Chapter 10

Banach Lattices and Positive Operators

In the remaining chapters we shall try to extend the theory of positive matrices to infinite-dimensional spaces. One of the first questions is how to generalize concepts like positivity of vectors, or positivity, irreducibility, and imprimitivity of matrices. We have tried to have an abstract look at the finite-dimensional case, to motivate infinite-dimensional concepts. Still, the transition from finite to infinite dimensions is not easy. This is why we decided to focus in this chapter only on the order relation and explore basic properties of infinite-dimensional ordered vector spaces, more precisely, Banach lattices.

We also continue the investigation of positive operators and positive exponential functions on Banach lattices. We shall be guided by the finite-dimensional situation and there will be many results and proofs which will be essentially reappearances from previous chapters.

10.1 Ordered Function Spaces

Let us first summarize the order structure of $\mathbb{R}^n$. Note that vectors in $\mathbb{R}^n$ can be identified with functions:

$$\mathbb{R}^n \equiv \{f : \{1, \ldots, n\} \to \mathbb{R}\}.$$

Positivity of a vector is thus nothing but pointwise positivity of the representing function:

$$f \geq 0 \text{ if and only if } f(k) \geq 0 \text{ for all } k = 1, \ldots, n.$$

Hence, if we have a vector space of real-valued functions, it is natural to introduce an order relation by pointwise ordering. Let us illustrate this with the most important example.

A. Bátkai et al., *Positive Operator Semigroups*, Operator Theory:
Advances and Applications 257, DOI 10.1007/978-3-319-42813-0_10

For a compact Hausdorff space K we take the space of continuous functions

$$X := \mathrm{C}(K,\mathbb{R}) := \{f : K \to \mathbb{R} \,:\, f \text{ is continuous}\},$$

which is a Banach space with the norm

$$\|f\| = \|f\|_\infty = \max_{x\in K} |f(x)|.$$

The pointwise ordering in this case is

$$f \geq g \iff f(x) \geq g(x) \text{ for all } x \in K.$$

This clearly generalizes the finite-dimensional case with $K = \{1,\dots,n\} \subset \mathbb{R}$ and the usual maximum norm.

It is straightforward from the definition that the ordering is compatible with the vector space operations in the sense that

$$f \leq g \text{ implies } f+h \leq g+h \text{ for all } h \in \mathrm{C}(K,\mathbb{R})$$

and

$$0 \leq f \text{ implies } 0 \leq tf \text{ for all } t \geq 0.$$

We can also define the supremum and infimum of two functions as

$$(f\vee g)(x) := \max\{f(x), g(x)\} \quad \text{and} \quad (f\wedge g)(x) := \min\{f(x), g(x)\}$$

for all $x \in K$. The *positive part*, *negative part*, and *absolute value* of a function can be then given as

$$f^+ := f\vee 0, \quad f^- := (-f)\vee 0, \quad |f| := f \vee (-f).$$

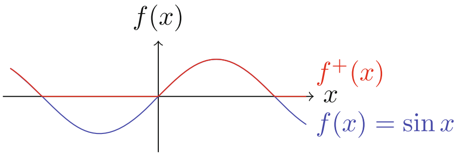
An important property of the positive and negative part of a function is that they live separate lives: if $f^+(x) \neq 0$, then $f^-(x) = 0$ and vice versa. This property is sometimes called *orthogonality* or *disjointness*.

Note that the following properties also follow from the fact that we defined the order relation pointwise and that the order behaves well on the real numbers:

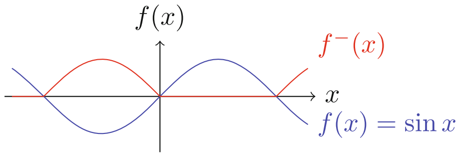
$$\begin{array}{rcl} f & = & f^+ - f^-, \\ |f| & = & f^+ + f^-, \\ f \leq g & \iff & f^+ \leq g^+ \text{ and } g^- \leq f^-, \\ |f-g| & = & (f\vee g) - (f \wedge g), \\ |f| \leq |g| & \implies & \|f\| \leq \|g\|. \end{array} \tag{10.1}$$

Recall that for reducibility in Chapter 5 (see Definition 5.8) we needed the invariance of a subspace of the form

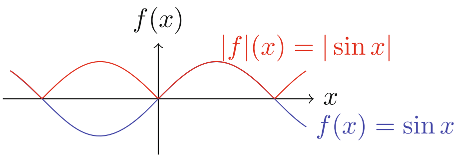
$$J_M := \left\{(\xi_1,\dots,\xi_n)^\top : \xi_i = 0 \text{ for } i \in M\right\} \subset \mathbb{R}^n$$



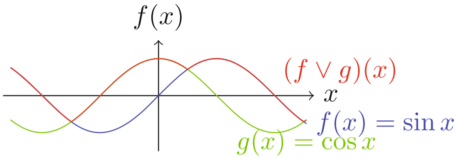
(a) The positive part f^+.



(b) The negative part f^-.



(c) The absolute value $|f|$.



(d) Supremum $f \vee g$.

Figure 10.1: Examples of f^+, f^-, $|f|$, and $f \vee g$.

for some $\emptyset \neq M \subsetneq \{1, \ldots, n\}$. In analogy, we define the following. Suppose that $F \subset K$ is a closed set and set

$$J_F := \{f \in \mathrm{C}(K, \mathbb{R}) \,:\, f(x) = 0 \text{ for all } x \in F\}\,. \tag{10.2}$$

Subspaces of the above form are also called *ideals*. It is important that such ideals can be characterized by order theoretic concepts.

Proposition 10.1. *For a closed subspace $I \subset \mathrm{C}(K, \mathbb{R})$ the following assertions are equivalent.*

(i)
$$f \in I \text{ implies } |f| \in I,$$
and
$$0 \leq g \leq f \in I \text{ implies } g \in I.$$

(ii) *There is a closed set $F \subset K$ such that $I = J_F$.*

Proof. Since the case $I = \{0\}$ (where 0 stands here for the constant zero function) is obvious, we may assume that $I \neq \{0\}$.

Clearly, if $I = J_F$ for a closed subset F, then the properties listed in (i) hold.

For the other direction, define

$$F := \{x \in K \,:\, f(x) = 0 \text{ for all } f \in I\}$$

and for $\alpha \in \mathbb{R}$ and $f \in \mathrm{C}(K, \mathbb{R})$ denote

$$[f \geq \alpha] := \{x \in K \,:\, f(x) \geq \alpha\}.$$

Obviously, $I \subset J_F$. Take now a positive nonzero function $0 \neq f \in J_F$. Our aim is to show that $f \in I$.

For $\varepsilon > 0$ let $B_\varepsilon^f := [f \geq \varepsilon]$. Observe that B_ε^f is a closed set satisfying $B_\varepsilon^f \cap F = \emptyset$. Thus, for every $x \in B_\varepsilon^f$ there is $0 \leq g_x \in I$ such that $g_x(x) > 0$. Since B_ε^f is compact, there are finitely many $x_1, \dots, x_r \in B_\varepsilon^f$ such that

$$B_\varepsilon^f \subset [g_{x_1} > 0] \cup [g_{x_2} > 0] \cup \dots \cup [g_{x_r} > 0].$$

We construct now an approximation of f in the set I. First observe that (i) and (10.1) show that $f_1, f_2 \in I$ implies $f_1 \vee f_2 \in I$ and $f_1 \wedge f_2 \in I$. We define

$$g := g_{x_1} \vee g_{x_2} \vee \dots \vee g_{x_r} \in I,$$

and take $\delta > 0$ such that $g(x) \geq \delta$ for all $x \in B_\varepsilon^f$. Then the function

$$h := f \wedge \left(\frac{\|f\|}{\delta} g \right) \in I$$

satisfies $0 \leq h \leq f$ and $h(x) = f(x)$ for all $x \in B_\varepsilon^f$. By the definition of the set B_ε^f, we see that $\|f - h\| \leq \varepsilon$. Hence, for every $f \in J_F$ and every $\varepsilon > 0$ we found $h \in I$ such that h approximates f with an error less than ε. By the closedness of I we obtain the desired conclusion. □

Thus, a closed subspace I of $\mathrm{C}(K, \mathbb{R})$ is an ideal if any of the equivalent conditions in Proposition 10.1 is satisfied. Let us only remark that this is also equivalent to saying that I is an *algebraic ideal* of the Banach algebra $\mathrm{C}(K, \mathbb{R})$.

An operator T on $\mathrm{C}(K, \mathbb{R})$ is called *reducible* if there exists a nontrivial ideal which is invariant under T. An operator which is not reducible, is called *irreducible*.

Another important observation concerning ideals is the following. Taking $f \geq 0$, we build the smallest ideal containing f, and denote it by E_f. It is then straightforward to check using Proposition 10.1 that

$$E_f = \bigcup_{k \in \mathbb{N}} [-kf, kf]$$

holds, where $[f_1, f_2] := \{g : f_1 \leq g \leq f_2\}$ denotes the *order interval* determined by f_1 and f_2, see Figure 10.2. We call E_f *the ideal generated by* f.

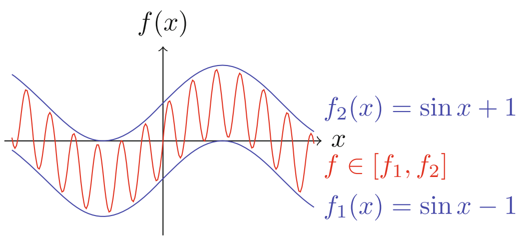


Figure 10.2: The order interval.

In some proofs (like in Corollary 7.4) strictly positive vectors (or the vector $\mathbb{1}$) played an important role. A natural observation is that the ideal generated by a strictly positive function is the whole space $\mathrm{C}(K,\mathbb{R})$. A function with this property is sometimes also called an *order unit*. Unfortunately, as we shall see in the next section, not all function spaces possess order units. We will be able to introduce a weaker notion that will be almost as satisfactory for our proofs, see Example 10.16 and the considerations before that.

Finally, let us note that for statements in spectral theory we need complex vector spaces. Observe that we can make the identification

$$\mathrm{C}(K,\mathbb{C}) \cong \mathrm{C}(K,\mathbb{R}) \oplus \mathrm{i} \cdot \mathrm{C}(K,\mathbb{R}),$$

meaning that for a complex-valued continuous function its real and imaginary parts are real-valued continuous functions.

10.2 Vector Lattices

Now we take an abstract point of view and try to axiomatize what we have seen in the previous section. Our main examples, besides the finite-dimensional vector spaces, are $\mathrm{C}(K)$ spaces, $\mathrm{L}^p(\Omega,\mu)$ spaces, and $\mathrm{C}_0(\Omega)$ spaces (see Example 10.6 later on). If you are uncomfortable with abstract terminology, you should pick one of these spaces and keep it in mind for the rest of this chapter.

We start by ordering. A non empty set M with a relation $\leq$ is said to be an *ordered set* if the following conditions are satisfied:

a) $f \leq f$ for every $f \in M$,
b) $f \leq g$ and $g \leq f$ imply $f = g$, and
c) $f \leq g$ and $g \leq h$ imply $f \leq h$.

First examples of ordered sets are the number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

Having an ordering at hand, we can consider order boundedness. Let F be a subset of an ordered set M. The element $f \in M$ (resp. $h \in M$) is called *an upper bound* (resp. *lower bound*) of F if $g \leq f$ for all $g \in F$ (resp. $h \leq g$ for all $g \in F$). Moreover, if there is an upper bound (resp. lower bound) of F, then F is said to be *bounded from above* (resp. *bounded from below*). If F is bounded from above and from below, then it is called an *order bounded set*.

We can introduce the concept of an order interval analogous to the intervals on the real line. Let $f, h \in M$ such that $f \leq h$. We denote by

$$[f,h] := \{g \in M : f \leq g \leq h\}$$

the *order interval* between f and g. We infer that a subset F is order bounded if and only if it is contained in some order interval.

Definition 10.2. A real vector space E which is ordered by some order relation $\leq$ is called a *vector lattice* if any two elements $f, g \in E$ have a least upper bound, denoted by $f \vee g = \sup(f, g) \in E$, and a greatest lower bound, denoted by $f \wedge g = \inf(f, g) \in E$, and the following properties are satisfied:

a) if $f \leq g$, then $f + h \leq g + h$ for all $f, g, h \in E$,
b) if $0 \leq f$, then $0 \leq tf$ for all $f \in E$ and $0 \leq t \in \mathbb{R}$.

Let E be a vector lattice. We denote by $E_+ := \{f \in E : 0 \leq f\}$ the *positive cone* of E. For $f \in E$, we define

$$f^+ := f \vee 0, \quad f^- := (-f) \vee 0, \quad \text{and} \quad |f| := f \vee (-f)$$

the *positive part*, the *negative part*, and the *absolute value* of f, respectively. Two elements $f, g \in E$ are called *orthogonal* (or *lattice disjoint*) (denoted by $f \perp g$) if $|f| \wedge |g| = 0$.

For a vector lattice E we have the following properties, which we will use frequently.

Proposition 10.3. *For all $f, g, h \in E$ the following assertions hold true.*

a) $f + g = (f \vee g) + (f \wedge g)$.
b) $f \vee g = -(-f) \wedge (-g)$.
c) $(f \vee g) + h = (f + h) \vee (g + h)$ *and* $(f \wedge g) + h = (f + h) \wedge (g + h)$.
d) $(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h)$ *and* $(f \wedge g) \vee h = (f \vee h) \wedge (g \vee h)$.
e) *For all $f, g, h \in E_+$ we have* $(f + g) \wedge h \leq (f \wedge h) + (g \wedge h)$.

Proof. We shall only prove a). The proof of the other properties is left to the reader (see Exercise 1). We have $f \wedge g \leq g \Longrightarrow f \leq f + g - f \wedge g$. In a similar way we have $g \leq f + g - f \wedge g$. Hence, $f \vee g \leq f + g - f \wedge g$, which gives

$$f \vee g + f \wedge g \leq f + g.$$

For the reverse inequality we note that $g \leq f \vee g \Longrightarrow f + g - f \vee g \leq f$, and similarly $f + g - f \vee g \leq g$. Thus,

$$f + g - f \vee g \leq f \wedge g. \qquad \square$$

For the positive part, negative part, and absolute value of $f \in E$ we have the following properties (compare with Properties (10.1) of functions in $\mathrm{C}(K, \mathbb{R})$).

Proposition 10.4. *If $f, g \in E$, then*

a) $f = f^+ - f^-$.
b) $|f| = f^+ + f^-$.
c) $f^+ \perp f^-$ *and the decomposition of f into the difference of two orthogonal positive elements is unique.*
d) $f \leq g$ *is equivalent to* $f^+ \leq g^+$ *and* $g^- \leq f^-$.
e) $|f - g| = (f \vee g) - (f \wedge g)$.

Proof. a) Using Proposition 10.3 a) and b), we obtain

$$\begin{aligned} f &= f + 0 = f \vee 0 + f \wedge 0 \\ &= f \vee 0 - (-f) \vee 0 = f^+ - f^-. \end{aligned}$$

b) Applying Proposition 10.3.c) and a) proved above, we have

$$\begin{aligned} |f| &= f \vee (-f) = (2f \vee 0) - f = 2(f \vee 0) - f \\ &= 2f^+ - f^+ + f^- = f^+ + f^-. \end{aligned}$$

c) Let us prove first that $f^+ \wedge f^- = 0$. To this purpose we apply Proposition 10.3.c) and deduce

$$\begin{aligned} f^+ \wedge f^- &= (f^+ - f^-) \wedge 0 + f^- = (f \wedge 0) + f^- \\ &= -[(-f) \vee 0] + f^- = 0. \end{aligned}$$

Let now $f = g - h$ with $g \wedge h = 0$. By c) and a) of Proposition 10.3, we have

$$f^+ = (g - h) \vee 0 = g \vee h - h = (g + h - (g \wedge h)) - h = g.$$

In a similar way we obtain $f^- = h$.

d) Using a), this is straightforward.

e) This can be proved using the identities

$$f \vee g = \frac{1}{2}(f + g + |f - g|) \quad \text{and} \quad f \wedge g = \frac{1}{2}(f + g - |f - g|)$$

(see Exercise 2). □

10.3 Banach Lattices

We finally arrived at the main objects of this chapter and consider Banach spaces which are ordered and whose norm is compatible with this ordering. First, let us explain what we mean by compatible.

A norm on a vector lattice E is called a *lattice norm* if

$$|f| \le |g| \text{ implies } \|f\| \le \|g\| \quad \text{for } f, g \in E.$$

Definition 10.5. A *Banach lattice* is a real Banach space E endowed with an ordering $\le$ such that $(E, \le)$ is a vector lattice and the norm on E is a lattice norm.

We will see that this combination of properties of a complete normed vector space and a compatible ordering leads to many fruitful results.

As already mentioned, apart from finite-dimensional vector spaces (such as $\mathbb{R}$ or $\mathbb{R}^n$), there are many interesting infinite-dimensional examples of Banach lattices.

Examples 10.6. The following Banach spaces are Banach lattices for the pointwise (almost everywhere) ordering.

a) Let (Ω, μ) be a measure space and take $\mathrm{L}^p(\Omega, \mu; \mathbb{R})$, $1 \leq p \leq \infty$, endowed with the norm

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, \mathrm{d}\mu\right)^{1/p} \qquad \text{if } 1 \leq p < \infty,$$
$$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ for } \mu\text{-a.e. } x \in \Omega\} \qquad \text{if } p = \infty,$$

and with the order

$$f \geq 0 \iff f(x) \geq 0 \text{ for } \mu\text{-a.e. } x \in \Omega.$$

We furthermore define

$$(f \vee g)(x) := \max\{f(x), g(x)\} \quad \text{and} \quad (f \wedge g)(x) := \min\{f(x), g(x)\} \tag{10.3}$$

for μ-a.e. $x \in \Omega$, which are both measurable functions. Note that for the absolute value of f this imposes

$$|f|(x) = (f \vee (-f))(x) = \max\{f(x), -f(x)\} = |f(x)| \text{ for } \mu\text{-a.e. } x \in \Omega.$$

Since

$$|(f \vee g)(x)| \leq |f(x)| + |g(x)| \quad \text{and} \quad |(f \wedge g)(x)| \leq |f(x)| + |g(x)| \tag{10.4}$$

for μ-a.e. $x \in \Omega$, we see, that

$$\|f \vee g\|_p \leq \|f\|_p + \|g\|_p \quad \text{and} \quad \|f \wedge g\|_p \leq \|f\|_p + \|g\|_p,$$

hence $f \vee g, f \wedge g \in \mathrm{L}^p(\Omega, \mu)$ for every $1 \leq p \leq \infty$. Clearly, the properties in Definition 10.2 are fulfilled and the p-norm is a lattice norm.

b) For a locally compact noncompact Hausdorff topological space Ω we take $\mathrm{C}_0(\Omega)$, the space of all real-valued continuous functions vanishing at infinity, endowed with the supremum norm

$$\|f\|_\infty = \sup_{x \in \Omega} |f(x)|,$$

and with the natural order

$$f \geq 0 \iff f(x) \geq 0 \text{ for all } x \in \Omega.$$

We define $f \vee g$, $f \wedge g$ as in (10.3), but now for every $x \in \Omega$. We obtain continuous functions and using inequalities (10.4) we see that $f \vee g, f \wedge g \in \mathrm{C}_0(\Omega)$. Again, the properties in Definition 10.2 are fulfilled and the supremum norm is a lattice norm.

c) The space of real-valued continuous functions $\mathrm{C}(K)$ on a compact Hausdorff space K, endowed with the supremum norm and with the order defined above was already investigated in Section 10.1.

Note that there are many ordered function spaces which are not Banach lattices. Let us give the following simple example.

Examples 10.7.

a) Consider the Banach space $\mathrm{C}^1([0,1])$ of continuously differentiable functions on $[0,1]$ with the norm

$$\|f\| = \max_{s\in[0,1]} |f(s)| + \max_{s\in[0,1]} |f'(s)|$$

and the natural order $f \geq 0$ if $f(s) \geq 0$ for all $s \in [0,1]$. Since $\sup\{t, 1-t\} \notin \mathrm{C}^1([0,1])$, the space $\mathrm{C}^1([0,1])$ is not a vector lattice. Moreover the above norm is not compatible with the order. In fact, let $f \equiv 1$ and $g(s) = \sin(2s)$, $s \in [0,1]$. Then, $0 \leq g \leq f$ and $\|g\| \geq |g'(0)| = 2 > 1 = \|f\|$.

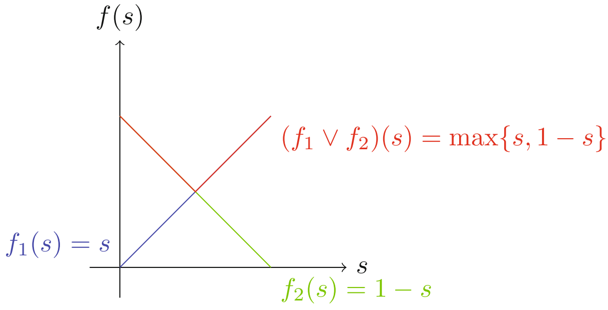


Figure 10.3: $C^1[0,1]$ with the norm in Example 10.7.a) is not a lattice.

b) Consider the Sobolev space $\mathrm{H}^1(0,1)$. Using similar arguments as in the previous example, we see that the norm is again not compatible with the order. As a difference, however, note that $\mathrm{H}^1(0,1)$ is a vector lattice, see Exercise 4.

Now we list some further properties of Banach lattices.

Proposition 10.8. *For a Banach lattice E the following hold.*

a) *The lattice operations are continuous.*
b) *The positive cone E_+ is closed.*
c) *The order intervals are closed and bounded.*

Proof. a) Consider (f_k), $(g_k) \subset E$ and f, $g \in E$ such that $\lim_{k\to\infty} f_k = f$ and $\lim_{k\to\infty} g_k = g$. Applying Birkhoff's inequality, see Exercise 2.f), we have

$$\begin{aligned} |f_k \wedge g_k - f \wedge g| &\leq |f_k \wedge g_k - f_k \wedge g| + |f_k \wedge g - f \wedge g| \\ &\leq |g_k - g| + |f_k - f|. \end{aligned}$$

Thus,

$$\|f_k \wedge g_k - f \wedge g\| \leq \|g_k - g\| + \|f_k - f\|.$$

This yields the continuity of $\wedge$. Analogously, one obtains the continuity of $\vee$.

b) Take $(f_k) \subset E_+$ such that $\lim_{k\to\infty} f_k = f \in E$. Then, by a),

$$\lim_{k\to\infty} f_k = \lim_{k\to\infty} (f_k \vee 0) = f \vee 0.$$

Hence, $f = f \vee 0 \in E_+$.

c) Let f, g, $h \in E$ with $h \in [f, g]$. Then, $0 \leq h - f \leq g - f$. So, using the triangle inequality from Exercise 2.d), one has

$$\|h\| - \|f\| \leq \|h - f\| \leq \|g - f\|, \text{ whence } \|h\| \leq \|f\| + \|g - f\|.$$

Therefore, order intervals are bounded. We prove now that order intervals are closed. Take $(h_k) \subset E$ and f, $g \in E$ with $f \leq h_k \leq g$ for all $k \in \mathbb{N}$. Since E_+ is closed, by b), $\lim_{k\to\infty}(h_k - f) = h - f \geq 0$ and $\lim_{k\to\infty}(g - h_k) = g - h \geq 0$. Hence, $f \leq h \leq g$, which proves the closedness of $[f, g]$. □

The following property of Banach lattices is a consequence of the Hahn–Banach theorem.

Proposition 10.9. *In a Banach lattice E every weakly convergent increasing sequence (f_k) is norm convergent.*

Proof. Consider the convex hull of the set $\{f_k\}$,

$$F := \left\{ \sum_{i=1}^{m} a_i f_i : m \in \mathbb{N},\, a_i \geq 0,\, a_1 + \cdots + a_m = 1 \right\}.$$

By the Hahn–Banach theorem, Theorem A.27, the norm closure of F coincides with the weak closure. This implies that $f \in \overline{F}$, where $f :=$ weak-$\lim_{k\to\infty} f_k$. Thus, for $\varepsilon > 0$, there exist $g \in F$, i.e.,

$$g = a_1 f_1 + \cdots + a_m f_m, \text{ with } a_1, \ldots, a_m \geq 0 \text{ and } a_1 + \cdots + a_m = 1,$$

such that $\|g - f\| < \varepsilon$. Since $g \leq f_k \leq f$, we infer that $\|f - f_k\| \leq \|f - g\| < \varepsilon$ for all $k \geq m$. □

Here we state a result that we shall need later. A Banach lattice E is *totally ordered* if for every $f \in E$ one has either $0 \leq f$ or $f \leq 0$.

Lemma 10.10. *A totally ordered real Banach lattice E is at most one-dimensional.*

Proof. Take $e \in E_+$ with $e \neq 0$, and $f \in E$. Consider the closed subsets of $\mathbb{R}$

$$C_+ := \{\alpha \in \mathbb{R} : \alpha e \geq f\} \quad \text{and} \quad C_- := \{\alpha \in \mathbb{R} : \alpha e \leq f\}.$$

It is obvious that C_+ and C_- are non-empty and $C_+ \cup C_- = \mathbb{R}$. Since $\mathbb{R}$ is connected, it follows that $C_+ \cap C_- \neq \varnothing$. Hence, there is $\alpha \in \mathbb{R}$ such that $f = \alpha e$. □

10.4 Sublattices and Ideals

We want to equip a vector subspace of a vector or Banach lattice with some order structure. Therefore we define two kinds of subspaces (compare with Proposition 10.1).

Definition 10.11. A vector subspace F of a vector lattice E is a *vector sublattice* if for all $f \in F$ we have $|f| \in F$. A subspace I of a Banach lattice E is called an *ideal* if

$$f \in I \text{ implies } |f| \in I \text{ and } 0 \leq g \leq f \in I \text{ implies } g \in I.$$

Consequently, a vector sublattice F is an ideal in E if $f \in F$ and $0 \leq g \leq f$ implies $g \in F$. Note also that if F is a vector sublattice, then $f^+ \in F$ and $f^- \in F$ for all $f \in F$.

Since the notions of sublattice and ideal are invariant under the formation of arbitrary intersections, there exists, for any subset M of E, a unique smallest sublattice (resp. ideal) of E containing M. This will be called *the sublattice* (*resp. the ideal*) *generated by* M.

We summarize all properties of sublattices and ideals which we will need in the sequel.

Proposition 10.12. *If E is a Banach lattice, then the following properties hold.*

a) *The closure of every sublattice of E is a sublattice.*
b) *The closure of every ideal of E is an ideal.*
c) *For every $f \in E_+$, the ideal generated by $\{f\}$ is*

$$E_f = \bigcup\nolimits_{k\in\mathbb{N}} k[-f, f].$$

Proof. The first two assertions follow from the continuity of the lattice operations, see Proposition 10.8. For the last assertion one can see easily that $I = \bigcup_{k\in\mathbb{N}} k[-f, f]$ is an ideal while any ideal included in I and containing f equals I. This means that $I = E_f$. □

For examples of closed ideals we again pay a visit to our function spaces and start by restating Proposition 10.1 in this context.

Proposition 10.13. *If $E = \mathrm{C}(K)$, where K is a compact Hausdorff space, then a subspace J of E is a closed ideal if and only if there is a closed subset $F \subset K$ such that*

$$J = \{\varphi \in E : \varphi(x) = 0 \textit{ for all } x \in F\}.$$

The arguments of the proof of Proposition 10.1 can be modified accordingly to obtain the following characterization.

Proposition 10.14. *If $E = \mathrm{C}_0(\Omega)$, where Ω is a locally compact Hausdorff topological space, then a subspace J of E is a closed ideal if and only if there is a closed subset F of Ω such that*

$$J = \{\varphi \in E : \varphi(x) = 0 \ \textit{for all} \ x \in F\}.$$

Finally, we close this set of examples by characterizing closed ideals of L^p-spaces.

Proposition 10.15. *If $E = \mathrm{L}^p(\Omega, \mu)$, $1 \leq p < \infty$, where (Ω, μ) is a σ-finite measure space, then a subspace I of E is a closed ideal if and only if there exists a measurable subset Y of Ω such that*

$$I = \{\psi \in E : \psi(x) = 0 \ \textit{a.e.} \ x \in Y\}.$$

Proof. First we show that for a measurable set $Y \subset \Omega$, the set

$$I_Y := \{\psi \in E : \psi(x) = 0 \ \text{a.e.} \ x \in Y\}$$

is a closed ideal. Clearly, I_Y is a linear subspace and if $f \in I_Y$, then $|f| \in I_Y$. The definition implies directly that if $f \in I_Y$ and $0 \leq g \leq f$, then $g \in I_Y$. Hence it only remains to show the closedness.

Let $(f_k) \subset I_Y$ be a sequence such that $f_k \to f \in E$. Then there is a subsequence (f_{k_m}) such that $f_{k_m}(x) \to f(x)$ for a.e. $x \in \Omega$. In particular, $f_{k_m}(x) \to f(x)$ for a.e. $x \in Y$, hence, $f \in I_Y$.

Conversely, suppose that $\{0\} \neq I \subset E$ is an ideal. We have to show the existence of a measurable set $Y \subset \Omega$ such that $I = I_Y$. Since (Ω, μ) is a σ-finite measure space, there is an increasing sequence (Ω_k) of sets of finite measure with $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. For each $k \in \mathbb{N}$ we define

$$\mathcal{B}_k := \{M \subset \Omega_k \ : \ \chi_M \in I\}.$$

Since I is a non-trivial ideal, we infer that there is $k \in \mathbb{N}$ such that $\mathcal{B}_k \neq \emptyset$. Observe that if $\mathcal{M} \subset \mathcal{B}_k$ is a finite set, then

$$\sup_{M \in \mathcal{M}} \chi_M \in I.$$

We also have that

$$s_k := \sup_{\mathcal{M} \subset \mathcal{B}_k, \, \mathcal{M} \ \text{finite}} \Big\| \sup_{M \in \mathcal{M}} \chi_M \Big\| \leq \mu(\Omega_k)^{1/p} < \infty.$$

Take a sequence $\mathcal{M}_m \subset \mathcal{B}_m$, where $\mathcal{M}_m$ is finite and

$$\Big\| \sup_{M \in \mathcal{M}_m} \chi_M \Big\| \geq s_k - \frac{1}{m}$$

holds for every $m \in \mathbb{N}$. Observe that for $m_1 \le m_2$ one has $\mathcal{M}_{m_2} \subseteq \mathcal{M}_{m_1}$. Now we define

$$C_k := \bigcup_{m\in\mathbb{N},\, M\in\mathcal{M}_m} M \in \mathcal{B}_k, \quad C := \bigcup_{k\in\mathbb{N}} C_k, \text{ and } Y := \Omega \setminus C.$$

Clearly, the sets C_k, C, and Y are measurable. Moreover, the sequence

$$\left(\sup_{M\in\mathcal{M}_m} \chi_M \right) \subset I$$

is bounded and monotone, and since I is closed, the Dominated Convergence Theorem (see Theorem A.23) implies that its limit $\chi_{C_k} \in I$ for all $k \in \mathbb{N}$.

Take now $f \in I$ and show that $f \in I_Y$. Since I and I_Y are both ideals it suffices to consider positive f only. Assume on the contrary that there is $M \subset Y$ such that $\mu(M) > 0$ and $f(x) > 0$ for a.e. $x \in M$. Fix k_0 such that $\mu(M \cap \Omega_{k_0}) > 0$. Since f is strictly positive on $M \cap \Omega_{k_0}$, there exists $j \in \mathbb{N}$ such that

$$\mu(M \cap \Omega_{k_0} \cap \{f \ge 1/j\}) > 0.$$

For such j we introduce the function $g_j := \chi_{M\cap\Omega_{k_0}\cap\{f\ge 1/j\}}$. Then, $0 \le g_j \le jf$ and hence, $g_j \in I$ and $\tilde{\mathcal{B}} := M \cap \Omega_{k_0} \cap \{f \ge 1/j\} \in \mathcal{B}_{k_0}$. Since $\tilde{\mathcal{B}} \cap C_{k_0} = \emptyset$ for sufficiently large $m \in \mathbb{N}$, we must have

$$\left\| \chi_{\tilde{\mathcal{B}}} + \sup_{M\in\mathcal{M}_m} \chi_M \right\| > s_{k_0},$$

which is a contradiction. Hence, $I \subset I_Y$.

To show that $I_Y \subset I$, take $0 \le f \in I_Y$ and fix $\varepsilon > 0$. The sequences $(f - \chi_{\Omega_k} f)$ and $(f - \chi_{\{f\le k\}} f)$ of positive functions converge to zero almost everywhere. The Dominated Convergence Theorem (see Theorem A.23) implies that there is $k_0 \in \mathbb{N}$ such that

$$\|f - \chi_{\Omega_{k_0}} f\| \le \frac{\varepsilon}{2} \quad \text{and} \quad \|f - \chi_{\{f\le k_0\}} f\| \le \frac{\varepsilon}{2}.$$

Since $k_0 \chi_{C_{k_0}} \in I$, we infer that

$$h := \chi_{\Omega_{k_0}\cap\{f\le k_0\}} f \wedge k_0 \chi_{C_{k_0}} \in I.$$

By the choice of k_0 and the fact that $f = 0$ almost everywhere on Y, we have that $\|f - h\| \le \varepsilon$. The closedness of I now implies that $f \in I$. □

Sometimes a Banach lattice E is generated by a single positive element. If $E_e = E$ holds for some $e \in E_+$ then e is called an *order unit*. If $\overline{E}_e = E$, then $e \in E_+$ is called a *quasi-interior point* of E_+.

It follows that e is an order unit of E if and only if e is an interior point of E_+. Quasi-interior points of the positive cone exist, for example, in every separable Banach lattice.

Examples 10.16.

a) If $E = \mathrm{C}(K)$, where K is a compact Hausdorff space, then the constant function $\mathbb{1}$, $\mathbb{1}(x) \equiv 1$, is an order unit. In fact, for every $f \in E$, there is $k \in \mathbb{N}$ such that $\|f\|_\infty \le k$. Hence, $|f(s)| \le k\mathbb{1}(s)$ for all $s \in K$. This implies $f \in k[-\mathbb{1}, \mathbb{1}]$.

b) Let $E = \mathrm{L}^p(\Omega, \mu)$ with a σ-finite measure μ such that $\mu(\{x\}) = 0$ for every $x \in \Omega$ and $1 \le p < \infty$. Then the quasi-interior points of E_+ coincide with the μ-a.e. strictly positive functions, while E_+ does not contain any interior point.

10.5 Complexification of Real Banach Lattices

It is often necessary to consider complex vector spaces (for instance in spectral theory). Therefore, we introduce the concept of a *complex Banach lattice.*

The complexification of a real Banach lattice E is the complex Banach space $E_\mathbb{C}$ whose elements are pairs $(f, g) \in E \times E$, with addition and scalar multiplication defined by $(f_0, g_0) + (f_1, g_1) := (f_0 + f_1, g_0 + g_1)$ and $(a + \mathrm{i}b)(f, g) := (af - bg, ag + bf)$, and norm

$$\|(f, g)\| := \| \, |(f, g)| \, \|,$$

where

$$|(f, g)| := \sup_{0 \le \theta \le 2\pi} (f \sin\theta + g \cos\theta)$$

is the natural extension of the modulus $|\cdot|$ in E. Note that the existence of the above supremum in E is in this generality a nontrivial fact, but we accept it here. However, in the standard function spaces, which are our main examples, this is a straightforward fact.

By identifying $(f, 0) \in E_\mathbb{C}$ with $f \in E$, the space E is isometrically isomorphic to a real linear subspace $E_\mathbb{R}$ of $E_\mathbb{C}$. We write $0 \le f \in E_\mathbb{C}$ if and only if $f \in E_+$.

A complex Banach lattice is an ordered complex Banach space $(E_\mathbb{C}, \le)$ that arises as the complexification of a real Banach lattice E. The underlying real Banach lattice E is called the real part of $E_\mathbb{C}$ and is uniquely determined as the closed linear span of all $f \in (E_\mathbb{C})_+$.

Instead of the notation (f, g) for elements of $E_\mathbb{C}$, we usually write $f + \mathrm{i}g$. The complex conjugate of an element $h = f + \mathrm{i}g \in E_\mathbb{C}$ is the element $\overline{h} = f - \mathrm{i}g$. We use also the notation $\mathrm{Re}(h) := f$ for $h = f + \mathrm{i}g \in E_\mathbb{C}$. All concepts introduced for real Banach lattices have a natural extension to complex Banach lattices.

10.6 Positive Operators

This section is concerned with positive operators on Banach lattices, that is, operators that preserve positive cones.

Definition 10.17. Let E and F be two complex Banach lattices. A linear operator $T : E \to F$ is called *positive* if $TE_+ \subset F_+$. Notation: $T \geq 0$.

Let us immediately give an alternative characterization of a positive operator (compare with the matrix case given in Lemma 5.3).

Lemma 10.18. *The following assertions for a linear operator $T : E \to F$ between the Banach lattices E and F are equivalent.*

(i) *T is positive.*
(ii) *For all $f \in E_{\mathbb{R}}$, we have $(Tf)^+ \leq Tf^+$ and $(Tf)^- \leq Tf^-$.*
(iii) *$|Tf| \leq T|f|$ for all $f \in E$.*

Proof. (i) $\Longrightarrow$ (ii): For $f \in E_{\mathbb{R}}$ we have $Tf = Tf^+ - Tf^- \leq Tf^+$ and $(Tf)^+ = Tf \vee 0$, which imply $(Tf)^+ \leq Tf^+$. The second property now follows since

$$Tf^+ - (Tf)^+ = Tf^- - (Tf)^-.$$

(ii) $\Longrightarrow$ (iii): Using $f = f^+ + f^-$ for $f \in E_{\mathbb{R}}$, and (ii) we obtain

$$|Tf| = (Tf)^+ + (Tf)^- \leq Tf^+ + Tf^- = T|f|.$$

For general $f \in E$ the assertion follows from the definition of $|f|$.

(iii) $\Longrightarrow$ (i): Let $f \in E_+$. Then $T|f| = Tf$ and by assumption we have

$$Tf = T|f| \geq |Tf| \geq 0. \qquad \square$$

We shall need a stronger property than the one given in Lemma 10.18.(iii), i.e., preserving the absolute value.

Definition 10.19. Let E and F be two complex Banach lattices. A linear operator $T : E \to F$ is called a *lattice homomorphism* if $|Tf| = T|f|$ for all $f \in E$.

All positive operators are bounded, as the following result shows.

Theorem 10.20. *Every positive linear operator $T : E \to F$ is continuous.*

Proof. Assume by contradiction that T is not bounded. Then there is $(f_k) \subset E$ such that $\|f_k\| = 1$ and $\|Tf_k\| \geq k^\gamma$ for each $k \in \mathbb{N}$ and some $\gamma > 2$. Since $|Tf_k| \leq T|f_k|$, one can assume that $f_k \geq 0$ for all $k \in \mathbb{N}$. From $\sum_{k=1}^\infty \frac{\|f_k\|}{k^{\gamma-1}} < \infty$ we infer that $\sum_{k=1}^\infty \frac{f_k}{k^{\gamma-1}}$ is norm convergent in E. Set $f = \sum_{k=1}^\infty \frac{f_k}{k^{\gamma-1}}$. Then

$$0 \leq \frac{f_k}{k^{\gamma-1}} \leq f \quad \text{for all } k \in \mathbb{N}.$$

So

$$k \le \left\| T\left(\frac{f_k}{k^{\gamma-1}}\right)\right\| \le \|Tf\| < \infty \quad \text{for all } k \in \mathbb{N},$$

which is a contradiction. Thus $T \in \mathcal{L}(E,F)$. ☐

As a consequence, we obtain the equivalence of Banach lattice norms (recall also Tikhonov's theorem, Theorem 1.5).

Corollary 10.21. *Let E be a vector lattice and $\|\cdot\|_1$ and $\|\cdot\|_2$ two norms such that $E_1 = (E, \|\cdot\|_1)$ and $E_2 = (E, \|\cdot\|_2)$ are both Banach lattices. Then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*

Proof. This follows from the positivity of the identity operators $I : E_1 \to E_2$ and $I : E_2 \to E_1$ and Theorem 10.20. ☐

We denote by $\mathcal{L}(E,F)_+$ the set of all positive linear operators from a Banach lattice E into a Banach lattice F. For positive operators one has

Proposition 10.22. *Let $T \in \mathcal{L}(E,F)_+$. Then the following properties hold.*

a) $\|T\| = \sup\{\|Tf\| : f \in E_+, \|f\| \le 1\}$.
b) *If $S \in \mathcal{L}(E,F)$ is such that $0 \le S \le T$ (this means that $0 \le Sf \le Tf$ for all $f \in E_+$), then $\|S\| \le \|T\|$.*

Proof. a) holds by Lemma 10.18 (iii).

b) Since $0 \le S \le T$ we have $|Sf| \le S|f| \le T|f|$ for all $f \in E$. The assertion now follows by a). ☐

Another property of positive operators is that they have positive resolvent. The converse in not always true, see also Proposition 10.29.

Proposition 10.23. *Let $T \in \mathcal{L}(E)$ be a positive operator with spectral radius $\mathrm{r}(T)$.*

a) *The resolvent $R(\mu, T)$ is positive whenever $\mu > \mathrm{r}(T)$.*
b) *If $|\mu| > \mathrm{r}(T)$, then*

$$\big|R(\mu,T)f\big| \le R\big(|\mu|, T\big)|f|, \quad f \in E.$$

Proof. We use the Neumann series representation

$$R(\mu,T) = \sum_{k=0}^{\infty} \frac{T^k}{\mu^{k+1}}$$

for the resolvent, which is valid for $|\mu| > \mathrm{r}(T)$, see Proposition 9.28.c).

a) If $T \ge 0$, then $T^k \ge 0$ for all k, hence for $\mu > \mathrm{r}(T)$, we have for every $f \in E_+$ that

$$R(\mu,T)f = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{T^k f}{\mu^{k+1}} \ge 0,$$

since the finite sums are positive.

b) We have for $|\mu| > \mathrm{r}(T)$ and $f \in E$ that

$$\big|R(\mu,T)f\big| = \left|\lim_{N\to\infty}\sum_{k=0}^{N}\frac{T^k f}{\mu^{k+1}}\right| \le \lim_{N\to\infty}\sum_{k=0}^{N}\left|\frac{T^k f}{\mu^{k+1}}\right|$$
$$\le \lim_{N\to\infty}\sum_{k=0}^{N}\frac{T^k}{|\mu|^{k+1}}|f| = R\big(|\mu|,T\big)|f|. \qquad \square$$

The following is an easy version of Perron's theorem (see Theorem 5.6) for the infinite-dimensional case.

Theorem 10.24. *If $T \in \mathcal{L}(E)$ is positive, then $\mathrm{r}(T) \in \sigma(T)$.*

Proof. Assertion b) of Proposition 10.23 implies that

$$\big\|R(\mu,T)\big\| \le \big\|R(|\mu|,T)\big\| \quad \text{for } |\mu| > \mathrm{r}(T).$$

Let now $\lambda \in \sigma(T)$ such that $|\lambda| = \mathrm{r}(T)$. Then, Proposition 9.28 implies that $\|R(\mu,T)\| \to \infty$ whenever μ approaches λ. Putting $\mu = s\lambda$ with $s > 1$ the above estimate yields

$$\big\|R(s\mathrm{r}(T),T)\big\| \ge \big\|R(s\lambda,T)\big\| \longrightarrow \infty \quad \text{as } s \downarrow 1,$$

hence, by Corollary 9.30, $\mathrm{r}(T)$ must be in the spectrum of T. $\square$

Combining Proposition 10.23 and Theorem 10.24 we have the following useful characterization of positivity of the operator $R(1,T) = (I-T)^{-1}$.

Lemma 10.25. *Let T be a positive linear operator on E. Then*

$$r(T) < 1 \iff 1 \in \rho(T) \text{ and } R(1,T) \ge 0.$$

Proof. The implication is a consequence of Proposition 10.23. For the converse, assume that $1 \in \rho(T)$ and $R(1,T) \ge 0$. For any $k \in \mathbb{N}$ we have

$$(I-T)\sum_{j=0}^{k} T^j = I - T^{k+1}.$$

Hence,

$$\sum_{j=0}^{k} T^j = R(1,T)(I-T^{k+1}) \le R(1,T), \tag{10.5}$$

since $T \ge 0$. So, in particular $T^k \le R(1,T)$ for all $k \in \mathbb{N}$. Now Proposition 10.22 implies that

$$\|T^k\| \le \|R(1,T)\|, \quad k \in \mathbb{N}.$$

Using the above estimate and the definition of $r(T)$ we obtain $r(T) \le 1$. If $r(T) = 1$, then Theorem 10.24 yields $1 \in \sigma(T)$, which contradicts our assumption. $\square$

Now we define irreducible operators on a Banach lattice.

Definition 10.26. An operator $T \in \mathcal{L}(E)$ is called *reducible* if there exists a non-trivial ideal which is invariant under T. Operators that are not reducible are called *irreducible*.

As in the finite-dimensional situation, positive irreducible operators enjoy some special spectral properties (see, e.g., Theorem A.38). However, we shall not discuss these properties here. We study them in the case of semigroups of positive irreducible operators in Section 14.3.

We end this section by reconsidering the Banach lattice of continuous functions on a compact Hausdorff space K.

Lemma 10.27. *Suppose that K is a compact Hausdorff topological space and $T : C(K) \to C(K)$ is a linear operator satisfying $T\mathbb{1} = \mathbb{1}$. Then $0 \le T$ if and only if $\|T\| \le 1$.*

Proof. If $0 \le T$, then

$$|Tf| \le T|f| \le T(\|f\|_\infty \mathbb{1}) = \|f\|_\infty \mathbb{1}.$$

Hence $\|T\| \le 1$.

To prove the converse, we first observe that

$$-\mathbb{1} \le f \le \mathbb{1} \iff \|f - \mathrm{i}r\mathbb{1}\|_\infty \le \rho_r := \sqrt{1+r^2} \quad \text{for all } r \in \mathbb{R}. \tag{10.6}$$

Let $0 \le f \in C(K)$. Then there is $k \in \mathbb{N}$ such that $0 \le f \le k\mathbb{1}$. Set $g = \frac{2}{k} f$. Then $0 \le g \le 2\mathbb{1}$, and so $-\mathbb{1} \le g - \mathbb{1} \le \mathbb{1}$. By (10.6), $\|g - \mathbb{1} - \mathrm{i}r\mathbb{1}\|_\infty \le \rho_r$ for all $r \in \mathbb{R}$. Since $T\mathbb{1} = \mathbb{1}$ and $\|T\| \le 1$, $\|Tg - \mathbb{1} - \mathrm{i}r\mathbb{1}\|_\infty \le \rho_r$ for all $r \in \mathbb{R}$. So by (10.6) we obtain $-\mathbb{1} \le Tg - \mathbb{1} \le \mathbb{1}$. This implies $0 \le Tg \le 2\mathbb{1}$ and hence $Tf \ge 0$. ☐

Indeed, operators satisfying $T\mathbb{1} = \mathbb{1}$ occur quite often and have a special name. Recall that in the finite-dimensional case we have shown this property for the transition matrix P of a Markov chain (see Lemma 6.6).

Definition 10.28. Let K and L be compact Hausdorff spaces. A linear operator $T : C(K) \to C(L)$ is called a *Markov operator* if $T\mathbb{1}_K = \mathbb{1}_L$.

10.7 Positive Exponential Functions

In the following, let E be a Banach lattice and $A \in \mathcal{L}(E)$. We investigate the positivity and asymptotic properties of the exponential function of A, and start with a characterization through the resolvent of A.

Proposition 10.29. *The semigroup $T(t) = \mathrm{e}^{tA}$ is positive if and only if*

$$R(\lambda, A) = (\lambda - A)^{-1} \ge 0$$

for all $\lambda > \omega_0(T)$.

Proof. When $T(\cdot)$ is a positive semigroup, then $R(\lambda, A)$ is positive for $\lambda > \omega_0(T)$ by the Laplace transform representation in (9.11).

For the other direction notice that, by Exercise 9.10.2, the Euler formula

$$\lim_{k\to\infty}\left(I - \frac{t}{k}A\right)^{-k} = \mathrm{e}^{tA}$$

holds for $t \geq 0$. Since $(I - \frac{t}{k}A)^{-k} = \left(\frac{k}{t}R(\frac{k}{t}, A)\right)^k \geq 0$ for k sufficiently large by assumption, the positivity of the operators $T(t)$ follows. For an alternative proof where the Euler formula is not needed we refer to Remark 11.3 and Corollary 11.4. □

Recall the notation already used in the case of matrices. For $A \in \mathcal{L}(E)$ we define its *spectral bound* as

$$\mathrm{s}(A) := \sup\{\operatorname{Re}\lambda \,:\, \lambda \in \sigma(A)\}. \tag{10.7}$$

The following is a fundamental technical result on positive exponential functions. It tells us that the Laplace transform representation in the case of positive bounded generators holds on an even larger set.

Proposition 10.30. *For a positive exponential function $T(t) = \mathrm{e}^{tA}$ we have*

$$R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)\,\mathrm{d}s$$

for all $\lambda > \mathrm{s}(A)$. Hence, for all $\lambda > \mathrm{s}(A)$ we have $0 \leq R(\lambda, A)$.

Proof. It is clear from the previous considerations that $R(\lambda, A)$ is positive for $\lambda > \|A\|$. The Neumann series representation from Proposition 9.28.c) implies that $R(\lambda, A) \geq 0$ for $\lambda > s(A)$.

Since we can always consider $A - \mu$ instead of A for $\mu > 0$, for simplicity we restrict our proof to the case where $\mathrm{s}(A) < 0$ and $\operatorname{Re}\lambda > 0$. The assumption implies

$$0 \leq V(t) = \int_0^t T(s)\,\mathrm{d}s = R(0, A) - R(0, A)T(t) \leq R(0, A),$$

so $\|V(t)\| \leq M$ for some constant M for all $t \geq 0$. Hence the improper integral

$$\int_0^\infty \mathrm{e}^{-\lambda s} V(s)\,\mathrm{d}s$$

exists for all $\operatorname{Re}\lambda > 0$. Integration by parts yields

$$\int_0^t \mathrm{e}^{-\lambda s} T(s)\,\mathrm{d}s = \mathrm{e}^{-\lambda t} V(t) + \lambda \int_0^t \mathrm{e}^{-\lambda s} V(s)\,\mathrm{d}s.$$

This last expression converges as $t \to \infty$, hence we infer that

$$R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)\, \mathrm{d}s$$

holds for all $\operatorname{Re} \lambda > 0$. □

As a corollary we obtain a version of Perron's theorem for the positive exponential function.

Corollary 10.31. *For a positive exponential function $T(t) = \mathrm{e}^{tA}$ we have*

$$\mathrm{s}(A) \in \sigma(A).$$

Proof. The positivity of the operators $T(t)$ means that

$$|T(t)f| \le T(t)|f|$$

for all $f \in E$ and $t \ge 0$. Therefore,

$$|R(\lambda, A)f| \le \int_0^\infty \mathrm{e}^{-\operatorname{Re} \lambda s} T(s)|f|\, \mathrm{d}s$$

for all $\operatorname{Re} \lambda > \mathrm{s}(A)$ and $f \in E$. Hence,

$$\|R(\lambda, A)\| \le \|R(\operatorname{Re} \lambda, A)\|.$$

Recall that since $A \in \mathcal{L}(E)$ is a bounded operator, we have that $\sigma(A) \neq \emptyset$. Further, there is $\lambda_k \in \rho(A)$ such that $\operatorname{Re} \lambda_k \to \mathrm{s}(A)$, $\operatorname{Re} \lambda_k > \mathrm{s}(A)$ and $\|R(\lambda_k, A)\| \to \infty$. This implies $\|R(\operatorname{Re} \lambda_k, A)\| \to \infty$, and hence $\mathrm{s}(A) \in \sigma(A)$ (see Corollary 9.30). □

Compare the following with Corollary 7.4 from Chapter 7.

Corollary 10.32. *Let K be a compact Hausdorff topological space and $E = \mathrm{C}(K)$. For a positive exponential function $T(t) = \mathrm{e}^{tA}$, $A \in \mathcal{L}(E)$, the following assertions are equivalent.*

(i) $\mathrm{s}(A) < 0$.

(ii) $-A^{-1}$ *exists and it is positive.*

(iii) *There exists $0 \le f \in E$ such that $Af = -\mathbb{1}$.*

Proof. The equivalence (i) $\iff$ (ii) follows from Proposition 10.30. Since $-A^{-1} = R(0, A)$, (ii) $\Longrightarrow$ (iii) follows by taking $f := -A^{-1}\mathbb{1}$.

We close the loop by showing (iii) $\Longrightarrow$ (i). Assume that $Af = -\mathbb{1}$ for some $0 \le f \in E$. Then for $\lambda > \max\{\mathrm{s}(A), 0\}$ we have

$$\begin{aligned} 0 \le R(\lambda, A)\mathbb{1} &= -AR(\lambda, A)f \\ &= f - \lambda R(\lambda, A)f \le f. \end{aligned}$$

Hence

$$\sup_{\lambda>\max\{\mathrm{s}(A),0\}} \|R(\lambda, A)\| \le \|f\|_\infty .$$

Since by Corollary 10.31, $\mathrm{s}(A) \in \sigma(A)$, it follows from Corollary 9.30 that $\mathrm{s}(A) < 0$. □

We close this chapter by a minimum principle characterization of positive exponential functions.

Theorem 10.33. *Let Ω be a locally compact Hausdorff space and let $A \in \mathcal{L}(E)$, where $E = \mathrm{C}_0(\Omega)$. Then the following are equivalent.*

(i) *A generates a positive exponential function, i.e., $\mathrm{e}^{tA} \ge 0$ for $t \ge 0$.*

(ii) *For $0 \le f \in E$ and $x \in \Omega$, $f(x) = 0$ implies that $(Af)(x) \ge 0$.*

(iii) *$A + \|A\|I \ge 0$.*

Proof. (i) $\Longrightarrow$ (ii): Take $0 \le f \in E$ and $x \in \Omega$ with $f(x) = 0$. Then

$$(Af)(x) = \lim_{t\downarrow 0} \frac{\mathrm{e}^{tA} f - f}{t}(x) = \lim_{t\downarrow 0} \frac{\mathrm{e}^{tA} f(x) - f(x)}{t} = \lim_{t\downarrow 0} \frac{\mathrm{e}^{tA} f(x)}{t} \ge 0.$$

(ii) $\Longrightarrow$ (iii): Consider $x \in \Omega$. We have to show that $(Af)(x) + \|A\| f(x) \ge 0$ for all $f \in E$. Define

$$A^* \delta_x = \mu + c\delta_x ,$$

where $\mu \in \mathrm{M}(\Omega)$ is such that $\mu(\{x\}) = 0$, and $c \in \mathbb{R}$. We claim that $\mu \ge 0$. Take $0 \le f \in E$ such that $f(x) = 0$. Then

$$\langle f, \mu\rangle = \langle f, A^*\delta_x\rangle = \langle Af, \delta_x\rangle = (Af)(x) \ge 0.$$

It can be shown (see Exercise 9) that this implies that $\langle g, \mu\rangle \ge 0$ for all $0 \le g \in E$. Hence $\mu \ge 0$.

Moreover,

$$|c| = \|c\delta_x\| \le \|c\delta_x + \mu\| = \|A^*\delta_x\| \le \|A\|.$$

Hence, for $0 \le f \in E$, we have that

$$\begin{aligned}(Af)(x) + \|A\| f(x) &= \langle Af + \|A\| f, \delta_x\rangle = \langle f, A^*\delta_x + \|A\|\delta_x\rangle \\ &= \langle f, \mu + (c + \|A\|)\delta_x\rangle \ge 0.\end{aligned}$$

(iii) $\Longrightarrow$ (i): The same argument as in the proof of Theorem 7.1 applies. We know that if $B := A + \|A\|I \in \mathcal{L}(E)$ is positive, then

$$\mathrm{e}^{tB} = \sum_{k=0}^{\infty} \frac{(tB)^k}{k!} \ge 0.$$

Hence,

$$\mathrm{e}^{tA} = \mathrm{e}^{-t\|A\|}\mathrm{e}^{tB} \ge 0.$$

□

Remark 10.34. The equivalence of (i) and (iii) in Theorem 10.33 is in complete analogy to Theorem 7.1 in the matrix case. This result is true in general Banach lattices, but the proofs are more involved. Conditions (ii) and (iii) are nothing but generalizations of the “positive off-diagonal” property for matrices.

10.8 Notes and Remarks

The investigation of ordered algebraic structures is a classical subject and of great interest in the literature, we mention here the monograph by Fuchs [50]. Most results of this chapter can be found for example in the monographs by Schaefer [126], Meyer-Nieberg [95] or Aliprantis and Burkinshaw [2]. For Proposition 10.12 see [95, Propositions 1.1.5, 1.2.3, and 1.2.5]. For complexification of real Banach lattices, we refer to Schaefer [126, Section II.11] or Meyer-Nieberg [95, Section 2.2].

For Theorem 10.33 we refer to Nagel (ed.) [101, Theorem B-II.1.3]. For the generalization to arbitrary Banach lattices, see [101, Theorem C-II.1.11].

10.9 Exercises

1. Prove the properties b)–e) in Proposition 10.3 and d) in Proposition 10.4.
2. Let E be a vector lattice and $f,\ g,\ h \in E$.
   a) Prove that $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|)$.
   b) Show that $|f| \vee |g| = \frac{1}{2}(|f + g| + |f - g|)$ and deduce that
   $$|f| \wedge |g| = \frac{1}{2}\,||f + g| - |f - g||\,.$$
   c) Deduce that $f \perp g$ is equivalent to $|f - g| = |f + g|$.
   d) Prove this variant of the triangle inequality:
   $$||f| - |g|| \le |f + g| \le |f| + |g|.$$
   e) Deduce that $f \perp g$ is equivalent to $|f| \vee |g| = |f| + |g|$, and that in this case
   $$||f| - |g|| = |f + g| = |f| + |g|.$$
   f) Show Birkhoff’s inequalities:
   $$|f \vee h - g \vee h| \le |f - g| \quad \text{and} \quad |f \wedge h - g \wedge h| \le |f - g|.$$
3. Prove that a subspace I of a Banach lattice is an ideal if and only if
   $$(\, f \in I,\ |g| \le |f|\,) \Longrightarrow g \in I.$$

4. Prove that $\mathrm{H}^1(0,1)$ endowed with the natural order, $f \geq 0$ if $f(s) \geq 0$ for a.e. $s \in [0,1]$, is a vector lattice.
5. Consider $E := \mathrm{C}^1([0,1])$ equipped with the norm

$$\|f\| = \max_{s\in[0,1]} |f'(s)| + |f(0)|$$

and the order $f \geq 0$ whenever $f(0) \geq 0$ and $f' \geq 0$. Show that E is a Banach lattice.
6. Let E be a Banach lattice. Use the Hahn–Banach theorem to prove that
   a) $0 \leq f$ is equivalent to $\langle f, f^* \rangle \geq 0$ for all $f^* \in E^*_+$;
   b) for each $f \in E$ there exists $f^* \in E^*_+$ such that $\|f^*\| \leq 1$ and $\langle f, f^* \rangle = \|f^+\|$.
7. Consider the Banach lattice $\mathrm{C}^1([0,1])$ as in Exercise 5 and define the operator

$$(Tf)(t) := \int_0^t g(s)f(s)\,\mathrm{d}s$$

with a given $g \in \mathrm{C}([0,1])$. Calculate $\|T\|$. For which g is T positive?
8. Let $T \in \mathcal{L}(E,F)$, where E and F are two Banach lattices. Show that T is a lattice homomorphism if and only if one of the following equivalent properties holds.
   (i) $T(f \vee g) = Tf \vee Tg$ and $T(f \wedge g) = Tf \wedge Tg$ for all $f, g \in E$.
   (ii) $Tf^+ \wedge Tf^- = 0$ for all $f \in E$.
9. Let Ω be a locally compact Hausdorff space, $x \in \Omega$, and μ a regular bounded Borel measure on Ω such that $\mu(\{x\}) = 0$. Show that $\mu \geq 0$ if and only if $\langle f, \mu \rangle \geq 0$ for all $f \in \mathrm{C}_0(\Omega)$ satisfying $f \geq 0$ and $f(x) = 0$.