Chapter 11

A Taste of Linear System Theory and Signal Processing

In the present chapter, we briefly discuss some links between the theory of analytic functions and the theory of linear systems. We refer to the books [89], [117], [170], [171], [178] for more information. The reader should be aware that more recent advances in linear system theory, in the setting of several complex variables, non-commuting variables, or stochastic setting, to name a few, require much more involved tools. Still it is necessary to master the elementary setting outlined here before going to these more advanced areas.

We recall that we denote by $\mathbf{L}_2(\mathbb{R})$ and $\mathbf{L}_2(-F,F)$ the Lebesgue spaces of functions measurable and square summable with respect to the Lebesgue measure, on $\mathbb{R}$ and on $(-F,F)$ respectively.

11.1 Continuous signals

A continuous signal of finite energy is modeled by a continuous complex-valued function f defined on the real line, and its energy will be by definition

$$\int_{\mathbb{R}} |f(t)|^2 dt.$$

The integral is a Riemann integral, but the fact that we consider f with this norm forces us to consider measurable functions and the Lebesgue space $\mathbf{L}_2(\mathbb{R})$. See Chapter 17 for a brief review of these notions.

The spectrum of the signal f is by definition its inverse Fourier transform (13.5.3):

$$\check{f}(u) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iut} f(t)dt,$$

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so that

$$f(t) = \int_{\mathbb{R}} e^{-itu} \check{f}(u) du.$$

The above expression is the decomposition of f along frequencies (technically, it would be better to have $2\pi u$ rather than u for frequencies, but we will stick to the present definition of the Fourier transform). We are interested in signals which have spectrum with finite support. It then follows that the signal itself is the restriction on the real line of an entire function. If the spectrum has support in the closed interval $[-F, F]$, the signal can be written as

$$f(t) = \frac{1}{2F} \int_{[-F,F]} e^{-itu} m(u) du, \tag{11.1.1}$$

where $m \in \mathbf{L}_2(-F, F)$ denotes the spectrum. The representation (11.1.1) expresses that the signal f is built from frequencies in a bounded domain (that is, f is a *band limited signal*). This is a characteristic of physical systems. The factor $\frac{1}{2F}$ is a normalization to have nicer formulas in the sequel. We recognize with (11.1.1) a function similar to the ones appearing in Exercises 3.4.13 and 4.2.14. In particular, f is the restriction to the real line of the entire function

$$f(z) = \frac{1}{2F} \int_{[-F,F]} e^{-izu} m(u) du, \quad z \in \mathbb{C}.$$

Besides being entire, this function has a special property:

Exercise 11.1.1. *Show that there exists $K > 0$ such that*

$$|f(z)| \leq K e^{F|z|}, \quad \forall z \in \mathbb{C}. \tag{11.1.2}$$

Entire functions which admit a bound of the form (11.1.2) are called of exponential type, and the smallest F in (11.1.2) is the exponential type of the function. That every entire function which admits a bound of the form (11.1.2) can be written as (11.1.1) with $m \in \mathbf{L}_1(-F, F)$ is a deep result, called the Paley–Wiener theorem. See for instance [71, § 3.3, p. 158], [72, §2.2, p. 28]. Here we restrict

$$m \in \mathbf{L}_2(-F, F) \subset \mathbf{L}_1(-F, F)$$

because we want an underlying Hilbert space structure.

To summarize, physical considerations in modeling signals (having a band limited spectrum) make it natural to consider a very special class of entire functions (entire functions of exponential type).

11.2 Sampling

Since the function f in (11.1.1) has an analytic extension to the whole complex plane, one can ask the question of reconstructing f from a discrete set of values. From an engineering point of view this is an important issue. The surprising answer to this question is a result called the sampling theorem, which we present in this section; see Theorem 11.2.1. The sampling theorem has a long history, and we refer to the paper [159] for a historical account. We mention that a version of the sampling theorem already appears in the 1915 paper [220] of E.T. Whittaker. We refer to the papers of Claude Shannon [196], [197]. This last paper refers in particular to the 1935 book [221, Ch. IV] of J.M. Whittaker for an earlier version of the sampling theorem. See also [26, p. 258].

We note that there is no need of analytic functions to prove the sampling theorem. On the other hand, the result is somewhat of a mystery to students who have no background in analytic functions.

We consider $\mathbf{L}_2(-F,F)$ with the normalized inner product (17.7.3)

$$\langle m,n\rangle = \frac{1}{2F}\int_{(-F,F)} m(t)\overline{n(t)}dt.$$

Theorem 11.2.1. *Let $m \in \mathbf{L}_2(-F,F)$, with $F \in (0,\infty)$, and let f be defined by* (11.1.1)

$$f(t) = \frac{1}{2F}\int_{(-F,F)} e^{-itu}m(u)du.$$

Then

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{\pi n}{F}\right)\frac{\sin(Ft-n\pi)}{Ft-n\pi}, \tag{11.2.1}$$

where the limit is pointwise, and uniformly on compact subsets of $\mathbb{C}$ (with $t \in \mathbb{C}$). Finally

$$\int_{\mathbb{R}} |f(t)|^2 dt = \frac{\pi}{F}\sum_{n\in\mathbb{Z}}\left|f\left(\frac{\pi n}{F}\right)\right|^2. \tag{11.2.2}$$

For instance consider the choice $F = 2$ and

$$m(u) = \begin{cases} 1, & u \in [-1,1],\\ 0, & u \in [-2,2]\setminus[-1,1].\end{cases}$$

Then

$$f(t) = \frac{\sin t}{2t}$$

and (11.2.2) becomes

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{R}}\left(\frac{\sin t}{t}\right)^2 dt &= \frac{\pi}{2}\sum_{n\in\mathbb{Z}}\left|f\left(\frac{\pi n}{2}\right)\right|^2\\ &= \frac{\pi}{2}\left(\frac{1}{4} + 2\sum_{k\in\mathbb{N}_0}\left|\frac{1}{2}\frac{\sin(\frac{(2k+1)\pi}{2})}{\frac{(2k+1)\pi}{2}}\right|^2\right)\\ &= \frac{\pi}{2}\left(\frac{1}{4} + 2\sum_{k\in\mathbb{N}_0}\frac{1}{(2k+1)^2\pi^2}\right).\end{aligned}$$

Using Exercise 5.3.3, this leads to (6.7.1)

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Exercise 11.2.2. *Give a direct proof of* (6.7.1) *taking into account* (1.3.14).

To prove Theorem 11.2.1 we use the expansion of an $\mathbf{L}_2(-F,F)$ function along an orthogonal basis. To characterize functions which admit a representation (11.1.1) is a more delicate matter, and uses the Phragmén–Lindelöf principle (we will not recall its definition here). See [72, p. 28] for more information.

Exercise 11.2.3. *The space $\mathcal{H}_F$ of functions of the form*

$$f(z) = \frac{1}{2F} \int_{(-F,F)} m(u) e^{-izu} du, \quad m \in \mathbf{L}_2(-F,F), \tag{11.2.3}$$

with norm

$$\|f\|_{\mathcal{H}_F} = \|m\|_{\mathbf{L}_2(-F,F)} \tag{11.2.4}$$

is the reproducing kernel Hilbert space of entire functions with reproducing kernel

$$K_F(z,w) = \frac{\sin(Fz - F\overline{w})}{Fz - F\overline{w}}, \quad z, w \in \mathbb{C}.$$

In view of the isometry property (13.5.2) of the Fourier transform, we see that the space $\mathcal{H}_F$ defined in the preceding exercise is in fact, up to a unitary constant, isometrically included in $\mathbf{L}_2(\mathbb{R}, dx)$. More precisely, we have

$$f(t) = \frac{\widehat{m}(t)}{2F},$$

and so

$$\|f\|^2_{\mathbf{L}_2(\mathbb{R},dx)} = \frac{2\pi}{2F} \|m\|^2_{\mathbf{L}_2(-F,F)}.$$

Thus

$$\begin{aligned} \|f\|^2_{\mathcal{H}_F} &= \|m\|^2_{\mathbf{L}_2(-F,F)} \\ &= \frac{2F}{2\pi} \|f\|^2_{\mathbf{L}_2(\mathbb{R},dx)}. \end{aligned}$$

Exercise 11.2.4. *Prove formula* (11.2.1).

11.3 Time-invariant causal linear systems

A linear continuous operator T,

$$u \in \mathbf{L}_2(\mathbb{R}) \mapsto Tu \in \mathbf{L}_2(\mathbb{R})$$

from $\mathbf{L}_2(\mathbb{R})$ into itself, is called a *linear system* when one views the elements of $\mathbf{L}_2(\mathbb{R})$ as signals with finite energy. The function u is then called the *input signal* and the function Tu is called the *output signal*. The (linear) system is called *dissipative* if the norm of the operator is less than or equal to 1:

$$\forall u \in \mathbf{L}_2(\mathbb{R}), \quad \|Tu\|_{\mathbf{L}_2(\mathbb{R})} \le \|u\|_{\mathbf{L}_2(\mathbb{R})}.$$

It will be called causal if the following property holds for every $t \in \mathbb{R}$: If the input function u has support $(-\infty, t)$, then the output function has also support in $(-\infty, t)$.

We are in particular interested in operators which have a *kernel representation* in the form

$$Tf(t) = \int_{\mathbb{R}} k(t,s)f(s)ds, \tag{11.3.1}$$

or as convolution operators

$$Tf(t) = \int_{\mathbb{R}} k(t-s)f(s)ds, \tag{11.3.2}$$

when the kernel $k(t,s)$ is required to depend only on the difference $t-s$.

Not every continuous linear operator from $\mathbf{L}_2(\mathbb{R})$ admits such a representation. To ensure such a representation for *every* continuous operator, one has to restrict the domain to a set of test functions and extend the range to the dual space of distributions. Continuity is then understood with respect to the topology of the Schwartz space and of its dual, and Schwartz' kernel theorem insures then a counterpart of (11.3.1) with a distribution $k(t,s)$. This is a fascinating line of research (see [109], [110] for instance for the background of the kernel theorem, and Zemanian's book [227] for applications to the theory of linear systems). Here we are interested in a simpler kind of linear systems, namely systems $y = Tu$ given by

$$(L(y))(z) = h(z)(L(u))(z)$$

where L denotes the Laplace transform. Such systems are time-invariant and characterized by a convolution in continuous time.

Exercise 11.3.1. *Let $(A,B,C,D) \in \mathbb{C}^{N\times N} \times \mathbb{C}^{N\times p} \times \mathbb{C}^{q\times N} \times \mathbb{C}^{q\times p}$, and consider the equations*

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \ge 0 \end{aligned} \tag{11.3.3}$$

where the functions x, u and y are respectively $\mathbb{C}^N$-valued, $\mathbb{C}^p$-valued and $\mathbb{C}^q$-valued. Assume that $x(0) = 0$ and that the Laplace transform $L(u)$ has a positive axis of convergence. Show that the function $L(y)$ has a positive axis of convergence and that

$$(L(y))(z) = h(z)(L(u))(z), \tag{11.3.4}$$

where

$$h(z) = D + C(zI_N - A)^{-1}B. \tag{11.3.5}$$

The equations (11.3.4) are called *state space equations*, and the vector $x(t)$ is called the state at time t. The expression (11.3.5) is called a *realization* of the rational matrix-valued function h. See Chapter 11 for more on this notion.

11.4 Discrete signals and systems

A discrete signal will be a sequence $(u_n)_{n\in\mathbb{N}_0}$ of complex numbers, indexed by $\mathbb{N}_0$ (or sometimes by $\mathbb{Z}$). Its z-transform is the power series

$$u(z) = \sum_{n=0}^{\infty} u_n z^n.$$

The energy of the signal is its ℓ_2 norm

$$\|u\|_{\ell_2} = \sqrt{\left(\sum_{n=0}^{\infty} |u_n|^2\right)},$$

and we see that the space of signals of finite energy is nothing else than the Hardy space $\mathbf{H}_2(\mathbb{D})$. See Definition 5.6.11 for the latter. It is therefore reasonable to think that function theory in $\mathbf{H}_2(\mathbb{D})$ should have implications, and applications, in signal theory.

A bounded linear system will be a linear bounded operator from ℓ_2 into itself. It translates into a linear bounded operator T from $\mathbf{H}_2(\mathbb{D})$ into itself. The linear system will be called dissipative if it is moreover a contraction

$$\|Tu\|_{\mathbf{H}_2(\mathbb{D})} \leq \|u\|_{\mathbf{H}_2(\mathbb{D})}, \quad \forall u \in \mathbf{H}_2(\mathbb{D}).$$

An important class of linear systems is defined by multiplication operators: The input sequence $(u_n)_{n\in\mathbb{N}_0}$ and the output sequence $(y_n)_{n\in\mathbb{N}_0}$ are related by

$$y(z) = h(z)u(z), \tag{11.4.1}$$

where $h(z) = \sum_{n=0}^{\infty} h_n z^n$ is convergent in $\mathbb{D}$. Therefore, $(y_n)_{n\in\mathbb{N}_0}$ is the convolution of $(h_n)_{n\in\mathbb{N}_0}$ and $(u_n)_{n\in\mathbb{N}_0}$. See (4.4.14) for the latter. The function h is called the *transfer function* of the system, and its Taylor coefficients at the origin are called the *impulse response*.

Not every h will lead to a bounded operator. We have:

Theorem 11.4.1. *The relation* (11.4.1) *defines a bounded linear operator from* $\mathbf{H}_2(\mathbb{D})$ *into itself if and only if* h *is analytic and bounded in the open unit disk. It defines a dissipative linear operator from* $\mathbf{H}_2(\mathbb{D})$ *into itself if and only if* h *is analytic and contractive in the open unit disk.*

The proof of Theorem 11.4.1 relies on the characterization (5.6.7) of the space $\mathbf{H}_2(\mathbb{D})$. If s is analytic and contractive in the open unit disk, then for every $f \in \mathbf{H}_2(\mathbb{D})$ and every $r \in (0,1)$

$$|s(re^{it})f(re^{it})|^2 \le |f(re^{it})|^2,$$

and thus

$$\int_0^{2\pi} |s(re^{it})f(re^{it})|^2 dt \le \int_0^{2\pi} |f(re^{it})|^2 dt.$$

It follows that $\|sf\|_{\mathbf{H}_2(\mathbb{D})} \le \|f\|_{\mathbf{H}_2(\mathbb{D})}$. We refer for instance to [6] for a proof of the converse statement.

Functions analytic and contractive (in modulus) in the open unit disk played an important role in Section 6.4 and were called there Schur functions.

The preceding discussion focused on scalar-valued signals and systems, but one can also consider the matrix-valued case. Then for a sequence $(u_n)_{n\in\mathbb{N}_0}$ of $\mathbb{C}^N$ vectors, the series

$$\sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} \begin{pmatrix} u_{n1} \\ u_{n2} \\ \vdots \\ u_{nN} \end{pmatrix} z^n$$

with

$$u_n = \begin{pmatrix} u_{n1} \\ u_{n2} \\ \vdots \\ u_{nN} \end{pmatrix}$$

is a column vector with each entry being a scalar power series. The radius of convergence of this series is by definition the smallest of the radiuses of convergence of the N power series

$$\sum_{n=0}^{\infty} u_{nj} z^n, \quad j = 1, \ldots, N.$$

See also Exercise 12.2.4.

11.5 The Schur algorithm

In Section 6.5 we have first met the recursion (6.5.7)

$$\begin{aligned} f_0(z) &= f(z), \\ f_{n+1}(z) &= \begin{cases} \dfrac{f_n(z) - f_n(0)}{z(1 - \overline{f_n(0)} f_n(z))}, & z \in \mathbb{D} \setminus \{0\}, \\ f_n'(0), & z = 0, \end{cases} \qquad n = 0, 1, \ldots \end{aligned}$$

where f is analytic and contractive in the open unit disk. The coefficients $\rho_n = f_n(0)$ are called the Schur coefficients, or reflection coefficients of the function f.

The Schur algorithm allows to solve in an iterative way classical interpolation problems such as:

Problem 11.5.1 (The Carathéodory–Fejér interpolation problem)**.** *Given numbers* $a_0, \ldots, a_N$*, find all (if any) Schur functions* f *such that*

$$\frac{f^{(n)}(0)}{n!} = a_n, \quad n = 0, \ldots, N.$$

Problem 11.5.2 (The Nevanlinna–Pick interpolation problem)**.** *Given* N *pairs of numbers* $(z_1, w_1), \ldots, (z_N, w_N)$ *in* $\mathbb{D}^2$*, find all (if any) Schur functions* f *such that*

$$f(z_n) = w_n, \quad n = 1, \ldots, N.$$

Exercise 11.5.3. *Let* $f \in \mathcal{S}$*. Then, show that the Schur algorithm applied to* f *ends after a finite number of times* $(N \geq 0)$ *if and only if* f *is a finite Blaschke product, or a unitary constant (this being the case when* $N = 0$*).*

For instance, if

$$f(z) = \frac{z-a}{1-z\overline{a}} \frac{z-b}{1-z\overline{b}},$$

then

$$f_1(z) = \frac{z-c}{1-z\overline{c}},$$

where c is given by (1.1.47),

$$c = \frac{(1-|a|^2)b + (1-|b|^2)a}{1-|ab|^2},$$

and

$$f_2(z) \equiv 1.$$

Indeed, we have for $z \neq 0$,

$$\begin{aligned}
f_1(z) &= \frac{1}{z} \frac{\dfrac{z-a}{1-z\overline{a}} \dfrac{z-b}{1-z\overline{b}} - ab}{1 - \overline{a}\overline{b}\dfrac{z-a}{1-z\overline{a}} \dfrac{z-b}{1-z\overline{b}}} \\
&= \frac{1}{z} \frac{(z-a)(z-b) - ab(1-z\overline{a})(1-z\overline{b})}{(1-z\overline{a})(1-z\overline{b}) - \overline{a}\overline{b}(z-a)(z-b)} \\
&= \frac{1}{z} \frac{z^2(1-|ab|^2) - z(a+b-ab(\overline{a}+\overline{b}))}{1-|ab|^2 - z(\overline{a}+\overline{b}) + \overline{a}\overline{b}(z(a+b)+ab)} \\
&= \frac{z-c}{1-\overline{c}z},
\end{aligned}$$

and

$$f_2(z) = \frac{1}{z} \frac{\frac{z-c}{1-z\overline{c}} + c}{1 + \overline{c}\frac{z-c}{1-z\overline{c}}} = \frac{1}{z} \frac{z(1-|c|^2)}{1-|c|^2} \equiv 1.$$

Theorem 11.5.4. *Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be a power series converging in a neighborhood of the origin. Then, f is analytic and contractive in the open unit disk if and only if either:*

(a) *Applying the Schur algorithm to f, we have*

$$|f_n(0)| < 1, \quad \forall n \in \mathbb{N}_0,$$

or

(b) *$f(0)$ has modulus 1 (and then f is a unitary constant), or the numbers $f_n(0)$ are strictly contractive up to a finite rank, say N_0, and $f_{N_0+1}(z)$ is a unitary constant.*

In view of the following question, we recall the notation (2.3.4)

$$T_M(z) = \frac{az+b}{cz+d},$$

where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Question 11.5.5. *Let us assume that the Schur function f in the recursion (6.5.7) is such that*

$$|f_n(0)| < 1, \quad n = 0, 1, \ldots, N.$$

Then, setting

$$\rho_n = f_n(0), \quad n = 0, 1, \ldots, N,$$

and using the notation (2.3.4) show that

$$f(z) = T_{M_N(z)}(f_{N+1}(z)) \tag{11.5.1}$$

where

$$M_N(z) = \overset{\curvearrowright}{\prod_{n=0}^{N}} \begin{pmatrix} 1 & \rho_n \\ \overline{\rho_n} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}. \tag{11.5.2}$$

Assume that $|\rho_n| < 1$, $n = 0, 1, \ldots$. The infinite product $\lim_{N\to\infty} M_N(z)$ diverges for every point z, with the possible exception of $z = 1$. A related infinite product, which plays a key role in the theory, converges on the unit circle:

Exercise 11.5.6. *Assume that* $|\rho_n| < 1$, $n = 0, 1, \ldots$, *and that, moreover*

$$\sum_{n=0}^{\infty} |\rho_n| < \infty.$$

Then, for every z *of modulus* 1, *the limit*

$$\lim_{N\to\infty} M_N(z) \begin{pmatrix} z^{-N-1} & 0 \\ 0 & 1 \end{pmatrix}$$

exists.

The following result gives four equivalent characterizations of Schur functions. The first one is on the level of a first complex variable course, while the second, third and fourth characterizations require (easy) functional analysis tools. These last three characterizations are much more conducive to defining counterparts of Schur functions for the extensions of linear system theory mentioned in the introduction of the chapter.

Theorem 11.5.7. *Let* f *be a function defined in the open unit disk. The following are equivalent:*

(1) f *is analytic and contractive in the open unit disk.*

(2) *The kernel*

$$k_f(z, w) = \frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}}$$

is positive definite in the open unit disk.

(3) *There exist a Hilbert space* $\mathcal{H}$ *and a coisometric operator matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \; : \; \mathcal{H} \oplus \mathbb{C} \longrightarrow \mathcal{H} \oplus \mathbb{C},$$

such that

$$f(z) = D + zC(I_{\mathcal{H}} - zA)^{-1}B.$$

(4) *The Taylor coefficients of* f *are of the form*

$$f_n = \begin{cases} D, & n = 0, \\ CA^{n-1}B, & n = 1, 2, \ldots, \end{cases}$$

where A, B, C, D *are as in* (3).

11.6 Solutions

Solution of Exercise 11.1.1. Let

$$f(z) = \frac{1}{2F}\int_{[-F,F]} e^{-izu}m(u)du.$$

Using (1.2.5) we have

$$|e^{izt}| \le e^{|z|F}.$$

Therefore, using the Cauchy–Schwarz inequality (16.1.5), we have

$$\begin{aligned}|f(z)| &\le \frac{e^{|z|F}}{2F}\int_{(-F,F)} |m(u)|du \\ &\le \frac{e^{|z|F}}{2F}\left(\int_{(-F,F)} |m(u)|^2 du\right)^{1/2}\left(\int_{(-F,F)} 1du\right)^{1/2} \\ &= Ke^{|z|F}\end{aligned}$$

with

$$K = \frac{\int_{(-F,F)} |m(u)|^2 du)^{1/2}}{\sqrt{2F}} < \infty,$$

since $m \in \mathbf{L}_2(-F,F)$. □

Solution of Exercise 11.2.2. We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{k=1}^{\infty}\frac{1}{(2k)^2} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} \\ &= \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.\end{aligned}$$

Taking into account (1.3.14) we have

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}.$$ □

Solution of Exercise 11.2.3. From the estimate in the previous exercise we see that the integral (11.2.3) is well defined for every $z \in \mathbb{C}$. The function is entire. For continuous m this follows from the same arguments as for Exercise 4.4.19. As explained after the proof of that exercise for the interval $(0,1)$, the statement is still true for functions $m \in \mathbf{L}_2(-F,F)$.

Let now $f \in \mathcal{H}_F$ be such that $f(z) \equiv 0$. Then, the choice $z = \frac{\pi n}{F}$ gives

$$\int_{(-F,F)} m(u)e^{\frac{-\pi i n u}{F}}du = 0, \quad \forall n \in \mathbb{Z}.$$

But the functions

$$f_n(u) = e^{\frac{\pi i n u}{F}}, \quad n \in \mathbb{Z}, \tag{11.6.1}$$

form an orthonormal basis of $\mathbf{L}_2(-F, F)$ (see Exercise 17.7.5). It follows that $m \equiv 0$ (as an element of $\mathbf{L}_2(-F, F)$). Therefore (11.2.4) indeed defines a norm, and $\mathcal{H}_F$ is a Hilbert space since $\mathbf{L}_2(-F, F)$ is a Hilbert space. Let for $z, w \in \mathbb{C}$,

$$K_F(z, w) = \frac{1}{2F} \int_{(-F,F)} e^{-i\overline{w}u} e^{izu} du = \frac{\sin(Fz - F\overline{w})}{Fz - F\overline{w}}.$$

Then for $f \in \mathcal{H}_F$ and $w \in \mathbb{C}$ we have that

$$f(w) = \frac{1}{2F} \int_{(-F,F)} m(u) e^{-iwt} dt = \langle f(\cdot), K_F(\cdot, w) \rangle_{\mathcal{H}_F}. \qquad \square$$

Solution of Exercise 11.2.4. Take $m \in \mathbf{L}_2(-F, F)$. Then

$$m(u) = \sum_{n \in \mathbb{Z}} \left(\frac{1}{2F} \int_{(-F,F)} m(s) e^{-\frac{i\pi s n}{F}} ds \right) e^{\frac{i\pi u n}{F}}$$

where the limit is in the norm of $\mathbf{L}_2(-F, F)$. By Parseval's equality, this sum becomes

$$f(\cdot) = \sum_{n \in \mathbb{Z}} f\left(\frac{\pi n}{F}\right) K_F\left(\cdot, \frac{\pi n}{F}\right), \tag{11.6.2}$$

where the equality is in the norm of $\mathcal{H}_F$. Let $z \in \mathbb{C}$ and $e_z(u) = e^{izu}$. Using the continuity of the inner product or Parseval equality we have with f_n as in (11.6.1)

$$\langle m, e_z \rangle_{\mathbf{L}_2(-F,F)} = \sum_{n \in \mathbb{Z}} \langle m, f_n \rangle_{\mathbf{L}_2(-F,F)} \langle f_n, e_z \rangle_{\mathbf{L}_2(-F,F)}.$$

In other words

$$f(z) = \sum_{n \in \mathbb{Z}} f\left(\frac{\pi n}{F}\right) K_F\left(z, \frac{\pi n}{F}\right), \quad z \in \mathbb{C}. \tag{11.6.3}$$

Here the convergence is pointwise, and uniform on bounded sets since the kernel is bounded on bounded sets. $\square$

Equation (11.6.3) can also be obtained directly from (11.6.2) since convergence in norm implies pointwise convergence in a reproducing kernel Hilbert space (see Exercise 16.3.13).

Solution of Exercise 11.3.3. It suffices to apply the Laplace transform on both sides of the state space equations. $\square$

Note that the transfer function is analytic at infinity. In the discrete case, the transfer function is analytic at the origin. See Exercise 12.2.4.

Solution of Exercise 11.5.3. Suppose that f is not a unitary constant and that the Schur algorithm ends after a finite number of steps. Then, there is an $N \in \mathbb{N}_0$ such that

$$f_n(0) \in \mathbb{D}, \quad n = 0, 1, 2, \ldots, N,$$

and $f_{N+1}(z)$ is a unitary constant. Formula (11.5.2) leads to

$$f(z) = T_{M_N(z)}(f_{N+1}).$$

For $|z| = 1$, the Moebius transform with matrix

$$\begin{pmatrix} 1 & \rho_n \\ \overline{\rho_n} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

sends the unit circle onto itself, and so does $T_{M_n(z)}$. Therefore, the function f is unitary on the unit circle. It follows from Exercise 6.3.4 that f is a finite Blaschke product, that is

$$f(z) = cz^L \prod_{n=1}^{M} b_{w_n}(z), \tag{11.6.4}$$

where $|c| = 1$, $L, M \in \mathbb{N}_0$ and the factors b_{w_n} are defined by (1.1.44), with $w_n \neq 0$.

Conversely, assume that f is a finite Blaschke product. We show that applying the Schur algorithm to f we obtain a finite Blaschke product with one less factor. If $L > 0$ in (11.6.4) this is clear. Assume now $L = 0$, and set

$$p(z) = c \prod_{n=1}^{M} (z - w_n) \quad \text{and} \quad q(z) = \prod_{n=1}^{M} (1 - \overline{w_n} z). \tag{11.6.5}$$

We have

$$f_1(z) = \frac{\overline{q(0)}}{q(0)} \frac{\left(c \dfrac{p(z)q(0) - p(0)q(z)}{z} \right)}{q(z)\overline{q(0)} - p(z)\overline{p(0)}} = c \frac{\left(\dfrac{p(z) - p(0)q(z)}{z} \right)}{q(z) - p(z)\overline{p(0)}}.$$

The coefficient of the power z^M in the polynomial $p(z) - p(0)q(z)$ is equal to

$$c \left(1 - \prod_{n=1}^{M} (-w_n) \prod_{n=1}^{M} (-\overline{w_n}) \right) = c \left(1 - \prod_{n=1}^{M} |w_n|^2 \right) \neq 0.$$

Thus the polynomial $p(z) - p(0)q(z)$ has degree M. It vanishes at the origin, and so the function

$$\frac{p(z) - p(0)q(z)}{z}, \quad z \neq 0,$$

defines a polynomial of degree $M-1$ (with value at the origin equal to $p'(0)-p(0)q'(0)$). The coefficient of the power z^M in the polynomial $q(z)$ is equal to $\prod_{n=1}^{M}(-\overline{w_n})$. Therefore, the coefficient of the power z^M in the polynomial

$$q(z)-p(z)\overline{p(0)}$$

is equal to

$$\prod_{n=1}^{M}(-\overline{w_n})-c\overline{p(0)}=0.$$

Therefore,

$$\deg(q(z)-p(z)\overline{p(0)})\leq M-1.$$

We want to show that $\deg f_1=M-1$. Since f_1 is unitary on the unit circle, it will then follow that f_1 is also a finite Blaschke product (see Exercise 6.3.4), but with one less factor.

To check that $\deg f_1=M-1$, we will show that the polynomials

$$\frac{p(z)-p(0)q(z)}{z}\quad\text{and}\quad q(z)-p(z)\overline{p(0)}$$

have no common zeros. Since $q(z)-p(z)\overline{p(0)}$ has value $1-|p(0)|^2>0$ at the origin, it is enough to check that the polynomials

$$p(z)-p(0)q(z)\quad\text{and}\quad q(z)-p(z)\overline{p(0)}$$

have no common zeros. If $z_0\in\mathbb{C}$ is such that

$$p(z_0)=p(0)q(z_0)\quad\text{and}\quad q(z_0)=p(z_0)\overline{p(0)},\tag{11.6.6}$$

we obtain

$$p(z_0)(1-|p(0)|^2)=0,$$

and hence $p(z_0)=0$, and hence, by (11.6.6), we also have $q(z_0)=0$. But this is not possible since, by (11.6.5), p and q have no common zero. It follows that $\deg f_1=M-1$. □

Solution of Exercise 11.5.6. Set

$$S_n(z)=\begin{pmatrix}z^n&0\\0&1\end{pmatrix}\begin{pmatrix}1&\rho_n\\\overline{\rho_n}&1\end{pmatrix}\begin{pmatrix}z^{-n}&0\\0&1\end{pmatrix}.$$

We have

$$\begin{pmatrix}z&0\\0&1\end{pmatrix}M_N(z)\begin{pmatrix}z^{-N-1}&0\\0&1\end{pmatrix}=\overset{\curvearrowright}{\prod_{n=0}^{N}}S_n(z).\tag{11.6.7}$$

Furthermore,

$$S_n(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & z^n \rho_n \\ z^{-n}\overline{\rho_n} & 0 \end{pmatrix} = I_2 + A_n(z),$$

with

$$A_n(z) = \begin{pmatrix} 0 & z^n \rho_n \\ z^{-n}\overline{\rho_n} & 0 \end{pmatrix}.$$

With

$$\|A\|_\infty = \max_{i,j=1,2} |a_{ij}|$$

we have, for $|z| = 1$,

$$\|A_n(z)\|_\infty = \rho_n.$$

Therefore

$$\sum_{n=0}^{\infty} \|A_n(z)\|_\infty$$

converges for every point on the unit circle. Since all norms are equivalent in $\mathbb{C}^{2\times 2}$ (see (16.1.2) for the definition of equivalent norms), we have that the infinite product (11.6.7) also converges in view of Theorem 3.7.3, $\square$