# A Logical Investigation of Heterogeneous Reasoning with Graphs in Elementary Economics

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**Abstract.** Heterogeneous reasoning is a salient component of logic, mathematics, and computer science. Another remarkable field it applies to is economics. In this paper, we apply the proof-theoretic techniques developed in our previous studies [7,8] to heterogeneous reasoning with graphs in elementary economics. We apply the natural deduction-style formalization, which makes it possible to apply well-developed proof-theoretic techniques to the analysis of heterogeneous reasoning with graphs. We also apply the proof-theoretic analysis of free rides developed in [7], and analyze the efficiency of heterogeneous reasoning with graphs. We further discuss abductive reasoning in elementary economics. Abduction has been discussed by philosophers and logicians, and has been extensively studied in the literature on artificial intelligence (see, for example, [2]). In the context of heterogeneous reasoning, we are able to formalize abductive reasoning in elementary economics in the style we employ in our actual reasoning.

## 1 Reasoning with Graphs in Economics

Because of space limitations, we omit some details. For them, see an extended version of this paper: http://abelard.flet.keio.ac.jp/person/takemura/.

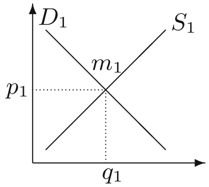
Let us examine the following example of reasoning with graphs in elementary economics, which is a slight modification of an example given in [5].

*Example 1 ([5] p. 94 by Krugman and Wells).* When a new, faster computer chip is introduced, (1) demand for computers using the older, slower chips decreases. (This graphically corresponds to a leftward shift of the demand curve from the original $D_1$ to $D_2$, which we express as $D_2 \leftarrow D_1$.) Simultaneously, (2) computer makers increase their production of computers containing the old chips in order to clear out their stocks of old chips. (Graphically, this corresponds to a rightward shift of the supply curve from the original $S_1$ to $S_2$; $S_1 \rightarrow S_2$.) Furthermore, (3) it is widely known that there is only a minor change in the new computer chip, and it does not make computers dramatically faster. That is, the decrease in demand is small relative to the increase in supply. What happens to the equilibrium price and quantity of computers?

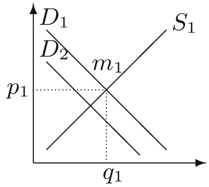
In economics, graphs of demand and supply functions are conventionally drawn in a two-dimensional plane, where the vertical axis represents price and the


M. Jamnik et al. (Eds.): Diagrams 2016, LNAI 9781, pp. 98–104, 2016.
DOI: 10.1007/978-3-319-42333-3_8

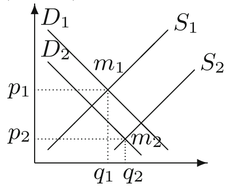
horizontal axis represents quantity demanded or supplied. In the above example, there are no concrete demand and supply functions. Hence, we draw demand and supply curves in the simplest manner, i.e., as straight lines with slopes of $-1$ and 1, respectively, as in the following $G_1$. We assume that $G_1$ represents the initial state of the given market, and its equilibrium is $m_1(q_1, p_1)$.




$G_1$




$G_2$




$G_3$

From premise (1), the original demand curve $D_1$ shifts to $D_2$, as in $G_2$. From (2), the supply curve $S_1$ shifts to $S_2$, as in $G_3$. Although we do not know how much $D_1$ (resp. $S_1$) shifts to $D_2$ (resp. $S_2$), we can infer from (3) that the horizontal shift of the supply curve is greater than that between $D_1$ and $D_2$, as expressed in $G_3$. These shifts lead to the new equilibrium $m_2(q_2, p_2)$. By comparing $m_2$ and the original $m_1$, we find $q_1 < q_2$ and $p_1 > p_2$.

Based on the above example, let us investigate the structure of reasoning with graphs in elementary economics. The reasoning in Example 1 goes as follows.

1. An appropriate graph is given, which describes the initial state of a market.
2. We shift a curve based on the given premise, which represents an increase or decrease in demand or supply. This step may be repeated several times. This shifting operation may be considered as the addition of a new curve, since it is convenient to keep the original curve to compare equilibriums at a later point.
3. With this shift in a curve, a new intersection (equilibrium) arises between the demand and supply curves.
4. We compare the new intersection and the original one, and read off the changes in price and quantity.

Let us compare the above graphical reasoning with algebraic reasoning, where we solve simultaneous equations describing given demand and supply functions.

1. Let the given demand function $D_1$ be $y = -x + \gamma$, and the supply function $S_1$ be $y = x + \delta$, where $\gamma, \delta$ are real numbers.
2. For some real numbers $\alpha > 0$ and $\beta > 0$ such that $\alpha < \beta$, $D_2$ can be expressed as $y = -x + \gamma - \alpha$, and $S_2$ as $y = x + \delta - \beta$.
3. By solving the simultaneous equations $D_1$ and $S_1$, we find $q_1 = \frac{\gamma-\delta}{2}$ and $p_1 = \frac{\gamma+\delta}{2}$, which represent the original equilibrium quantity and price.
4. Similarly, by solving $D_2$ and $S_2$, we find $q_2 = \frac{\gamma-\delta-\alpha+\beta}{2}$ and $p_2 = \frac{\gamma+\delta-\alpha-\beta}{2}$, which represent the new equilibrium quantity and price.
5. By comparing the equilibrium quantities, we find that $q_1 - q_2 = \frac{\alpha-\beta}{2} < 0$ (since $\alpha < \beta$), and hence, we have $q_1 < q_2$.
6. By comparing the equilibrium prices, we find $p_1 - p_2 = \frac{\alpha+\beta}{2} > 0$, i.e., $p_1 > p_2$.

Although the above calculation is not difficult, it is slightly cumbersome compared with our graphical reasoning. Furthermore, if we formalize it in the framework of mathematical logic, a considerable number of steps are required.

Economic reasoning similar to our example has been studied in the framework of qualitative reasoning, e.g., [3,4]. In qualitative reasoning studies, with the aim of implementation, economic reasoning "without graphs" is investigated. Such a formalization in the framework of qualitative reasoning is considered as another symbolic or linguistic counterpart of our graphical formalization. In some aspects, the economic reasoning we investigate here is an extension of previous research, where either the demand or supply curve is allowed to shift just once. Such an analysis has been extended to a more complicated, multivariable setting [3]. However, we concentrate on analyzing the basic demand and supply market, but allow *simultaneous shifts* of the demand and supply curves.

## 2 Heterogeneous Logic with Graphs in Economics HLGe

We assume the shift size is specified when we consider the shift in a curve. However, in our qualitative framework, the exact value of the shift is not as significant as the relation between the magnitude of the shifts. Thus, we do not express the shift size as a numeral, but as a constant $a$ that represents some real number. A formula $C \xrightarrow{a} C'$ then means "$C$ shifts rightward to $C'$ with shift width $a$."

For our heterogeneous system HLGe, we use the following symbols: *Connectives* $\&, \vee, \Rightarrow, \Leftrightarrow, \neg, \forall, \exists$; *Constants for widths* $a, b, c$; *Constants for coordinates* $p, q, r$; *Variables for coordinates* $x, y$; *Curves* $D, S, C, B$. We also use typical mathematical function symbols such as $+$ and $-$ and predicates such as $=$ and $<$.

Among the usual mathematical formulas, we distinguish the following special formulas in HLGe. A **demand (resp. Supply) curve** is written as $D(x) = -x + r$ (resp. $S(x) = x + r$) for some $r$. When $(q, p)$ is an **intersection point** of $C$ and $C'$, we write $C \cap C'(q) = p$. We define **shift formulas** as follows:

- $D \xrightarrow{a} D' := \forall x(D(x) = -x + r \Leftrightarrow D'(x) = -x + r + a)$
- $D' \xleftarrow{a} D := \forall x(D(x) = -x + r \Leftrightarrow D'(x) = -x + r - a)$

Similarly for $S \xrightarrow{a} S'$ and $S' \xleftarrow{a} S$.

**Definition 1 (Graph).** A **graph** in HLGe consists of the following items:

- The first quadrant of the $xy$-coordinate space.
- Straight lines of slope 1, called **supply curves** and named $S, S', S_1, \ldots$; and of slope $-1$, called **demand curves** and named $D, D', D_1, \ldots$. When we do not distinguish between them, we denote a curve by $C, C', C_1, \ldots$.
- Every point of intersection of straight lines is accompanied by its coordinates.

**Definition 2 (Width).** Let $C_i$ and $C_j$ be a pair of lines that are parallel in a graph. Let $q_i$ (and $q_j$) be the intersection point of $C_i$ (resp. $C_j$) and the vertical axis when $C_i$ (resp. $C_j$) is extended as necessary. We define the **width** $w(C_i, C_j)$ between $C_i$ and $C_j$ as $|q_i - q_j|$.
When $G$ is a graph, by $w(G)$, we denote the set of all widths in $G$.

In contrast to a graph drawn as a diagram, we consider the *type* of a graph, which is a symbolic specification. The type of a graph also defines what kind of information we can extract from it; cf. our inference rule Observe in Definition 6.

**Definition 3 (Type).** The **type of a graph** $G$ is $(\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$, where:

- $\mathcal{C}$ is two sequences $D_i \to D_j \to \cdots \to D_n$; $S_k \to S_l \to \cdots \to S_m$ of demand curves and supply curves in $G$, respectively, which are ordered from left to right as they are in the drawn graph $G$.
- By allowing equality, i.e., some elements are equal, $l_w$ is the linearly ordered set $w(C_i, C_j) < \cdots\cdots < w(C_k, C_l) < \cdots$ of all widths in $G$.
- $\mathcal{E}$ is the set of points of intersection in $G$ of the form $D_i \cap S_j(q_k) = p_k$.
- $l_p$ is the linearly ordered set $p_i < p_j < \cdots$ of all $y$-coordinates of intersections.
- $l_q$ is the linearly ordered set $q_i < q_j < \cdots$ of all $x$-coordinates of intersections.

The translation of our graphs into first-order formulas is straightforward based on the type of graph.

**Definition 4 (Translation of graphs).** A graph $G$ of $(\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$ is translated into a conjunctive formula $\bigwedge \mathcal{C} \& \bigwedge l_2 \& \bigwedge \mathcal{E} \& \bigwedge l_p \& \bigwedge l_q$, where $\bigwedge X$ denotes the conjunction of all corresponding formulas contained in the set $X$.

For the set-theoretical semantics of HLGe, it is sufficient to employ a domain of real numbers in which arithmetic operations such as $+$ and $-$ are defined. Hence, we provide the real closed field with the ordering relation $<$ as our model. Then, graphs in HLGe are interpreted as follows.

**Definition 5 (Interpretation of graphs).** Let $M$ be a model. Let $G$ be a graph of $(\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$, where $\mathcal{C} = D_1 \to D_2 \to \cdots D_n$; $S_1 \to S_2 \to \cdots S_m$, and $l_w = w(C_1, C_2) < w(C_3, C_4) < \cdots < w(C_k, C_l)$. Then, $M \models G$ if and only if

- $M \models D_1 \xrightarrow{w(D_1,D_2)} D_2 \,\&\cdots\&\, D_{n-1} \xrightarrow{w(D_{n-1},D_n)} D_n$; and
- $M \models S_1 \xrightarrow{w(S_1,S_2)} S_2 \,\&\cdots\&\, S_{m-1} \xrightarrow{w(S_{m-1},S_m)} S_m$; and
- $M \models w(C_1, C_2) < w(C_3, C_4) < \cdots < w(C_k, C_l)$; and
- $M \models \mathcal{E}$, that is, $M \models C \cap C'(q) = p$ for all $C \cap C'(q) = p \in \mathcal{E}$.

The inference rules for HLGe consist of the usual natural deduction rules for first-order formulas and rules for graphs. Our rules for graphs are the following Apply and Observe as in Hyperproof [1].

**Definition 6 (Inference rules for graphs of HLGe).**

Apply: Let $G$ be a graph, that contains a curve $C$ but does not contain $C'$. Let $C \xrightarrow{a} C'$ be a shift formula. Let $l$ be an ordering condition that specifies a linear ordering of all widths in $w(G) + C' = w(G) \cup \{w(C', B) \mid B$ is a curve parallel to $C$ (including $C$ ) in $G\}$:

$$\frac{G \quad C \xrightarrow{a} C' \quad l}{G'} \text{ Apply}$$

where $G'$ is obtained from $G$ by adding the curve $C'$ so that (1) $C'$ is parallel to $C$; (2) $C'$ is orthogonal to every curve that is orthogonal to $C$; (3) the width between $C'$ and $C$ is $a$; (4) the widths including $a$ satisfy $l$.
Similarly for $C' \overset{a}{\leftarrow} C$.

Observe: From a given graph $G$, we can extract, as a conclusion, any corresponding formula contained in the type of $G$.

When the given ordering condition $l$ does not fully specify a linear ordering among $w(G) + C'$, we cannot apply Apply. In such a case, we enumerate all possible linear orderings of $w(G) + C'$ and apply the $\vee$-elimination rule ($\vee E$) of natural deduction: Let $\{l_1, \ldots, l_n\}$ be the enumeration of all possible linear orderings of $w(G) + C'$ that satisfies the given $l$. Since $l_1 \vee \cdots \vee l_n$ is provable from $l$, we divide the cases according to $l_1, \ldots, l_n$ by using $\vee E$, and then, apply Apply in every case as follows:

$$\dfrac{\dfrac{l}{l_1 \vee \cdots \vee l_n} \qquad \begin{array}{c}\dfrac{G \quad C \overset{a}{\rightarrow} C' \quad [l_1]^m}{G_1}\,\text{Apply} \\ \vdots \\ G'/\psi\end{array} \qquad \cdots \qquad \begin{array}{c}\dfrac{G \quad C \overset{a}{\rightarrow} C' \quad [l_n]^m}{G_n}\,\text{Apply} \\ \vdots \\ G'/\psi\end{array}}{G'/\psi}\ \vee E, m$$

where $G'/\psi$ denotes that either a graph $G'$ or a first-order formula $\psi$ is obtained, and $[l_i]^m$ denotes the assumption $l_i$ is closed as usual in natural deduction. By regarding the above part of a proof as an inference rule, we call it the rule of Cases.

*Example 2 (A proof in* HLGe*).* Figure 1 is an example of a proof in HLGe.

It is shown that HLGe can handle simultaneous curve shifts even though Apply (and Cases) is applied in order during a proof.

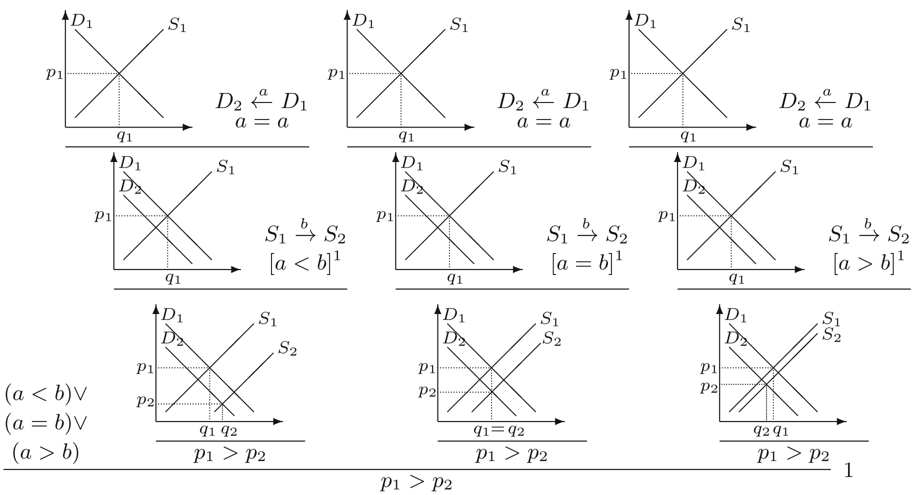



**Fig. 1.** A proof of $D_2 \overset{a}{\leftarrow} D_1$ , $S_1 \overset{b}{\rightarrow} S_2$ , $D_1 \cap S_1(p_1) = q_1$ , $D_2 \cap S_2(p_2) = q_2 \vdash p_1 > p_2$, which describes the situation of Example 1 without condition (3).

The soundness theorem for HLGe is proved, after dividing several cases, in a similar way to that described in Sect. 1 by algebraic calculation.

**Theorem 1 (Soundness).** *Let $\mathcal{S}$ be a set of shift formulas; $\mathcal{E}$ be a set of intersections; $\mathcal{O}$ be a set of ordering conditions among widths; and $\mathcal{A}$ be a conjunction of formulas comparing x- and y-coordinates. If $\mathcal{S}, \mathcal{E}, \mathcal{O} \vdash \mathcal{A}$, then $\mathcal{S}, \mathcal{E}, \mathcal{O} \models \mathcal{A}$.*

By slightly extending the notion of *free ride* [6], we refer to diagrammatic objects, or the translated formulas thereof, as **free rides** if they do not appear in the given premise diagrams or sentences, but (automatically) appear in the conclusion after adding pieces of information to the given premise diagrams. The notion of free rides enables us to analyze the effectiveness of each inference rule. (Cf. [7,8].) Let us consider our Apply. We compare the types, or translated formulas, of graphs of premises and the conclusion. Let $G = (\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$. Then, $G'$ is $(\mathcal{C}', l'_w, \mathcal{E}', l'_p, l'_q)$, where:

- $\mathcal{C}' = \mathcal{C} \cup \{C \to C'\}$, and $l'_w = l$,
- $\mathcal{E}' = \mathcal{E} \cup \{C' \cap B(q) = p \mid B \text{ is orthogonal to } C \text{ in } G\}$,
- $l'_p$ is the linear ordering of $l_p \cup \{p \mid C' \cap B(q) = p \in \mathcal{E}'\}$,
- $l'_q$ is the linear ordering of $l_q \cup \{q \mid C' \cap B(q) = p \in \mathcal{E}'\}$.

Observe that $\mathcal{C}'$ and $l'_w$ are already given in the premises of Apply. On the other hand, the differences between $\mathcal{E}'$ and $\mathcal{E}$, $l'_p$ and $l_p$, and $l'_q$ and $l_q$, respectively are free rides of Apply, as they do not appear in the premises.

## 3 Abduction in Economic Reasoning

*Example 3.* When a new, faster computer chip is introduced, (1) demand for computers using the older, slower chips decreases (i.e., $D_2 \overset{a}{\leftarrow} D_1$). Simultaneously, (2) computer makers increase their production of computers containing the old chips in order to clear out their stocks of old chips (i.e., $S_1 \overset{b}{\to} S_2$). When the equilibrium quantity falls in response to these events, what possible explanations are there for this change?

Let $D_1 \cap S_1(q_1) = p_1$ and $D_2 \cap S_2(q_2) = p_2$. First, note that we cannot prove $q_1 > q_2$ under the given premises (1) and (2), as observed in Example 2. Thus, our task in this question is to find a possible explanation $H$ such that $D_2 \overset{a}{\leftarrow} D_1$, $S_1 \overset{b}{\to} S_2, D_1 \cap S_1(q_1) = p_1, D_2 \cap S_2(q_2) = p_2, H \vdash q_1 > q_2$ holds. In Example 2, the two given premises (1) and (2) provide three graphs, according to whether $a < b, a = b$, or $a > b$, as shown in Fig. 1. Among these three graphs, we find a graph (the third one) in which $q_1 > q_2$ holds. Thus, we know that $q_1 > q_2$ holds when $a > b$ holds for the shift widths of the demand and supply curves. Hence, we can propose $a > b$ as a possible explanation $H$.

This type of reasoning is called *abduction*, and frequently appears in scientific reasoning. Abduction has been extensively studied in the literature on artificial intelligence AI. In the framework of AI, abduction is usually formalized as the task of finding a hypothesis $H$ that explains a given observation $O$ under a theory $T$ such that $O$ is a logical consequence of $T$ and $H$, i.e., $T, H \vdash O$, and $T, H$ are consistent. To solve abductive problems, the usual strategy such as resolution and proof-search to construct deductive proofs are applied. (See, for example, [2] for surveys of abduction in AI.) Our strategy in this paper can be considered as a kind of model enumeration. Our inference using graphs in HLGe essentially corresponds to model construction by regarding our graph as a certain kind of representative model. When there is insufficient information on the shift widths of curves, we enumerate all possible cases (i.e., models) by using Cases. We can then determine the required explanation from among these cases. To describe our abductive reasoning more formally, we modify Cases as follows:

$$\dfrac{\dfrac{l}{\overline{\overline{l_1 \vee \cdots \vee l_n}}} \quad \dfrac{G \quad C \xrightarrow{a} C' \quad \underline{l_i}}{G_i}\,\textsf{Apply}}{G_i}\,\textsf{AbCases}$$

where $l_i$ is one of the linear orderings of $l_1, \ldots, l_n$, and the underline indicates a proposed explanation. Similarly for $C' \xleftarrow{a} C$.

We formalize our procedure as follows. Let $\mathcal{S}, \mathcal{E}, \mathcal{O}$ be given premises, and $\mathcal{A}$ be a given conclusion or observation. Our task is to find an explanation $H$ such that $\mathcal{S}, \mathcal{E}, \mathcal{O}, H \vdash \mathcal{A}$ holds, where we restrict $H$ to be an ordering condition. (1) We construct a proof of $\mathcal{S}, \mathcal{E}, \mathcal{O} \vdash \mathcal{A}$ by using AbCases as well as our Apply, Observe (and Cases) for HLGe. (2) Among the applications of AbCases, we choose a linear ordering $\underline{l_i}$ that has the maximal length, and set $H = l_i$.

In this paper, we concentrated on a competitive market described by supply and demand models. However, extending our HLGe would enable the investigation of economic reasoning with graphs employed in various other analyses, such as a consumer's optimal consumption analysis and IS-LM analysis.

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