Chapter 6
Variations

6.1 Partial Interleavings

In some practical data analysis situations, one considers persistence modules which are only partially interleaved. One such scenario is presented by Chazal et al. in the context of clustering by mode-seeking [18]. A filtered simplicial complex on an input point cloud is compared with the sublevelset filtration of the density function it was sampled from. In low-density regions, the sample is too sparse to expect there to be an interleaving. Nevertheless, there is interleaving when the density is sufficiently high.

This leads to the following notion of partial interleaving, adapted from [18]. Two persistence modules $\mathbb{U}$ and $\mathbb{V}$ are said to be **δ-interleaved up to time** t_0 if there are maps $\phi_t : U_t \to V_{t+\delta}$ and $\psi_t : V_t \to U_{t+\delta}$ defined for all $t \leq t_0$, such that the diagrams (4.1) commute for all values $s < t \leq t_0$; that is, for all values where the maps are defined.

We can prove a modified version of the stability theorem; see Fig. 6.1 (left).

Theorem 6.1 (from [18]) *Let $\mathbb{U}$ and $\mathbb{V}$ be two q-tame persistence modules that are δ-interleaved up to time t_0. Then, there is a partial matching $\mathsf{M} \subset \mathsf{dgm}(\mathbb{U}) \times \mathsf{dgm}(\mathbb{V})$ with the following properties:*

- *Points (p, q) in either diagram for which $\frac{1}{2}|p - q| \leq \delta$ are not required to be matched.*
- *Points (p, q) in either diagram for which $p \geq t_0 - \delta$ are not required to be matched.*

All other points must be matched. Then:

- *If α, β are matched, then the p-coordinates of α, β differ by at most δ.*
- *If α, β are matched and one of α, β lies below the line $q = t_0$, then we have $\mathrm{d}^\infty(\alpha, \beta) \leq \delta$.*

F. Chazal et al., *The Structure and Stability of Persistence Modules*,
SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-42545-0_6

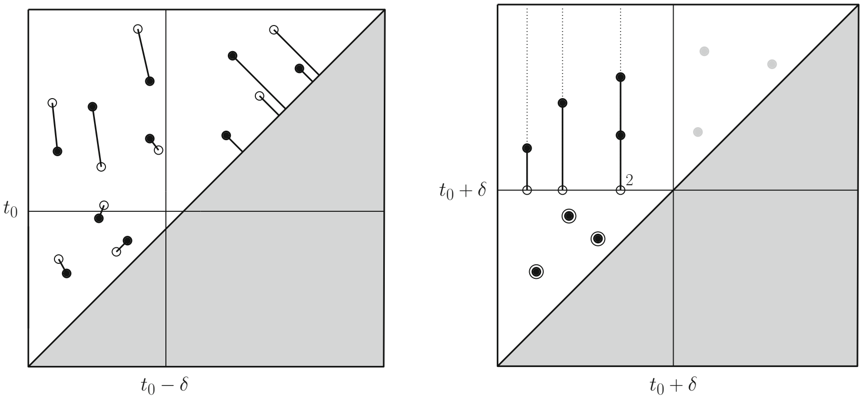

Fig. 6.1 *Left:* The partial matching of Theorem 6.1 between $\mathsf{dgm}(\mathbb{U})$ (•) and $\mathsf{dgm}(\mathbb{V})$ (○). *Right:* The projection from $\mathsf{dgm}(\mathbb{U})$ (• and •) to $\mathsf{dgm}(\tilde{\mathbb{U}})$ (○). The *grey dots* are the points that disappear

For the proof, we introduce two new persistence modules $\tilde{\mathbb{U}}, \tilde{\mathbb{V}}$.

$$\begin{array}{llll} \tilde{U}_t = U_t & \text{if } t \leq t_0 + \delta & \text{and} & \tilde{U}_t = 0 \text{ otherwise} \\ \tilde{V}_t = V_t & \text{if } t \leq t_0 + \delta & \text{and} & \tilde{V}_t = 0 \text{ otherwise} \end{array}$$

with maps

$$\begin{array}{llll} \tilde{u}_t^s = u_t^s & \text{if } t \leq t_0 + \delta & \text{and} & \tilde{u}_t^s = 0 \text{ otherwise} \\ \tilde{v}_t^s = v_t^s & \text{if } t \leq t_0 + \delta & \text{and} & \tilde{v}_t^s = 0 \text{ otherwise} \end{array}$$

for all $s \leq t$. We may call $\hat{\mathbb{U}}, \hat{\mathbb{V}}$ the **truncations** of $\mathbb{U}, \mathbb{V}$ to $(-\infty, T]$, where $T = t_0 + \delta$.

Proof There are three steps.

Step 1. The decorated diagram of a persistence module $\mathbb{U}$ determines the decorated diagram of its truncation $\tilde{\mathbb{U}}$, in a straightforward way. Specifically, transform each point $(p^*, q^*) \in \mathsf{Dgm}(\mathbb{U})$ as follows:

$$(p^*, q^*) \mapsto \begin{cases} (p^*, q^*) & \text{if } q^* < T^+ \\ (p^*, T^+) & \text{if } p^* < T^+ \leq q^* \\ \text{disappears} & \text{if } T^+ \leq p^* \end{cases} \tag{6.1}$$

Then $\mathsf{Dgm}(\tilde{\mathbb{U}})$ is the result of this transformation. The consequent relationship between the undecorated diagrams is illustrated in Fig. 6.1 (right).

Step 2. If $\mathbb{U}, \mathbb{V}$ are δ-interleaved up to time t_0, then $\tilde{\mathbb{U}}, \tilde{\mathbb{V}}$ are δ-interleaved.

Combining the first two steps we get the third.

Step 3. The stability theorem gives a δ-matching between $\mathsf{dgm}(\tilde{\mathbb{U}})$, $\mathsf{dgm}(\tilde{\mathbb{V}})$. This lifts to a matching between $\mathsf{dgm}(\mathbb{U})$, $\mathsf{dgm}(\mathbb{V})$ which has the properties stated in the theorem.

The second and third steps are straightforward. Only the (intuitively plausible) first step requires any technical input. The framework developed in [14] leads to a 2-page argument, presented in the appendix of [18]. Here is a shorter proof. Write $\mu = \mu_{\mathbb{U}}$ and $\tilde{\mu} = \mu_{\tilde{\mathbb{U}}}$. Let A denote the multiset obtained from $\mathsf{Dgm}(\mathbb{U})$ by applying the transformation in Eq. (6.1). Consider an arbitrary rectangle $[a, b] \times [c, d] \in \mathrm{Rect}(\overline{\mathcal{H}})$. We easily see:

$$\mathrm{card}(\mathsf{A}|_{[a,b]\times[c,d]}) = \begin{cases} \mu([a, b] \times [c, d]) & \text{if } d \leq T \\ \mu([a, b] \times [c, +\infty]) & \text{if } c \leq T < d \\ 0 & \text{if } T < c \end{cases}$$

To show that we have correctly determined $\mathsf{Dgm}(\tilde{\mathbb{U}})$, it suffices to show that $\mathrm{card}(\mathsf{A}|_{[a,b]\times[c,d]}) = \tilde{\mu}([a, b] \times [c, d])$ for all rectangles. And indeed:

- If $d \leq T$, then:

$$\begin{aligned} \tilde{\mu}([a, b] \times [c, d]) &= \langle \circ_a\!\!-\!\!\bullet_b\!\!-\!\!\bullet_c\!\!-\!\!\circ_d \mid \tilde{\mathbb{U}} \rangle \\ &= \langle \circ_a\!\!-\!\!\bullet_b\!\!-\!\!\bullet_c\!\!-\!\!\circ_d \mid \mathbb{U} \rangle = \mu([a, b] \times [c, d]) \end{aligned}$$

- If $c \leq T < d$, then:

$$\begin{aligned} \tilde{\mu}([a, b] \times [c, d]) &= \langle \circ_a\!\!-\!\!\bullet_b\!\!-\!\!\bullet_c\!\!-\!\!\circ_d \mid \tilde{\mathbb{U}} \rangle \\ &= \langle \circ_a\!\!-\!\!\bullet_b\!\!-\!\!\bullet_c\!\!-\!\!\!- \mid \tilde{\mathbb{U}} \rangle \\ &= \langle \circ_a\!\!-\!\!\bullet_b\!\!-\!\!\bullet_c\!\!-\!\!\!- \mid \mathbb{U} \rangle = \mu([a, b] \times [c, +\infty]) \end{aligned}$$

  since $\tilde{U}_d = 0$.
- If $T < c$, then:

$$\tilde{\mu}([a, b] \times [c, d]) = \langle \circ_a\!\!-\!\!\bullet_b\!\!-\!\!\bullet_c\!\!-\!\!\circ_d \mid \tilde{\mathbb{U}} \rangle = 0$$

  since $\tilde{U}_c = 0$.

It follows that $\mathsf{Dgm}(\tilde{\mathbb{U}}) = \mathsf{A}$ as claimed. □

6.2 Extended Persistence

Cohen-Steiner, Edelsbrunner and Harer [20] introduced extended persistence to capture the homological information carried by a pair (X, f). Some but not all of this information is recovered by the sublevelset persistence $\mathrm{H}(\mathbb{X}_{\mathrm{sub}})$. The idea is to grow the space from the bottom up, through sublevelsets; and then to relativise the space from the top down, with superlevelsets. Extended persistence is the persistent homology of this sequence of spaces and pairs.

It is usually assumed that (X, f) has finitely many homological critical points (a_i). One applies a homology functor to the finite sequence[1]

$$X^{a_0} \to X^{a_1} \to \cdots \to X^{a_{n-1}} \to X \to (X, X_{a_n}) \to \cdots \to (X, X_{a_2}) \to (X, X_{a_1})$$

to get a quiver representation. The indecomposable summands of this representation are interpreted as features, and are drawn as points in the 'extended persistence diagram'. There are three kinds of feature:

- ordinary features (which are born and die before the central X);
- relative features (which are born and die after the central X);
- extended features (which are born before the X and die after it).

We refer to [20] for the interpretation of these three types of features. The finiteness assumption is satisfied when (X, f) is a compact manifold with a Morse function, or a compact polyhedron with a piecewise-linear map. In the former situation, there are extra symmetries (Poincaré, Lefschetz) which are explored in [20].

In practice, it is straightforward to define the extended persistence diagram under a weaker hypothesis. Suppose X is a compact polyhedron and f is a continuous real-valued function on X. Then:

- rank $\left(\mathrm{H}(X^s) \to \mathrm{H}(X^t)\right) < \infty$ whenever $s < t$; and
- rank $(\mathrm{H}(X, X_s) \to \mathrm{H}(X, X_t)) < \infty$ whenever $s > t$.

The first of these facts is Theorem 3.33. The second fact is proved similarly, by factorising the map $\mathrm{H}(X, X_s) \to \mathrm{H}(X, X_t)$ through some $\mathrm{H}(X, Y)$, where Y is a subpolyhedron of X nested between X_s, X_t. Since $\mathrm{H}(X, Y)$ is finite-dimensional the result follows.

Define the ordered set

$$\boldsymbol{Я} = \{\underline{t} \mid t \in \mathbf{R}\} \quad \text{ordered by} \quad \underline{s} \leq \underline{t} \;\Leftrightarrow\; s \geq t,$$

thought of as a 'backwards' copy of the real line, with bars under numbers to remind us. For extended persistence we may work with the set

$$\mathbf{R}_{\mathrm{EP}} = \mathbf{R} \cup \{+\infty\} \cup \boldsymbol{Я}$$

with the ordering $s < +\infty < \underline{t}$ for all $s, \underline{t}$.

[1] We write $X^t = (X, f)^t = f^{-1}(-\infty, t]$ and $X_t = (X, f)_t = f^{-1}[t, +\infty)$ for sublevelsets and superlevelsets.

The extended persistence module $\mathbb{X}_{\mathrm{EP}} = \mathbb{X}^f_{\mathrm{EP}}$ for (X, f) is defined as follows:

$$\begin{array}{lll} V_t & = \mathrm{H}(X^t) & \text{for } t \in \mathbf{R} \\ V_{+\infty} & = \mathrm{H}(X) & \\ V_{\underline{t}} & = \mathrm{H}(X, X_t) & \text{for } \underline{t} \in \text{Я} \end{array}$$

Note that $\mathrm{H}(X^{+\infty}) = \mathrm{H}(X) \cong \mathrm{H}(X, \emptyset) = \mathrm{H}(X, X_{+\infty})$.

Since $\mathbf{R}_{\mathrm{EP}}$ is order-isomorphic to the real line, we may interpret $\mathbb{X}_{\mathrm{EP}}$ it as a persistence module over $\mathbf{R}$. The two facts cited above imply that it is q-tame, so the decorated diagram is defined away from the diagonal.

Alternatively, we can define the extended persistence diagram in three pieces:

$$\begin{array}{lll} \mu_{\mathrm{ord}}([a, b] \times [c, d]) = \langle \circ_a \!-\!\!-\! \bullet_b \!-\!\!-\! \bullet_c \!-\!\!-\! \circ_d \rangle & \text{when } a < b \leq c < d \\ \mu_{\mathrm{rel}}([\underline{a}, \underline{b}] \times [\underline{c}, \underline{d}]) = \langle \circ_{\underline{a}} \!-\!\!-\! \bullet_{\underline{b}} \!-\!\!-\! \bullet_{\underline{c}} \!-\!\!-\! \circ_{\underline{d}} \rangle & \text{when } a > b \geq c > d \\ \mu_{\mathrm{ext}}([a, b] \times [\underline{c}, \underline{d}]) = \langle \circ_a \!-\!\!-\! \bullet_b \!-\!\!-\! \bullet_{\underline{c}} \!-\!\!-\! \circ_{\underline{d}} \rangle & \text{when } a < b \text{ and } c > d \end{array}$$

taking $V_{-\infty} = 0$ and $V_{\underline{-\infty}} = 0$ whenever needed.

The measures $\mu_{\mathrm{ord}}, \mu_{\mathrm{rel}}$ are defined over the half-plane $\overline{\mathcal{H}}$, whereas μ_{ext} is defined over $\overline{\mathbf{R}}^2$.

Stability for $\mathsf{dgm}_{\mathrm{ord}}$, $\mathsf{dgm}_{\mathrm{rel}}$ and $\mathsf{dgm}_{\mathrm{ext}}$ may be proved individually for each diagram. Given two functions f, g which are δ-close in the supremum norm, there are inclusions

$$\begin{array}{ll} (X, f)^t \subseteq (X, g)^{t+\delta} & (X, f)_t \subseteq (X, g)_{t-\delta} \\ (X, g)^t \subseteq (X, f)^{t+\delta} & (X, g)_t \subseteq (X, f)_{t-\delta} \end{array}$$

which imply the box lemma (Lemma 5.26) for each measure. Since linear combinations of continuous functions are continuous, we can interpolate between f and g to satisfy the hypotheses required by the measure stability theorem (Theorem 5.29).

Remark 6.1 In the spirit of Theorem 3.37, one may treat the case where X is a locally compact polyhedron and f is proper. The exercise of locating the possible singularities of the three measures is left to the persistent reader.