.e., for every $g \in E$ there is a $\lambda \in \mathbb{C}$ such that $P_{\mathrm{s}(A)} g = \lambda f$. By the Hahn–Banach theorem (see Exercise 10.9.6), there exists

$$0 \leq g^* \in (\ker(\mathrm{s}(A) - A))^* \text{ with } \|g^*\| = 1 \text{ and } \langle f, g^* \rangle = \|f\| = 1.$$

Hence,

$$\langle P_{\mathrm{s}(A)} g, g^* \rangle = \lambda = \langle g, P_{\mathrm{s}(A)}^* g^* \rangle.$$

Putting $u^* := P_{\mathrm{s}(A)}^* g^* \geq 0$, we obtain $P_{\mathrm{s}(A)} = u^* \otimes f$ and $\langle f, u^* \rangle = \langle P_{\mathrm{s}(A)} f, g^* \rangle = \langle f, g^* \rangle = 1$. Moreover, $0 \lneqq u^* \in P_{\mathrm{s}(A)}^* E^* \subseteq \ker(\mathrm{s}(A) - A^*)$, so $u^* > 0$ by b).

In the proof of c) we have seen that for every $g \in \ker(\mathrm{s}(A) - A)$ we have either $g^+ = 0$, or $g^- = 0$. So, we may assume that our eigenvector f is strictly positive. This ends the proof of the proposition. □

Now we study the boundary spectrum of irreducible semigroups on Banach lattices. The results resemble the properties of imprimitive matrices obtained in Chapter 5.

Before going on, we need some auxiliary results on the structure of Banach lattices and their quasi-interior points. The following result, due to Kakutani, shows that for every $e \in E_+$ the generated ideal satisfies $E_e \cong \mathrm{C}(K)$ for some compact Hausdorff space K. Here, E_e is equipped with the norm

$$\|f\|_e := \inf\{\lambda > 0 : f \in \lambda[-e, e]\}, \quad f \in E_e.$$

We recall that $T \in \mathcal{L}(E, F)$ is called a *lattice homomorphism* if $|Tf| = T|f|$ for every $f \in E$, where F is a complex Banach lattice (see Definition 10.19).

Theorem 14.13 (Kakutani)**.** *Let $e \in E_+$ and let E_e be the ideal generated by $\{e\}$. Further, take $B := \{f^* \in (E_e)^*_+ : \langle e, f^*\rangle = 1\}$ and denote by K the set of all extreme points of B. Then K is $\sigma(E^*, E)$-compact and the mapping*

$$U_e : E_e \ni f \longmapsto \varphi_f \in \mathrm{C}(K), \quad \varphi_f(f^*) = \langle f, f^*\rangle, f^* \in K,$$

is an isometric lattice isomorphism.

Now, if $|h|$ is a quasi-interior point of E_+, then $E_{|h|}$ is a dense subspace of E, isomorphic to a space of continuous functions $\mathrm{C}(K)$ on some K. Let $U_{|h|}$ be the lattice isomorphism obtained from Kakutani's theorem and let $\widetilde{h} := U_{|h|}h$. Then $|\widetilde{h}| = U_{|h|}|h| = \mathbb{1}$. Consider the operator

$$\widetilde{S}_0 : \mathrm{C}(K) \longrightarrow \mathrm{C}(K), \quad f \longmapsto (\operatorname{sign} \widetilde{h})f := \frac{\widetilde{h}}{|\widetilde{h}|} f = \widetilde{h}f, \tag{14.17}$$

and put $S_h := U^{-1}_{|h|}\widetilde{S}_0 U_{|h|}$. Then S_h is a linear mapping from $E_{|h|}$ into itself satisfying

a) $S_h\overline{h} = |h|$, where $\overline{h} = \operatorname{Re} h - \mathrm{i} \operatorname{Im} h$,
b) $|S_h f| \le |f|$ for every $f \in E_{|h|}$.

Since b) implies the continuity of S_h for the norm induced by E and $|h|$ is a quasi-interior point of E_+, S_h can be uniquely extended to E. This extension is also denoted by S_h and is called the *signum operator* with respect to h.

In the following we generalize Wielandt's lemma (see Lemma 5.18) and its consequences to irreducible semigroups on Banach lattices.

Lemma 14.14. *Let E be a Banach lattice and $|h|$ a quasi-interior point of E_+. Suppose that for $T, R \in \mathcal{L}(E)$ we have $Rh = h$, $T|h| = |h|$, and $|Rg| \le T|g|$ for all $g \in E$. Then $T = S_h^{-1}RS_h$, where S_h is the signum operator.*

Proof. First observe that for $g \in E_+$ we have

$$Tg = T|g| \ge |Rg| \ge 0,$$

so T is a positive operator. Since $T|h| = |h|$, the ideal $E_{|h|}$ is T-and R-invariant. Consider the operators $\widetilde{T} := U_{|h|}TU^{-1}_{|h|}$ and $\widetilde{R} := U_{|h|}RU^{-1}_{|h|}$, and put $\widetilde{h} := U_{|h|}h$. We then have

$$\widetilde{R}\widetilde{h} = \widetilde{h}, \quad \widetilde{T}\mathbb{1} = \mathbb{1}, \quad \text{and} \quad |\widetilde{R}f| \le \widetilde{T}|f| \text{ for all } f \in \mathrm{C}(K). \tag{14.18}$$

Define $T_1 := \widetilde{S}_0^{-1}\widetilde{R}\widetilde{S}_0$, where $\widetilde{S}_0$ is the multiplication operator by $\widetilde{h}$ on $\mathrm{C}(K)$ defined in (14.17). By (14.18), we have

$$\begin{aligned} &T_1\mathbb{1} = \mathbb{1} \quad \text{and} \\ &|T_1 f| = |\widetilde{S}_0^{-1}\widetilde{R}\widetilde{S}_0 f| = |\widetilde{R}\widetilde{S}_0 f| \le \widetilde{T}|\widetilde{S}_0 f| = \widetilde{T}|f| \end{aligned} \tag{14.19}$$

for all $f \in \mathrm{C}(K)$. Hence, $\|T_1\| \le \|\widetilde{T}\| = \|\widetilde{T}\mathbb{1}\|_\infty = 1$. So by Lemma 10.27, T_1 is a positive operator and (14.19) implies that $0 \le T_1 \le \widetilde{T}$. Therefore,

$$\|\widetilde{T} - T_1\| = \|(\widetilde{T} - T_1)\mathbb{1}\|_\infty = 0,$$

thus $T_1 = \widetilde{T}$ and hence $T = S_h^{-1} R S_h$. □

The following result describes the eigenvalues of an irreducible semigroup which are contained in the boundary spectrum $\sigma_{\mathrm{b}}(A) = \{\lambda \in \sigma(A) : \operatorname{Re}\lambda = \mathrm{s}(A)\}$, where A is the corresponding generator. Compare this with Theorem 5.19 for finite imprimitive matrices.

Proposition 14.15. *Let $(T(t))_{t\ge 0}$ be an irreducible C_0-semigroup with generator A on a Banach lattice E. Assume that $\mathrm{s}(A) = 0$ and there is $0 \lneqq f^* \in D(A^*)$ with $A^* f^* = 0$. If $\sigma_{\mathrm{p}}(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$, then the following assertions hold.*

a) *For $0 \neq h \in D(A)$ and $\alpha \in \mathbb{R}$ with $Ah = \mathrm{i}\alpha h$, $|h| \in \ker A$ is a quasi-interior point,*
$$S_h(D(A)) = D(A), \quad \text{and} \quad S_h^{-1} A S_h = A + \mathrm{i}\alpha,$$
where S_h is the signum operator defined above.

b) $\dim\ker(\lambda - A) = 1$ *for every $\lambda \in \sigma_{\mathrm{p}}(A) \cap \mathrm{i}\mathbb{R}$.*

c) $\sigma_p(A) \cap \mathrm{i}\mathbb{R}$ *is an additive subgroup of* $\mathrm{i}\mathbb{R}$.

d) 0 *is the only eigenvalue of A admitting a positive eigenvector.*

Proof. We first remark that by Proposition 14.12.b) we have $f^* > 0$, and by Corollary 9.32, $f^* = T(t)^* f^*$ for all $t \ge 0$.

a) Assume that $Ah = \mathrm{i}\alpha h$ for some $0 \neq h \in D(A)$ and $\alpha \in \mathbb{R}$. Then, by Corollary 9.32, $T(t)h = \mathrm{e}^{\mathrm{i}\alpha t} h$ and hence $|h| = |T(t)h| \le T(t)|h|$. This implies that

$$T(t)|h| - |h| \ge 0$$

for every $t \ge 0$. On the other hand,

$$\langle T(t)|h| - |h|, f^* \rangle = \langle |h|, T(t)^* f^* \rangle - \langle |h|, f^* \rangle = 0$$

for all $t \ge 0$. Since $f^* > 0$, we obtain $T(t)|h| = |h|$ for every $t \ge 0$, which implies that $A|h| = 0$. So, by Proposition 14.12.a), the vector $|h|$ is a quasi-interior point. If we set

$$T_\alpha(t) := \mathrm{e}^{-\mathrm{i}\alpha t} T(t)$$

for $t \ge 0$, then $T(t)$ and $T_\alpha(t)$ satisfy the assumptions of Lemma 14.14 and hence

$$T(t) = S_h^{-1} T_\alpha(t) S_h, \quad t \ge 0.$$

Therefore, $S_h(D(A)) = D(A)$ and $A = S_h^{-1}(A - \mathrm{i}\alpha) S_h$.

b) The calculations in the proof of a) imply that $\ker A \neq \{0\}$ and $\dim\ker(\mathrm{i}\alpha - A) = \dim\ker A$, so Proposition 14.12.c) yields the claim.

c) Let $\alpha, \beta \in \mathbb{R}$ be such that $Ah = \mathrm{i}\alpha h$ and $Ag = \mathrm{i}\beta g$ for some $0 \neq h, g \in D(A)$. By a), we have

$$S_h^{-1} A S_h = A + \mathrm{i}\alpha \quad \text{and} \quad S_g^{-1} A S_g = A + \mathrm{i}\beta.$$

Thus

$$A + \mathrm{i}(\alpha + \beta) = S_h(A + \mathrm{i}\beta)S_h^{-1} = S_h S_g^{-1} A S_g S_h^{-1},$$

which implies that

$$\ker(A + \mathrm{i}(\alpha + \beta)) = S_h S_g^{-1} \ker A \neq \{0\}.$$

Therefore $\mathrm{i}(\alpha + \beta) \in \sigma_p(A) \cap \mathrm{i}\mathbb{R}$.

d) If $Af = \lambda f$, where $0 \lneqq f \in D(A)$, then

$$\lambda \langle f, f^* \rangle = \langle Af, f^* \rangle = \langle f, A^* f^* \rangle = 0.$$

Since $f^* > 0$, we see that $\langle f, f^* \rangle > 0$. Hence, $\lambda = 0$. □

The following result, which we recall without proof, states that the boundary spectrum of the generator A of an irreducible C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach lattice E is always contained in the point spectrum $\sigma_{\mathrm{p}}(A)$ if $\mathrm{s}(A)$ is a pole of $R(\cdot, A)$.

Lemma 14.16. *Let A be the generator of an irreducible C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach lattice E. If $\mathrm{s}(A)$ is a pole of $R(\cdot, A)$, then $\sigma_{\mathrm{b}}(A) \subset \sigma_{\mathrm{p}}(A)$.*

As a consequence, we obtain the following description of the boundary spectrum of irreducible semigroups.

Theorem 14.17. *Let $(T(t))_{t \geq 0}$ be an irreducible C_0-semigroup with generator A on a Banach lattice E and assume that $\mathrm{s}(A)$ is a pole of the resolvent. Then there is $\alpha \geq 0$ such that*

$$\sigma_{\mathrm{b}}(A) = \mathrm{s}(A) + \mathrm{i}\alpha\mathbb{Z}.$$

Moreover, $\sigma_{\mathrm{b}}(A)$ consists of simple poles.

Proof. Without loss of generality we may suppose that $\mathrm{s}(A) = 0$. It can be shown that $\sigma_{\mathrm{b}}(A) \subseteq \sigma_{\mathrm{p}}(A)$, see Lemma 14.16. Hence

$$\sigma_{\mathrm{b}}(A) = \sigma_{\mathrm{p}}(A) \cap \mathrm{i}\mathbb{R}.$$

Proposition 14.12.d) yields the existence of a positive eigenvector $f^* \in D(A^*)$ corresponding to the eigenvalue $\mathrm{s}(A) = 0$. Proposition 14.15.c) implies that $\sigma_{\mathrm{b}}(A)$ is a subgroup of $(\mathrm{i}\mathbb{R}, +)$. Since $\sigma_{\mathrm{b}}(A)$ is closed and $\mathrm{s}(A) = 0$ is an isolated point, we have

$$\sigma_{\mathrm{b}}(A) = \mathrm{i}\alpha\mathbb{Z}$$

for some $\alpha \geq 0$. Proposition 14.12.d) implies that 0 is a simple pole and by Proposition 14.15.a) we have, for $\lambda \in \rho(A)$,

$$R(\lambda + \mathrm{i}k\alpha, A) = S_h^k R(\lambda, A) S_h^{-k}$$

for all $k \in \mathbb{Z}$. Therefore, $\mathrm{i}k\alpha$ is a simple pole for each $k \in \mathbb{Z}$. This ends the proof of the theorem. □

14.4 Asymptotic Behavior

In many concrete examples one can observe some regularity in the long-term behavior of the orbits of a semigroup. We will encounter two types of such behavior that are interesting for applications: balanced exponential growth, and asymptotic periodicity.

Let us start with the first kind of behavior. We say that a semigroup $(T(t))_{t\geq 0}$ with a generator A possesses a *balanced exponential growth* if there are a rank-one projection P and constants $\varepsilon > 0$ and $M \geq 1$ such that

$$\|\mathrm{e}^{-\mathrm{s}(A)t}T(t) - P\| \leq M\mathrm{e}^{-\varepsilon t} \quad \text{for all } t \geq 0.$$

We will present an example of such a semigroup in Chapter 17, see also Exercise 2. Using our spectral results, we can prove such behavior for certain class of irreducible semigroups.

Theorem 14.18. *Let $(T(t))_{t\geq 0}$ be an irreducible C_0-semigroup with the generator A on a Banach lattice E. If $\omega_{\mathrm{ess}}(T) < \omega_0(T)$, then there exist a quasi-interior point $0 \leq f \in E$ and $0 < f^* \in E^*$ with $\langle f, f^*\rangle = 1$ such that*

$$\|\mathrm{e}^{-\mathrm{s}(A)t}T(t) - f^* \otimes f\| \leq M\mathrm{e}^{-\varepsilon t} \quad \textit{for all } t \geq 0$$

and appropriate constants $M \geq 1$ and $\varepsilon > 0$.

Proof. Since $\omega_{\mathrm{ess}}(T) < \omega_0(T)$, Proposition 14.3 implies that $\mathrm{s}(A) = \omega_0(T)$. On the other hand, $\omega_{\mathrm{ess}}(T) < \omega_0(T)$ implies that $\mathrm{r}_{\mathrm{ess}}(T(1)) < \mathrm{r}(T(1))$. Hence, by Proposition A.34, $\mathrm{r}(T(1))$ is a pole of the resolvent of $T(1)$ and thus $\omega_0(T) = \mathrm{s}(A)$ is a pole of $R(\cdot, A)$.

Now, by Theorem 14.17, there exists $\alpha \geq 0$ such that $\sigma_{\mathrm{b}}(A) = \mathrm{s}(A) + \mathrm{i}\alpha\mathbb{Z}$ and therefore $\sigma_{\mathrm{b}}(A - \omega_0(T)) = \mathrm{i}\alpha\mathbb{Z}$, where $A - \omega_0(T)$ is the generator of the rescaled semigroup

$$T_{-\omega_0(T)}(t) := \mathrm{e}^{-\omega_0(T)t}T(t), \quad t \geq 0.$$

Since

$$\omega_{\mathrm{ess}}\left(T_{-\omega_0(T)}\right) = \omega_{\mathrm{ess}}(T) - \omega_0(T) < 0 \quad \text{and} \quad \omega_0\left(T_{-\omega_0(T)}\right) = 0,$$

we have, by Theorem 14.4, that the set

$$\{\lambda \in \sigma(A - \omega_0(T)) : \operatorname{Re}\lambda \geq 0\} = \{\lambda \in \sigma(A - \omega_0(T)) : \operatorname{Re}\lambda = 0\} = \sigma_{\mathrm{b}}(A - \omega_0(T))$$

is finite. Therefore, $\sigma_{\mathrm{b}}(A-\omega_0(T))=\{0\}$. The theorem is now proved by applying Theorem 14.4 and Proposition 14.12 to the rescaled semigroup $(T_{-\omega_0(T)}(t))_{t\geq 0}$. □

Without the quasi-compactness assumption for the rescaled semigroup, i.e., $\omega_{\mathrm{ess}}(T)<\omega_0(T)$, one obtains that the semigroup $(T(t))_{t\geq 0}$ behaves in the long run like a rotation group. Here we assume $\mathrm{s}(A)>-\infty$. So, by considering the rescaled semigroup $(\mathrm{e}^{-\mathrm{s}(A)t}T(t))_{t\geq 0}$ instead of $(T(t))_{t\geq 0}$, one may without loss of generality assume $\mathrm{s}(A)=0$. Compare the following theorem with Definition 4.14 and Theorem 4.15 for the finite-dimensional case.

Theorem 14.19. *Let $(T(t))_{t\geq 0}$ be a bounded and irreducible C_0-semigroup with the generator A on a Banach lattice $E:=\mathrm{L}^p(\Omega,\mu)$, $1\leq p<\infty$. If $\mathrm{s}(A)=0$ is a pole of $R(\cdot,A)$ and there is $\xi\in\mathbb{R}$ such that $\mathrm{i}\xi\in\sigma(A)$, then there exists a positive projection P commuting with $(T(t))_{t\geq 0}$ such that the following holds.*

a) *We have*

$$E=\operatorname{im}P\oplus\ker P,\quad T(t)=T_{\mathrm{r}}(t)\oplus T_{\mathrm{s}}(t),\quad t\geq 0,\quad \textit{and}\quad A=A_{\mathrm{r}}\oplus A_{\mathrm{s}},$$

corresponding to the decomposition $\sigma(A)=\sigma_{\mathrm{r}}\cup\sigma_{\mathrm{s}}$, where $\sigma_{\mathrm{r}}=\mathrm{i}\alpha\mathbb{Z}$ and $\sigma_{\mathrm{s}}=\sigma(A)\setminus\sigma_{\mathrm{r}}$ for some $\alpha>0$.

b) *The subspace $\operatorname{im}P$ is a closed sublattice of E and $(T_r(t))_{t\geq 0}$ is a periodic and irreducible C_0-semigroup on $\operatorname{im}P$.*

c) *For every $f\in E$ we have*

$$\lim_{t\to\infty}\|T(t)f-T_r(t)f\|=0.$$

Proof. First observe that by Theorem 14.17 we have

$$\sigma_{\mathrm{b}}(A)=\sigma(A)\cap\mathrm{i}\mathbb{R}=\mathrm{i}\alpha\mathbb{Z}\quad\text{for some }\alpha>0. \tag{14.20}$$

Next, from Proposition 14.15.a) and its proof, we see that there is a quasi-interior point $h\in E_+$ which is also a fixed point of $T(t)$. Hence, $|T(t)f|\leq T(t)h=h$ for all $f\in[-h,h]$. Since h is a quasi-interior point and order intervals in E are weakly compact, $(T(t))_{t\geq 0}$ is relatively weakly compact. Thus, by the Jacobs–de Leeuw–Glicksberg splitting theorem, see Theorem A.39, there is a projection $P\in\mathcal{L}(E)$ commuting with $T(t)$ such that $E=\operatorname{im}P\oplus\ker P$. Moreover,

$$\operatorname{im}P=\overline{\operatorname{span}}\{f\in D(A):\exists k\in\mathbb{Z}\text{ such that }Af=\mathrm{i}\alpha kf\}$$

and

$$\ker P=\{f\in E: 0\text{ belongs to the weak closure of }\{T(t)f:t\geq 0\}\}.$$

Furthermore, from (14.20) and Proposition A.40, it follows that

$$\ker P=\{f\in E:\lim_{t\to\infty}\|T(t)f\|=0\}. \tag{14.21}$$

Since P commutes with each $T(t)$, it splits $(T(t))_{t\geq 0}$ into $(T_{\mathrm{r}}(t))_{t\geq 0}$ on $\operatorname{im} P$ and $(T_{\mathrm{s}}(t))_{t\geq 0}$ on $\ker P$. Moreover, by Corollary 9.32, $T\left(\frac{2\pi}{\alpha}\right) f = f$ for all $f \in D(A)$ such that $Af = \mathrm{i}\alpha k f$ for some $k \in \mathbb{Z}$. Hence, $T_{\mathrm{r}}\left(\frac{2\pi}{\alpha}\right) = I$ and $T_{\mathrm{r}}(\cdot)$ is a periodic C_0-semigroup on $\operatorname{im} P$.

Theorem A.39 tells us that P belongs to the weak closure of $(T(t))_{t\geq 0}$. Since $(T(t))_{t\geq 0}$ is irreducible, we see that $Pf \gneqq 0$ whenever $f \gneqq 0$. So, by Lemma A.41, $\operatorname{im} P$ is a closed sublattice of E. This and the irreducibility of $(T(t))_{t\geq 0}$ imply that $(T_r(t))_{t\geq 0}$ is a periodic and irreducible C_0-semigroup on $\operatorname{im} P$. Denote its generator by A_{r}. Then, by Theorem 14.6 and equation (14.14), we have $\sigma(A_{\mathrm{r}}) = \mathrm{i}\alpha\mathbb{Z}$.

The family $T_{\mathrm{s}}(t) := T(t)|_{\ker P}$, $t \geq 0$, defines a C_0-semigroup on $\ker P$. We denote its generator by A_{s}. Then, by the spectral decomposition, we have $\sigma(A_{\mathrm{s}}) = \sigma(A) \setminus \mathrm{i}\alpha\mathbb{Z}$. This ends the proof of assertions a) and b). Assertion c) follows from (14.21). □

Remark 14.20.

a) Denote by G the closure of the set $\{T_r(t) : t \geq 0\}$ in the weak operator topology. Using abstract results from harmonic analysis and Theorem 14.19, one can prove that $\operatorname{im} P$ is lattice isomorphic to an $(R_\tau(t))_{t\in\mathbb{R}}$-invariant Banach function space $\mathcal{C}$ on G satisfying $C(G) \subset \mathcal{C} \subset \mathrm{L}^1(G, m)$ such that $(T_r(t))_{t\in\mathbb{R}}$ is similar to the group induced by $(R_\tau(t))_{t\in\mathbb{R}}$ on $\mathcal{C}$. Here m is the Haar measure on G and $(R_\tau(t))_{t\in\mathbb{R}}$ is the rotation group defined in Example 14.8 with period $\tau = \frac{2\pi}{\alpha}$. Moreover, if $E = \mathrm{L}^1(\Omega, \mu)$, then $\mathcal{C}$ can be identified with $L^1(G)$.

b) It can be seen that Theorem 14.19 holds if E is any Banach lattice with *order continuous norm*, see Remark 13.12 for the definition. In fact, it holds that a Banach lattice E has order continuous norm if and only if every order interval in E is weakly compact. This gives the weak compactness needed in the proof above.

c) One obtains from the proof above that $\mathrm{s}(A_{\mathrm{s}}) < 0$.

An example of a C_0-semigroup that behaves asymptotically periodic will be presented in Chapter 18.

14.5 Notes and Remarks

For the general spectral theory of operators we refer to monographs by Kato [73] or by Gohberg, Goldberg and Kaashoek [53]. More on spectral theory of irreducible semigroups can be found in the monograph edited by Nagel [101, Section B-III.3].

Kakutani's theorem, Theorem 14.13, originates from Kakutani [70]. We cited it from Meyer-Nieberg [95, Theorem 2.1.3], where you can also find a proof.

Concerning Lemma 14.14 and signum operators we refer to Nagel (ed.) [101, Chapter B-III]. For the proof of Lemma 14.16 see [101, p. 315].

Remark 14.20.a) is connected to abstract Halmos–von Neumann type theorems. We refer to Schaefer [126, Section III.10] for the corresponding abstract results. See also Keicher and Nagel [74]. For Remark 14.20.b) see Meyer-Nieberg [95, Theorem 2.4.2].

14.6 Exercises

1. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Prove that the following assertions are equivalent.
   (i) $\omega_{\text{ess}}(T) < 0$.
   (ii) $\|T(t_0) - K\| < 1$ for some $t_0 > 0$ and $K \in \mathcal{L}(X)$ compact.
2. On the Banach space $\mathrm{C}(K)$ with $K = [-\infty, 0]$ consider the operator
$$\begin{aligned} Af &= f' + mf \\ D(A) &= \{f \in \mathrm{C}(K) : f \text{ is differentiable, } f' \in \mathrm{C}(K) \text{ and } f'(0) = Lf\}, \end{aligned}$$
   where $m \in \mathrm{C}(K)$ is real-valued and $L : \mathrm{C}(K) \to \mathbb{R}$ a continuous linear form.
   a) Show that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathrm{C}(K)$.
   b) Prove that $(T(t))_{t\geq 0}$ is given by
$$T(t)f(s) = \mathrm{e}^{\int_s^0 m(\nu)\mathrm{d}\nu} \left(\mathrm{e}^{(s+t)m(0)} f(0) + \int_0^{t+s} \mathrm{e}^{\tau m(0)} LT(s+t-\tau)f \mathrm{d}\tau \right)$$
   for $s + t > 0$ and
$$T(t)f(s) = \mathrm{e}^{\int_s^{t+s} m(\tau)\mathrm{d}\tau} f(t+s)$$
   for $s + t \leq 0$.
   c) Using Exercise 1, prove that $\omega_{\text{ess}}(T) < 0$ provided that $m(-\infty) < 0$.
   d) Describe the asymptotic behavior of $(T(t))_{t\geq 0}$.
3. Consider the transport operator
$$D(A) = \left\{ f \in \mathrm{L}^1(I \times V) : v\frac{\partial f}{\partial x} \in \mathrm{L}^1(I \times V) \text{ and } \left\{ \begin{array}{l} f(0,v) = 0 \text{ if } v > 0, \\ f(1,v) = 0 \text{ if } v < 0, \end{array} \right\} \right.,$$
$$(Af)(x,v) = -v\frac{\partial f}{\partial x}(x,v),$$
   where $I = [0,1]$ and $V = \{v \in \mathbb{R} : 1 \leq |v| \leq 2\}$. Prove that A generates a reducible C_0-semigroup on $\mathrm{L}^1(I \times V)$.
4. On the Banach lattice $\mathrm{C}([0,1])$ consider the Laplace operator with Neumann boundary conditions:
$$\begin{aligned} (Af)(x) &= f''(x), \quad x \in [0,1], \\ f \in D(A) &= \{f \in \mathrm{C}^2([0,1]) : f'(0) = f'(1) = 0\}. \end{aligned}$$
   Prove that A generates an irreducible C_0-semigroup on $\mathrm{C}[0,1]$.

5. On $E = \mathrm{L}^1([-1,0])$ and for $0 \le g \in \mathrm{L}^\infty[-1,0]$ define the operator

$$Af := f', \quad D(A) = \left\{ f \in E : f' \in E \text{ and } f(0) = \int_{-1}^{0} f(s)g(s)\,\mathrm{d}s \right\}.$$

   a) Show that A generates a positive C_0-semigroup $(T(t))_{t\ge0}$ on E.
   b) Prove that $(T(t))_{t\ge0}$ is reducible if and only if there exists $\varepsilon > 0$ such that g vanishes a.e. on $[-1,-1+\varepsilon]$.