Digital Technologies and a Modeling Approach to Learn Mathematics and Develop Problem Solving Competencies

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Abstract. This study is framed within a conceptual approach that integrates modeling, problem solving, and the use of digital technologies perspectives in mathematical learning. It focuses on the use of a Dynamic Geometry System (GeoGebra) to construct mathematical models as a means to represent and explore mathematical relationships. In particular, we analyze and document what ways of reasoning high school students exhibit as a result of working on a mathematical task in problem solving sessions. Results show that the students rely on a set of technology affordances to dynamically visualize, represent and explore mathematical relations. In this process, the students' discussions became relevant not only to explain their approaches; but also to contrast, and eventually refine, their initial models and ways of reasoning.

Keywords: Digital technologies · Modeling · Problem solving · Mathematics learning

1 Introduction

Significant developments and uses of digital technologies permeate the ways people communicate, share information, interact and carry out social events. Educational systems face a challenge to incorporate the use of several digital technologies in both the ways of structuring and approaching the content to be studied and in the design of learning scenarios for students to construct disciplinary knowledge. In the learning of mathematics, both multiple purpose (Internet, YouTube, Wikipedia, etc.) and ad hoc technologies (Dynamic Geometry Systems, WolframAlpha, etc.) are transforming the mathematics curriculum (all levels) and the design of learning scenarios. In this context, mathematical tasks are essential to analyze what changes the use of technology brings to the content and to document the type of thinking that learners develop or construct as a result of problem solving instruction. The National Council Teachers of Mathematics (NCTM) [15] recognize that tasks are the vehicle for high school students to engage in mathematical reasoning and making sense activities in order to understand mathematical concepts and to develop problem solving competencies. Similarly,

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Schoenfeld [18] stated that "… curricula could be enlivened with more interesting tasks, with a greater focus on sense making, and with a more coherent attempt to build problem-solving skills: not as "add-ons" but as a core component of mathematical activity" (p. 506). Thus, the design or selection of mathematical tasks is crucial for instruction to help and promote the students' construction of mathematical knowledge and a way of thinking that is consistent with mathematical practice. Cai [2] stated "[o] nly worthwhile problems give students the chance to both solidify and extend what they know and to stimulate their learning" (p. 252). It is also relevant to recognize that students' understanding of mathematical ideas is an ongoing process in which they gradually construct a network of concepts. Engelbrecht [5] mentioned: "the dynamic process of understanding new mathematics takes place in layers. With every layer you understand a little deeper" (p. 152).

English and Sriraman [6] argued that if students are to develop a problem solving approach in a productive way, they need to engage, during their interaction with the tasks, in iterative cycles of describing, representing, exploring, testing, and revising mathematical interpretation, as well as identifying, modifying, and refining mathematical concepts to be used in problem solving.

2 Conceptual Framework

The notion of "bricolage", taken as the convergence of several perspectives, is used to structure and frame the study. That is, "…rather than adhering to one particular theoretical perspective, we act as bricoleurs by adapting ideas from a range of theoretical sources" ([4], p. 29). To this end, the foundations, in this study, incorporate concepts associated with modeling approaches [1, 11]; and principles associated with problem-solving approaches and the use of digital technologies to support and frame the study [19, 21]. Simon [20] pointed out that "the availability of multiple explanatory theories and the use of multiple layers of analysis can, depending on the research, provide a richer set of constructs for accounting for observed phenomena" (p. 484). Under these frameworks and perspectives, the students' construction of mathematical knowledge is conceptualized as a process in which they gradually construct an interrelated network of concepts (conceptual comprehension); utilize different procedures and algorithms; participate in the process of formulation of questions and conjectures; propose various ways to explore, explain, and support mathematical results; and develop a disposition to solve problems and to exhibit reasoning habits[1] associated with mathematical thinking [9].

Kelly & Lesh [8] have recognized that researchers, teachers, and students rely on models to represent, organize, examine, and explain situations or phenomena. Thus, models are not only key ingredients and tools for researchers to explain the students' development of mathematical thinking; they also are essential for teachers to structure and promote their students' construction of mathematics knowledge. For instance,

[1] The NCTM (2009) stated that "a reasoning habit is a productive way of thinking that becomes common in the process of mathematical inquiry and sense making" (p. 9).

researchers construct models to analyze and interpret teachers and students' mathematical behaviors. Similarly, teachers use models to foster, describe, examine, and predict students' mathematical competencies. In this context, model construction becomes a crucial activity for students to describe, explain, justify, and refine their ways of thinking. Thus, modeling is essential for students to develop sense making and to use mathematics. Lesh and Sriraman [11] pointed out that "the subjects need to express their thinking in the form of some thought-revealing artifact (or conceptual tool), which goes through a series of iterative design cycles of testing and revision in order to be sufficiently useful for specified purposes" (p. 125). Thus, a model is conceived of as a conceptual unit or entity to foster and document both the teachers' construction of instructional routes and the students' development of mathematical knowledge. As Lester [12] pointed out, a perspective based on modeling might be considered as "a system of thinking about problems of mathematical learning that integrates ideas from a variety of theories" (p. 73).

The model construction process involves examining the situation or problem to be modeled in order to identify essential elements that need to be represented and scrutinized through operations and rules with the aim of identifying and exploring mathematical relations [16]. Mathematical models need to be contrasted and refined. That is, "models evolve by being sorted out, refined, or reorganized at least as often as they evolve by being assembled (or constructed)" ([10], p. 365). In addition, model construction relies on using diverse strategies to represent and explore patterns and relations. Schoenfeld [19] recognizes the complexity in selecting and implementing a particular strategy: "For example, to use the strategy "Make sense of the problem by looking at examples" one must (a) think to use the strategy, (b) know which version of the strategy to use, (c) generate the appropriate examples, (d) gain the insight needed from the examples, and (e) use that insight to solve the original problem" (p. 106).

Another crucial element of the framework is to explain the role played by digital technologies during the constructions and refinements of models associated with posed problems. Specifically, the use of the tools becomes relevant to sustain and promote students' mathematical activities and in this study the technology affordances offered the participants the opportunity to participate in activities for:

> …(a) gaining insight and intuition, (b) discovering new patterns and relationships, (c) graphing to expose mathematical principles, (d) testing and especially falsifying conjectures, (e) exploring a possible result to see whether it merits formal proof, (f) suggesting approaches for formal proof, (g) replacing lengthy hand derivations with tool computations, and (h) confirming analytically derived results (Borwein & Bailey, 2003, cited in [22], p. 1170).

The tool or artifact itself does not provide affordances needed for students to use it in problem solving activities; it requires an appropriation process in which the students transform an artifact into an instrument. Trouche [21] argued: "an instrument can be considered an extension of the body, a functional organ made up of an artifact component (an artifact, or the part of an artifact mobilized in the activity) and a psychological component" (p. 285). That is, the artifact or technology characteristics (ergonomics and constrains) and the schemata developed by the subjects during the activities are important for students to transform the artifact into a problem-solving instrument. Thus, the use of a dynamic geometry system plays an important role in constructing models of situations and tasks where the movement of particular elements can be examined and explained in terms of mathematical relations [17]. That is, models might involve configurations made from simple mathematical objects (points, segments, lines, triangles, squares, etc.) in which some elements of the models can be moved within the configuration in order to identify and explore mathematical relations. These relations and conjectures become a source that engages students in mathematical inquiry and reflection.

2.1 Participants, Research Questions, Design, and General Procedures

The study is part of an ongoing project that aims to analyze and document what ways of reasoning high school teachers and students develop as a result of systematically using various computational tools in problem solving environments [14, 17]. The project involves orienting high school teachers and students in using several digital technologies in problem solving approaches. The focus of this study is on analyzing the extent to which six volunteer senior high school students rely on technology affordances to reason and solve mathematical tasks. The participants worked on a set of mathematical tasks during three-hour weekly problem solving sessions for a full semester. In each session, the six students received a written statement of the tasks. Afterwards, they were asked to read and make sense of the problem individually, and later they worked on the problem in pairs. Then, each pair of students had the opportunity to present to the others their approaches to the problems. At this stage, all students and the teacher could ask for concept explanation or clarification. At the end of the session the teacher encouraged the students to contrast the different models used to solve the problems. In general terms, an inquiring approach to the task was key for the participants in eliciting individuals' ideas that later were refined in pairs and small group discussions. For example, when students introduced an idea or used a representation, the teacher questioned and encouraged them to reflect on their ideas and other related concepts. Similarly, when students ran out of ideas the teacher either oriented the discussion (through questions), or asked other students for suggestions to consider. The aim was to encourage the students themselves to formulate and then pursue the questions. In this perspective, the organization of the sessions is consistent with the activities that Mason and Johnston-Wilder [13] recommend for students to participate during problem-solving discussions. In particular, the authors identify four ways to organize students' participation to develop and discuss their mathematical knowledge:

> *Individual* work allows learners to review, consolidate, and develop their facility, as well as to reconstruct for themselves.
> Work in *pairs* allows learners to try out ideas on each other before offering them to a wider group; it also provides an opportunity for learners to consider something that has happened or been said, and to generate more ideas about this [the problem] than an individual is likely to produce when working alone.
> Work in *small groups* allows a multitude of ideas to be generated, and also allows a large task to be split up amongst several people; with discipline, small groups can provide a forum for discussing ideas, modifying conjectures, and coming to a consensus with supporting reasons and justifications.

> *Collective* and *plenary* work allows everyone to hear about novel ideas and approaches, and to see teachers or peers displaying their mathematical thinking (p. 52, italics in the original).

The research questions used to guide and structure the development of the study were:

1. What ways of reasoning did the participants exhibit to construct a model, including dynamic models, to represent and explore the problems? The students' previous knowledge and resources play an important role in representing and constructing models. Thus, it becomes relevant to document the extent to which the dynamic representation of mathematical objects shapes the students' ways of reasoning about the problems, and as a consequence their model construction process.
2. What were the stages or cycles of comprehension that students showed during the process of refining their initial models? According to Lesh & Sriraman [11] the students' model construction involves a series of iterative cycles where initial models are refined or transformed into powerful, shareable, and reusable entities. Thus, it is important to discuss the extent to which the use of the tools, the students' interactions, and group discussions help the participants move from the construction of partial or limited models to more robust ones.
3. What type of mathematical resources and strategies emerge during the students' construction of mathematical models associated with the phenomena? The goal here is to analyze the extent to which the use of a dynamic geometric system helps the participants retrieve conceptual knowledge that might appear when moving mathematical objects within the model. In addition, it is important to document the patterns of productive or nonproductive behaviors that students exhibit during their interaction with the tasks.

Each pair of students handed in for each session the computer files in which they showed their problem approaches. Additionally, all pairs' presentations and groups' discussions were videotaped and transcribed. Thus, sources of information came from the pairs' files, teachers' notes, and videotapes of the group discussions. The videotapes were analyzed during the research group meetings. The group includes a mathematician, two mathematics educators, and four doctoral students.

The analysis of the students' work was done by first focusing on individual contributions and pairs' presentations, and later by discussing the phases involved in the modeling process. A task that involves deciding the position of a camera to take a picture to capture a certain part of a mural was chosen to discuss ways in which students constructed their models. The problem is similar to those that were discussed throughout the sessions. The participants worked on this problem during two problem solving sessions (six hours total) held at the end of the semester. At this stage, the participants had developed a significant experience in the use of GeoGebra.

The Task. Alan wants to take a picture of a mural painting and sets his camera lens with a fixed horizontal viewing angle of 12 degrees. From his position, he focuses on a view of the mural that he wants to use as a reference to take the picture. That is, he wants to fit the viewing angle and an imaginary segment joining two points of the mural (Fig. 1). Are there other positions for the camera where Alan can fit the viewing angle and the mural segment to take the picture? (Justify your answer).

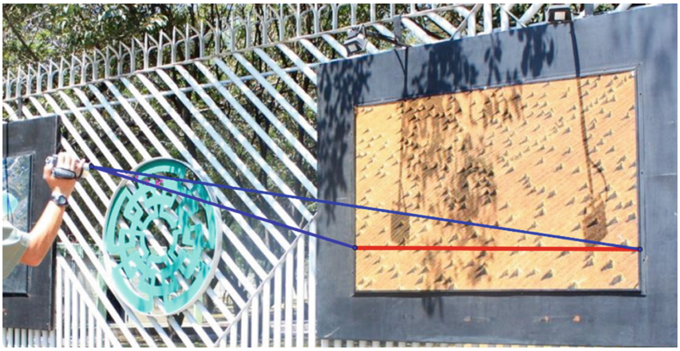

Fig. 1. Taking a picture of the mural by adjusting a fixed horizontal viewing angle to a bottom segment of the mural.

3 Presentation of Results

To present and discuss the approaches exhibited by the students during the problem solving sessions, we focus on relevant phases around the model construction. The conceptual framework discussed previously provided the structure to recognize model construction cycles that appeared during the students' work.

Comprehension and Making Sense of the Task. Students spent some time reading the problem individually and later they worked in pairs. In general, they initially focused their attention to the identification of key elements associated with the situation. Some questions that the students posed included: What does it mean that a camera has a fixed horizontal angle? Can that angle be changed? How can the camera position be adjusted to capture the view of the mural painting? What is the difference between focusing the camera on a point from focusing it on a segment of the mural? What does it happen to the view of the mural when the camera is moved closer or farther from the position where the angle fits the mural segment? These and similar questions were discussed during the initial session by all participants and were important for the students to identify and visualize relevant information for the task. At this stage, students commenced to address the problem in pairs. For example, Anna and Patricia, who used their mobile phone camera to capture a part of a wall, stated that for a point on the perpendicular bisector there should be a point where the horizontal fixed angle captures the view of the segment. The position was standing up in front of the picture.

– Anna: I want to use my phone to take a picture of that wall, but I don't see the angle I need to fix. I see the mural here (pointing at her phone screen).
– Patricia: Right… But if you get closer to the mural you will see different parts of the wall. The angle changes when you change your position.
– Anna: I see that, OK…
– Patricia: There, you move back and forth to fit the angle with the target, as the problem figure.

Problem Representation and Model Construction. Students observed that the camera horizontal angle vertex formed a triangle with the points that determine the segment located at the bottom of the mural (Fig. 1). Thus, their idea was to identify other positions for the camera in which they could fit the fixed angle with the mural bottom segment. The three pairs of students decided to use GeoGebra (a dynamic geometric system) to represent the task. Anna and Patricia first tried to draw by hand a segment and its perpendicular bisector to identify a point (vertex) where the angle was 12 degrees; but when they wanted to measure the angle for different positions of the vertex, they switched to the use of GeoGebra. Then, they drew segment AB (the mural bottom), its perpendicular bisector and chose point D on it. Then, they moved point C along the perpendicular bisector to identify a position where angle APD measured 12 degrees (Fig. 2). To this end, the students observed that the angle decreased when point D was further from the segment. So, they found a position for D where angle APD measured 12 degrees. They also noticed that when point P was located at the position C then angle ACB measured 24 degrees and they recognized that at this location point C was the center of a circle that passes by points A, B, and D (Fig. 2). Without providing any argument or explanation, they drew the circle with center at C and radius CA. Here, they sated that "the camera eye could be located at any point on the arc of the circle ADB."

Thus, Anna and Patricia reported that the camera lens could be located at any point on the arc of the circle ADB (Except points between A and B) because the angle to capture the segment AB was always the same (Fig. 2). However, their explanation was based on using the problem representation they had constructed.

Comment: Anna and Patricia model construction focused on identifying a point D on the perpendicular bisector of segment AB such that angle ADB was the desired angle (Fig. 2). Their empirical approach relied on using the software to measure the angle and moving point D along the perpendicular bisector to identify the position of D where the angle ADB measured 12 degrees. When they identified the position for D in which the angle was 12 degrees, then they also recognized that there was a point on the perpendicular where the central angles measured 24 degrees. Based on this information they drew a circle with center at C and radius CA and showed the camera could be situated at any point along the arc ADB. Thus, this empirical model was built in terms of recognizing the relationship that exists between the central and inscribed angles in a circle (although they never mentioned it). That is, *the inscribed angle in a circle measures half of the central angle measure*. Regarding the use of the tool, there is evidence that they thought of the problem in terms of moving a point on the perpendicular bisector and matching its corresponding angle measure with the desired value (12 degrees). That is, they relied on a functional approach in which a point on the bisector was associated with an angle value, but without explicitly defining that function.

Rob and Lionel (another pair of students) approached the problem by drawing segment AB to represent the bottom view of the mural. They drew a circle with center at point A and any radius AP (the circle is a heuristic to move a line on the plane) and chose point C on the circle to draw line AC. On line AC, they chose point D that represents the position of the camera and line DE was drawn by rotating line AD an angle of 12 degrees around point D (Fig. 3).

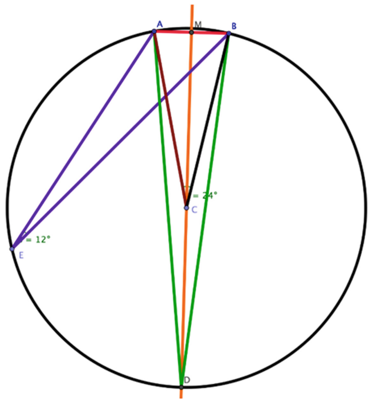

Fig. 2. Finding a position for P on the perpendicular bisector where angle APB measures 12 degrees.

Based on Fig. 3, the students noticed that the angle ADE did not capture the complete segment AB. Here, they started to move line AC (by moving point C along the circle) and found that for a certain position of point D the angle covered the segment AB (Fig. 4). This empirical model was constructed by adjusting the angle view to the length of segment AB. This partial solution allowed the students to identify triangle ADB and to think of drawing a circle that passes by its vertices.

– Lionel: Look, there is a triangle ADB(E) and we can draw a circle that passes through the three vertices, OK?
– Rob: why do we need to draw a circle if we got the solution already?
– Lionel: Because, if we draw it, we can find others…You know how can we find the center of the circle?
– Rob: I see it, we need to draw the perpendicular bisectors…

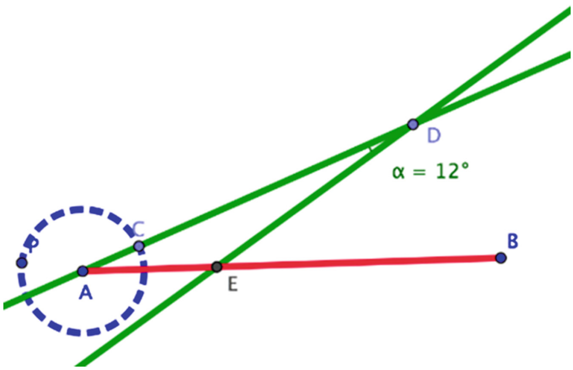

Fig. 3. A dynamic representation of the problem where point C can be moved along circle with center A and radius AC.

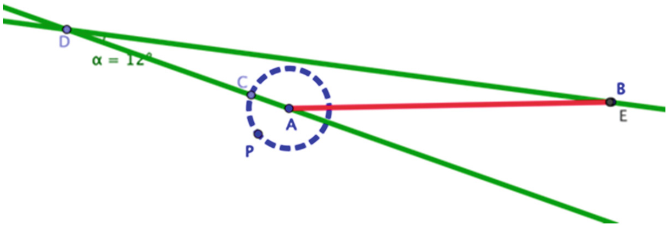

Fig. 4. For a certain position of point D, the angle includes segment AB.

Based on this visual approach, the students decided to identify the center of a circle that passes through points A, D, and B. Here they recognized that the Circumcenter of triangle ADB is the point where the perpendicular bisectors of its sides intersect (Fig. 5). Thus, this pair reported that they had solved the problem and were ready to present their solution to the class. That is, they wrote that the camera could be located at any point on the arc ADB (Fig. 5). They also mentioned that the camera could not be situated on the arc that completes the circle because from there they would observe the rear side of the mural.

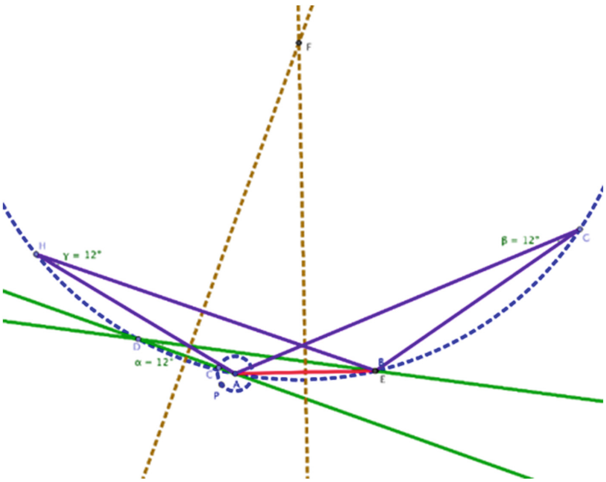

Fig. 5. Identifying different position for point D.

Comment: To construct this model, the students focused on drawing an angle with a side one of the extreme of the targeted segment. To draw the angle with the given measure, they used the command "rotate". The key component in this construction was that the angle could be moved (by moving line AC along the circle) without altering the angle measure. That is, by moving line AC, they observed that there was a position for point D (angle vertex) where the angle included the whole segment AB (Fig. 5). Again, when they found this position, then they recognized that points A, D and B formed a triangle and decided to find the circle that passes through the three vertices. Thus, this visual model was useful for students to identify relevant information needed to locate the camera position.

A Locus Model: Dianne and Petra (the last pair of students) also used the software to represent the problem dynamically (Fig. 6). They also represented the bottom view of the mural as segment AB. From A they drew a line AC (via a circle with center at A and radius AP) and rotated this line around D (D was a point on line AC) at an angle of 12 degrees. Thus, angle ADE measures 12 degrees. They noticed that angle ADE did not capture the whole bottom view (segment AB). And then they drew a parallel line to ED that passes through point B. This line intersects line AD at M. Here, they mentioned that angle AMB is congruent to angle ADE. Therefore, point M is the position where the camera can be situated to completely capture the segment AB (Fig. 6). They also observed that when line AD is moved (by moving point C along the circle with center A and radius AC) then point M describes a path or locus. With the help of the software, they recognized that the locus of point M when line AP is moved is a circle. Thus, these students reported that the camera could be situated at any point on the arc BMA (Fig. 6).

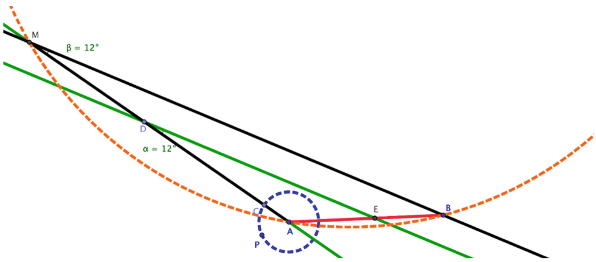

Fig. 6. What is the locus of point M when point C is moved along line the circle with center at A and radius AC?

- Dianne: Angle ADE does not fit the segment and if I move point D along line AC, then it seems that the angle (she is referring to angle ADE) can be moved to fit the segment.
- Petra: I see that line DE is parallel to the line that passes through point B.
- Dianne: Right, you mean that the parallel line to DE that passes through point B forms an angle BMA that is the same as angle ADE?
- Petra: Yes, line AD is transversal to both parallels, and they are corresponding angles.
- Dianne: OK… What about if we move line AC? …you see point M moves in a certain path.

Comment: In this model, an interesting feature is that the participants observed that point M followed a path when line AC is moved along the circle (Fig. 6). This path was determined with the help of the tool. These students showed evidence that they had a good appropriation of the tool. They were the only pair that introduced the *locus* concept to solve the problem. In addition, they asked the others pairs to complement their empirical approaches with more formal arguments.

Students' Model Presentations. Each pair of students presented its work to the whole group and had an opportunity to explain and defend each approach to the problem; but also an opportunity for the others to question and contrast their solutions. For example, when Anna and Patricia explained to the group their solution, they were asked about whether they could draw the inscribed angle without moving the point along the perpendicular bisector. To respond, they examined triangle ADB (Fig. 2) and argued that it was isosceles. Indeed, they mentioned that all the family of triangles that were generated by moving point D along the perpendicular bisector was isosceles. Therefore, the two congruent angles of triangle ADB (Fig. 2) would measure 78 degrees each. And they explained that to draw angle BAD with a measure of 78 degrees, it was sufficient to rotate 78 degrees segment AB around point A. Similarly, when Dianne and Petra presented their solution to the class, all agreed that the way how they identified the initial position of the camera did not depend on moving a particular point or line; rather, it was obtained by using parallel lines properties. However, they were questioned on why they thought that locus generated by point M when point C was moved along the circle was a circle (Fig. 6). Their argument was that the locus passed through points M, A, and B and these points were vertices of triangle MAB, and as a consequence, there is only one circle that passes through those points. Indeed, they mentioned that the center of that circle was the intersection point of the perpendicular bisector of two sides of triangle MBA. Rob and Lionel recognized that their solution was similar to what Dianne and Petra had presented. They mentioned that Dianne and Petra's approaches to find the camera angle to include the whole segment was more general than their construction. This was because in their model, Rob and Lionel moved the line (by moving point C on the circle) manually in order to identify the position for point D to match the angle with the given segment (Fig. 3). They also recognized that they focused on triangle ADB to draw the circle instead of determining the locus of point D. Thus, all recognized that Dianne and Petra's model does not rely on moving a point to identify a determined angle measure or moving a line to visualize the solution, rather it is based on relating properties of parallel lines to the angle position.

4 Discussion of Results and Remarks

The models constructed by the students while interacting with the task showed interesting mathematical features. Indeed, the students' use of the tools provides consistent information around the cycle nature of a problem solving approach [3]. Although two of these models (Anna and Patricia's and Rob and Lionel's) followed different paths, both relied on students' empirical reasoning to identify and relate key concepts to solve the problem. Anna and Patricia identified the vertex of the given angle by moving a point on the perpendicular bisector of the segment that represented the view of the mural. This point was obtained by observing the variation of the angle's measure to match the given value. Similarly, Rob and Lionel identified the position of the camera by moving a lined around a circle. In this case, they identified visually the point in which the angle included the entire segment. For both pairs, this stage was crucial to relate other concepts that they used to identify other camera positions to take the picture. Anna and Patricia relied on the relationship between the central and inscribed angles of a circle; while Rob and Lionel related the triangle to the construction of the circle that passed through its vertices. The third model also involved the construction of the given angle that later was translated to a position from where its vertex covers the segment. The translation was made by drawing a parallel line to one side of the initial angle that passes through one extreme of the segment (view of the mural). Here, the students observed that the vertex of the angle followed a path when one side of the initial angle is moved. They used the software to identify the locus of the vertex because they thought that the path was a circle. Although, at that moment they did not explain why that locus was circle, later on when these students presented their solution to the class, they were asked to justify that conjecture. In terms of problem solving behaviors there is evidence that the students grasped the structural relations associated with the situations and those relations were first identified and later explored with the help of the tool [7]. For example, the construction of a circle that passed by three points (vertices of a triangle) or finding the locus of a particular point became a crucial stage to find different positions for the camera.

During the group discussion, the teacher asked the students to identify the relevant content and processes that were important during the solution of the task. In terms of contents they identify the concept of perpendicular bisector, the relationship between inscribed and central angles in a circle, properties of triangles (isosceles, sum of interior angles), the circumcenter, parallel lines, and corresponding angles. They mentioned that the processes involved in their approaches included the construction of dynamic models, the visualization of relations, observing variation patterns, exploring functional relations, justification of conjectures, etc. From this perspective, students recognized that the use of the tool was relevant to initially construct a model to relate and explore the situation in terms of mathematical objects. That exploration led them to identify concepts and to look for relations to solve the problem. In addition, they recognized that their presentations and discussions within the whole group were important not only to acknowledge the work done by others, but also to refine their own models. It became evident that arguments based on geometric properties to justify their constructions were more formal and solid than those based on visual results. Here, they accepted that both types of arguments are important while exploring the situation. They also recognized that often visual explorations, together with empirical analysis, can lead the problem solver to identify interesting relations that are not easy to find in formal approaches.

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