Chapter 2
Theoretical Preliminaries of Acoustics

In this chapter, we review some of the fundamentals of acoustics and introduce the spherical harmonic expansion of a sound field, which is the basis for the spherical harmonic processing framework used with spherical microphone arrays.

This chapter intends to introduce the key theory and equations required in the rest of the book. For a more comprehensive introduction to acoustics, the reader is referred to [2, 12], or [17, 20] for a thorough treatment of acoustics in spherical coordinates.

2.1 Fundamentals of Acoustics

The propagation of acoustic waves through a material is described by a second-order partial differential equation known as the *wave equation*. The homogeneous wave equation describes the evolution of the sound pressure p as a function of time t and position $\llcorner\mathbf{r} = (x, y, z)$ in a homogeneous, source-free medium.[1] In three dimensions it is given by [12, Eq. 1.5]

$$\nabla^2 p(\llcorner\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 p(\llcorner\mathbf{r}, t)}{\partial t^2} = 0, \tag{2.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.2}$$

[1] In this section, vectors in Cartesian coordinates are denoted with a corner mark $\llcorner$ to distinguish them from vectors in spherical coordinates, which will be introduced in Sect. 2.2.

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is the Laplace operator in Cartesian coordinates (x, y, z) and c denotes the speed of sound. The separation of variables method is used to simplify the analysis. The time-harmonic solution to the wave equation can then be written in the form

$$p(\mathbf{r}, t) = P(\mathbf{r}, k)e^{i\omega t}, \tag{2.3}$$

where $i = \sqrt{-1}$, and $P(\mathbf{r}, k)$, to be defined later in this section, is a function of the position $\mathbf{r}$ and the wavenumber k. The wavenumber is related to the angular frequency ω, ordinary frequency f and speed of sound c via the dispersion relation

$$k = \frac{\omega}{c} = \frac{2\pi f}{c}. \tag{2.4}$$

The acoustic waves are assumed to be propagating in a non-dispersive medium, such that the propagation speed c is independent of the wavenumber k. Throughout this book, the speed of sound is assumed to be constant; when a numerical value is required, we will use $c = 343$ m/s, obtained when the medium is air at a temperature of approximately 19 °C [12, Eq. 1.1].

The function $P(\mathbf{r}, k)e^{i\omega t}$ in (2.3) can be represented in the complex plane by a rotating vector or a *phasor*. The time-independent vector, represented by the complex number $P(\mathbf{r}, k)$, is the *complex amplitude*. The complex amplitude is multiplied by the unit vector $e^{i\omega t}$ rotating anti-clockwise at speed ω (in rad $\cdot$ s^{-1}), which is the angular frequency of the harmonic function.

Warning:
Throughout this book, $e^{i\omega t}$ represents the time dependence of a positive-frequency wave; a convention that is commonly adopted in electrical and mechanical engineering. In Sect. 2.3, we will summarize the effect of the choice of convention on the key equations of this chapter.

The Fourier transform of a time-domain signal $f(t)$ is defined as

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t. \tag{2.5}$$

As a consequence, the $e^{i\omega t}$ term in the time-harmonic solution to the wave equation (2.3) is eliminated when applying the Fourier transform. Using (2.5), the frequency-domain homogeneous wave equation, also known as the *homogeneous Helmholtz equation*, is obtained [12, Eq. 3.1]:

$$\nabla^2 P(\mathbf{r}, k) + k^2 P(\mathbf{r}, k) = 0, \tag{2.6}$$

where $P(\mathbf{r}, k) = \mathcal{F}\{p(\mathbf{r}, t)\}$ denotes the temporal Fourier transform of $p(\mathbf{r}, t)$.

The homogeneous wave equation and Helmholtz equation assume a source-free medium. If waves are being produced by a harmonic disturbance, a source function of the form $s(\llcorner\mathbf{r}, t) = S(\llcorner\mathbf{r}, k)e^{i\omega t}$ is added to the right-hand side of the homogeneous wave equation (2.1) to obtain the *inhomogeneous* wave equation

$$\nabla^2 p(\llcorner\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 p(\llcorner\mathbf{r}, t)}{\partial t^2} = -s(\llcorner\mathbf{r}, t), \tag{2.7}$$

and by taking the temporal Fourier transform $\mathcal{F}$, we obtain the inhomogeneous Helmholtz equation

$$\nabla^2 P(\llcorner\mathbf{r}, k) + k^2 P(\llcorner\mathbf{r}, k) = -S(\llcorner\mathbf{r}, k). \tag{2.8}$$

In the presence of a unit-amplitude harmonic point source at a position $\llcorner\mathbf{r}_\mathrm{s}$, the solution to the wave equation is known as the *Green's function* and is denoted by $G(\llcorner\mathbf{r}|\llcorner\mathbf{r}_\mathrm{s}, k)$. Alternatively it is termed an *acoustic transfer function* (ATF) from the point $\llcorner\mathbf{r}_\mathrm{s}$ to the point $\llcorner\mathbf{r}$. The frequency-domain source function is then given by $S(\llcorner\mathbf{r}, k) = \delta_3(\llcorner\mathbf{r} - \llcorner\mathbf{r}_\mathrm{s})$, where $\delta_3(\cdot)$ denotes the three dimensional Dirac delta function, and the Green's function can be found by solving the following equation:

$$\nabla^2 G(\llcorner\mathbf{r}|\llcorner\mathbf{r}_\mathrm{s}, k) + k^2 G(\llcorner\mathbf{r}|\llcorner\mathbf{r}_\mathrm{s}, k) = -\delta_3(\llcorner\mathbf{r} - \llcorner\mathbf{r}_\mathrm{s}). \tag{2.9}$$

The Green's function must also satisfy a boundary condition at infinity, the *Sommerfeld radiation condition*, which ensures that sources radiate energy instead of absorbing it. It is given by [20, Eq. 8.28]

$$\lim_{||\llcorner\mathbf{r}-\llcorner\mathbf{r}_\mathrm{s}||\to\infty} ||\llcorner\mathbf{r} - \llcorner\mathbf{r}_\mathrm{s}|| \left(\frac{\partial G(\llcorner\mathbf{r}|\llcorner\mathbf{r}_\mathrm{s}, k)}{\partial||\llcorner\mathbf{r} - \llcorner\mathbf{r}_\mathrm{s}||} - ikG(\llcorner\mathbf{r}|\llcorner\mathbf{r}_\mathrm{s}, k)\right) = 0, \tag{2.10}$$

where $||\cdot||$ denotes the 2-norm (Euclidean norm).

For a source at a position $\llcorner\mathbf{r}_\mathrm{s}$ and a receiver at a position $\llcorner\mathbf{r}$, a solution to the inhomogeneous Helmholtz equation satisfying the Sommerfeld radiation condition is given by the *free-space Green's function*, where free-space indicates that the only boundary condition that applies is the Sommerfeld radiation condition, that is, the waves are not propagating within an enclosure. The free-space Green's function is given by [20, Eq. 8.5]

$$G(\llcorner\mathbf{r}|\llcorner\mathbf{r}_\mathrm{s}, k) = \frac{e^{-ik||\llcorner\mathbf{r}-\llcorner\mathbf{r}_\mathrm{s}||}}{4\pi||\llcorner\mathbf{r} - \llcorner\mathbf{r}_\mathrm{s}||}. \tag{2.11}$$

From (2.11) it is clear that $G(\llcorner\mathbf{r}|\llcorner\mathbf{r}_\mathrm{s}, k) = G(\llcorner\mathbf{r}_\mathrm{s}|\llcorner\mathbf{r}, k)$. This equality represents one of the most fundamental examples of the principle of acoustic reciprocity because the pressure at a receiver point is unchanged when exchanging the source and receiver positions.

2.2 Sound Field Representation Using Spherical Harmonic Expansion

To describe the sound field on the surface of a sphere, we need to find the solutions of the Helmholtz differential equation, as described in the previous section, on the surface of the sphere. In the following, we introduce spherical harmonics, which are a series of special functions defined on the surface of a sphere and are commonly used to solve such differential equations. After introducing the spherical harmonics, we introduce a spherical harmonic expansion of the free-space Green's function that underpins the spherical harmonic domain (SHD) processing in this book.

We adopt the spherical coordinate system used in [5, 13, 18, 20], which is illustrated in Fig. 2.1. The spherical coordinates are related to Cartesian coordinates x, y, z via the expressions [20, Eq. 2.47]

$$x = r \sin\theta \cos\phi, \tag{2.12a}$$

$$y = r \sin\theta \sin\phi, \tag{2.12b}$$

$$z = r \cos\theta, \tag{2.12c}$$

where r, θ and ϕ respectively denote the radius, inclination and azimuth. Conversely, the spherical coordinates may be obtained from the Cartesian coordinates using

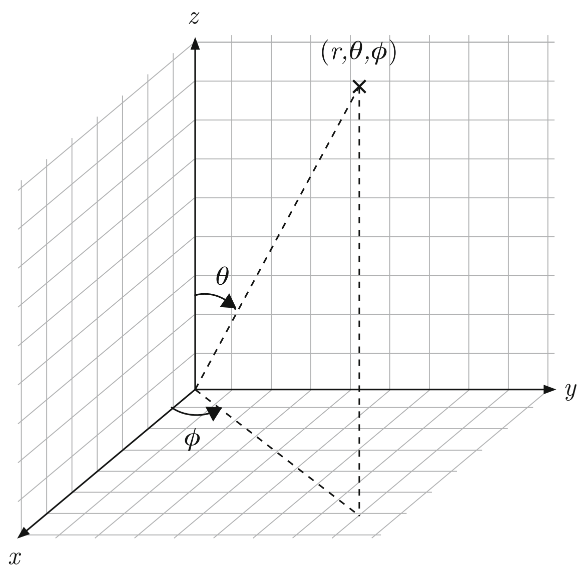

Fig. 2.1 Spherical coordinate system used in this book, defined relative to Cartesian coordinates. The radial distance r is the distance between the observation point and the origin of the coordinate system. The inclination angle θ (a.k.a. co-latitude, polar angle, or normal angle) is measured from the positive z-axis, and the azimuth angle ϕ is measured in the xy-plane from the positive x-axis. Copyright © Daniel Jarrett. Used with permission

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{2.13a}$$

$$\theta = \arccos\left(\frac{z}{r}\right), \tag{2.13b}$$

$$z = \arctan\left(\frac{y}{x}\right), \tag{2.13c}$$

where arctan is the four-quadrant inverse tangent (implemented using the function `atan2()` in many computational environments including, for example, MATLAB).

We express the vectors $\mathbf{r}$ and $\mathbf{r}_\mathrm{s}$ in spherical coordinates as $\mathbf{r} = (r, \Omega) = (r, \theta, \phi)$ and $\mathbf{r}_\mathrm{s} = (r, \Omega_\mathrm{s})$. It is hereafter assumed that when the addition, scalar product and 2-norm operators are applied to vectors in spherical coordinates, these operations will in fact be performed in the Cartesian space by first performing a conversion from spherical to Cartesian coordinates using (2.12).

The *spherical harmonic* of order $l \geq 0$ and degree or mode m (satisfying $|m| < l$) is denoted by Y_{lm} and defined as

$$Y_{lm}(\Omega) = Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\mathcal{P}_{lm}(\cos\theta)e^{im\phi}, \tag{2.14}$$

where $\mathcal{P}_{lm}$ denotes the associated Legendre function of order[2] l and degree m.

The spherical harmonics, derived in [1, 20], represent the *angular component* of the solutions to the Helmholtz equation in spherical coordinates, and are involved in solving many problems in spherical coordinates. A number of zero-, first- and second-order spherical harmonics are plotted for illustrative purposes in Fig. 2.2.

For positive degrees m, the associated Legendre functions are related to the Legendre polynomials $\mathcal{P}_l(x)$ by the formula

$$\mathcal{P}_{lm}(x) = (-1)^m\,(1-x^2)^{m/2}\,\frac{\mathrm{d}^m}{\mathrm{d}\,x^m}\mathcal{P}_l(x), \tag{2.15}$$

where the factor $(-1)^m$ is known as the Condon-Shortley phase. For negative degrees m, the associated Legendre functions can be obtained from

$$\mathcal{P}_{l(-m)}(x) = (-1)^m\,\frac{(l-m)!}{(l+m)!}\,\mathcal{P}_{lm}(x), \tag{2.16}$$

[2] In this book, for consistency with spherical array processing literature, we refer to l as the order and m as the degree of the spherical harmonics and associated Legendre functions (or polynomials). However, it should be noted that in other fields, l is referred to as the degree, and m as the order. This reflects the fact that the words *degree* and *order* are used interchangeably when referring to polynomials.

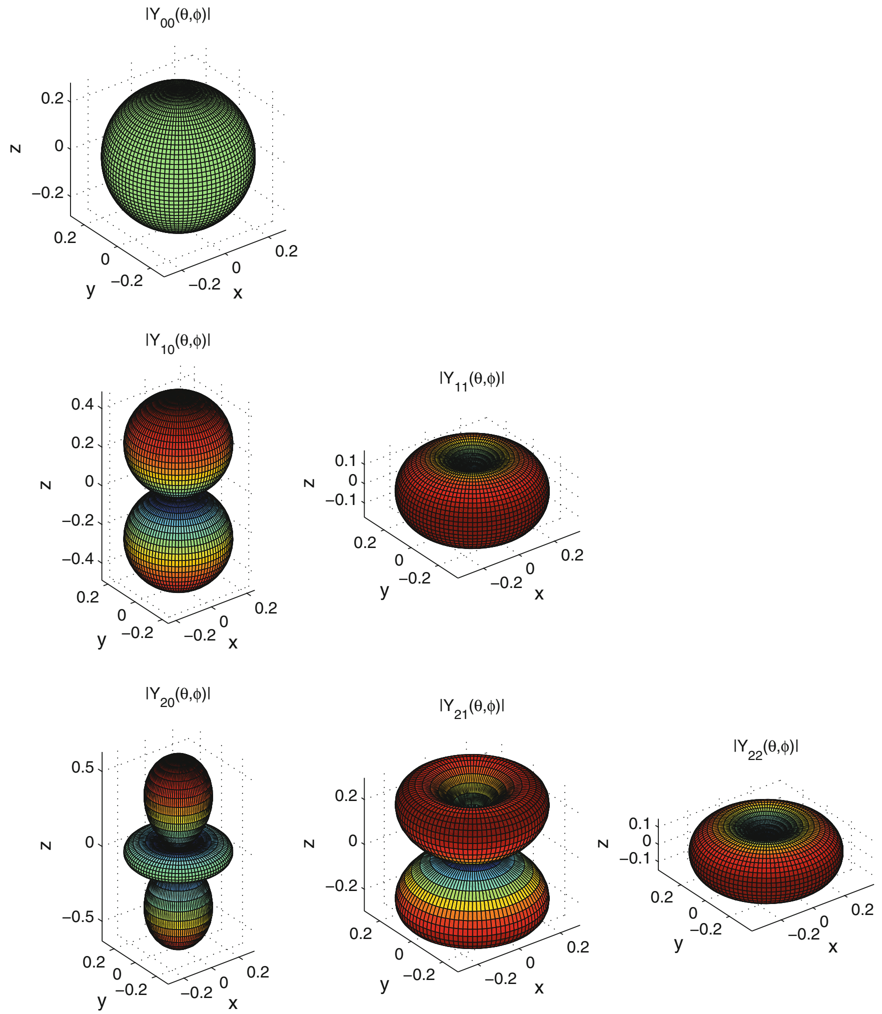

Fig. 2.2 Magnitude $|Y_{lm}(\theta, \phi)|$ of the complex spherical harmonics for $\{l \in \mathbb{Z}|0 \leq l \leq 2\}$, $\{m \in \mathbb{Z}|0 \leq m \leq l\}$. The plots for $m < 0$ are omitted as they are identical to those for $m > 0$. Copyright © Daniel Jarrett. Used with permission

where $m > 0$. From (2.16) it follows that the spherical harmonics for corresponding negative degrees m can be computed using

$$Y_{l(-m)}(\Omega) = (-1)^m \, Y^*_{lm}(\Omega), \tag{2.17}$$

where $m > 0$.

The spherical harmonics constitute an *orthonormal* set of solutions to the Helmholtz equation in spherical coordinates, that is [20, Eq. 6.45]:

$$\int_{\Omega \in \mathcal{S}^2} Y_{lm}(\Omega) Y^*_{l'm'}(\Omega) \mathrm{d}\Omega = \delta_{l,l'} \delta_{m,m'}, \tag{2.18}$$

where the notation $\int_{\Omega \in \mathcal{S}^2} \mathrm{d}\Omega$ is used to denote compactly the solid angle[3] $\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \mathrm{d}\theta \mathrm{d}\phi$, and the Kronecker delta $\delta_{i,j}$ is defined as

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.19}$$

In addition, they constitute a *complete* set of solutions, or equivalently they satisfy the completeness relation [20, Eq. 6.47]

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi') = \delta(\cos\theta - \cos\theta')\delta(\phi - \phi'), \tag{2.20}$$

where δ denotes the Dirac delta function. As a result, any function on a sphere can be represented using a *spherical harmonic expansion* (SHE).

In particular, the free-space Green's function (2.11) can be expanded using the following SHE [20, Eqs. 8.22 and 8.76]:

$$G(\mathbf{r}|\mathbf{r}_\mathrm{s}, k) = \frac{e^{-ik||\mathbf{r}-\mathbf{r}_\mathrm{s}||}}{4\pi||\mathbf{r} - \mathbf{r}_\mathrm{s}||} \tag{2.21}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \underbrace{-i\,k\,j_l(kr)\,h_l^{(2)}(kr_\mathrm{s}) Y^*_{lm}(\Omega_\mathrm{s})}_{\text{expansion coefficients}} Y_{lm}(\Omega), \tag{2.22}$$

where $(\cdot)^*$ denotes the complex conjugate, j_l is the spherical Bessel function of order l, and $h_l^{(2)}$ is the spherical Hankel function of the second kind and of order l. The spherical Bessel function forms the real part of the Hankel function, and the spherical Neumann function forms its imaginary part. The spherical Hankel function of the first kind $h_l^{(1)}$, used in Sect. 2.3, is the complex conjugate of $h_l^{(2)}$. The spherical Bessel and Neumann functions represent the *radial component* of the solutions to the Helmholtz equation in spherical coordinates.

In many cases it is convenient to remove the sum over all degrees m in (2.22) using the *spherical harmonic addition theorem* [1], which states that

$$\sum_{m=-l}^{l} Y^*_{lm}(\Omega_\mathrm{s}) Y_{lm}(\Omega) = \frac{2l+1}{4\pi} \mathcal{P}_l \left(\frac{\mathbf{r} \cdot \mathbf{r}_\mathrm{s}}{r r_\mathrm{s}} \right) \tag{2.23a}$$

$$= \frac{2l+1}{4\pi} \mathcal{P}_l(\cos\Theta), \tag{2.23b}$$

[3] The factor $\sin\theta$ compensates for the denser sampling near the poles ($\theta = 0$ and $\theta = \pi$).

where $\mathbf{r} \cdot \mathbf{r}_s$ denotes the scalar product[4] of the vectors $\mathbf{r}$ and $\mathbf{r}_s$, $\mathcal{P}_l$ is the Legendre polynomial of order l and Θ is the angle between $\mathbf{r}$ and $\mathbf{r}_s$. Using (2.23), the SHE of the free-space Green's function (2.22) then becomes

$$G(\mathbf{r}|\mathbf{r}_s, k) = -\frac{ik}{4\pi} \sum_{l=0}^{\infty} j_l(kr) h_l^{(2)}(kr_s)(2l+1)\mathcal{P}_l(\cos\Theta). \tag{2.24}$$

Under farfield conditions, when $r_s \to \infty$, the spherical wave represented by the free-space Green's functions (2.21) and (2.24) can be approximated as a plane wave. Although plane waves are only an approximation of spherical waves in the far field, plane waves are actually of utmost importance, since any complex wavefield can be represented as a superposition of plane waves [12, Chap. 1].

To obtain a far-field approximation of the Green's function given by (2.21), the denominator $||\mathbf{r} - \mathbf{r}_s||$ is first approximated by r_s. The phase term cannot be approximated so simply since it oscillates with respect to $||\mathbf{r} - \mathbf{r}_s||$. Instead, this term is approximated as [18, Eq. 2.29]

$$||\mathbf{r} - \mathbf{r}_s|| \approx r_s - \frac{\mathbf{r} \cdot \mathbf{r}_s}{r_s} \tag{2.25a}$$

$$\approx r_s - r\cos\Theta. \tag{2.25b}$$

Applying these approximations to (2.21) then yields

$$\begin{aligned} G(\mathbf{r}|\mathbf{r}_s, k) &= \frac{e^{-ik||\mathbf{r}-\mathbf{r}_s||}}{4\pi||\mathbf{r}-\mathbf{r}_s||} \\ &\approx \frac{e^{-ikr_s}}{4\pi r_s} e^{+ikr\cos\Theta}. \end{aligned} \tag{2.26}$$

A farfield approximation for the SHE of the Green's function given by (2.24) can be obtained by making use of the large argument approximation of the spherical Hankel function [20, Eqs. 6.68 and 6.58]:

$$h_l^{(2)}(kr_s) \approx i^{l+1} \frac{e^{-ikr_s}}{kr_s} \quad \text{for} \quad kr_s \gg 1 \tag{2.27}$$

Applying this approximation to the free-space Green's function (2.24), we obtain

$$G(\mathbf{r}|\mathbf{r}_s, k) \approx \frac{e^{-ikr_s}}{4\pi r_s} \sum_{l=0}^{\infty} i^l j_l(kr)(2l+1)\mathcal{P}_l(\cos\Theta). \tag{2.28}$$

[4] As noted earlier in the chapter, the scalar product of vectors in spherical coordinates is applied after these vectors have been converted to Cartesian coordinates using (2.12).

The second term in (2.28) is equal to $e^{+ikr\cos\Theta}$ [20, Eq. 6.174], that is,

$$e^{+ikr\cos\Theta} = \sum_{l=0}^{\infty} i^l j_l(kr)(2l+1)\mathcal{P}_l(\cos\Theta). \tag{2.29}$$

This is the expression for the pressure measured at a position $\mathbf{r}$ due to a unit-amplitude plane wave incident from a direction Ω_s, which we will use on multiple occasions later in this book.

2.3 Sign Convention

As mentioned in Sect. 2.1 the way the time dependence of a positive-frequency wave is defined affects many of the equations in this chapter, such as (2.3). In order to avoid any confusion, we have listed the key equations under each convention in Table 2.1.

Table 2.1 Key equations under the two sign conventions: one common in the acoustics literature and the other common in the engineering literature. The engineering convention is adopted in this book

Acoustics convention	Engineering convention
Temporal Fourier transform	
$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{+i\omega t}\mathrm{d}t$	$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t$
Free-space Green's function	
$G(\mathbf{r}\|\mathbf{r}_s, k) = \dfrac{e^{+ik\|\|\mathbf{r}-\mathbf{r}_s\|\|}}{4\pi\|\|\mathbf{r}-\mathbf{r}_s\|\|}$	$G(\mathbf{r}\|\mathbf{r}_s, k) = \dfrac{e^{-ik\|\|\mathbf{r}-\mathbf{r}_s\|\|}}{4\pi\|\|\mathbf{r}-\mathbf{r}_s\|\|}$
Free-space Green's function (expansion)	
$G(\mathbf{r}\|\mathbf{r}_s, k) = \dfrac{ik}{4\pi}\sum_{l=0}^{\infty} j_l(kr)h_l^{(1)}(kr_s) \times(2l+1)\mathcal{P}_l(\cos\Theta)$	$G(\mathbf{r}\|\mathbf{r}_s, k) = \dfrac{-ik}{4\pi}\sum_{l=0}^{\infty} j_l(kr)h_l^{(2)}(kr_s) \times(2l+1)\mathcal{P}_l(\cos\Theta)$
Farfield approximation for the free-space Green's function	
$G(\mathbf{r}\|\mathbf{r}_s, k) \approx \dfrac{e^{+ikr_s}}{4\pi r_s}e^{-ikr\cos\Theta}$	$G(\mathbf{r}\|\mathbf{r}_s, k) \approx \dfrac{e^{-ikr_s}}{4\pi r_s}e^{+ikr\cos\Theta}$
Farfield approximation for the free-space Green's function (expansion)	
$G(\mathbf{r}\|\mathbf{r}_s, k) \approx \dfrac{e^{+ikr_s}}{4\pi r_s}\sum_{l=0}^{\infty}(-i)^l j_l(kr) \times(2l+1)\mathcal{P}_l(\cos\Theta)$	$G(\mathbf{r}\|\mathbf{r}_s, k) \approx \dfrac{e^{-ikr_s}}{4\pi r_s}\sum_{l=0}^{\infty} i^l j_l(kr) \times(2l+1)\mathcal{P}_l(\cos\Theta)$
Plane wave	
$e^{-ikr\cos\Theta} = \sum_{l=0}^{\infty}(-i)^l j_l(kr)(2l+1) \times\mathcal{P}_l(\cos\Theta)$	$e^{+ikr\cos\Theta} = \sum_{l=0}^{\infty} i^l j_l(kr)(2l+1) \times\mathcal{P}_l(\cos\Theta)$

The engineering convention is adopted in this book. A more detailed discussion of sign conventions in spherical microphone array processing and, in particular, the effects of inconsistent use of sign conventions, can be found in [19].

2.4 Sound Intensity

In acoustic signal processing, the signals of interest are usually sound pressure signals, the sound pressure being the physical quantity that is perceived by the ear. However, a sound field can also be analyzed in terms of the acoustic energy that is radiated, transmitted and absorbed [4], allowing sound sources to be located and their power to be determined.

The *sound intensity vector* describes the magnitude and direction of the flow of acoustic energy per unit area, and has units of watts per square metre. The *instantaneous* sound intensity vector at a position $\mathbf{r}$ and time t is defined as [12, Eq. 1.26]

$$\boldsymbol{\mathcal{I}}(\mathbf{r}, t) = p(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t), \tag{2.30}$$

where $p(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ respectively denote the sound pressure and particle velocity vector at a position $\mathbf{r}$.

The *time-averaged* intensity vector has been found to be of more practical significance [4], and is defined as [4, 20]

$$\boldsymbol{\mathcal{I}}(\mathbf{r}) = \langle\, p(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t)\,\rangle\,, \tag{2.31}$$

where $\langle\,\cdot\,\rangle$ denotes the time-averaging operation. The time-average of the net flow of energy out of a closed surface $\mathcal{S}$ is zero unless power is generated (or dissipated) within this surface [4], in which case it is equal to the power P_{src} of the sound source enclosed, or equivalently [4, Eq. 5]

$$\oint_{\mathcal{S}} \boldsymbol{\mathcal{I}} \cdot \mathrm{d}\mathbf{S} = P_{\text{src}}, \tag{2.32}$$

where $\mathrm{d}\mathbf{S}$ denotes the differential surface area vector normal to $\mathcal{S}$.

For a simple harmonic sound field with constant angular frequency ω, the time-averaged sound intensity vector can be expressed in complex notation as [4, 20]

$$\boldsymbol{\mathcal{I}}(\mathbf{r}, \omega) = \frac{1}{2}\Re\left\{p(\mathbf{r}, \omega)\mathbf{v}^*(\mathbf{r}, \omega)\right\}, \tag{2.33}$$

where $p(\mathbf{r}, \omega)$ and $\mathbf{v}(\mathbf{r}, \omega)$ are complex exponential quantities, and $\Re\{\cdot\}$ denotes the real part of a complex number.

In general, there is no simple relationship between the intensity vector and sound pressure [7]. Nevertheless, for a plane progressive wave, the sound pressure p is related to the particle velocity $\mathbf{v}$ via the relationship [14–16]

$$\mathbf{v}(\mathbf{r}, t) = -\frac{p(\mathbf{r}, t)}{\rho_0\, c}\mathbf{u}(\mathbf{r}, t), \tag{2.34}$$

where ρ_0 and c respectively denote the ambient density of the medium and speed of sound, and $\mathbf{u}$ is a unit vector pointing from $\mathbf{r}$ towards the source. In this case, the direction of arrival of a sound source can be determined as the direction opposite to that of the intensity vector $\mathcal{I}$.

Point sources produce spherical waves, but when sufficiently far from these sources, in the farfield, these waves can be considered as plane waves so that p and $\mathbf{v}$ are in phase. In contrast, in the nearfield, p and $\mathbf{v}$ are out of phase [4]. This phase relationship can be described by next introducing the concept of *active* and *reactive* sound fields. All time-stationary fields can be split into two components, described by [6]:

- An active intensity vector, given by the product of the pressure p and the in-phase component of the particle velocity vector $\mathbf{v}$, which is the intensity vector we have described thus far. The active intensity vector has a non-zero time average [8], computed using (2.33).
- A reactive intensity vector, given by the product of the pressure p and the out-of-phase component of the particle velocity vector $\mathbf{v}$, which measures the energy stored in a sound field. The time-average of the reactive intensity vector is zero; to quote Fahy: there is "*local oscillatory transport of energy*" [6].

In the nearfield, the reactive field is stronger than the active field [4, 11]. In an anechoic environment, where there are no reflections, the strength of the reactive field decreases rapidly as the distance from the source increases [4, 11], such that in the farfield the sound field is essentially an active field. In Chap. 5, we will also take advantage of the fact that in a diffuse sound field, often used to model reverberation, the time-averaged active intensity vector is zero [9].

In practice, measurement of the intensity vector is difficult: typically it is measured with two closely-spaced matched pressure microphones using a finite-difference approximation of the pressure gradient (the *p*–*p* method [10]), although this method is very sensitive to mismatches in the phase response of the two microphones. The alternative (the *p*-*u* method [10]) is to combine a pressure transducer and a particle velocity transducer; this can be done using the Microflown [3]. In Sect. 5.1.3, we will see that the intensity vector can also be measured using a spherical microphone array.

2.5 Chapter Summary

The main aim of this chapter has been to introduce some of the relevant elements of the fundamentals of acoustics. The chapter reviewed the key equations that govern the propagation of sound waves in a medium, more specifically, the wave equation, Helmholtz equation, and free-space Green's function. We also presented the SHE of the Green's function; the SHE forms the basis of a processing framework that advantageously exploits the spherical symmetry of spherical microphone arrays. Finally, we introduced the sound intensity vector, which describes the magnitude and direction of the flow of acoustic energy. In Chap. 5, it will be seen that the intensity vector can be employed in the estimation of two acoustic parameters: the direction of arrival (DOA) of a sound source, and the diffuseness of a sound field.

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