

# Chapter 4

## Signs and Their Use: Peirce and Wittgenstein

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### 4.1 Introductory Remarks

It is obvious that mathematicians throughout history have used signs of various kinds, such as symbols, diagrams, graphs, and formulae, but they also occur in everyday language and scientific language. The technical symbols and formulas of mathematics have contributed in particular to its specific status, and many learning difficulties have been attributed to these characteristics, which are viewed as turning mathematics into a highly abstract and inaccessible field of scientific enquiry. Even for the most basic mathematical activities such as arithmetic calculations, the use of number symbols is unavoidable, and it can also be said that much of the strength and relevance of mathematics for applications derives from its symbolic techniques. One could express this by stating that the “formula” was one of the great cultural inventions and intellectual innovations, comparable in its ramifications to those of the wheel. The use of symbolic techniques within mathematics, such as in proofs, needs no further discussion. It is also virtually impossible to translate mathematics into any kind of vernacular, and most mathematical “narratives” are rather misleading or missing the mathematical point. To understand mathematics, one has to do it; this doing in a very deep sense is an activity with signs and based on signs, as should become even clearer from the following considerations.

So far most people concerned in some way or the other with mathematics will agree with what was stated above. Pronounced differences show up when one turns to what one can term the meaning of the signs and symbols of mathematics. In the common understanding, signs are used to designate something that is different from and independent of the sign, namely, the object of the sign, and this object is viewed as the source of the meaning of the sign. Often the signs are considered as

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being secondary to what they designate and arbitrary and neutral with respect to the mathematical content. Their main use in this view is to communicate and express the mathematical ideas. Hersh (1986, p. 19), for instance, compared mathematics with music, where according to him the score has solely the role of noting the music which is already there before the score. The signs and notations in this view have no influence on invention and creation in mathematics or music. An extreme position in this vein was taken by Brouwer see Shapiro (2000), who considered mathematics to be a purely mental “construction” not dependent on any sign system. In a general way, in all these positions of mathematical realism mathematical signs and notations have been viewed as describing what have been termed mathematical objects, whatever those might be and wherever they might be located. Thus numerals denote numbers and diagrams denote geometric objects. Only algebraic formulas have sometimes been spared this descriptive role, yet they have then been reduced to a purely technical means for calculations and proofs. I will not continue these ontological and philosophical issues any further, but these short hints should serve to make the possible impact of the views taken by Peirce and Wittgenstein more conspicuous.

## 4.2 Charles Sanders Peirce

Peirce (1839–1914) was an American mathematician, logician, and philosopher. From among his comprehensive works, only his fundamental work in semiotics can very briefly be considered here. Peirce developed a complex and comprehensive theory of signs by devising a multilevel categorization of signs, starting with the differentiation into index, icon, and symbol. With Peirce, the sign in itself has a triadic structure of “object-representamen-interpretant,” but we will not go into any details here. Interestingly, for decades mathematics educators apparently have not taken note of the potential of the theories presented by Peirce. Yet Peirce was interested in educational questions and has written a very interesting draft for a textbook on elementary arithmetic [see the two articles by Radu in Hoffmann (2003)].

To the best of my knowledge, it was due to the initiative taken by Michael Otte in some of his papers (see Otte 1997, 2011) that the relevance of the semiotics of Peirce was recognized by a growing number of mathematics educators in Germany and elsewhere. It is impossible to adequately present the work by Otte with regard to Peirce here because it is very complex and comprehensive. He puts Peirce and his semiotics into the context of philosophy, epistemology and ontology by relating it to many other strands of thought in this realm but pays less attention to the concrete mathematical activities on and with signs. Rather, the papers by Otte furnish a powerful background and basis for more detailed investigations into/about how and which signs are used in mathematics and especially in mathematics learning. On the other hand, his papers show and explicate deliberations in Peirce that may be more general and fundamental. But it is also sensible to investigate—as

it will be done here—a Peircean notion, such as diagrammatic reasoning, independently from other dimensions of Peircean semiotics and its philosophical ramifications. In a pointed way, one could say that in Michael Otte the purview of a sign is the whole of life, experience, and cognition, whereas here we focus on its important role in doing and learning mathematics. To give the reader a flavour of the work by Otte, it is instructive to cite from the abstracts of Otte (1997; my translation from the German original) and of Otte (2011):

Peirce treats the concepts meaning, (natural) law, continuum—and some others like representation or mind—as synonyms. By that they all acquire those paradoxical qualities which have been since long discussed for the example of the continuum and which recently have been addressed in different contexts, as in systems theory. The meaning of a sign, for example, for sure cannot be separated from its application—what is already stipulated by the Pragmatic Maxim of Peirce. On the other hand, it cannot be identified either with a single application or with some well-defined set of applications but it rather rests on the general conditions for possible applications. The notion of sign and the concept of the continuum are the two pillars on which Peirce’s phenomenological epistemology is based. The latter shall be elucidated first, through the relation to the history of mathematics; and second, through the comparison with other phenomenological positions during the foundational crisis of mathematics. The significance of mathematics results from the fact that in mathematics, the two pillars mentioned most deeply confront each other. (Otte 1997, p. 175)

One of the most salient arguments in favour of a semiotic approach... claims that semiotics is most appropriate for treating the interaction between socio-cultural and objective aspects of knowledge problems. If we want to take such claims seriously, however, we have to revise our basic conceptions about reality, existence, cognition, and cultural development. The semiotic evolutionary realism of Charles S. Peirce provides—or appears to provide—an appropriate basis for such intentions. Man is a sign, Peirce famously said, and “thought is more without us than within. It is we that are in it, rather than it in any of us” (Peirce CP 8.256). As there is no thought without a sign, we have to accept thoughts, concepts, theories, or works of art as realities *sui generis*. Concepts or theories have to be recognized as real before we ask for their meaning or relevance. (Otte 2011, p. 313)

An important early contribution to the dissemination of the semiotics of Peirce was Hoffmann (Hoffmann 2005a), which explicated many aspects of Peircean semiotics, especially with emphasis on mathematics. A related work is that by Stjernfelt (2000), which also contains very worthwhile interpretations of ideas and notions in Peirce. Much of this work was concerned with Peirce’s general sign theory and its philosophical dimensions. Within mathematics education, in addition to the triadic structure of sign, the notion of diagram and diagrammatic thinking was mainly exploited. It should be noted that for Peirce, signs always possessed an object that they explored in an ongoing semiotic process; the only exception was diagrams, for which Peirce allowed the object to be fictional or ideal, especially with respect to mathematics. Before concentrating on the concept of diagrammatic thinking, which appears to be of special value for mathematics and the learning of mathematics, some more references on the work on (Peircean) semiotics within German mathematics education are included: Hoffmann (2003, 2005), Hoffmann et al. (2005), Kadunz (2010, 2015). Of course, on an international level semiotics in general and Peircean notions in particular have also received growing attention. Some publications illustrate this ever-extending tendency: Rotman (2000); the

contributions to special issues in the journals *ESM*, *ZDM*, and *JMD* by authors including Presmeg, Saenz-Ludlow, and Radford; and Radford et al. (2008). In addition to the publications that have an explicit focus on semiotics, one could refer to the vast literature on visualization and representation. Yet because in these the signs have mostly been considered in their descriptive and representational function (see below), this is beyond the scope of this contribution. In addition to Peirce, there have been other semiotic traditions and theories which have been exploited in mathematics education; for instance, Duval (1995). We now turn to the notion of diagram and diagrammatic thinking in the form of a liberal interpretation of the ideas of Peirce based on Dörfler (2004, 2006, 2008), where one can find a host of examples for diagrams and diagrammatic reasoning.

### 4.3 Diagrams and Diagrammatic Thinking

Peirce (3.363 in *Collected Papers*, this means paragraph 363 in Volume 3 according to the standard way of citing from the papers by Peirce) made the following comment, among others, on a basic feature of mathematics:

It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts.... As for algebra, the very idea of the art is that it presents formulae, which can be manipulated and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries, which are embodied in general formulae. These are patterns, which we have the right to imitate in our procedure, and are the icons par excellence of algebra.

I have chosen to stick to the term *diagram* as it has been used by Peirce and others, though I am aware that this term might cause some misunderstandings and arouse false expectations. First of all, the reader should dismiss all geometric connotations. This can be seen from the above reference to Peirce, who includes formulas of all kinds in his notion of diagram (or icon). What is important are the spatial structure of a diagram, the spatial relationships of its parts to one another, and the operations and transformations of and with the diagrams. The constitutive parts of a diagram can be any kind of inscriptions such including letters, numerals, special signs, or geometric figures.

Peirce did not nor will I give a general definition of the notion of diagram. Instead, several descriptive features of diagrams are presented. Diagrams are based on a kind of permanent inscription (paper, sand, screen, etc.). Those inscriptions are

mostly planar, but some are 3-dimensional, such as models of geometric solids or manipulatives in school mathematics. Mathematics at all levels abounds with such inscriptions: number lines, Venn diagrams, geometric figures, Cartesian graphs, point-line graphs, arrow diagrams (mappings), arrows in the Gaussian plane or as vectors, and commutative diagrams (category theory); but there are also inscriptions with a less geometric flavour: arithmetic or algebraic terms, function terms, fractions, decimal fractions, algebraic formulas, polynomials, matrices, systems of linear equations, continued fractions, and many more. There are features common to some of these inscriptions that contribute to their diagrammatic quality as it is understood here. However, I emphasize that not every inscription that occurs in mathematical reasoning, learning, or teaching has a diagrammatic quality. Quite a few of what are taken as visualizations or representations of mathematical notions and ideas do not qualify as diagrams since they lack some of the essential features. This is mostly the precise operative structure that for genuine diagrams permits and invites their investigation and exploration as mathematical objects. Some widely shared qualities of diagrams, or rather of inscriptions when used as diagrams, are proposed in the following:

- Diagrammatic inscriptions have a structure consisting of a specific spatial arrangement of and spatial relationships among their parts and elements. This structure often has a conventional character.
- Based on this diagrammatic structure, there are rule-governed operations on and with the inscriptions by transforming, composing, decomposing, or combining them (calculations in arithmetic and algebra, constructions in geometry, and derivations in formal logic). These operations and transformations could be called the internal meaning of the respective diagram (compare to Wittgenstein on meaning). Depending on the operations and transformations applied, an inscription might give rise to essentially different diagrams. Thus, a triangular inscription will be a general or isosceles triangle, depending on which of those properties is used in diagrammatic arguments; this is similar to the same card playing different roles in different card games.
- Another set of conventionalized rules governs the application and interpretation of the diagram within and outside of mathematics, i.e., what the diagram can be taken to denote or model. These rules could be termed the external or referential meaning (algebraic terms standing for calculations with numbers or a graph depicting a network or a social structure). The two meanings closely inform and depend on each other.
- Diagrammatic inscriptions express (or can be viewed as expressing) relationships by their very structure, from which those relationships must be inferred based on the given operation rules. Diagrams are not to be understood in a figurative but rather in a relational sense (such as a circle expressing the relation of its peripheral points to the midpoint).
- There is a type-token relationship between the individual and specific material inscription and the diagram of which it is an instance (such as between a written letter and the letter as such).

- Operations with diagrammatic inscriptions are based on the perceptive activity of the individual (such as pattern recognition) that turns mathematics into a perceptive and material activity.
- Diagrammatic reasoning is a rule-based but inventive and constructive manipulation of diagrams for investigating their properties and relationships.
- Diagrammatic reasoning is not mechanistic or purely algorithmic; it is imaginative and creative. Analogy: the music of Bach is based on strict rules of counterpoint but is highly creative and variegated.
- Many steps and arguments of diagrammatic reasoning have no referential meaning, nor do they need any.
- In diagrammatic reasoning the focus is on the diagrammatic inscriptions irrespective of what their referential meaning might be. The objects of diagrammatic reasoning are the diagrams themselves and their established properties.
- Diagrammatic inscriptions arise from many sources and for many purposes: as models of structures and processes, by deliberate design and construction, by idealization and abstraction from experiential reality, etc., and they are used accordingly for many purposes.
- Efficient and successful diagrammatic reasoning presupposes intensive and extensive experience with manipulating diagrams. A comprehensive “inventory” of diagrams, their properties, and relationships supports and facilitates the creative and inventive usage of diagrams. An analogy: expert chess players have command over a great supply of chess diagrams that guide their strategic problem solving. Consequence: learning mathematics has to comprise diagrammatic knowledge of a great variety.
- Diagrams can be viewed as ideograms, such as those in Chinese writing systems. They are not translations from any natural language or abbreviations of names and definitions; by their diagrammatic structure, they “directly” present (to the initiated user!) their intended meaning. The latter usually is a system of relationships (between the elements or the parts of the diagram) and of operations and transformations.
- Diagrams are composed of signs of different characters in the sense of Peirce. There are icons, indices, and symbols as well, and a whole diagram has iconic and symbolic functionality if in itself it is considered to be a sign in the sense of Peirce.

To be understood and used appropriately, diagrams need to be described in natural language and specific terms relating to the diagram. These descriptions and explanations cannot be substituted for the diagram and its various uses, however. In relation to the diagram and its intended relations and operations, this is a meta-language about the diagrams, which also focuses attention and interest on its relevant aspects and activities. It is similar to the way in which the legend on a map of a city explains how to use that map appropriately. Generally, diagrams are imbedded in a complex context and discourse, which is better viewed as a social practice.

- Diagrams are extra-linguistic signs. One cannot speak the diagram, but one can speak about the diagram. In this sense, diagrams are irreducible entities of mathematics (there is no mathematics without “formulas”), yet their properties can be named by words and formulated as theorems. Thus, on the other hand (specialized) language (as extension of natural language) is equally indispensable.

As a final remark: it would be misleading to consider diagrams as mathematical objects. They are the objects and the means of mathematical activity for which we do not have to view them as designating mathematical objects. This emphasis on activity and concrete operations with signs leads us to Wittgenstein’s views.

#### 4.4 Wittgenstein: Meaning as Use

The Austrian philosopher Ludwig Wittgenstein (1889–1951) dedicated a great part of his work to the philosophy of mathematics (e.g., Wittgenstein 1999), proposing radically alternative views on the basic character of mathematics. Together with other features of his writings, this might have prevented any notable recognition within mathematics education. Therefore, this contribution will (also) try to alert the community of mathematics education to the potential of the ideas of Wittgenstein which might (also) influence general attitudes and basic orientations of the concrete teaching in the classroom. A caveat is, of course, that only a few aspects can be treated here and these in only a rather superficial way. The interested reader is referred to Dörfler (2013a, 2014) and the vast literature on Wittgenstein’s philosophy of mathematics, for instance, Kienzler (1997) or Mühlhölzer (2010).

Contrary to the traditional view, Wittgenstein views the meaning of many signs, words, and symbols in general and of mathematics as well to reside in the use made of those signs in what he calls language games or sign games. Thus, signs do not express a meaning that exists independently of the sign game and that is given by something outside of the sign game that the signs refer to and denote. For mathematics, then, the meaning of the signs, symbols, and diagrams does not come from outside of mathematics but is created by a great variety of activities with the signs within mathematics. This resonates very closely with the diagrammatic reasoning described above (though Peirce would hold that thereby some independent “object” is investigated, contrary to the position taken by Wittgenstein, which is strongly non-metaphysical and anti-platonistic). Wittgenstein introduces the metaphor of mathematics as a game, in particular by pointing to chess. In chess, the figures receive all their meaning from the rules of the game, and they do not refer to anything outside of the system of rules. The figures correspond to the signs in mathematics and the game rules correspond to the rules in mathematics for calculating, manipulating, and deriving (i.e., the diagrammatic rules in the above sense). This game metaphor helps to solve many puzzles in math: Consider the

“number” zero. There has been and continues to be a great deal of discussion about what this sign denotes and how it could designate a number. In the Wittgensteinian sense, the meaning of “0” is determined and presented by the rules for how we calculate with it;  $5 + 0 = 5$  or  $0 \times 6 = 0$ , for example, reflect the origin of zero from the place value systems. Thus, there is no mystery and no miracle about zero if you do not ask questions that are outside the purview of math (what Wittgenstein in a telling way calls the prose of mathematics). Very similar considerations apply to the empty set, the “number”  $-1$  and first and foremost to  $i$ , the imaginary unit which simply is determined by the rule that  $i \times i = -1$ . It is a very helpful and sober way of thinking in this way to consider the respective number systems as number games where the meaning of the number signs flows from how they are calculated and not from a mystical reference, say, to “nothing” or negative or imaginary magnitudes. About those mathematical entities we can only know what is shown to us by the results of the calculations within the number games.

To pursue this line of thinking further, we turn to the notion of grammar and grammatical proposition as used by Wittgenstein. He says that mathematical propositions do not describe factual situations as do propositions in science because there are no independent mathematical objects those propositions could be about. In his view, mathematical propositions are instead rules for how to use the terms and signs involved in their formulation as they are developed within the various sign games of mathematics. Or to put in still another way, in mathematics the propositions are used as rules, e.g., in proofs and calculations, though in mathematical prose they are interpreted as accounts of mathematical facts in a mathematical world. Examples for nonmathematical grammatical propositions would be: “White is brighter than black,” “Every rod has a length,” “Nothing can be red and blue at the same place,” or “Every finite set has a number.” Every arithmetic “fact” in this view is just another rule and not the description of an eternal and absolute truth about numbers. The concerns of many philosophers and sociologists about the status of, say, “ $2 + 2 = 4$ ” dissolve when one takes this as a rule, which of course then can neither be verified nor falsified in an empirical interpretation. For Wittgenstein, the whole notion of truth against this background makes no sense since rules are neither true nor false. Rules have to be accepted; they require consent, which is often motivated by a kind of practicability and viability. Rules are outside of all aspects of time (in addition to questions such as when they were established or abolished) or at least this is the way we use rules. Think again of chess as a metaphor for the sign games of mathematics. The rules of chess usually are not viewed as being true or eternal, and one can refuse to accept them but then one will not be playing chess anymore. Such a view fundamentally changes one’s attitudes and relations to mathematics and the learning of mathematics. The practice and fluency in sign games is now the centrepiece and not the mental grasp of ideal and abstract objects or of “ideas” which are just denoted and represented by the mathematical signs. The learner has to indulge in the mathematical “games” whereby meaning and understanding gradually will develop. In mathematics, meaning cannot be imported from outside but emerges inside it through manifold activities. Wittgenstein was often blamed for the apparent conventional and thus



possibly arbitrary character of mathematics derived from his views. Yet to counter this, one can point to the fact that many basic rules (axioms) are motivated by practical or theoretical demands and that many other rules are then derived from given ones by proofs and calculations. On the other hand, there is in fact a great liberty regarding the rules according to which one wants to do the mathematics and this holds as well for the logic involved.

As Wittgenstein says, mathematics can be viewed as the grammar or the grammatical study of its signs and terms. This proves especially helpful wherever a notion of the “infinite” turns up, which notoriously poses great obstacles for learners. Historically it is interesting that Leibniz remarked that for his infinitesimals such as  $dx$ , one should not look for referents, that is, objects that are denoted by them. He took the view that they are completely determined by the rules governing how to operate with them. These rules, on the other hand, were motivated by the problems that Leibniz wanted to solve. In Wittgenstein’s terms, the infinitesimals make sense and have meaning within the sign game developed by Leibniz but are meaningless outside of it. Similarly, a chess figure has no isolated meaning as such, no absolute meaning independent of the whole game and its rules. Meaning always depends on the respective language game or sign game and also reference of the signs to objects will be controlled by the language game. An extreme case in mathematics is the notion of infinite set and infinite cardinal number. It might be difficult for the learner to take a naïve Platonist stance viewing set theory as descriptive of a universe of prefabricated sets (as in Gödel or in Deiser 2010). With Wittgenstein, one can interpret set theory as one possible answer to the question of how one could sensibly talk about infinity. That not every such talk is sensible was shown by the well-known paradoxes. The definitions and the propositions of set theory then are the rules within a language game that develop the grammar of “infinity,” and as is known, different such grammars are possible and sensible. Researching the infinite then becomes the more mundane activity of exploring rule systems in regard to their consequences, which is still a wonderful intellectual achievement. The “infinitely large” becomes part of the prose of math. Again we find that it is not some external object (infinite set) that regulates how mathematics is done but mathematics itself that determines how one can view the infinite, which in a way emerges in the respective language game. It should be clear that such views and attitudes bring mathematics back to the purview of human beings, which does not make it any easier to learn but possibly arouses less fear and anxiety about an inaccessible realm far beyond one’s reach.

The final notion in Wittgenstein to be mentioned briefly is that of “norm” or “paradigm.” In connection with the notions of language game, grammar, and rule use, it permits the dissolution of some of the notorious enigmas ascribed to mathematics: the necessity or unavoidability of mathematics. Mathematics cannot be otherwise and alternatives are not conceivable as is possible for statements, say, about nature. There is no change in mathematics, mathematics is timeless, and its propositions are eternally true and they are exactly true, not only approximately. Furthermore, there is the puzzle of the applicability of mathematics to nature,

though the latter is seen as categorically different from mathematics. The way out of many of these enigmas proposed by Wittgenstein is to recognize that mathematical notions and propositions in many cases are used as a norm, as a measuring stick against which something is judged and evaluated. We use established arithmetic rules to judge the correctness of calculations and of counting: Only what conforms to the rules is considered to be acceptable. Those arithmetic propositions and relations are not used as descriptions of eternally true properties of numbers but as templates to carry out and to check the correctness of other calculations, even if the prose tells us otherwise. The mathematical circle, or the mathematical sphere in this sense, is not used as an object but again as a rule to which something to be called a circle or a sphere has to conform. Those uses of math are not descriptive but rather prescriptive or evaluative. Again as with rules, norms or paradigms have no truth value and all the conundrums about mathematical objects for them simply do not make sense. It is the use made of mathematics that makes it timeless, eternal, apodictic, necessary, and, in a trivial sense, true, since that truth results from accepting something as a rule, a norm, or a paradigm. Mathematical propositions are not used as descriptions of facts but are used as rules for description. There is therefore no need to ascribe to them or to mathematical objects any ontological status, since their “reality” resides in their uses within the sign games of mathematics. At least this Wittgenstein would very likely agree with.

## 4.5 Conclusion

The main purpose of this contribution is to arouse more interest in the views on mathematics and mathematical activity proposed by Peirce and especially Wittgenstein, whose ideas were often overlooked within math education. For some possible consequences of Wittgensteinian ideas for learning mathematics, see Dörfler (2014). A common theme for both of these men is that human intellectual and linguistic activity is fundamentally based on signs of all sorts, and this applies all the more to mathematics. The signs are not just a means or a tool for mathematical activity and creativity, but they are essential and constitutive for mathematics, its notions, and propositions and their meanings. Thus for Peirce, to learn mathematics would be to acquire expertise in diagrammatic reasoning, and for Wittgenstein, it would be to participate in the many various sign games and their techniques. In both cases, which are closely related, it is of great importance to stick meticulously to established rules. This holds for pure mathematics and its proof techniques and for the manifold ways of applying mathematics to other fields. Importantly, mathematics is thereby fundamentally shown to be a deeply social and socially shared cultural activity and product: sign activity can be executed with others and shown to others in a public form. This is very different from imagining mathematics as a kind of abstract and mental activity.

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