Chapter 13

Special Functions and Transforms

In this short chapter we present some exercises on elliptic functions and on the Mellin transform. We also briefly discuss some aspects of the Fourier transform pertaining to the Bargmann transform.

13.1 Elliptic functions

The first exercise is taken from the book of Choquet on topology [46, p. 315], [47, p. 299]. The purpose of the exercise is to build a *meromorphic* bi-periodic function on $\mathbb{C}$ (thus it has a lattice of periods). Such functions are called elliptic. For more on elliptic functions expressed as infinite products, see for instance [167, pp. 286–290]. See also Exercise 7.2.15.

Exercise 13.1.1. *Let $k \in \mathbb{C}$ with $|k| > 1$.*

(a) *Show that the infinite product*

$$P(z) = \prod_{\ell=1}^{\infty} \left(1 + \frac{z}{k^\ell}\right)$$

converges for all $z \neq -k^\ell$, $\ell = 1, 2, \ldots$.

(b) *Show that*

$$P(kz) = (1 + z)P(z).$$

(c) *Set $S(z) = P(z)P(1/z)(1 + z)$. Show that $S(kz) = kzS(z)$.*

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(d) *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be distinct points in $\mathbb{C}$ such that*

$$a_1 \cdots a_n = b_1 \cdots b_n, \tag{13.1.1}$$

and let $M(z) = \dfrac{S(a_1 z) \cdots S(a_n z)}{S(b_1 z) \cdots S(b_n z)}$. Show that $M(kz) = M(z)$.

(e) *Set $G(z) = M(e^z)$. What can be said about G?*

Remark 13.1.2. An additive analog of (13.1.1) comes into play in Exercise 13.3.3. See equation (13.3.2) there.

Exercise 13.1.3. *Using Exercise* 3.6.2, *show that the function*

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{p,q \in \mathbb{Z} \\ (p,q) \neq (0,0)}} \frac{1}{(z - (p + iq))^2} - \frac{1}{(p + iq)^2}$$

is analytic in $\mathbb{C} \setminus \mathbb{Z} + i\mathbb{Z}$.

The function $\wp$ is called the Weierstrass function (associated to the lattice $\mathbb{C} \setminus \mathbb{Z} + i\mathbb{Z}$). It has only poles and satisfies

$$\wp(z + 1) = \wp(z + i) = \wp(z),$$

and hence is an elliptic function. It follows as a consequence of Exercise 7.2.15 that the function $\wp$ satisfies a differential equation of the form

$$(\wp')^2 = g_0 \wp^3 + g_1 \wp^2 + g_2 \wp + g_3$$

for complex numbers g_0, g_1, g_2 and g_3 such that $g_0 \neq 0$.

The function $\wp$ is closely related to the function ϑ appearing in Exercise 13.2.1. See [162, p. 25].

Question 13.1.4.

(1) *Find the decomposition* (12.1.4) *for $f(z) = \wp''(z)$.*
(2) *Compare the decompositions* (12.1.4) *for a general elliptic function and its derivative.*

In contrast with the case of rational functions we have:

Question 13.1.5. *Show that the composition of two* (*non-trivial*) *elliptic functions is not elliptic.*

13.2 The ϑ function

Exercise 13.2.1. *Let $\tau \in \mathbb{C}$ be such that* $\operatorname{Im} \tau > 0$. *Show that the function*

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau + 2\pi i n z}$$

is entire (as a function of z), and that it satisfies

$$\vartheta(z+1,\tau) = \vartheta(z,\tau), \tag{13.2.1}$$

$$\vartheta(z+\tau,\tau) = e^{-i\pi\tau-2\pi iz}\vartheta(z,\tau). \tag{13.2.2}$$

Show that

$$\vartheta\left(\frac{1+\tau}{2},\tau\right) = 0. \tag{13.2.3}$$

The function ϑ is called the theta function with characteristic τ. See [162] for a thorough study of these functions and of their applications.

In Exercise 13.2.2 we now show that $\frac{1+\tau}{2}$ is the only zero of ϑ modulo $\mathbb{Z}+\tau\mathbb{Z}$.

Exercise 13.2.2. *Show that the zeros of the function*

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{in^2\tau+2\pi inz}$$

are

$$\frac{1+\tau}{2} + m + \tau n, \quad n,m\in\mathbb{Z}.$$

13.3 An application to periodic entire functions

Exercise 13.3.1. *Let f be an entire function and assume that*

$$f(z+1) = f(z).$$

Show that there is a function g analytic in $\mathbb{C}\setminus\{0\}$ such that

$$f(z) = g(e^{2\pi iz}).$$

Show that there exist complex numbers $c_n, n\in\mathbb{Z}$ such that

$$f(z) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inz},$$

where the convergence is uniform on every closed strip inside every closed horizontal strip.

Exercise 13.3.2. *Let $\tau\in\mathbb{C}$ be such that $\operatorname{Im}\tau>0$. Apply the previous result to find all entire functions f such that, for some pre-assigned complex numbers a and b,*

$$\begin{aligned} f(z+1) &= f(z),\\ f(z+\tau) &= e^{az+b}f(z). \end{aligned} \tag{13.3.1}$$

See [162, pp. 2–3].

Exercise 13.3.3. *Let f be a non-identically vanishing entire function satisfying the conditions* (13.3.1)*, and let $a_1, \ldots, a_N, b_1, \ldots, b_N$ be complex numbers such that*

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} b_n. \tag{13.3.2}$$

Show that the function

$$q(z) = \prod_{n=1}^{N} \frac{f(z - a_n)}{f(z - b_n)}$$

is elliptic.

13.4 The Γ function and the Mellin transform

The Mellin transform is defined by the formula

$$(M(f))(z) = \int_0^\infty t^{z-1} f(t) dt \tag{13.4.1}$$

for appropriate functions f defined on $(0, \infty)$, and where for $t > 0$ and $z \in \mathbb{C}$ we set

$$t^z = e^{z \ln t}.$$

We refer to [50, Chapitre II] for more information. The case $f(t) = e^{-t}$ leads to the important Gamma function (see (3.1.11)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

In the following exercise, the convergence of the integral (3.1.11) is studied. In Exercise 13.4.2 we will see that the function Γ defined in the following exercise is in fact analytic in $\operatorname{Re} z > 0$ (and in fact by analytic continuation, in $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Exercise 13.4.1. *Show that the integral* (3.1.11) *converges for every z such that* $\operatorname{Re} z > 0$*. Show that, for real $x > 0$, it holds that*

$$\Gamma(x + 1) = x\Gamma(x). \tag{13.4.2}$$

We now turn to a proof of the analyticity of the Gamma function (see (3.1.11) and the previous exercise).

Exercise 13.4.2. *Show that the* Γ *function*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

is analytic in $\operatorname{Re} z > 0$.

Hint. Consider compact sets of the form

$$K = \{(x,y)\,;\, m \le x \le M \text{ and } -R \le y \le R\},$$

with $m > 0$ and $R > 0$. Show that the series of functions

$$\Gamma_n(z) = \int_{1/n}^{n} t^{z-1} e^{-t} dt, \quad n = 1, 2, \ldots,$$

converges uniformly on K to Γ.

Exercise 13.4.3. *Let* Γ *denote the Gamma function defined by* (3.1.11)*. Show that*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}, \quad \operatorname{Re} z > 0. \tag{13.4.3}$$

Hint (See for instance [23, Exercise 2.6.2, p. 119].)**.** Apply the dominated convergence theorem (see Theorem 17.5.2) to the series of functions

$$f_n(t) = 1_{[0,n]}(t) \left(1 - \frac{t}{n}\right)^n t^{z-1},$$

where we have denoted by $1_{[0,n]}(t)$ the indicator function of the interval $[0, n]$:

$$1_{[0,n]}(t) = \begin{cases} 1, & \text{if } t \in [0,n], \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 13.4.4 (see [50, pp. 49–50])**.**

(a) *Show that the Mellin transform of* e^{-t^2} *is equal to* $\frac{1}{2}\Gamma(z/2)$.

(b) *Show that the Mellin transforms of* $\cos t$ *and* $\sin t$ *are respectively*

$$\Gamma(z)\cos\frac{\pi z}{2} \quad \textit{and} \quad \Gamma(z)\sin\frac{\pi z}{2}, \quad \textit{with } \operatorname{Re} z \in (0,1).$$

In the following exercise implicit is the hypothesis that there exist real numbers c_1 and c_2 such that $\int_0^\infty u^{c_j - 1}|f_j(u)|du < \infty$ for $j = 1, 2$.

Exercise 13.4.5. *Let* f_1 *and* f_2 *be functions with Mellin transforms* F_1 *and* F_2 *respectively.*

(1) *Show that the Mellin transform of the function*

$$\int_0^\infty f_1(u) f_2(t/u) \frac{du}{u} \tag{13.4.4}$$

is $F_1 F_2$.

(2) *compute* (13.4.4) *when* $f_1(u) = f_2(u) = e^{-u}$.

13.5 The Fourier transform

The Fourier transform is defined by

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x} f(x)dx, \tag{13.5.1}$$

first for functions in $\mathbf{L}_1(\mathbb{R}, dx)$. In general $\widehat{f}$ will not belong to $\mathbf{L}_2(\mathbb{R}, dx)$. The Fourier transform maps the Schwartz space of rapidly vanishing smooth functions onto itself in an isometric way up to a multiplicative constant, and extends, up to a multiplicative constant, to an isometry from $\mathbf{L}_2(\mathbb{R}, dx)$ onto itself:

$$\|f\|_{\mathbf{L}_2(\mathbb{R},dx)} = \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_{\mathbf{L}_2(\mathbb{R},dx)}. \tag{13.5.2}$$

Note that $\widehat{f}$ is not, in general, a function but rather an equivalence class of functions. Furthermore, the Fourier transform of an arbitrary element $f \in \mathbf{L}_2(\mathbb{R}, dx)$ is not given directly by formula (13.5.1) (which will not make sense in general), but is defined in terms of limits. Its inverse is given by the formula

$$\check{f}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x} f(\lambda)d\lambda, \tag{13.5.3}$$

and we have

$$\|f\|_{\mathbf{L}_2(\mathbb{R},dx)} = \frac{1}{\sqrt{2\pi}}\|\check{f}\|_{\mathbf{L}_2(\mathbb{R},dx)}. \tag{13.5.4}$$

As an illustration of the preceding inversion formula, consider the function $g(x) = \frac{1}{x^2+1}$. Its Fourier transform was computed to be $h(\lambda) = \pi e^{-|\lambda|}$. See (8.6.10). Thus, from (13.5.3),

$$\begin{aligned}
\check{h}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x} h(\lambda)d\lambda \\
&= \frac{1}{2}\left\{\int_0^{\infty} e^{-\lambda}e^{i\lambda x}d\lambda + \int_{-\infty}^{0} e^{\lambda}e^{i\lambda x}d\lambda\right\} \\
&= \frac{1}{2}\left\{\frac{-1}{ix-1} + \frac{1}{ix+1}\right\} \\
&= g(x).
\end{aligned}$$

We follow [206, pp. 42–43] for the next exercise.

Exercise 13.5.1. *For $R > 0$, consider the closed contour*

$$\gamma_R = \gamma_{1,R} + \gamma_{2,R} + \gamma_{3,R} + \gamma_{4,R},$$

defined as follows:

(i) $\gamma_{1,R}$ *is the interval* $[-R, R]$.

(ii) $\gamma_{2,R}$ *is the interval* $[R, R+iy]$.

(iii) $\gamma_{3,R}$ *is the interval* $[R+iy, -R+iy]$.

(iv) $\gamma_{4,R}$ *is the interval* $[-R+iy, -R]$.

(1) *By computing the integral of the function* e^{-z^2} *along this rectangle and using the value of the Gaussian integral* (5.2.6), *show that, for* $y \in \mathbb{R}$,

$$\int_{\mathbb{R}} e^{-t^2} e^{-2ity} dt = \sqrt{\pi} e^{-y^2}. \tag{13.5.5}$$

(2) *Using* (13.5.5) *compute the even moments* (5.2.7).

We now discuss some aspects of the theory of Hermite functions. More exercises and details can be found in [CAPB2]. By making the change of variables $z \mapsto \sqrt{2}z$ and $t \mapsto \sqrt{2}t$, and a normalization we first rewrite (5.6.4) as

$$e^{2tz-t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n. \tag{13.5.6}$$

We have

$$H_n(z) = (-1)^n e^{z^2} \left(e^{-z^2}\right)^{(n)}, \tag{13.5.7}$$

as is seen by writing $e^{2tz-t^2} = e^{z^2} e^{-(t-z)^2}$ and considering the Taylor expansion centered at $t = 0$ of the function $t \mapsto e^{-(t-z)^2}$.

Question 13.5.2. *Prove that*

$$\int_{\mathbb{R}} e^{-u^2} H_n(u) H_m(u) du = \sqrt{\pi} 2^n n! \delta_{n,m}. \tag{13.5.8}$$

Hint. Denoting by α_{nm} the left side of (13.5.8) compute, using (13.5.6), the generating function

$$\sum_{n,m=0}^{\infty} \alpha_{nm} z^n w^n.$$

The functions $\eta_0, \eta_1, \ldots$ with

$$\eta_n(z) = \frac{e^{\frac{z^2}{2}}}{\sqrt[4]{\pi} 2^{n/2} \sqrt{n!}}, \quad n = 0, 1, \ldots \tag{13.5.9}$$

are called the Hermite functions. They belong to the Schwartz space, and form an orthonormal basis of $\mathbf{L}_2(\mathbb{R}, dx)$.

The map which to η_n associates the function $\frac{z^n}{\sqrt{n!}}$ extends to a unitary operator from $\mathbf{L}_2(\mathbb{R}, dx)$ onto the Fock space. It is called the Bargmann transform.

Question 13.5.3. *The Bargmann transform can be written as*

$$F(z) = \frac{1}{\sqrt[4]{\pi}} \int_{\mathbb{R}} e^{\left\{-\frac{1}{2}(z^2+u^2)+\sqrt{2}zu\right\}} f(u)du.$$

We conclude by mentioning that

$$\widehat{\eta_n} = (-i)^n \eta_n, \quad n = 0, 1, \ldots.$$

13.6 Solutions

Solution of Exercise 13.1.1. Since $|k| > 1$ the series with general term z/k^n is absolutely convergent for any $z \in \mathbb{C}$. Thus, by Theorem 3.7.1, the infinite product converges for every z not equal to $-k^n$, $n = 1, 2, \ldots$ (and the corresponding function, extended to be 0 at these points, is entire).

To prove (b) we write

$$\begin{aligned} P(kz) &= \prod_{n=1}^{\infty} \left(1 + \frac{kz}{k^n}\right) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{k^{n-1}}\right) \\ &= \prod_{n=0}^{\infty} \left(1 + \frac{z}{k^n}\right) \\ &= (1+z)P(z). \end{aligned}$$

We now turn to (c). From (b) we have $P(k/z) = (1 + 1/z)P(1/z)$, and replacing z by kz in the above expression,

$$P(1/z) = \left(1 + \frac{1}{kz}\right) P(1/kz) \quad \text{and hence} \quad P(1/kz) = \frac{kz}{1+kz} P(1/z).$$

Thus

$$\begin{aligned} S(kz) &= P(kz)P(1/kz)(1+kz) \\ &= (1+z)P(z)P(1/z)\frac{kz}{1+kz}(1+kz) \\ &= (1+z)P(z)P(1/z)kz \\ &= kzS(z). \end{aligned}$$

(d) Using (c) we have

$$\begin{aligned} M(kz) &= \frac{S(a_1kz)\cdots S(a_nkz)}{S(b_1kz)\cdots S(b_nkz)} \\ &= \frac{ka_1zS(a_1z)\cdots ka_nzS(a_nz)}{kb_1zS(b_1z)\cdots kb_nzS(b_nz)} \\ &= M(z)\frac{a_1\cdots a_n}{b_1\cdots b_n} = M(z) \end{aligned}$$

since we assumed $a_1 \cdots a_n = b_1 \cdots b_n$.

(e) Let $\omega \in \mathbb{C}$ be such that $k = \exp \omega$. Since $|k| > 1$ the numbers ω and $2\pi i$ are linearly independent over $\mathbb{Z}$. We cannot find m and n such that $m\omega + 2\pi i n = 0$. Indeed, if there are such m and n, then $e^{m\omega} = e^{-2\pi i n} = 1$ and so $k^m = 1$ contradicting the assumption $|k| > 1$. Moreover, we have

$$G(z + m\omega + n2\pi i) = M(e^{z+m\omega+n2\pi i}) = M(e^{z+m\omega}) = M(k^m e^z) = M(e^z) = G(z)$$

where we used (d) with e^z in place of z. Thus, $G(z)$ is bi-periodic since ω and 2π are linearly independent over $\mathbb{Z}$. □

Solution of Exercise 13.1.3. It follows from the proof of Exercise 3.6.2 that the convergence of the family of functions is uniform on compact sets, and therefore the limit is analytic. □

Solution of Exercise 13.2.1. Let $L > 0$. We have, with $z = x + iy$,

$$|e^{i\pi n^2\tau + 2\pi i n z}| = e^{-\pi n^2 \operatorname{Im} \tau} \cdot e^{-2\pi n y} \le e^{-\pi n^2 \operatorname{Im} \tau} \cdot e^{2\pi |n| L}$$

for $|y| \le L$. We now show that the series converge uniformly in every band of the form $|\operatorname{Im} z| \le L$, $L > 0$. For L fixed, there exists $n_0 \in \mathbb{N}$ such that

$$|n| \ge n_0 \longrightarrow |\frac{2\pi L}{n}| \le \frac{\pi \operatorname{Im} \tau}{2}.$$

Thus for $|n| \ge n_0$ we have

$$|e^{i\pi n^2\tau + 2\pi i n z}| = e^{-\pi n^2 \operatorname{Im} \tau} \cdot e^{-2\pi n y} \le e^{-\frac{n^2 \pi \operatorname{Im} \tau}{2}}.$$

Therefore the series converge uniformly on each band of the asserted form, and ϑ is an entire function of z.

(13.2.1) follows from the periodicity of the exponentials $e^{2\pi i n z}$. Equality (13.2.2) is proved as follows:

$$\begin{aligned}
\vartheta(z + \tau, \tau) &= \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau + 2i\pi n(z+\tau)} \\
&= \sum_{n \in \mathbb{Z}} e^{i\pi\tau(n^2+2n) + 2\pi i n z}, \quad \text{and, completing the square,} \\
&= \sum_{n \in \mathbb{Z}} e^{i\pi\tau(n+1)^2 + 2\pi i n z - i\pi\tau} \\
&= e^{-i\pi\tau - 2\pi i z} \cdot \sum_{n \in \mathbb{Z}} e^{i\pi\tau(n+1)^2 + 2\pi i(n+1)z} \\
&= e^{-i\pi\tau - 2\pi i z} \vartheta(z, \tau).
\end{aligned}$$

We now prove (13.2.3). Using (13.2.2) with $z = \frac{1-\tau}{2}$ we obtain

$$\begin{aligned}\vartheta\left(\frac{1+\tau}{2},\tau\right) &= \vartheta\left(\frac{1-\tau}{2}+\tau,\tau\right)\\ &= e^{-i\pi\tau-2\pi i\frac{1-\tau}{2}}\vartheta\left(\frac{1-\tau}{2},\tau\right)\\ &= e^{-i\pi}\vartheta\left(\frac{1-\tau}{2},\tau\right), \quad \text{and, using (13.2.1),}\\ &= -\vartheta\left(\frac{1-\tau}{2}-1,\tau\right)\\ &= -\vartheta\left(\frac{1+\tau}{2},\tau\right),\end{aligned}$$

and hence the result since ϑ is an even function of z. □

Solution of Exercise 13.2.2. We already know from Exercise 13.2.1 that ϑ vanishes at the point $\frac{1+\tau}{2}$, and hence, because of (13.2.1) and (13.2.2) at all the points

$$\frac{1+\tau}{2}+m+\tau n, \quad m,n\in\mathbb{Z}.$$

The entire function $\vartheta(z,\tau)$ may vanish *a priori* for some points on the parallelogram with nodes $0, 1, \tau$ and $1+\tau$. By making a small translation by a complex number a, we obtain a parallelogram P_a, with nodes $a, 1+a, \tau+a, 1+\tau+a$, which still contains $\frac{1+\tau}{2}$, but on which ϑ does not vanish. We have

$$\begin{aligned}\int_{P_a}\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz &= \int_{[a,1+a]}\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz + \int_{[1+a,1+\tau+a]}\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz\\ &+\int_{[1+a+\tau,a+\tau]}\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz + \int_{[a+\tau,a]}\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz\end{aligned} \tag{13.6.1}$$

since ϑ has period 1 with respect to z (see (13.2.1)), the function $\frac{\vartheta'}{\vartheta}$ is also periodic with period 1 with respect to z and we have

$$\int_{[a+\tau,a]}\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz = \int_{[1+a,1+\tau+a]}\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz = -\int_{[1+a+\tau,1+a]}\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz.$$

Thus the second and fourth integrals on the right side of (13.6.1) cancel each other. We now compare the first and the third integral, taking into account (13.2.2). Using for instance the property (4.2.3) of the logarithmic derivative, (13.2.2) leads to

$$\frac{\vartheta'(z+\tau,\tau)}{\vartheta(z+\tau,\tau)} = \frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)} - 2\pi i.$$

It follows that

$$\begin{aligned}\int_{[1+a+\tau,a+\tau]} \frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz &= \int_{[1+a,a]} \frac{\vartheta'(z+\tau,\tau)}{\vartheta(z+\tau,\tau)}dz \\ &= \int_{[1+a,a]} \left(\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)} - 2\pi i\right) dz \\ &= -\int_{[a,1+a]} \frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz + 2\pi i.\end{aligned}$$

Thus the first and the third integral in (13.6.1) sum up to $2\pi i$, and so

$$\frac{1}{2\pi i}\int_{P_a} \frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}dz = 1.$$

Since ϑ is entire, it follows from (7.3.5) that $\frac{1+\tau}{2}$ is the only zero of ϑ in P_a, and hence the result. □

Solution of Exercise 13.3.1. We define a function g in $\mathbb{C}\setminus(-\infty,0]$ by

$$g(\zeta) = f\left(\frac{\ln\rho + i\theta}{2\pi i}\right), \quad \text{with} \quad \zeta = \rho e^{i\theta}, \quad \theta \in (-\pi,\pi).$$

For $\zeta = e^{2\pi i z}$ and z in the strip $|x| < 1/2$ we have

$$g(e^{2\pi i z}) = f(z).$$

The function g is analytic in $\mathbb{C}\setminus(-\infty,0]$. Take $x<0$ to be a point on the negative axis. We have

$$\lim_{\substack{\zeta\to x\\ \operatorname{Im}\zeta>0}} g(\zeta) = f\left(\frac{\ln x + i\pi}{2\pi i}\right),$$

and

$$\lim_{\substack{\zeta\to x\\ \operatorname{Im}\zeta<0}} g(\zeta) = f\left(\frac{\ln x - i\pi}{2\pi i}\right).$$

The fact that f is periodic with period 1 leads to the continuity of g on $(-\infty,0)$. Using Morera's theorem we conclude that g is analytic in $\mathbb{C}\setminus\{0\}$, and therefore has a Laurent expansion, which converges uniformly in every ring of the form $r<|\zeta|<R$ (r and R are strictly positive numbers such that $r<R$):

$$g(\zeta) = \sum_{n\in\mathbb{Z}} c_n\zeta^n.$$

Thus,

$$f(z) = g(e^{2\pi i z}) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n z},$$

where by analytic continuation, z is arbitrary in $\mathbb{C}$, and where the convergence is uniform in every closed horizontal strip. □

For more on the subject, see for instance [42, Exercise 11.10, p. 365], [75, Exercice 34.10, p. 307], [193, pp. 106–107]. As an application of the previous exercise, prove the following result (see [193, (2.23-12) and (2.23-13), p. 108]):

$$\frac{1}{\tan \pi z} = \begin{cases} -i(1+2\sum_{n=1}^{\infty} e^{2\pi i n z}), & \operatorname{Im} z > 0, \\ i(1+2\sum_{n=1}^{\infty} e^{-2\pi i n z}), & \operatorname{Im} z < 0. \end{cases}$$

Solution of Exercise 13.3.2. We follow [162, pp. 3-4]. In view of Exercise 13.3.1 we look for f, not identically vanishing, and of the form

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(\tau) e^{2\pi i n z}. \tag{13.6.2}$$

The condition

$$f(z+\tau) = e^{az+b} f(z)$$

leads to

$$\sum_{n\in\mathbb{Z}} c_n(\tau) e^{2\pi i n(z+\tau)} = e^{az+b} \sum_{n\in\mathbb{Z}} c_n(\tau) e^{2\pi i n z}.$$

Replacing z by $z+1$ in this expression we obtain (since we assume $f \not\equiv 0$)

$$e^a = 1,$$

that is, $a = 2\pi i k_0$ for some $k_0 \in \mathbb{Z}$. Comparing the coefficient of $e^{2\pi i n z}$ we have

$$c_n(\tau) = c_{n-k_0}(\tau) e^{-2\pi i n \tau} e^b = c_{n-k_0}(\tau) e^{b+2\pi n \operatorname{Im} \tau} e^{-2\pi i n \operatorname{Re} \tau}.$$

When $k_0 > 0$, the coefficients $c_n(\tau)$ go exponentially fast in modulus to infinity, and the series (13.6.2) diverges. We leave it to the student to consider the cases $k_0 = 0$ and $k_0 < 0$. □

Solution of Exercise 13.3.3. The function q is meromorphic in the plane since it is the quotient of two entire functions. Since f has period 1, all the functions $f(z-a_n)$ and $f(z-b_n)$ have also period 1, and so has the function q. We now show, using the second equality in (13.3.1), that q has also period τ. We have

$$\begin{aligned} q(z+\tau) &= \prod_{n=1}^{N} \frac{f(z+\tau-a_n)}{f(z+\tau-b_n)} \\ &= \prod_{n=1}^{N} \frac{e^{a(z-a_n)+b} f(z-a_n)}{e^{a(z-b_n)+b} f(z-b_n)} \\ &= \frac{e^{aNz - a(\sum_{n=1}^{N} a_n) + Nb}}{e^{aNz - a(\sum_{n=1}^{N} b_n) + Nb}} q(z) \\ &= q(z), \end{aligned}$$

in view of (13.3.2). □

The student will recognize in (13.3.2) a condition similar to (13.1.1) in Exercise 13.1.1.

Solution of Exercise 13.4.1. Let $z = x + iy$. We have

$$|t^{z-1}| = |e^{\{(z-1)\ln t\}}| = e^{(x-1)\ln t} = t^{x-1}.$$

The integral $\int_0^1 t^{x-1}dt$ converges for $x > 0$, and so the integral $\int_0^1 t^{z-1}e^{-t}dt$ converges absolutely for $\operatorname{Re} z > 0$. As for the convergence at infinity of the integral (3.1.11)

$$\int_0^\infty t^{x-1}e^{-t}dt,$$

we proceed as follows (the same argument will be used later in the solution of Exercise 13.4.2): Write

$$t^{x-1}e^{-t} = e^{\left\{\left((x-1)\frac{\ln t}{t}-1\right)t\right\}}.$$

For a given $x > 0$, there exists $M > 0$ such that

$$t \geq M \Longrightarrow \left|(x-1)\frac{\ln t}{t}\right| \leq \frac{1}{2}.$$

Then,

$$(x-1)\frac{\ln t}{t} - 1 \leq \left|(x-1)\frac{\ln t}{t}\right| - 1 \leq -\frac{1}{2},$$

and we have

$$\int_M^\infty t^{x-1}e^{-t}dt \leq \int_M^\infty e^{-\frac{t}{2}}dt < \infty.$$

Finally, equation (13.4.2) is proved by integration by parts. □

Solution of Exercise 13.4.2. We follow the method given in the hint after the exercise. By Theorem 6.2.3 each of the functions Γ_n is analytic in $\operatorname{Re} z > 1$. Furthermore, for $z \in K$ we have

$$\left|\int_n^\infty t^{z-1}e^{-t}dt\right| \leq \int_n^\infty e^{(M-1)\ln t - t}dt = \int_n^\infty e^{\left(\frac{(M-1)\ln t}{t}-1\right)t}dt.$$

For a given M there exists n_0 such that

$$t \geq n_0 \Longrightarrow 0 < \frac{(M-1)\ln t}{t} < \frac{1}{2},$$

and therefore, for $n \geq n_0$,

$$\left|\int_n^\infty t^{z-1}e^{-t}dt\right| \leq \int_n^\infty e^{-\frac{t}{2}}dt \to 0, \quad \text{as} \quad n \to \infty.$$

Similarly, still for $z = x + iy \in K$, we have

$$\begin{aligned}\left|\int_0^{1/n} t^{z-1}e^{-t}dt\right| &\leq \int_0^{1/n} e^{(x-1)\ln t}dt\\ &= \int_0^{1/n} t^{x-1}dt\\ &= \frac{1}{xn^x} \leq \frac{1}{m\cdot n^m}.\end{aligned}$$

It follows that, for $n \geq n_0$,

$$|\Gamma(z) - \Gamma_n(z)| \leq \int_n^\infty e^{-\frac{t}{2}}dt + \frac{1}{m\cdot n^m}$$

uniformly in K (and in fact uniformly in the band $m \leq x \leq M$), and so Γ is analytic as the uniform limit on compact sets of analytic functions. □

Solution of Exercise 13.4.3. We follow [23, p. 119]. In view of (1.2.6), we have that, for every $t \in [0, \infty)$,

$$\lim_{n\to\infty} f_n(t) = e^{-t}t^{z-1}.$$

Moreover, in view of item (a) in Exercise 3.2.6,

$$|f_n(t)| \leq \left(1 - \frac{t}{n}\right)^n t^{x-1} \leq e^{-t}t^{x-1}.$$

The dominated convergence theorem (see Theorem 17.5.2) leads to

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty f_n(t)dt &= \int_0^\infty (\lim_{n\to\infty} f_n(t))dt\\ &= \int_0^\infty e^{-t}t^{z-1}dt\\ &= \Gamma(z).\end{aligned}$$

It remains to show that

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}dt = \frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

As suggested in [23] this is done by repeated integration by parts. Indeed, we have

$$\begin{aligned}
\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt &= \frac{n}{n}\int_0^n \left(1-\frac{t}{n}\right)^{n-1} \frac{t^z}{z}dt \\
&= \frac{n(n-1)}{n^2}\int_0^n \left(1-\frac{t}{n}\right)^{n-2} \frac{t^{z+1}}{z(z+1)}dt \\
&\vdots \\
&= \frac{n(n-1)\cdots 2}{n^{n-1}}\int_0^n \frac{t^{z+(n-1)}}{z(z+1)\cdots(z+n-1)}dt \\
&= \frac{n!}{n^n}\frac{n^{z+n}}{z(z+1)\cdots(z+n-1)(z+n)} \\
&= \frac{n!n^z}{z(z+1)\cdots(z+n)}.
\end{aligned}$$

□

Solution of Exercise 13.4.4. (a) The first equality follows directly from the change of variable $t=\sqrt{u}$. Indeed,

$$\int_0^\infty e^{-t^2}t^{z-1}dt = \int_0^\infty e^{-u}u^{\frac{z-1}{2}}\frac{du}{2\sqrt{u}} = \frac{\Gamma(\frac{z}{2})}{2}.$$

(b) The other two integrals are computed using Cauchy's theorem as follows. Consider the function of the complex variable s defined by

$$f(s) = e^{is+(z-1)\ln s},$$

where $\ln s$ is the principal branch of the logarithm in $\mathbb{C}\setminus(-\infty,0]$, that is

$$\ln s = \ln\rho + i\theta,$$

where $s=\rho e^{i\theta}$ with $\theta\in(-\pi,\pi)$. We consider the closed path consisting of the following four parts:

(i) The interval $[r,R]$, with $0<r<R<\infty$.

(ii) The arc of circle C_R parametrized by

$$\gamma_R(u) = Re^{iu}, \quad u\in\left[0,\frac{\pi}{2}\right].$$

(iii) The interval $[iR,ir]$.

(iv) The arc of circle c_r parametrized by

$$\gamma_r(u) = re^{iu}, \quad u\in\left[\frac{\pi}{2},0\right].$$

By Cauchy's theorem, the integral of f on this closed path is equal to 0. On the other hand,

$$\int_{[r,R]} f(s)ds = \int_r^R e^{it}t^{z-1}dt \to \int_0^\infty e^{it}t^{z-1}dt,$$

as $r \to 0$ and $R \to \infty$, and, with the parametrization $\gamma(t) = it$, with $t \in [R, r]$,

$$\begin{aligned}
\int_{[iR,ir]} f(s)ds &= \int_R^r e^{-t+(z-1)(\ln t + i\frac{\pi}{2})} idt \\
&= -e^{i(z-1)\frac{\pi}{2}} \int_r^R e^{-t}t^{z-1}idt \\
&= -e^{-i\frac{\pi}{2}}e^{iz\frac{\pi}{2}} \int_r^R e^{-t}t^{z-1}idt \\
&\to -e^{iz\frac{\pi}{2}}\Gamma(z)
\end{aligned}$$

as $r \to 0$ and $R \to \infty$. We now show that

$$\lim_{r\to 0}\int_{c_r} f(s)ds = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(s)ds = 0. \tag{13.6.3}$$

The first of these limits is computed as follows:

$$\begin{aligned}
\left|\int_{c_r} f(s)ds\right| &= \left|-\int_0^{\pi/2} e^{ire^{iu}+(z-1)(\ln r+iu)}rie^{iu}du\right| \\
&\le e^{|\operatorname{Im} z|\frac{\pi}{2}} \int_0^{\pi/2} e^{-r\sin u}r^x du \\
&\le \frac{\pi}{2}r^x e^{|\operatorname{Im} z|\frac{\pi}{2}} \\
&\to 0,
\end{aligned} \tag{13.6.4}$$

as $r \to 0$. In the computation we have used that, with $z = x + iy$,

$$|e^{(z-1)(\ln r+iu)}| \cdot r = e^{(x-1)\ln r - yu} \cdot r \le r^x e^{|\operatorname{Im} z|\frac{\pi}{2}},$$

since $e^{-yu} \le e^{|y|u} \le e^{|\operatorname{Im} z|\frac{\pi}{2}}$. In computing the limit (13.6.4) we have used that $x > 0$. To show that the second limit goes to 0 we make use of the fact that $x < 1$. Making use of (5.9.5) and of (13.6.4) with R instead of r we have

$$\left|\int_{c_R} f(s)ds\right| \le e^{|\operatorname{Im} z|\frac{\pi}{2}} \cdot R^x \cdot \frac{\pi}{R} \longrightarrow 0,$$

as $R \to \infty$ since $x < 1$. Therefore we have

$$\int_0^\infty e^{it}t^{z-1}dt = e^{iz\frac{\pi}{2}}\Gamma(z).$$

Take first $z = x$ real. Comparing the real and imaginary parts of this equality we obtain the asserted formulas for $x > 0$. They extend to complex z with $x \in (0,1)$ by analytic extension. $\square$

Solution of Exercise 13.4.5.
(1) To compute the integral

$$\int_0^\infty t^{z-1}\left(\int_0^\infty f_1(u)f_2(t/u)\frac{du}{u}\right)dt$$

we make the change of variable $(u,t) \mapsto (u,uv)$. The Jacobian matrix (see (4.2.7)) is equal to

$$J(u,v) = \begin{pmatrix} 1 & 0 \\ v & u \end{pmatrix}.$$

and $\det J(u,v) = u$. Thus, by the theorem on change of variables for double integrals, we can write:

$$\begin{aligned}\int_0^\infty t^{z-1}\left(\int_0^\infty f_1(u)f_2(t/u)\frac{du}{u}\right)dt &= \int_0^\infty \int_0^\infty u^{z-1}v^{z-1}u\frac{dudv}{u}\\ &= \left(\int_0^\infty u^{z-1}f_1(u)du\right)\left(\int_0^\infty v^{z-1}f_2(v)dv\right),\end{aligned}$$

where the various interchanges of integrals are done using the dominated convergence theorem.

(2) In the case $f_1(u) = f_2(u) = e^{-u}$ we have:

$$\begin{aligned}\int_0^\infty f_1(u)f_2(t/u)\frac{du}{u} &= \int_0^\infty e^{-u-\frac{t}{u}}\frac{du}{u}\\ &= \int_0^\infty e^{-\sqrt{t}(v+\frac{1}{v})}\frac{dv}{v} \quad \text{(with the change of variable } u = \sqrt{t}v)\\ &= \int_{-\infty}^\infty e^{-2\sqrt{t}\cosh a}da \quad \text{(with the change of variable } v = e^a)\\ &= 2K_0(2\sqrt{t}),\end{aligned}$$

with

$$K_0(x) = \int_0^\infty e^{-x\cosh a}da. \tag{13.6.5}$$

$\square$

Remark 13.6.1. The function K_0 defined in (13.6.5) is the K Bessel function of order 0. See, e.g., [50, p. 7 and p. 50]. We have

$$\int_0^\infty t^{z-1}2K_0(2\sqrt{t})dt = (\Gamma(z))^2. \tag{13.6.6}$$

Setting $z = n + 1$ in the previous expression gives

$$\int_0^\infty t^n 2K_0(2\sqrt{t})dt = (n!)^2. \tag{13.6.7}$$

This fact is used in [13, 16] to study (and in particular give a geometric characterization of the elements of) the reproducing kernel Hilbert space with reproducing kernel

$$\sum_{n=0}^\infty \frac{z^n \overline{w}^n}{(n!)^2}.$$

Solution of Exercise 13.5.1.
(1) For $y = 0$, (13.5.5) is the value of the Gaussian integral, which we assume known. See the discussion after (5.2.6). The integral under consideration is an even function of y, and we take $y > 0$. We give to Γ_R the positive orientation. We then have the following parametrizations for the components of Γ_R (we do not stress the dependence on y in the notation):

$$\begin{aligned}
\gamma_{1,R}(t) &= t, \quad t \in [-R, R],\\
\gamma_{2,R}(t) &= R + it, \quad t \in [0, y],\\
\gamma_{3,R}(t) &= -t + iy, \quad t \in [-R, R],\\
\gamma_{4,R}(t) &= -R + i(y - t), \quad t \in [0, y].
\end{aligned}$$

Since e^{-z^2} is defined by a power series centered at the origin, and converging in all of $\mathbb{C}$, it has a primitive in $\mathbb{C}$ and we can write

$$\int_{\Gamma_R} e^{-z^2} dz = 0, \quad \forall R > 0,$$

that is,

$$\int_{\gamma_{1,R}} e^{-z^2} dz + \int_{\gamma_{2,R}} e^{-z^2} dz + \int_{\gamma_{3,R}} e^{-z^2} dz + \int_{\gamma_{4,R}} e^{-z^2} dz = 0, \quad \forall R > 0. \tag{13.6.8}$$

We have

$$\begin{aligned}
\left| \int_{\gamma_{2,R}} e^{-z^2} dz \right| &= \left| \int_0^y e^{-(R^2 + 2Rti - t^2)} i dt \right|\\
&\le \int_0^y e^{-R^2 + t^2} dt\\
&= e^{-R^2} \int_0^y e^{t^2} dt \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.
\end{aligned}$$

Similarly,

$$\lim_{R \to \infty} \int_{\gamma_{4,R}} e^{-z^2} dz = 0.$$

Therefore letting $R \to \infty$ in (13.6.8) and using the value of the Gaussian integral we obtain

$$e^{-y^2} \int_{\mathbb{R}} e^{-t^2} dt = e^{-y^2} \sqrt{\pi} = \int_{\mathbb{R}} e^{-t^2} e^{-2ity} dt. \tag{13.6.9}$$

See for instance [206, p. 43].

(2) Using the dominated convergence theorem and the power series expansion of e^{-2ity} we rewrite (13.6.9) as

$$\sqrt{\pi} \left(\sum_{u=0}^{\infty} (-1)^u \frac{y^{2u}}{u!} \right) = \sum_{n=0}^{\infty} \frac{(-2iy)^n}{n!} \left(\int_{\mathbb{R}} e^{-t^2} t^n dt \right).$$

The odd moments vanish. Setting $n = 2u$ in the equality above and comparing the coefficient of y^{2u} we obtain the even moments:

$$\sqrt{\pi} \frac{(-1)^u}{u!} = \frac{(-1)^u (-2)^{2u}}{(2u)!} \left(\int_{\mathbb{R}} e^{-t^2} t^{2u} dt \right), \quad u = 0, 1, \ldots$$

and hence

$$\int_{\mathbb{R}} e^{-t^2} t^{2u} dt = \sqrt{\pi} \frac{(2u)!}{u! 2^{2u}}. \tag{13.6.10}$$

□

Remark 13.6.2. The right side of (13.6.10) can be rewritten as

$$\sqrt{\pi} \frac{(2u-1)!!}{2^u}$$

where $n!! = n(n-2)(n-3)\cdots$.