Chapter 10

Conformal Mappings

Riemann's mapping theorem asserts that a simply-connected domain different from $\mathbb{C}$ is conformally equivalent to the open unit disk: There exists an analytic bijection from Ω onto $\mathbb{D}$ (that the inverse is itself analytic is automatic; see Exercise 10.2.4). In this chapter we closely follow Chapters 5 and 6 of [45] and present some related exercises. The chapter is smaller than the previous ones, but is certainly of key importance in the theory of analytic functions. To quote [195, p. 1], Riemann's theorem *is one of those results one would like to present in a one-semester introductory course in complex variables, but often does not for lack of sufficient time.* The proof requires also some topology, which is not always known by students of a first complex variable course.

10.1 Uniform convergence on compact sets

The proof of Riemann's theorem is not constructive, and uses deep properties of the topology of the space of functions analytic in an open set. We review here some of these properties. The solutions of the following two questions will not be given here.

Question 10.1.1. *Let Ω be an open connected subset of $\mathbb{C}$. Show that there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of Ω with the following property: Given any compact subset K of Ω, there exists $N \in \mathbb{N}$ such that*

$$K \subset \bigcup_{n=1}^{N} K_n.$$

Question 10.1.2. *Let Ω and $(K_n)_{n\in\mathbb{N}}$ be as in the previous exercise. Show that* (see

D. Alpay, *A Complex Analysis Problem Book*,
DOI 10.1007/978-3-319-42181-0_10

[45, (3.3), p. 149])

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, \max_{z \in K_n} |f(z) - g(z)|) \tag{10.1.1}$$

defines a metric on the space $A(\Omega)$ *of functions analytic in* Ω.

Convergence of a sequence in the metric (10.1.1) is equivalent to uniform convergence on every compact subspace of Ω. The space $A(\Omega)$ endowed with this metric has a key property: A subset of $A(\Omega)$ is compact if and only if it is both closed and bounded. Locally convex Hausdorff barreled topological vector spaces for which this property holds are called Montel spaces. See, e.g., [214, Definition 34.2, p. 356]. We also refer to [CAPB2], where some of these definitions and concepts are reviewed. Bounded here does not mean boundedness with respect to the metric, but boundedness in a topological vector space. Recall:

Definition 10.1.3. Let V denote a topological vector space on the complex numbers or on the real numbers. The set $U \subset V$ is called *bounded* if for every neighborhood W of the origin there exists $\lambda > 0$ such that

$$U \subset \lambda W.$$

The above characterization of compact sets is the key in the proof of Riemann's theorem. We refer to [109] for a thorough study of the metric spaces where (sequential) compactness is equivalent to being closed and bounded.

10.2 One-to-oneness

It is an important fact that an analytic function is one-to-one in a neighborhood of a point where its derivative does not vanish. For the following exercise, see [148, p. 372].

Exercise 10.2.1. *Let* f *be analytic in a convex open set* Ω *and assume that* $\operatorname{Re} f'(z) > 0$ *in* Ω. *Show that* f *is one-to-one in* Ω.

Note that an analytic function which is one-to-one on an open set Ω is said to be univalent in that set. As a corollary of this exercise we get the following very important result. For the converse statement, namely that when $f'(z_0) = 0$ there is no neighborhood of z_0 in which the function is one-to-one, see Exercise 7.3.8.

Theorem 10.2.2. *An analytic function is univalent in a neighborhood of any point where its derivative does not vanish.*

Indeed, if $f'(z_0) \neq 0$, then at least one of the numbers $\operatorname{Re} f'(z_0)$ and $\operatorname{Im} f'(z_0)$ is not zero. Without loss of generality we may assume that $\operatorname{Re} f'(z_0) > 0$ (otherwise replace f by $-f$ or $\pm if$ depending on the case). By continuity, $\operatorname{Re} f'(z) > 0$ in an open disk around z_0. We can then apply the precedent result since a disk is in particular convex.

Exercise 10.2.3. *Give a solution of Exercise* 5.2.9 *using Exercise* 10.2.1*.*

We note the following:

$$f'(z) = 1 + \sum_{n=2}^{\infty} n a_n z^{n-1},$$

and in particular

$$\begin{aligned} |f'(z)| &\geq 1 - \left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \\ &\geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \\ &\geq 1 - \sum_{n=2}^{\infty} n |a_n| > 0, \quad \forall z \in \mathbb{D}. \end{aligned}$$

Thus, by Theorem 10.2.2, f is one-to-one in a neighborhood of every point in $\mathbb{D}$. This is a local result. We want a direct solution of a global result: f is one-to-one in $\mathbb{D}$.

Exercise 10.2.4. *Assume that the analytic function f is one-to-one in Ω. Show that the formula* (see, e.g., [42, p. 180])

$$g(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{s f'(s)}{f(s) - z} ds, \tag{10.2.1}$$

where γ is a closed simple contour, defines the inverse of f inside γ.

Formula (10.2.1) shows in particular that f^{-1} is analytic.

We now consider the case where the derivative vanishes at a given point. It is no loss of generality to assume that the function itself also vanishes at that point.

Exercise 10.2.5. *Let f be analytic in the open subset Ω and assume that $z_0 \in \Omega$ is a zero of order N of f. Show that there is a function g which is analytic and one-to-one in some open neighborhood $U \subset \Omega$ of z_0 and such that*

$$f(z) = g(z)^N, \quad z \in U. \tag{10.2.2}$$

With the preceding exercises at hand we can state the following key result, called the *open mapping theorem* (see also Exercise 7.4.9).

Theorem 10.2.6. *Let Ω be an open subset of $\mathbb{C}$ and let f be analytic in Ω. Then, $f(\Omega)$ is an open subset of $\mathbb{C}$.*

Proof. Take first a point $\omega \in \Omega$ where $f'(\omega) \neq 0$. By Exercise 10.2.4 the function f is one-to-one in an open neighborhood U of ω, with analytic inverse. The inverse h of f,

$$h: \ f(U) \longrightarrow U,$$

is in particular continuous, and therefore

$$f(U) = h^{-1}(U) \subset f(\Omega)$$

is an open neighborhood of $f(\omega)$ which lies inside $f(\Omega)$. If $f'(\omega) = 0$, we first remark that for any $N \in \mathbb{N}$ the map $z \mapsto z^N$ maps open balls into open balls, and therefore open sets into open sets. Write f in the form (10.2.2). There is an open neighborhood U of ω where $g(z)$ is one-to-one. By the above argument, $g(U)$ is open, and so is $f(U) = g(U)^N$. $\square$

Exercise 10.2.7. *In the notation and hypothesis of Exercise* 6.1.9*, show that the set* Ω_0 *contains uncountably many points.*

We conclude this section with an important fact on univalent functions, which comes into play in the proof of Riemann's mapping theorem. See [45, Proposition 2.2, p. 147, p. 191].

Exercise 10.2.8. *Let* Ω *be open and connected, and let* $(s_n)_{n\in\mathbb{N}}$ *be a sequence of functions univalent in* Ω*, which converge uniformly on compact subsets of* Ω*. The limit is then either a constant or univalent.*

10.3 Conformal mappings

Simply-connected sets have already been characterized in a number of ways. Geometrically, Riemann's mapping theorem expresses the following characterization:

Definition 10.3.1. A connected open subset Ω of $\mathbb{C}$ which is different from $\mathbb{C}$ is *simply-connected* if it is conformally equivalent to the open unit disk.

Question 10.3.2. *Show that any open disk is conformally equivalent to any open half-plane.*

We recall that the Blaschke factors (1.1.44), possibly multiplied by a constant of modulus 1,

$$\varphi(z) = c\frac{z-a}{1-\overline{a}z}$$

are the only conformal mappings from the open unit disk onto itself. Taking into account this fact allows to solve the following exercise.

Exercise 10.3.3.

(1) *Show that the conformal maps from the open upper half-plane* $\mathbb{C}_+$ *onto itself are exactly the Moebius maps which can be written in the form*

$$\varphi(z) = \frac{az+b}{cz+d},$$

where a, b, c, d *are real and such that* $ad - bc = 1$.

(2) *Show that any two points in* $\mathbb{C}_+$ *can be related by such a conformal map.*

The proof of Riemann's mapping theorem (see for instance H. Cartan's [45]) uses the fact that a connected open subset Ω of the complex plane is simply connected if and only if every non-vanishing function analytic in Ω admits an analytic logarithm. The proof can be divided into three steps (and here, we follow [45]):

(a) Reduce to the case where $\Omega \subset \mathbb{D}$ and $0 \in \Omega$.

(b) Show that the existence of a conformal map is equivalent to the solution of a maximum problem.

(c) Show that the maximum problem has a solution.

Steps (a) and (b) use, each once only once, the assumed existence of an analytic logarithm. Step (c) uses topology tools which are somewhat beyond the scope of the present book. The content of the following question is Step (b).

Question 10.3.4. *Let* Ω *be an open subset of* $\mathbb{D}$, *containing the origin, and with the property that every non vanishing function analytic in* Ω *has an analytic logarithm. Let* M *denote the set of univalent functions from* Ω *into* $\mathbb{D}$ *such that* $f(0) = 0$. *Show that the range of* f *is* $\mathbb{D}$ *if and only if*

$$|f'(0)| = \max_{g \in M} |g'(0)|.$$

Hints: One direction is relatively easy, and uses the Schwarz lemma. For the other direction, proceed by contradiction, and use Theorem 5.7.6 (see [45]).

Question 10.3.5. *Show that* $\tan z$ *is a conformal map from the strip*

$$L_1 = \{(x, y)\,;x \in (-\pi/4, \pi/4) \quad \text{and} \quad y \in \mathbb{R}\}$$

onto the open unit disk.

Exercise 10.3.6. *Find a conformal map between the open right half-plane and the quarter-plane*

$$\{(x, y)\,;\, 0 < x < |y|\}\,.$$

Exercise 10.3.7. *Let* $\mathbb{D}$ *denote the open unit disk and* $\mathbb{C}_+$ *denote the open upper half-plane. Show that the map*

$$\varphi(z) = \frac{z - i(z^2+1)}{z + i(z^2+1)}$$

is a conformal mapping from $\mathbb{D}_+ = \mathbb{D} \cap \mathbb{C}_+$ *onto* $\mathbb{D}$. *What happens on the boundary?*

Exercise 10.3.8 (see [75, Exercice 35.33, p. 329])**.** *Let* $\alpha \in (0, \pi/2)$ *and define*

$$D_\alpha = \left\{ z \in \mathbb{C}\,;\ |z \pm i \cot \alpha| < \frac{1}{\sin \alpha} \right\}.$$

Show that the map

$$c(z) = \frac{\left(\frac{1+z}{1-z}\right)^{\frac{\pi}{2\alpha}} - 1}{\left(\frac{1+z}{1-z}\right)^{\frac{\pi}{2\alpha}} + 1}$$

is conformal from D_α *onto* $\mathbb{D}$*, and that its inverse is given by*

$$c^{-1}(z) = \frac{\left(\frac{1+z}{1-z}\right)^{\frac{2\alpha}{\pi}} - 1}{\left(\frac{1+z}{1-z}\right)^{\frac{2\alpha}{\pi}} + 1}.$$

The following exercise can be found for instance in [53, p. 203], [168, Exercise 2, p. 196], and [18, § 10.4.4, pp. 308–311]. We follow the solution of that latter reference. In the statement the function $\sqrt{1 - s^4}$ is defined via (4.4.9).

Exercise 10.3.9. *Show that the map*

$$z \mapsto c(z) = \int_{[0,z]} \frac{ds}{\sqrt{1 - s^4}} \tag{10.3.1}$$

is conformal from $\mathbb{D}$ *onto a square.*

Hint. Following [18, § 10.4.4, pp. 308–311] we suggest to solve the exercise along the steps below:

Step 1: Show that the map c extends continuously to the closed unit disk, and that (see [18, p. 310])

$$c(e^{i\theta}) = M + e^{i\frac{3\pi}{4}} \int_0^\theta \frac{du}{\sqrt{2\sin(2u)}}, \quad \theta \in \left[0, \frac{\pi}{4}\right]. \tag{10.3.2}$$

for some constant $M > 0$.

Step 2: Show that the image of the unit circle is the boundary of a square. Exercise 3.5.7 plays an important role in this step. It is also useful to note that

$$c(iz) = ic(z), \quad z \in \mathbb{D}. \tag{10.3.3}$$

Step 3: Compute $\frac{1}{2\pi i} \int_{|z|=1} \frac{c'(z)}{c(z)} dz$.

10.4 Solutions

Solution of Exercise 10.2.1. Let z_1 and z_2 be in Ω. Since Ω is convex, the closed interval

$$[z_1, z_2] = \{z_1 + t(z_2 - z_1)\,;\, t \in [0,1]\} \subset \Omega.$$

By the fundamental theorem of calculus for analytic functions,

$$\begin{aligned}
f(z_2) - f(z_1) &= \int_{[z_1,z_2]} f'(z)dz \\
&= (z_2 - z_1)\int_0^1 f'(z_1 + t(z_2 - z_1))dt \\
&= (z_2 - z_1)\left\{\mathrm{Re}\left(\int_0^1 f'(z_1 + t(z_2 - z_1))dt\right)\right. \\
&\qquad \left. + i\,\mathrm{Im}\left(\int_0^1 f'(z_1 + t(z_2 - z_1))dt\right)\right\}.
\end{aligned} \tag{10.4.1}$$

Since $\mathrm{Re}\, f'(z) > 0$ in Ω we have

$$\mathrm{Re}\left(\int_0^1 f'(z_1 + t(z_2 - z_1))dt\right) > 0. \tag{10.4.2}$$

It follows from (10.4.2) that

$$\begin{aligned}
f(z_2) - f(z_1) = (z_2 - z_1)&\left\{\mathrm{Re}\left(\int_0^1 f'(z_1 + t(z_2 - z_1))dt\right)\right. \\
&\qquad \left. + \, i\ \mathrm{Im}\left(\int_0^1 f'(z_1 + t(z_2 - z_1))dt\right)\right\}
\end{aligned}$$

that $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. □

Solution of Exercise 10.2.3. We have, for $z \in \mathbb{D}$,

$$\mathrm{Re}\, f'(z) = 1 - \mathrm{Re}\sum_{n=2}^{\infty} n a_n z^{n-1} \geq 1 - \left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \geq 1 - \sum_{n=2}^{\infty} n|a_n| > 0.$$

It suffices then to apply the previous exercise. □

Solution of Exercise 10.2.4. We have, for z_0 inside γ,

$$\begin{aligned}
g(f(z_0)) &= \frac{1}{2\pi i}\int_\gamma \frac{sf'(s)}{\frac{f(s) - f(z_0)}{s - z_0}} \frac{1}{s - z_0} ds \\
&= \left(\frac{sf'(s)}{\frac{f(s) - f(z_0)}{s - z_0}}\right)_{s=z_0} \\
&= z_0.
\end{aligned}$$

□

Solution of Exercise 10.2.5. By definition of a zero of order N we can write in some neighborhood $W \subset \Omega$ of z_0,

$$f(z) = (z - z_0)^N h(z),$$

where h is analytic in W and does not vanish there. We can always assume W to be convex (for instance, W may be chosen to be an open disk with center z_0 and small enough radius). Then, the function h has an analytic logarithm in W, and therefore also an analytic root of order N: There is a function h_0 analytic in W and such that

$$h(z) = h_0(z)^N, \quad z \in W.$$

We therefore have $f(z) = ((z - z_0)h_0(z))^N, \quad z \in W$. The function $g(z) = (z - z_0)h_0(z)$ is analytic in W. It is one-to-one in a neighborhood $U \subset W$ of z_0 since

$$g'(z)|_{z=z_0} = ((z - z_0)h_0'(z) + h_0(z))\,|_{z=z_0} = h_0(z_0) \neq 0. \qquad \square$$

The following solution is taken from [10, pp. 4–5].

Solution of Exercise 10.2.7. We use the notation of Exercises 4.1.13 and 6.1.9. We know from Exercise 6.1.9 that there is a point $\mu \in \Omega$ such that

$$|a(\mu)| = |b(\mu)| \neq 0.$$

The map

$$\sigma(z) = \frac{b(z)}{a(z)}$$

is analytic in the open set $\Omega \setminus \mathcal{Z}(a)$. The image $\sigma(\Omega \setminus \mathcal{Z}(a))$ is an open set, and therefore there exists an $r > 0$ such that

$$B(\sigma(\mu), r) \subset \sigma(\Omega \setminus \mathcal{Z}(a)).$$

The image $\sigma(\Omega \setminus \mathcal{Z}(a))$ contains in particular an arc of a circle, and the claim follows. $\square$

Solution of Exercise 10.2.8. We first remark that the limit function s is indeed analytic, since the convergence is uniform on compact subsets of Ω. Assume that s is not a constant, but that there are two points a_1 and a_2 in Ω such that

$$s(a_1) = s(a_2) \stackrel{\text{def.}}{=} c.$$

The function $s(z) - c$ has isolated zeroes (since it is not a constant), and therefore we can find two closed neighborhoods

$$B_c(a_1, \rho_1) = \{z \in \Omega\,;\, |z - a_1| \leq \rho_1\}$$

and

$$B_c(a_2, \rho_2) = \{z \in \Omega\,;\, |z - a_2| \leq \rho_1\},$$

with ρ_1 and ρ_2 strictly positive, such that

$$B_c(a_1,\rho_1)\cap B_c(a_2,\rho_2)=\emptyset,$$

and such that

$$s(z)-c\neq 0,$$

both in $B_c(a_1,\rho_1)\setminus\{a_1\}$ and in $B_c(a_2,\rho_2)\setminus\{a_2\}$. Set

$$m_\ell=\min_{|a_\ell-z|=\rho_\ell}|s(z)-c|,\quad \ell=1,2.$$

We have that $m_1>0$ and $m_2>0$. Furthermore, since the neighborhoods $B_c(a_1,\rho_1)$ and $B_c(a_2,\rho_2)$ are compact, there exists $N\in\mathbb{N}$ such that

$$n\geq N\Longrightarrow \forall z\in B_c(a_1,\rho_1)\cup B_c(a_2,\rho_2),\quad |s_n(z)-s(z)|<m_\ell,\quad \ell=1,2.$$

Thus, for all $z\in B_c(a_\ell,\rho_\ell)$, $\ell=1,2$, we have

$$|s_n(z)-s(z)|<m_\ell\leq|s(z)-c|.$$

From Rouché's theorem (see Exercise 7.4.1), we have that $s_n(z)-c$ vanishes in $B_c(a_\ell,\rho_\ell)$ for $\ell=1,2$, contradicting the fact that the s_n are univalent since $B_c(a_1,\rho_1)\cap B_c(a_2,\rho_2)=\emptyset$. □

Solution of Exercise 10.3.3.
(1) The map $\varphi(z)=\frac{1+iz}{1-iz}$ sends conformally $\mathbb{C}_+$ onto $\mathbb{D}$. It follows that the conformal maps of $\mathbb{C}_+$ onto itself are, in terms of matrices, of the form

$$\begin{pmatrix} i & 1\\ -i & 1\end{pmatrix}^{-1}\begin{pmatrix} k & ku\\ \overline{u} & 1\end{pmatrix}\begin{pmatrix} i & 1\\ -i & 1\end{pmatrix}=\frac{1}{2i}\begin{pmatrix} i(k(1-u)+(1-\overline{u})) & k(1+u)-(1+\overline{u})\\ -k(1-u)+(1-\overline{u}) & i(k(1+u)+(1+\overline{u})).\end{pmatrix}$$

with $k\in\mathbb{T}$ and $u\in\mathbb{D}$. Let $k=e^{i\theta}$ with $\theta\in\mathbb{R}$. Dividing the entries of the above matrix by $e^{i\frac{\theta}{2}}\sqrt{1-|u|^2}$ we obtain the matrix

$$\frac{1}{\sqrt{1-|u|^2}}\begin{pmatrix}\operatorname{Re}e^{i\frac{\theta}{2}}(1-u)) & \operatorname{Im}(e^{i\frac{\theta}{2}}(1+u))\\ -\operatorname{Im}(e^{i\frac{\theta}{2}}(1-u)) & \operatorname{Re}(e^{i\frac{\theta}{2}}(1+u))\end{pmatrix},\tag{10.4.3}$$

which is of the required form. Conversely for any $\varphi(z)=\frac{az+b}{cz+d}$ where a,b,c,d are real and such that $ad-bc=1$ we have

$$\operatorname{Im}\varphi(z)=\frac{\operatorname{Im}z}{|cz+d|^2}$$

and so φ sends $\mathbb{C}_+$ onto itself.

(2) The result is a direct consequence of Exercise 2.3.5. □

Remark 10.4.1. When $u = 0$ the matrix (10.4.3) becomes

$$\begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}.$$

Solution of Exercise 10.3.6. The open right half-plane $\mathbb{C}_r$ consists of the complex numbers $z = re^{it}$ with $r > 0$ and $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The map $\sqrt{z} = \sqrt{r}e^{i\frac{t}{2}}$ is a conformal map from $\mathbb{C}_r$ onto the quarter-plane, with inverse map z^2. □

Solution of Exercise 10.3.7. The map $G(z) = \dfrac{1-z}{1+z}$ is conformal from the open right half-plane $\mathbb{C}_r = \{z = x + iy \in \mathbb{C}\ ;\ x > 0\}$ onto $\mathbb{D}$. It is therefore enough to check that $G^{-1} \circ \varphi$ is conformal from $\mathbb{D}_+$ onto the right half-plane. But $G^{-1}(z) = \dfrac{1-z}{1+z}$ and so

$$G^{-1} \circ \varphi(z) = i\frac{z^2+1}{z}.$$

Let us write

$$\psi(z) = i\frac{z^2+1}{z} = i\left(z + \frac{1}{z}\right). \tag{10.4.4}$$

We now proceed in a number of steps.

Step 1: *ψ is one-to-one from $\mathbb{D}_+$ onto its range.*

Indeed, assume that $\psi(z_1) = \psi(z_2)$. Then, in view of (10.4.4),

$$z_1 - z_2 + \frac{1}{z_1} - \frac{1}{z_2} = 0,$$

that is

$$(z_1 - z_2)(1 - \frac{1}{z_1 z_2}) = 0.$$

Thus $z_1 = z_2$ or $z_1 = \dfrac{1}{z_2}$. Since we assume that both z_1 and z_2 belong to $\mathbb{D}_+$ we have $z_1 = z_2$.

Step 2: *The range of ψ is inside $\mathbb{C}_r$.*

Indeed, with $z = x + iy$,

$$\begin{aligned}\psi(z) &= i\left((x+iy) + \frac{x-iy}{x^2+y^2}\right) \\ &= y\left(\frac{1}{x^2+y^2} - 1\right) + i\left(\frac{x}{x^2+y^2} + x\right).\end{aligned}$$

But for $z \in \mathbb{D}_+$ we have

$$y > 0 \quad \text{and} \quad \frac{1}{x^2+y^2} > 1,$$

and so

$$y\left(\frac{1}{x^2+y^2}-1\right)>0,$$

that is $\operatorname{Re}\psi(z)>0$.

Step 3: *ψ is onto $\mathbb{C}_r$.*

Indeed, for w such that $\operatorname{Re} w>0$ consider the equation $\psi(z)=w$. We have

$$z^2+izw+1=0,$$

and thus the product of the two roots of this second degree equation is equal to 1. Since, in view of the previous step,

$$0<\operatorname{Re} w=y\left(\frac{1}{x^2+y^2}-1\right),$$

we see that one of them is in $\mathbb{D}_+$. □

Solution of Exercise 10.3.8. We proceed in a number of steps:

(1) *The map*

$$z\mapsto\psi(z)=\frac{1+z}{1-z}$$

is conformal from $\mathbb{D}$ onto the open right half-plane, with inverse $\psi^{-1}(z)=\frac{z-1}{z+1}$.

This follows from

$$\frac{1+z}{1-z}+\frac{1+\overline{z}}{1-\overline{z}}=2\frac{1-|z|^2}{|1-z|^2}.$$

(2) *Let $z=re^{it}$ with $r>0$ and $t\in(-\pi,\pi)$. The map*

$$z\mapsto p_\alpha(z)=z^{\frac{\pi}{2\alpha}}=r^{\frac{\pi}{2\alpha}}e^{i\frac{\pi}{2\alpha}t} \tag{10.4.5}$$

is conformal from the domain

$$C_{r,\alpha}=\{z\in\mathbb{C}\,;0<x<(\tan\alpha)|y|\}$$

onto the open right half-plane.

This is because $z\in C_{r,\alpha}$ if and only if it is of the form $z=re^{i\theta}$, where $\theta\in(-\alpha,\alpha)$. Under the map (10.4.5) the angle has now range $(-\frac{\pi}{2},\frac{\pi}{2})$.

(3) *The map $\psi^{-1}(z)=\frac{z-1}{z+1}$ is conformal from $C_{r,\alpha}$ onto D_α.*

We note that the boundary of $C_{r,\alpha}$ consists of the two rays $re^{\pm i\alpha}$, with $r\in[0,\infty)$. We first check that this boundary is sent onto the boundary of D_α. We consider the ray $re^{i\alpha}$. The other one is treated in the same way. Let therefore

$$x+iy=\frac{re^{i\alpha}-1}{re^{i\alpha}+1}=\frac{r^2-1}{r^2+1+2r\cos\alpha}+i\frac{2r\sin\alpha}{r^2+1+2r\cos\alpha}$$

be in the image of this ray under ψ^{-1}. We have

$$y+\cot\alpha=\frac{1}{\sin\alpha}\frac{2r+(1+r^2)\cos\alpha}{r^2+1+2r\cos\alpha}.$$

Thus

$$\begin{aligned}|x+iy+i\cot\alpha|^2&=\left(\frac{r^2-1}{r^2+1+2r\cos\alpha}\right)^2+\frac{1}{\sin^2\alpha}\left(\frac{2r+(1+r^2)\cos\alpha}{r^2+1+2r\cos\alpha}\right)^2\\&=\frac{(r^2-1)^2\sin^2\alpha+(r^2+1)^2\cos^2\alpha+4r^2+4r(r^2+1)\cos\alpha}{(\sin^2\alpha)(r^2+1+2r\cos\alpha)^2}\\&=\frac{1}{\sin^2\alpha},\end{aligned}$$

and similarly when α is replaced by $-\alpha$. Since the image of $z=1$ under ψ^{-1} is $z=0$ we conclude that ψ^{-1} is conformal from $C_{r,\alpha}$ onto D_α. The claim on the inverse of c follows from the fact that $c=\psi^{-1}\circ p_\alpha\circ\psi$ (where p_α is defined by (10.4.5)). □

Solution of Exercise 10.3.9.

Step 1: Let $\alpha_0,\alpha_1,\alpha_2,\ldots$ be defined by

$$\frac{1}{\sqrt{1-z}}=\sum_{n=0}^\infty\alpha_nz^n,\quad z\in\mathbb{D}.$$

For $z\in\mathbb{D}$ we have

$$\begin{aligned}c(z)&=\int_{[0,z]}\frac{ds}{\sqrt{1-s^4}}=\int_0^1\frac{z}{\sqrt{1-z^4t^4}}dt\\&=\sum_{n=0}^\infty\alpha_nz^{4n+1}\int_0^1t^{4n}dt\end{aligned}$$

(where one can use, for instance, the dominated convergence theorem to interchange the sum and the integral)

$$=\sum_{n=0}^\infty\frac{\alpha_nz^{4n+1}}{n+1},\quad z\in\mathbb{D}.$$

The coefficients $\alpha_0,\alpha_1,\ldots$ satisfy (3.5.9), and so this last expression defines a function analytic in $\mathbb{D}$ (namely, $c(z)$) and continuous in the closed unit disk $\overline{\mathbb{D}}$. By Exercise 3.5.7, we have for $\theta\in[0,2\pi]$

$$c(e^{i\theta})=M+i\int_0^\theta\left(\sum_{n=0}^\infty\alpha_ne^{i(4n+1)u}\right)du,\quad\text{where}\quad M=\sum_{n=0}^\infty\frac{\alpha_n}{n+1}.$$

To conclude the first step we show that

$$i\sum_{n=0}^{\infty}\alpha_n e^{i(4n+1)u} = \frac{e^{i\frac{3\pi}{4}}}{\sqrt{2\sin(2u)}}, \quad u \in \left(0, \frac{\pi}{4}\right). \tag{10.4.6}$$

To that purpose, let $t \in (0,1)$. By Theorem 3.5.1 the sum $\sum_{n=0}^{\infty}\alpha_n t^n e^{4inu}$ converges for $u \in (0, \frac{\pi}{4})$. By Theorem 3.5.4, and for such u, we have:

$$ie^{iu}\lim_{\substack{t\to 1\\ t\in(0,1)}}\sum_{n=0}^{\infty}\alpha_n t^n e^{4inu} = ie^{iu}\sum_{n=0}^{\infty}\alpha_n e^{4inu}.$$

On the other hand,

$$i\sum_{n=0}^{\infty}\alpha_n t^n e^{i(4n+1)u} = \frac{ie^{iu}}{\sqrt{1-t^4e^{4iu}}}.$$

Consider the polar decomposition

$$\frac{ie^{iu}}{\sqrt{1-t^4e^{4iu}}} = \rho_t(u)e^{i\theta_t(u)},$$

with $\theta_t(u) \in (0, \frac{\pi}{4})$. We have

$$\begin{aligned}\rho_t(u) &= \frac{1}{|\sqrt{1-t^4e^{4iu}}|}\\ &= \frac{1}{\sqrt[4]{1+t^8-2t^4\cos(4u)}}\\ &\longrightarrow \frac{1}{\sqrt[4]{2-2\cos(4u)}} = \frac{1}{\sqrt{2\sin(2u)}}.\end{aligned}$$

as $t \to 1$. Moreover,

$$\rho_t(u)^2 e^{2i\theta_t(u)} = \frac{-e^{2iu}}{1-t^4e^{4iu}} \ \to\ \frac{-i}{2\sin(2u)},$$

as $t \to 1$, and so $\lim_{t\to 1} 2\theta_t(u) = \frac{3\pi}{2}$.

Step 2: It follows from (10.3.2) that c maps $[0, \frac{\pi}{4}]$ into a closed interval. On the other hand, the formula (10.3.3)

$$c(iz) = \int_{[0,iz]}\frac{ds}{\sqrt{1-s^4}} = \int_0^1 \frac{iz\,dt}{\sqrt{1-(iz)^4}} = ic(z), \quad z \in \mathbb{D},$$

still holds on the boundary using radial limits since $\lim_{\substack{r\to 1\\ r\in(0,1)}} c(re^{i\theta})$ exists for $\theta \in [0, 2\pi] \setminus \left\{0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi\right\}$, and shows that the image of $[\frac{\pi}{4}, \frac{\pi}{2}]$ is an interval of the same length, rotated by $\pi/2$ in the trigonometric sense. The same holds for the other two quadrants, and the image of the unit circle is a square.

Step 3: Let $w \in \mathbb{D}$ and let $r \in (|w|, 1)$. By Exercise 7.3.5

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{c'(z)}{c(z)-w}dz$$

is equal to the number of solutions of the equation $c(z) = w$ in $B(0, r)$. The function $w \mapsto \frac{1}{2\pi i}\int_{|z|=r}\frac{c'(z)}{c(z)-w}dz$ takes integer values and is continuous. It is constant on open connected sets, and so equal to its value at $w = 0$. On the other hand, by the dominated convergence theorem

$$\lim_{r\to 1}\int_{|z|=r}\frac{c'(z)}{c(z)}dz = \int_{|z|=1}\frac{c'(z)}{c(z)}dz.$$

To conclude, note that, by definition of the winding number,

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{c'(z)}{c(z)}dz = 1. \qquad \square$$

Remark 10.4.2. A variation of the preceding arguments will show that the application

$$c(z) = \int_{[0,z]}\frac{ds}{(1-s^n)^{\frac{2}{n}}}$$

maps conformally the open unit disk onto the interior of a regular polygon with n sides, the length of the side being equal to

$$\frac{2\pi}{n}\frac{\Gamma\left(1-\frac{2}{n}\right)}{\left(\Gamma\left(1-\frac{1}{n}\right)\right)^2} = \frac{1}{n}2^{1-\frac{4}{n}}\frac{\left(\Gamma\left(\frac{1}{2}-\frac{1}{n}\right)\right)^2}{\Gamma\left(1-\frac{2}{n}\right)}. \tag{10.4.7}$$

See [168, Exercise 4, p. 196], [195, Example 5.1, p. 48].

Using Legendre's duplication formula (see, e.g., [53, p. 212], [146, (1.2.3) p. 3])

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{10.4.8}$$

it is readily seen that both expressions in (10.4.7) coincide. Indeed, it is equivalent to prove that

$$\sqrt{\pi}\Gamma\left(1-\frac{2}{n}\right) = 2^{-\frac{2}{n}}\Gamma\left(\frac{1}{2}-\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right), \tag{10.4.9}$$

which is (10.4.8) with $z = \frac{1}{2} - \frac{1}{n}$.

Remark 10.4.3. We will not discuss here the Schwarz–Christoffel formula (see, e.g., [168, Chapter 5, § 6, p. 189], [195, p. 42]), which allows to build conformal maps onto certain polygons.