# Chapter 4
# Interleaving

Interleaving is a way of comparing two persistence modules. As with any category, two persistence modules $\mathbb{U}, \mathbb{V}$ are said to be isomorphic if there are maps

$$\Phi \in \text{Hom}(\mathbb{U}, \mathbb{V}), \quad \Psi \in \text{Hom}(\mathbb{V}, \mathbb{U}),$$

such that

$$\Psi\Phi = 1_{\mathbb{U}}, \quad \Phi\Psi = 1_{\mathbb{V}}.$$

This relation is too strong in situations where the persistence modules have been constructed from noisy or uncertain data. The natural response is to consider a weaker relation, **$\delta$-interleaving**, where $\delta \geq 0$ quantifies the uncertainty.

In this section, we define the interleaving relation and study its elementary properties. We prove the nontrivial result (from [14]) that if two persistence modules are $\delta$-interleaved, then they are connected in the space of persistence modules by a path of length $\delta$. This 'interpolation lemma' is a crucial step in the proof of the stability theorem in Chap. 5.

## 4.1 Shifted Homomorphisms

The first step is to consider homomorphisms which shift the value of the persistence index. Let $\mathbb{U}, \mathbb{V}$ be persistence modules over $\mathbf{R}$, and let $\delta$ be any real number. A **homomorphism of degree** $\delta$ is a collection $\Phi$ of linear maps

$$\phi_t : U_t \to V_{t+\delta}$$

for all $t \in \mathbf{R}$, such that the diagram



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$$\begin{array}{ccc} U_s & \xrightarrow{u_t^s} & U_t \\ \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi_t} \\ V_{s+\delta} & \xrightarrow{v_{t+\delta}^{s+\delta}} & V_{t+\delta} \end{array}$$

commutes whenever $s \leq t$.

We write

$$\begin{aligned} \mathrm{Hom}^\delta(\mathbb{U}, \mathbb{V}) &= \{\text{homomorphisms } \mathbb{U} \to \mathbb{V} \text{ of degree } \delta\}, \\ \mathrm{End}^\delta(\mathbb{V}) &= \{\text{homomorphisms } \mathbb{V} \to \mathbb{V} \text{ tof degree } \delta\}. \end{aligned}$$

Composition gives a map

$$\mathrm{Hom}^{\delta_2}(\mathbb{V}, \mathbb{W}) \times \mathrm{Hom}^{\delta_1}(\mathbb{U}, \mathbb{V}) \to \mathrm{Hom}^{\delta_1+\delta_2}(\mathbb{U}, \mathbb{W}).$$

For $\delta \geq 0$, the most important degree-$\delta$ endomorphism is the shift map

$$1_{\mathbb{V}}^{\delta} \in \mathrm{End}^\delta(\mathbb{V}),$$

which is the collection of maps $(v_{t+\delta}^t)$ from the persistence structure on $\mathbb{V}$. If $\Phi$ is a homomorphism $\mathbb{U} \to \mathbb{V}$ of any degree, then by definition $\Phi 1_{\mathbb{U}}^{\delta} = 1_{\mathbb{V}}^{\delta}\Phi$ for all $\delta \geq 0$.

*Remark 4.1* Here is another way to think of morphisms of non-zero degree. For any persistence module $\mathbb{V}$, and $\delta \in \mathbf{R}$, we let $\mathbb{V}[\delta]$ denote the **shifted module**

$$(V[\delta])_t = V_{t+\delta}, \quad (v[\delta])_t^s = v_{t+\delta}^{s+\delta}.$$

In other words, $\mathbb{V}[\delta]$ is obtained from $\mathbb{V}$ by shifting all the information downwards by $\delta$. Then there are obvious identifications

$$\mathrm{Hom}^\delta(\mathbb{U}, \mathbb{V}) = \mathrm{Hom}(\mathbb{U}, \mathbb{V}[\delta]) = \mathrm{Hom}(\mathbb{U}[a], \mathbb{V}[a+\delta])$$

for all $a \in \mathbf{R}$. To avoid excessive notation, we will use the same symbol for

$$\Phi = (\phi_t) : \mathbb{U} \to \mathbb{V}[\delta]$$

as for its shifted equivalent

$$\Phi = (\phi_{t+a}) : \mathbb{U}[a] \to \mathbb{V}[a+\delta]$$

since the constituent maps are the same.

## 4.2 Interleaving

Let $\delta \geq 0$. Two persistence modules $\mathbb{U}, \mathbb{V}$ are said to be $\delta$ **-interleaved** if there are maps

$$\Phi \in \mathrm{Hom}^{\delta}(\mathbb{U}, \mathbb{V}), \quad \Psi \in \mathrm{Hom}^{\delta}(\mathbb{V}, \mathbb{U})$$

such that

$$\Psi\Phi = 1_{\mathbb{U}}^{2\delta}, \quad \Phi\Psi = 1_{\mathbb{V}}^{2\delta}.$$

More expansively (with many more indices written out), there are maps

$$\phi_t : U_t \to V_{t+\delta} \quad \text{and} \quad \psi_t : V_t \to U_{t+\delta}$$

defined for all $t$, such that the following diagrams

$$\begin{array}{ccc} U_s & \xrightarrow{u_t^s} & U_t \\ {\scriptstyle \phi_s}\downarrow & & \downarrow{\scriptstyle \phi_t} \\ V_{s+\delta} & \xrightarrow{v_{t+\delta}^{s+\delta}} & V_{t+\delta} \end{array} \qquad \begin{array}{ccc} U_{s-\delta} & \xrightarrow{u_{s+\delta}^{s-\delta}} & U_{s+\delta} \\ & {\scriptstyle \phi_{s-\delta}}\searrow \quad \nearrow{\scriptstyle \psi_s} & \\ & V_s & \end{array}$$

$$\begin{array}{ccc} V_s & \xrightarrow{v_t^s} & V_t \\ {\scriptstyle \psi_s}\downarrow & & \downarrow{\scriptstyle \psi_t} \\ U_{s+\delta} & \xrightarrow{u_{t+\delta}^{s+\delta}} & U_{t+\delta} \end{array} \qquad \begin{array}{ccc} V_{s-\delta} & \xrightarrow{v_{s+\delta}^{s-\delta}} & V_{s+\delta} \\ & {\scriptstyle \psi_{s-\delta}}\searrow \quad \nearrow{\scriptstyle \phi_s} & \\ & U_s & \end{array} \tag{4.1}$$

commute for all eligible parameter values; that is, for all $s \leq t$.

*Remark 4.2* Where possible, we will be concise rather than expansive.

*Example 4.3* Let $X$ be a topological space and let $f, g : X \to \mathbf{R}$. Suppose $\|f - g\|_\infty < \delta$. Then the persistence modules $\mathrm{H}(\mathbb{X}_{\mathrm{sub}}^f)$, $\mathrm{H}(\mathbb{X}_{\mathrm{sub}}^g)$ are $\delta$-interleaved. Indeed, there are inclusions

$$(X, f)^t \subseteq (X, g)^{t+\delta}$$
$$(X, g)^t \subseteq (X, f)^{t+\delta}$$

for all $t$, which induce maps

$$\Phi : \mathrm{H}(\mathbb{X}_{\mathrm{sub}}^f) \to \mathrm{H}(\mathbb{X}_{\mathrm{sub}}^g)$$
$$\Psi : \mathrm{H}(\mathbb{X}_{\mathrm{sub}}^g) \to \mathrm{H}(\mathbb{X}_{\mathrm{sub}}^f)$$

of degree $\delta$. Since all the maps are induced functorially from inclusion maps, the interleaving relations are automatically satisfied.

This is the situation for which the stability theorem of Cohen-Steiner, Edelsbrunner and Harer [19] was originally stated: if two functions $f, g$ are close then the diagrams for their sublevelset persistent homology are close. Subsequently, stability has been formulated as a theorem about the diagrams of interleaved persistence modules [14, 15]. In the present work, we will come to view stability as a theorem about r-measures.

## 4.3 Interleaving (Continued)

An interleaving between two persistence modules can be thought of as a persistence module over a certain partially ordered set (poset). We develop this idea next.

Consider the standard partial order on the plane:

$$(p_1, q_1) \leq (p_2, q_2) \quad \Leftrightarrow \quad p_1 \leq p_2 \text{ and } q_1 \leq q_2.$$

For any real number $x$, define the corresponding shifted diagonal in the plane:

$$\Delta_x = \{(p, q) \mid q - p = 2x\} = \{(t - x, t + x) \mid t \in \mathbf{R}\}$$

As a poset, this is isomorphic to the real line. We will use the specific isomorphism by which $t \in \mathbf{R}$ corresponds to $(t - x, t + x) \in \Delta_x$. This gives a canonical identification between persistence modules over $\mathbf{R}$ and persistence modules over $\Delta_x$.

**Proposition 4.4** *Let $x, y$ be real numbers. Persistence modules $\mathbb{U}, \mathbb{V}$ are $|y - x|$-interleaved if and only if there is a persistence module $\mathbb{W}$ over $\Delta_x \cup \Delta_y$ such that $\mathbb{W}|_{\Delta_x} = \mathbb{U}$ and $\mathbb{W}|_{\Delta_y} = \mathbb{V}$. Here $\Delta_x \cup \Delta_y$ is regarded as a subposet of $\mathbf{R}^2$.*

*Proof* Assume $x < y$ without loss of generality. We claim that (i) the extra information carried by $(y - x)$-interleaving maps $\Phi, \Psi$ is equivalent to (ii) the extra information carried by $\mathbb{W}$. Let us describe both, more carefully:

(i) In addition to $\mathbb{U}, \mathbb{V}$ we have a system of maps $\Phi = (\phi_t)$, where

$$\phi_t : U_t \to V_{t+y-x},$$

and a system of maps $\Psi = (\psi_t)$, where

$$\psi_t : V_t \to U_{t+y-x}.$$

These are constrained by the relations (for all $\eta \geq 0$).

$$\Phi 1_{\mathbb{U}}^{\eta} = 1_{\mathbb{V}}^{\eta}\Phi, \quad \Psi 1_{\mathbb{V}}^{\eta} = 1_{\mathbb{U}}^{\eta}\Psi, \quad \Psi\Phi = 1_{\mathbb{U}}^{2y-2x}, \quad \Phi\Psi = 1_{\mathbb{V}}^{2y-2x}. \tag{4.2}$$

There are no other constraints.

(ii) In addition to $\mathbb{U}$, $\mathbb{V}$ the persistence module $\mathbb{W}$ carries maps between the two components $\Delta_x$, $\Delta_y$. These maps are constrained by the composition law

$$w_T^R = w_T^S \circ w_S^R$$

for all $R, S, T \in \Delta_x \cup \Delta_y$ with $R \leq S \leq T$.

First, observe that we recover the maps $\phi_t$, $\psi_t$ as vertical maps from $\Delta_x$ to $\Delta_y$, and horizontal maps from $\Delta_y$ to $\Delta_x$, respectively (see Fig. 4.1):

$$\begin{aligned} U_t &= W_{(t-x,t+x)} \to W_{(t-x,t+2y-x)} = V_{t+y-x} \\ V_t &= W_{(t-y,t+y)} \to W_{(t+y-2x,t+y)} = U_{t+y-x} \end{aligned}$$

Next, observe that the composition law implies all of the relations (4.2).

Finally, there is no additional information in $\mathbb{W}$, beyond the interleaving maps and relations. Indeed, all remaining maps $w_T^S$, where $S \leq T$, can all be factored in the form:

$$\begin{aligned} w_T^S &= v_t^{s+y-x} \circ \phi_s \quad \text{if } S \in \Delta_x \text{ and } T \in \Delta_y, \\ w_T^S &= u_t^{s+y-x} \circ \psi_s \quad \text{if } S \in \Delta_y \text{ and } T \in \Delta_x. \end{aligned}$$

Thus each map in $\mathbb{W}$ is an instance of one of

$$\begin{aligned} &1_{\mathbb{U}}^{\eta} && \text{from } \Delta_x \text{ to } \Delta_x, \\ &1_{\mathbb{V}}^{\eta} && \text{from } \Delta_y \text{ to } \Delta_y, \\ &1_{\mathbb{V}}^{\eta}\Phi && \text{from } \Delta_x \text{ to } \Delta_y, \\ &1_{\mathbb{U}}^{\eta}\Psi && \text{from } \Delta_y \text{ to } \Delta_x. \end{aligned}$$

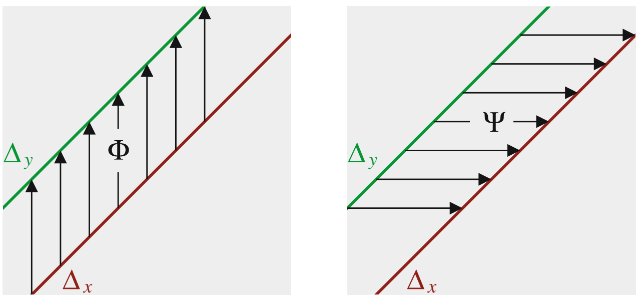



**Fig. 4.1** The maps $\Phi$, $\Psi$ recovered from the module $\mathbb{W}$ over $\Delta_x \cup \Delta_y$

It is a simple matter to verify that the composition law is satisfied for each composable pair of maps. For instance:

$$(1_{\mathbb{V}}^{\eta}\Phi)(1_{\mathbb{U}}^{\zeta}\Psi) = 1_{\mathbb{V}}^{\eta}\Phi 1_{\mathbb{U}}^{\zeta}\Psi = 1_{\mathbb{V}}^{\eta}1_{\mathbb{V}}^{\zeta}\Phi\Psi = 1_{\mathbb{V}}^{\eta+\zeta}1_{\mathbb{V}}^{2y-2x} = 1_{\mathbb{V}}^{\eta+\zeta+2y-2x}$$

This can be done using only the known relations, so there are no further constraints on the $w_T^S$. □

*Remark 4.5* This characterisation makes it clear (or, in another view, depends on the fact) that all composable combinations of the maps $u, v, \phi, \psi$ from a given domain to a given codomain must be equal: indeed, they must agree with the appropriate map $w_T^S$ of $\mathbb{W}$.

## 4.4 The Interpolation Lemma

In this section we prove a crucial result from [14]:

**Lemma 4.6** (interpolation lemma) *Suppose $\mathbb{U}$, $\mathbb{V}$ are a $\delta$-interleaved pair of persistence modules. Then there exists a 1-parameter family of persistence modules $(\mathbb{U}_x \mid x \in [0, \delta])$ such that $\mathbb{U}_0, \mathbb{U}_\delta$ are equal to $\mathbb{U}, \mathbb{V}$ respectively, and $\mathbb{U}_x, \mathbb{U}_y$ are $|y-x|$-interleaved for all $x, y \in [0, \delta]$. Moreover, if $\mathbb{U}, \mathbb{V}$ are q-tame then the $(\mathbb{U}_x)$ may be assumed q-tame also.*

We prove something sharper: given a specific pair of interleaving maps

$$\Phi \in \mathrm{Hom}^{\delta}(\mathbb{U}, \mathbb{V}) \qquad \Psi \in \mathrm{Hom}^{\delta}(\mathbb{V}, \mathbb{U})$$

we explicitly provide, for each $x < y$ in $[0, \delta]$, a pair of interleaving maps

$$\Phi_y^x \in \mathrm{Hom}^{y-x}(\mathbb{U}_x, \mathbb{U}_y) \qquad \Psi_x^y \in \mathrm{Hom}^{y-x}(\mathbb{U}_y, \mathbb{U}_x)$$

such that $\Phi_\delta^0 = \Phi$ and $\Psi_0^\delta = \Psi$, and moreover

$$\Phi_z^y \Phi_y^x = \Phi_z^x \qquad \Phi_x^y \Phi_y^z = \Phi_x^z$$

for all $x < y < z$. In view of Proposition 4.4, this sharp form of the interpolation lemma can be restated as follows.

**Theorem 4.7** (interpolation lemma, version 2) *Any persistence module $\mathbb{W}$ over $\Delta_0 \cup \Delta_\delta$ extends to a persistence module $\overline{\mathbb{W}}$ over the diagonal strip*

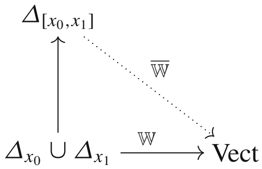
$$\Delta_{[0,\delta]} = \{(p, q) \mid 0 \le q - p \le 2\delta\} \subset \mathbf{R}^2.$$

*If $\mathbb{W}|_{\Delta_0}, \mathbb{W}|_{\Delta_\delta}$ are q-tame, then the extension may be chosen so that each $\overline{\mathbb{W}}|_{\Delta_x}$ is q-tame.*

*Remark 4.8* The extension is by no means unique.

Let us clarify how Theorem 4.7 implies Lemma 4.6. If $\mathbb{U}$, $\mathbb{V}$ are $\delta$-interleaved, then there exists a persistence module $\mathbb{W}$ over $\Delta_0 \cup \Delta_\delta$ such that $\mathbb{W}|_{\Delta_0} = \mathbb{U}$ and $\mathbb{W}|_{\Delta_\delta} = \mathbb{V}$. By Theorem 4.7, this extends to $\overline{\mathbb{W}}$ over the strip $\Delta_{[0,\delta]}$. If we define a 1-parameter family $\mathbb{U}_x = \overline{\mathbb{W}}|_{\Delta_x}$, then $\mathbb{U}_x$, $\mathbb{U}_y$ are $|x-y|$-interleaved for all $x, y \in [0, \delta]$.

For readers familiar with Kan extensions in category theory [43], here is a very short proof of the theorem. Let us regard the posets $\Delta_0 \cup \Delta_\delta$ and $\Delta_{[0,\delta]}$ as categories (see Remark 2.1); then persistence modules over these posets are the same as functors to the category of vector spaces. The theorem asserts the existence of an extension $\overline{\mathbb{W}}$




for any functor $\mathbb{W}$. Peter Bubenik has pointed out to us that the Kan extension theorem immediately provides two such extensions, since the category Vect is both complete (which yields the 'right-extension') and co-complete (which yields the 'left-extension').

We proceed now to a detailed proof, for those who would still like one.

*Proof (Theorem 4.7)* In order to express the proof more symmetrically, it is convenient to replace the interval $[0, \delta]$ by the interval $[-1, 1]$. This can be done by rescaling and translating the plane. Accordingly, suppose we are given a persistence module $\mathbb{W}$ over $\Delta_{-1} \cup \Delta_1$.

Our strategy is to construct two persistence modules over the strip $\Delta_{[-1,1]}$ and a module map between them. The image (or coimage) of this map is itself a persistence module over the strip, and will be the required extension.

By Proposition 4.4, $\mathbb{W}$ provides $\mathbb{U} = \mathbb{W}|_{\Delta_{-1}}$ and $\mathbb{V} = \mathbb{W}|_{\Delta_1}$, which we can view as persistence modules over $\mathbf{R}$ using the canonical parametrisation:

$$U_t = W_{(t+1,t-1)} \quad \text{and} \quad V_t = W_{(t-1,t+1)}$$

and corresponding linear maps $u_t^s$, $v_t^s$. The module $\mathbb{W}$ also provides interleaving maps $\Phi \in \mathrm{Hom}^2(\mathbb{U}, \mathbb{V})$ and $\Psi \in \mathrm{Hom}^2(\mathbb{V}, \mathbb{U})$ of degree 2:

$$\phi_t = w_{(t+1,t+3)}^{(t+1,t-1)} : U_t \to V_{t+2}, \quad \psi_t = w_{(t+3,t+1)}^{(t-1,t+1)} : V_t \to U_{t+2},$$

From $\mathbb{U}$, $\mathbb{V}$ we construct four persistence modules over $\mathbf{R}^2$:

$$\begin{array}{llll}
\mathbb{A} & \text{defined by} \quad A_{(p,q)} = U_{p-1} & \text{and} & a_{(r,s)}^{(p,q)} = u_{r-1}^{p-1} \\
\mathbb{B} & \text{defined by} \quad B_{(p,q)} = V_{q-1} & \text{and} & b_{(r,s)}^{(p,q)} = v_{s-1}^{q-1} \\
\mathbb{C} & \text{defined by} \quad C_{(p,q)} = U_{q+1} & \text{and} & c_{(r,s)}^{(p,q)} = u_{s+1}^{q+1} \\
\mathbb{D} & \text{defined by} \quad D_{(p,q)} = V_{p+1} & \text{and} & d_{(r,s)}^{(p,q)} = v_{r+1}^{p+1}
\end{array}$$

Note that $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ are the vertical and horizontal extensions of the modules $\mathbb{W}|_{\Delta_{\pm 1}}$ to the whole plane. In particular, we have

$$\mathbb{A}|_{\Delta_{-1}} = \mathbb{U}, \quad \mathbb{B}|_{\Delta_1} = \mathbb{V}, \quad \mathbb{C}|_{\Delta_{-1}} = \mathbb{U}, \quad \mathbb{D}|_{\Delta_1} = \mathbb{V},$$

with respect to our canonical parametrisations of each diagonal. Restricting each module to its 'other' diagonal, on the other hand, we have

$$\mathbb{A}|_{\Delta_1} = \mathbb{U}[-2], \quad \mathbb{B}|_{\Delta_{-1}} = \mathbb{V}[-2], \quad \mathbb{C}|_{\Delta_1} = \mathbb{U}[2], \quad \mathbb{D}|_{\Delta_{-1}} = \mathbb{V}[2],$$

using the 'shifted module' notation of Remark 4.1.

Next, we construct four module maps:

$$\begin{array}{lll}
\overline{1}_{\mathbb{U}} : \mathbb{A} \to \mathbb{C} & \text{defined at } (p,q) \text{ to be} & u_{q+1}^{p-1} : U_{p-1} \to U_{q+1} \\
\overline{\Phi} : \mathbb{A} \to \mathbb{D} & \text{defined at } (p,q) \text{ to be} & \phi_{p-1} : U_{p-1} \to V_{p+1} \\
\overline{\Psi} : \mathbb{B} \to \mathbb{C} & \text{defined at } (p,q) \text{ to be} & \psi_{q-1} : V_{q-1} \to U_{q+1} \\
\overline{1}_{\mathbb{V}} : \mathbb{B} \to \mathbb{D} & \text{defined at } (p,q) \text{ to be} & v_{p+1}^{q-1} : V_{q-1} \to V_{p+1}
\end{array}$$

The maps $\overline{\Phi}$, $\overline{\Psi}$ are defined over the whole plane, whereas $\overline{1}_{\mathbb{U}}$ is defined only where $p - 1 \leq q + 1$, and $\overline{1}_{\mathbb{V}}$ is defined only where $q - 1 \leq p + 1$. To verify that the four definitions give module maps, it is enough to observe that the required commutation relations involve composable combinations of the maps $u, v, \phi, \psi$, which always agree by Remark 4.5.

Note that the intersection of the regions of definition, where all four maps are defined, is precisely the strip $\Delta_{[-1,1]}$. Henceforth, we restrict $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ and the four maps to that strip.

Define $\Omega \in \mathrm{Hom}(\mathbb{A} \oplus \mathbb{B}, \mathbb{C} \oplus \mathbb{D})$ by the 2-by-2 matrix

$$\begin{bmatrix} \overline{1}_{\mathbb{U}} & \overline{\Psi} \\ \overline{\Phi} & \overline{1}_{\mathbb{V}} \end{bmatrix}$$

of module maps. Our claim is that $\overline{\mathbb{W}} = \mathrm{im}(\Omega)$ is the required extension. We may equivalently claim that $\overline{\mathbb{W}} = \mathrm{coim}(\Omega) = (\mathsf{A} \oplus \mathsf{B})/\ker(\Omega)$ is the required extension.[1]

**Step 1**. $\overline{\mathbb{W}}|_{\Delta_{-1}}$ is isomorphic to $\mathbb{U}$.

*Proof* On the diagonal $\Delta_{-1}$ we have

$$(\mathbb{A} \oplus \mathbb{B})|_{\Delta_{-1}} = \mathbb{U} \oplus \mathbb{V}[-2], \quad (\mathbb{C} \oplus \mathbb{D})|_{\Delta_{-1}} = \mathbb{U} \oplus \mathbb{V}[2],$$

and the homomorphism $\Omega|_{\Delta_{-1}}$ takes the matrix form

$$\begin{bmatrix} 1_{\mathbb{U}} & \Psi \\ \Phi & 1_{\mathbb{V}}^4 \end{bmatrix}.$$

Since $1_{\mathbb{V}}^4 = \Phi\Psi$, this factorises as

$$\mathbb{U} \oplus \mathbb{V}[-2] \xrightarrow{\Omega_1} \mathbb{U} \xrightarrow{\Omega_2} \mathbb{U} \oplus \mathbb{V}[2]$$

where

$$\Omega_1 = [\, 1_{\mathbb{U}} \ \Psi \,] \quad \text{and} \quad \Omega_2 = \begin{bmatrix} 1_{\mathbb{U}} \\ \Phi \end{bmatrix}$$

in matrix form. Thanks to the $1_{\mathbb{U}}$ entries, we see that $\Omega_1$ is surjective and $\Omega_2$ is injective. This implies that $\Omega_1, \Omega_2$ induce isomorphisms

$$\mathrm{coim}(\Omega|_{\Delta_{-1}}) \xrightarrow{\cong} \mathbb{U} \xrightarrow{\cong} \mathrm{im}(\Omega|_{\Delta_{-1}})$$

as required. □

**Step 2**. $\overline{\mathbb{W}}|_{\Delta_1}$ is isomorphic to $\mathbb{V}$.

*Proof* On the diagonal $\Delta_1$ we have

$$(\mathbb{A} \oplus \mathbb{B})_{\Delta_1} = \mathbb{U}[-2] \oplus \mathbb{V}, \quad (\mathbb{C} \oplus \mathbb{D})_{\Delta_1} = \mathbb{U}[2] \oplus \mathbb{V},$$

and the homomorphism $\Omega|_{\Delta_1}$ takes the matrix form

$$\begin{bmatrix} 1_{\mathbb{U}}^4 & \Psi \\ \Phi & 1_{\mathbb{V}} \end{bmatrix}.$$

[1]The image and the coimage are naturally isomorphic; the difference is whether we wish to think of $\overline{\mathbb{W}}$ as a submodule of $\mathbb{C} \oplus \mathbb{D}$ or as a quotient module of $\mathbb{A} \oplus \mathbb{B}$. In the following pages, we will treat the two points of view with equal emphasis.

Since $1^4_{\mathbb{U}} = \Psi\Phi$, this factorises as

$$\mathbb{U}[-2] \oplus \mathbb{V} \xrightarrow{\Omega_3} \mathbb{V} \xrightarrow{\Omega_4} \mathbb{U}[2] \oplus \mathbb{V}$$

where

$$\Omega_3 = \begin{bmatrix} \Phi & 1_{\mathbb{V}} \end{bmatrix} \quad \text{and} \quad \Omega_4 = \begin{bmatrix} \Psi \\ 1_{\mathbb{V}} \end{bmatrix}$$

in matrix form. Thanks to the $1_{\mathbb{V}}$ entries, we see that $\Omega_3$ is surjective and $\Omega_4$ is injective. This implies that $\Omega_3$, $\Omega_4$ induce isomorphisms

$$\operatorname{coim}(\Omega|_{\Delta_1}) \xrightarrow{\cong} \mathbb{V} \xrightarrow{\cong} \operatorname{im}(\Omega|_{\Delta_1})$$

as required. □

**Step 3**. The cross-maps of $\overline{\mathbb{W}}$ between $\Delta_{-1}$ and $\Delta_1$ correspond to $\Phi$ and $\Psi$ under the isomorphisms of Steps 1 and 2.

*Proof* The cross maps for $\overline{\mathbb{W}}$ are induced by the cross maps for $\mathbb{A} \oplus \mathbb{B}$ (if we view $\overline{\mathbb{W}}$ as a coimage) and equally by the cross maps for $\mathbb{C} \oplus \mathbb{D}$ (if we view $\overline{\mathbb{W}}$ as an image). The vertical cross-map for $\mathbb{A} \oplus \mathbb{B}$ is a map

$$(\mathbb{A} \oplus \mathbb{B})|_{\Delta_{-1}} \longrightarrow (\mathbb{A} \oplus \mathbb{B})|_{\Delta_1}$$

of degree 2 which we can identify as

$$\mathbb{U} \oplus \mathbb{V}[-2] \xrightarrow{1_{\mathbb{U}} \oplus 1^4_{\mathbb{V}}} \mathbb{U} \oplus \mathbb{V}[2].$$

Alternatively, the vertical cross-map for $\mathbb{C} \oplus \mathbb{D}$ is a map

$$(\mathbb{C} \oplus \mathbb{D})|_{\Delta_{-1}} \longrightarrow (\mathbb{C} \oplus \mathbb{D})|_{\Delta_1}$$

of degree 2 which we can identify as

$$\mathbb{U} \oplus \mathbb{V}[2] \xrightarrow{1^4_{\mathbb{U}} \oplus 1_{\mathbb{V}}} \mathbb{U}[4] \oplus \mathbb{V}[2].$$

The following diagram shows the vertical cross-maps for $\mathbb{A} \oplus \mathbb{B}$ (on the left) and $\mathbb{C} \oplus \mathbb{D}$ (on the right), the factorisations of Steps 1 and 2, and the map $\Phi$.

$$\begin{array}{ccccc}
\mathbb{U} \oplus \mathbb{V}[-2] & \xrightarrow{\Omega_1} & \mathbb{U} & \xrightarrow{\Omega_2} & \mathbb{U} \oplus \mathbb{V}[2] \\
\downarrow {\scriptstyle 1_{\mathbb{U}} \oplus 1^4_{\mathbb{V}}} & & \downarrow {\scriptstyle \Phi} & & \downarrow {\scriptstyle 1^4_{\mathbb{U}} \oplus 1_{\mathbb{V}}} \\
\mathbb{U} \oplus \mathbb{V}[2] & \xrightarrow{\Omega_3} & \mathbb{V}[2] & \xrightarrow{\Omega_4} & \mathbb{U}[4] \oplus \mathbb{V}[2]
\end{array}$$

It is enough to show that either square commutes. And indeed

$$\Omega_3(1_{\mathbb{U}} \oplus 1^4_{\mathbb{V}}) = \begin{bmatrix} \Phi & 1_{\mathbb{V}} \end{bmatrix} \begin{bmatrix} 1_{\mathbb{U}} & 0 \\ 0 & 1^4_{\mathbb{V}} \end{bmatrix} = [\, \Phi \;\; 1^4_{\mathbb{V}} \,] = [\Phi][\, 1_{\mathbb{U}} \;\; \Psi \,] = \Phi \Omega_1$$

for the left square, and

$$(1^4_{\mathbb{U}} \oplus 1_{\mathbb{V}})\Omega_2 = \begin{bmatrix} 1^4_{\mathbb{U}} & 0 \\ 0 & 1_{\mathbb{V}} \end{bmatrix} \begin{bmatrix} 1_{\mathbb{U}} \\ \Phi \end{bmatrix} = \begin{bmatrix} 1^4_{\mathbb{U}} \\ \Phi \end{bmatrix} = \begin{bmatrix} \Psi \\ 1_{\mathbb{V}} \end{bmatrix} [\Phi] = \Omega_4 \Phi$$

for the right square. Thus the induced vertical cross-map corresponds to $\Phi$.

A similar argument using the diagram

$$\begin{array}{ccccc}
\mathbb{U}[2] \oplus \mathbb{V} & \xrightarrow{\Omega_3} & \mathbb{V} & \xrightarrow{\Omega_4} & \mathbb{U}[2] \oplus \mathbb{V} \\
\downarrow {\scriptstyle 1^4_{\mathbb{U}} \oplus 1_{\mathbb{V}}} & & \downarrow {\scriptstyle \Psi} & & \downarrow {\scriptstyle 1_{\mathbb{U}} \oplus 1^4_{\mathbb{V}}} \\
\mathbb{U}[2] \oplus \mathbb{V} & \xrightarrow{\Omega_1} & \mathbb{U}[2] & \xrightarrow{\Omega_2} & \mathbb{U}[2] \oplus \mathbb{V}[4]
\end{array}$$

shows that the induced horizontal cross-map corresponds to $\Psi$. □

This completes the construction of the extension $\overline{\mathbb{W}}$. Now we verify the last assertion of theorem. Suppose that $\mathbb{U}$, $\mathbb{V}$ are q-tame, meaning that their non-identity structure maps have finite rank. On any diagonal $\Delta_x$, the restricted modules $\mathbb{A}|_{\Delta_x}$, $\mathbb{B}|_{\Delta_x}$ are shifted copies of $\mathbb{U}$, $\mathbb{V}$ so they are q-tame. It follows that the direct sum $\mathbb{A} \oplus \mathbb{B}|_{\Delta_x}$ and its homomorphic image $\mathbb{U}_x = \Omega(\mathbb{A} \oplus \mathbb{B})|_{\Delta_x}$ are q-tame.

This completes the proof of Theorem 4.7. □

We point out that Step 3 isn't necessary to deduce the interpolation Lemma 4.6. It is sufficient to show that $\mathbb{U} = \mathbb{U}_{-1}$ and $\mathbb{V} = \mathbb{U}_1$ for some 1-parameter family of persistence modules such that each pair $\mathbb{U}_x$, $\mathbb{U}_y$ admits an $|x - y|$-interleaving. This already follows from Steps 1 and 2. We do not need to know that the 2-interleaving of $\mathbb{U}$, $\mathbb{V}$ induced from $\overline{\mathbb{W}}$ is equal to the original 2-interleaving.

## 4.5 The Interpolation Lemma (Continued)

In this optional section, we study the interpolation lemma in greater depth. The results are not used elsewhere. Given two modules $\mathbb{U}$, $\mathbb{V}$ and a $\delta$-interleaving between them, there are at least three natural ways to construct an interpolation. We describe these constructions and some relationships between them.

As in the proof of Theorem 4.7, we may suppose that $\delta = 2$ and that $\mathbb{U}$, $\mathbb{V}$ and their interleaving are represented as a module over $\Delta_{-1} \cup \Delta_1$ in the plane, which we wish to extend to a module over $\Delta_{[-1,1]}$.

It will be helpful to introduce some temporary notation. Let $\mathbb{V}$ be a persistence module over $\mathbf{R}$. Then $\mathbb{V}^{\mathrm{P}}$, $\mathbb{V}^{\mathrm{Q}}$ are the persistence modules over $\mathbf{R}^2$ defined by

$$(V^{\mathrm{P}})_{(p,q)} = V_p, \quad (V^{\mathrm{Q}})_{(p,q)} = V_q,$$

and the canonical linear maps.

Now consider the sequence

$$\begin{matrix} \mathbb{U}[-3]^{\mathrm{Q}} \\ \oplus \\ \mathbb{V}[-3]^{\mathrm{P}} \end{matrix} \xrightarrow{\ \Omega'\ } \begin{matrix} \mathbb{U}[-1]^{\mathrm{P}} \\ \oplus \\ \mathbb{V}[-1]^{\mathrm{Q}} \end{matrix} \xrightarrow{\ \Omega\ } \begin{matrix} \mathbb{U}[1]^{\mathrm{Q}} \\ \oplus \\ \mathbb{V}[1]^{\mathrm{P}} \end{matrix} \xrightarrow{\ \Omega''\ } \begin{matrix} \mathbb{U}[3]^{\mathrm{P}} \\ \oplus \\ \mathbb{V}[3]^{\mathrm{Q}} \end{matrix} \tag{4.3}$$

of modules over $\Delta_{[-1,1]}$ with maps

$$\Omega' = \begin{bmatrix} \overline{1_{\mathbb{U}}} & -\overline{\Psi} \\ -\overline{\Phi} & \overline{1_{\mathbb{V}}} \end{bmatrix}, \quad \Omega = \begin{bmatrix} \overline{1_{\mathbb{U}}} & \overline{\Psi} \\ \overline{\Phi} & \overline{1_{\mathbb{V}}} \end{bmatrix}, \quad \Omega'' = \begin{bmatrix} \overline{1_{\mathbb{U}}} & -\overline{\Psi} \\ -\overline{\Phi} & \overline{1_{\mathbb{V}}} \end{bmatrix}$$

defined analogously to $\Omega$ from the proof of Theorem 4.7.

Notice that $\Omega$, $\Omega'$ and $\Omega''$ are essentially the same map. Certainly $\Omega'$, $\Omega''$ are formally identical, up to a translation $\tau$ of the strip. In fact, each of the modules in the sequence is related to the next by an isomorphism $\sigma$ which changes the sign of the $\mathbb{V}$-term and transforms indices by $(p, q) \mapsto (q + 2, p + 2)$. We have $\tau = \sigma^2$, and conjugacies $\Omega = \sigma\Omega'\sigma^{-1}$ and $\Omega'' = \sigma\Omega\sigma^{-1}$.

**Proposition 4.9** *Each of the three modules*

$$\mathrm{coker}(\Omega'), \quad \mathrm{coim}(\Omega) = \mathrm{im}(\Omega), \quad \ker(\Omega'')$$

*over $\Delta_{[-1,1]}$ defines an interpolating family between $\mathbb{U}$, $\mathbb{V}$.*

*Proof* We already know this for $\mathrm{coim}(\Omega) = \mathrm{im}(\Omega)$ from the proof of Theorem 4.7. Now we outline the proof that $\mathrm{coker}(\Omega')$ and $\ker(\Omega'')$ restrict on $\Delta_{-1}$ to modules isomorphic to $\mathbb{U}$.

On the diagonal $\Delta_{-1}$ the sequence (4.3) restricts to:

$$\begin{matrix} \mathbb{U}[-4] \\ \oplus \\ \mathbb{V}[-2] \end{matrix} \xrightarrow{\ \Omega'\ } \begin{matrix} \mathbb{U} \\ \oplus \\ \mathbb{V}[-2] \end{matrix} \xrightarrow{\ \Omega\ } \begin{matrix} \mathbb{U} \\ \oplus \\ \mathbb{V}[2] \end{matrix} \xrightarrow{\ \Omega''\ } \begin{matrix} \mathbb{U}[4] \\ \oplus \\ \mathbb{V}[2] \end{matrix}$$

and we have factorisations

$$\Omega' \text{ or } \Omega'' = \begin{bmatrix} 1_{\mathbb{U}}^4 & -\Psi \\ -\Phi & 1_{\mathbb{V}} \end{bmatrix} = \begin{bmatrix} -\Psi \\ 1_{\mathbb{V}} \end{bmatrix} \begin{bmatrix} -\Phi & 1_{\mathbb{V}} \end{bmatrix} = \Omega_1'\Omega_2' \text{ or } \Omega_1''\Omega_2''.$$

These reveal that $\mathrm{im}(\Omega') = \mathrm{im}(\Omega_1')$ is a complementary submodule to $\mathbb{U} \oplus 0$ in $\mathbb{U} \oplus \mathbb{V}[t-2]$, and that $\ker(\Omega'') = \ker(\Omega_2'')$ is a complementary submodule to $0 \oplus \mathbb{V}[t+2]$ in $\mathbb{U} \oplus \mathbb{V}[t+2]$. It follows that $\mathrm{coker}(\Omega')$ and $\ker(\Omega'')$ are each isomorphic to $\mathbb{U}$.

By a symmetric argument, the restriction of each module to $\Delta_1$ is isomorphic to $\mathbb{V}$. This completes the proof that $\mathrm{coker}(\Omega')$ and $\ker(\Omega'')$ interpolate between $\mathbb{U}$ and $\mathbb{V}$. □

Which of the three constructions should we prefer? It turns out that $\mathrm{coker}(\Omega')$ and $\ker(\Omega'')$ are respectively isomorphic to the Kan left- and right-extensions, so these are natural from the category theoretic point of view. Now observe that $\Omega\Omega' = 0$ and $\Omega''\Omega = 0$, meaning that (4.3) is a chain complex. It follows that there is a natural projection and a natural inclusion

$$\mathrm{coker}(\Omega') \twoheadrightarrow \mathrm{coim}(\Omega) = \mathrm{im}(\Omega) \hookrightarrow \ker(\Omega'')$$

by which we see that $\mathrm{coim}(\Omega) = \mathrm{im}(\Omega)$ is isomorphic to the image of the composite map $\mathrm{coker}(\Omega') \to \ker(\Omega'')$. In this sense, it is intermediate between the left- and right-extensions; and structurally it is the 'smallest' of the three, being a quotient of one and a subobject of the other.

The surplus information carried by the two Kan extensions may be measured as the kernel of the projection and the cokernel of the inclusion. These are precisely the homology at the second and third terms of (4.3). It follows from the conjugacies described above that the two homology modules are isomorphic upon translating the strip by 2 and interchanging $p$ and $q$ (i.e. reversing the interpolation parameter).

We can use the 'vineyard' technique of [23] to visualise the 1-parameter family of persistence modules produced by each of the three constructions. We obtained the vineyards by sketching the supports of the eight module summands in (4.3) and using the sketches to partition the interpolation parameter range $[-1, 1]$ into suitable intervals for case splitting. It is perhaps easier done than described, so we invite readers to conduct their own calculations and confirm that our vineyards are correct. As further corroboration, one verifies that the homology modules are isomorphic in the sense described above.

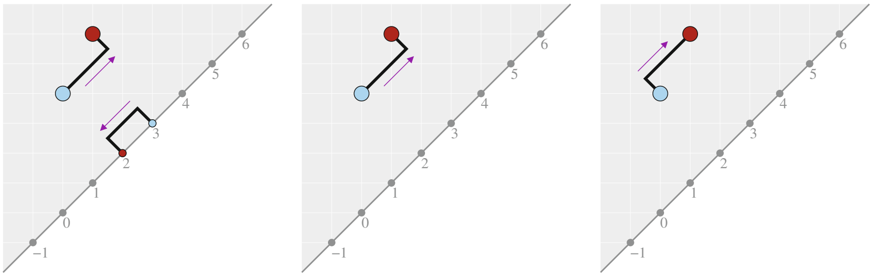



**Fig. 4.2** Vineyards of the cokernel (*left*), image (*middle*), and kernel (*right*) interpolations for the 2-interleaving between **k**[0, 4) and **k**[1, 6)

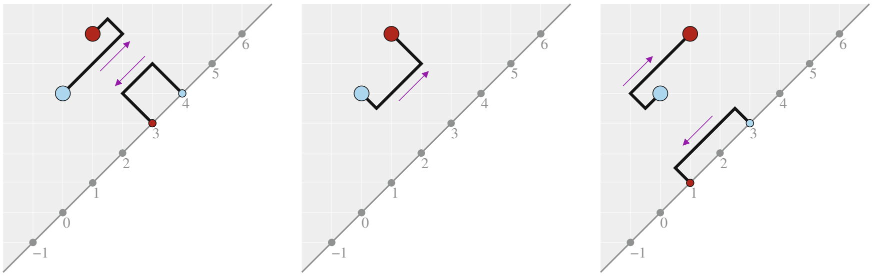



**Fig. 4.3** Vineyards of the cokernel (*left*), image (*middle*), and kernel (*right*) interpolations for the 3-interleaving between **k**[0, 4) and **k**[1, 6)

In Fig. 4.2, we consider the canonical 2-interleaving between interval modules **k**[0, 4) and **k**[1, 6). The thick black lines show how the points of the persistence diagram travel in the plane as we proceed along the interpolating family, for each of the three constructions. Each point travels with speed 1 and traverses a path of length 2 (in the $d^\infty$-metric). The cokernel interpolation has an extra 'ghost' summand which emerges from the diagonal at (3, 3) at the beginning of the interpolation, and is reabsorbed by the diagonal at (2, 2) at the end.

In Fig. 4.3 we repeat the exercise using the canonical 3-interleaving between **k**[0, 4) and **k**[1, 6). The thick black paths now have length three, and the kernel and cokernel interpolations both produce 'ghosts' at the diagonal.