Designing Technology-Based Tasks for Enhancing Mathematical Understanding Through Problem Solving

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Abstract. In this paper, we propose some organizing principles that can be useful for high schools or bachelor mathematics teachers when designing technology-based instructional tasks. It is widely accepted that tasks are the most important aspect to promote students' mathematical understanding, since tasks offer opportunities to attain relevant sensorial experiences fostering the construction of mental images as sources of meanings for mathematical concepts. In this vein, we reflect on the work developed by three bachelor mathematics teachers who participated in a problem solving seminar. The main points identified during task design involved recognizing how mathematical concepts are structured around the task, and which are needed to approach it, and proposing a hypothetical learning trajectory in which technology plays a role as amplifier and reorganizer of cognitive processes.

Keywords: Tasks design · Mathematical understanding · Technology

1 Introduction

Tasks design is important, since tasks orient the work that teachers and students develop in mathematics classrooms. Characteristics of tasks have a significant effect on levels of mathematical understanding that students can attain [1], since activities that students address in a classroom shape their mathematical view and influence the ways they behave when confronted with problem situations [2].

The main objective of mathematics educations is that students construct mathematical ways of thinking, as well as progressive levels of conceptual understanding. To achieve this objective, it is important that learning environments offer students opportunities to experience critical thinking and reasoning, and to develop specific skills through solving problems related to professional fields [3].

However, we generally observe passive and uninterested students in mathematics classrooms. We argue that it is due to the kind of tasks that are implemented in school, which are inadequate to enhance students' particular learning styles and potentiate their natural reasoning abilities. Alignment of mathematics learning with mathematical thinking is an important goal in school [4], which requires that teachers be able to design tasks based on mathematical principles. In this vein, it is relevant that teachers have a

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deep mathematical knowledge, knowledge about learning theories, mathematics pedagogy and brain functioning [5]. Related to the latter, teachers should know that sense making for mathematical ideas involves engaging learners in significant sensorial experiences (seeing, hearing, touching, moving, reasoning) and promoting reflection about those experiences as the source of meaning [6]. All the products of the mind come from the human body interactions with the world [7] and for this reason, sensible experiences help us to construct mental images to link previous and new knowledge [3] and as a consequence, attain higher levels of conceptual understanding.

Teaching new materials is often done without taking into account concepts or ideas related to a given context. Naaranoja and Uden [3] argue that there exists evidence that mathematical understanding requires constructions of relations between previous knowledge and new ideas, as well as clues to recover or reconstruct those links. People learn new material contextually fitting it into existing cognitive structures to develop robust conceptual networks. New information that cannot be linked to existing knowledge is not likely to be integrated into a robust network of concepts. No one can understand anything if it is not connected in some way to something they already know [7]. Additionally, once information is stored in the long term memory, cues in the form of mental images are required for recovering and mobilizing this knowledge.

2 Some Principles to Guide Task Design

One basic principle that supports this work is that intelligence is a product of the relation between mental structures and the actions we perform with tools provided by the culture. Tools help us transcend the biological limitations of mind [4, 8, 9] by externalizing the intermediate products of thinking through representations such as words, texts, graphs, tables, etc. These representations provide material records that are useful to organize thinking and reflect about our own thought processes [4].

From the previous arguments, it is important to consider and analyze the fundamental characteristics of learners' thinking that accompany the usage of these tools. What aspects of mathematical thinking can digital technologies enhance, catalyze or uncover? The dynamic and interactive media provided by digital tools make it possible to gain an intuitive understanding of the interrelations among several representations which are more accessible to learners.

An instruction based on problem solving that promotes the systematic use of digital tools helps mathematics becoming *functional*, since technologies prompt the development of mathematical thinking, and conceptual tools formation during the solution of tasks that represents an intellectual challenge for learners. Interactive technologies enhance construction of meaning and understanding, since they provide a means of intertwining multiple representations and the formation of mental images that help us organize and integrate conceptual, factual and procedural knowledge into a network of relationships [4].

In the same vein, construction of mathematical understanding requires learners to develop successive cycles of action, observation, formulation of conjectures and justification of results (Fig. 1). Each cycle is nurtured with the information and relations of the previous one. In this process, the learners' construction of progressive understanding

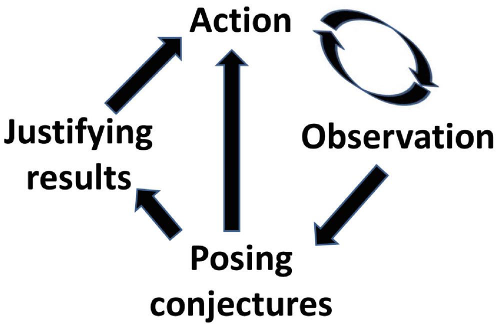

Fig. 1. Basic cycle to develop conceptual understanding during problem solving processes (Adapted from [7]).

levels is stimulated through problem situations which encourage acting on and integrating contents, resources and strategies in learners' problem solving approaches.

The action and observation phases are closely intertwined. In the action phase learners focus on representing the information and reflecting on the mathematical objects involved in a problem, in terms of its fundamental properties. At this stage, it is important to quantify some objects' attributes or adding auxiliary elements in order to get insights into the problem. During the observation phase, it is important that learners identify relationships among the elements considered in the previous phase. The posing conjectures phase is oriented so that learners can formulate their observations in mathematical terms. Some learners possibly need to go back to a new *action-observation-posing conjectures* cycle in order to gain better understanding of the problem, while others can go directly to the justification phase. This latter phase includes the use of visual, empirical and formal arguments, as well as extending and generalizing the task.

Addressing problem solving with digital tools demands that learners think about mathematical objects in terms of their basic properties, and that they consider problems as a set of dilemmas which should be approached by means of mathematical resources: How are the objects involved in a problem defined? Is there any relation between those objects? What are the characteristics, if any, of those relationships? A key element to promote the construction of connections among mathematical ideas is to take advantage of visualization and exploration affordances offered by digital tools to explore properties of mathematical objects, to formulate and justify conjectures, to communicate results and pose new problems [10].

It is well known that using different ways of representing data in a problem are crucial for the generation of solution routes, as are strategy selection, prior knowledge and the tools used to organize a solution path [11]. The more senses we use in an activity, the more fruitful the learning experience will become, since it increases the neuronal connections in students' brain and therefore the knowledge is more accessible and usable [6]. All students have prior knowledge that affects how they approach the activities that are proposed to them.

In this paper, a technology-based task is conceptualized as a guide to help teachers organize classroom activities by engaging learners in significant sensorial experiences (seeing, hearing, touching, moving, and reasoning) promoted by the systematic use of digital artifacts and reflection about those experiences. The experiences and reflection about technology based tasks have the aim to promote the construction of mathematical ways of thinking and mental images, which have a twofold function: structuring previous and new concepts into a network or mental schema and act as clues to recover and use the knowledge. The design of technology-based tasks rests heavily on a complex relationship between teacher knowledge about mathematics, learning, teaching, tool affordances and a kind of feedback offered by the tools [12]. A technology-based task has the objective of promote constructing new connections or relations between concepts, starting from students' prior knowledge, through sensorial experiences and reflection promoted during problem solving activity (Fig. 1).

The design of technology-based task involves determining the core mathematical concept that students must learn and reflect about the relations needed to structure prior and new knowledge, mental images that give meaning to the concepts, as well as the sensory experiences, and reflection activities useful to achieve such organization and structuration of mathematical ideas, considering potentialities and limitations of available tools.

3 The Research Context

The task proposed in this paper was discussed in a context of problem-solving seminar in which participated mathematics teachers, mathematicians and mathematics educators. The aim of this seminar is to promote the discussion and reflection about mathematics teaching and learning, specifically about how students develop progressive levels of mathematical understanding, reasoning and mathematical ways of thinking when they are engaged in problem solving activities.

In the seminar development, at the end of the first semester of the academic year 2015-2016, three mathematics teachers, who teach Precalculus courses to engineering students, solved a task using GeoGebra and reflected on the developed problem solving process from a didactic point of view with the aim to design a similar task to be implemented in their classrooms. All teachers have gained a bachelor degree in applied mathematics; two of them are enrolled in a master program in mathematics and one is a PhD mathematics students. GeoGebra is a Dynamic Geometry System that integrates different tools of geometry, calculus, and statistics, among other and can be used to explore different kind of mathematical problems from elementary school to university mathematics courses.

An important element in this paper is to explore how each question or reflection was posed and pursued during the seminar sessions. To this end, we identify some routes that the participants, as a group, identified as important and discussed while delving into the task. The departure point in each session was to conceptualize solving problems as an opportunity to understand the different ways that students think about and learn. Thus, our unit of analysis is what the group as a whole proposed and discussed during each of four sessions of an hour and a half.

Data included teachers' written records, electronic files, and video recordings of each session, which were transcribed. First of all, data analysis was mainly oriented to identify stages of basic cycle to develop conceptual understanding (Fig. 1). Based on these, we draw some organizing principles that might be useful to design instructional tasks.

The seminar discussion was oriented to determine some of the core mathematical concepts in a Precalculus course. Teachers agreed that one of the central concepts in calculus is the concept of function and that this is grounded on the idea of covariation of two quantities. Then, teachers suggested that some mental images to give meaning to this idea can be approached by posing problems related to uniform motion, filling liquid containers, population growth, simple and compound interest or problems of maxima and minima.

3.1 The Task

With the aim of promoting problem solving activities as well as didactical reflection, the research team proposed the teachers to solve the following problem: find the rectangle of maximum area that can be inscribed in a given triangle. The resources needed to approach the problem consisted of basic geometric ideas such as triangle, inscribed rectangle and area calculation. In a first approach, teachers proposed solving the task by means of standard calculus procedures, which include determining a function of one variable, calculate the derivate, solving an equation to obtain the critical points and so on. However, teachers were encouraged to think and reflect about the minimum mathematical requirements to approach the task and ways to promote construction of mental images about covariation of two quantities using GeoGebra and the basic cycle to develop conceptual understanding (Fig. 1). The use of Geogebra was suggested since dynamic characteristics of this tool are useful to explore phenomena of change and variation.

First Episode: Understanding the Problem and Gaining Insights about Students' Mathematical Action. In this phase teachers realized that to approach the task, students should know that inscribing a rectangle in a triangle means that the vertices of the rectangle must lie on the sides of the triangle. Teachers constructed the dynamic configuration easily, however to engaged them thinking how students could construct such a configuration required a deep reflection about problem solving activity. Teachers agreed that sketching a drawing on paper and pencil can be useful to organize the data of the problem, but the construction of the figure in GeoGebra also involves thinking about the inscribed rectangle in terms of mathematical properties such as perpendicularity and parallelism of lines. Additionally, teachers realized that using GeoGebra can allow students considering non-prototypical examples, which is useful to generate robust mental images associated with the idea of a triangle.

Some useful questions that teachers identified as important to promoting students' action are: For what kind of triangles are there two vertices of the inscribed rectangle on two sides of the triangle? Is it possible to inscribe a unique rectangle for any given triangle? Can each side of the triangle contain the base of an inscribed rectangle?

Teachers also considered that constructing several particular cases of rectangles inscribed in a given triangle could be useful as a first approach to note the covariation of two quantities (relating the change of some length and the area of the rectangle).

The teachers realized that dragging affordances of GeoGebra can be employed to direct students' attention to the construction of families of geometrical objects and to highlight the differences of area among elements of this family. During the discussion, teachers pointed out the importance that students communicate their observation about area variation orally or in writing form. Ideally, reflection about extreme cases could allow students to realize that there exists a rectangle of a maximum area. An important element that was also recognized is that students should identify some other variable quantities as well as possible ways to relate them.

Second Episode: Analyzing Problem Information as a Way to Observe Relations, Patterns and Invariants. This phase was oriented to formulate questions about the problem data as a means to detect relations, patterns and invariants among them. In this vein, teachers commented about the importance that students realize that translating the triangle on the plane does not change the area of an inscribed rectangle or other variables of the problem. This knowledge is a necessary antecedent to introduce a coordinate system as a tool for representing symbolically two variable quantities.

Teachers commented that using appropriately a coordinate system requires to relate points in a plane with ordered pairs of real numbers. On the other hand, managing and operating equations of loci requires that students approached previously sub-tasks that allow them to realize that the coordinates of loci satisfy some relations and that these relations can be generalized and represented in symbolic language. It is important that students identify those changes of position of a locus result in changes on the equation representing it. In this line of ideas, it is important that students realize that placing geometric objects in a suitable position can be useful to simplify equations that describe those geometrical objects.

Teachers expressed that students have several difficulties to explore the general case, and for this reason they proposed to explore particular cases of triangles with integer coordinates, for example triangle ABC whose vertices coordinates are A = (0,0), B = (8,0), C = (5,3). Students can be encouraged to develop an empirical route dragging point D (Fig. 2) to approach the rectangle of maximum area and then, the teacher could orient students' attention to observe some relation between vertex C and position of D that approximates the maximum area. Students might be able to conjecture that the x-coordinate of point D is a half of the x-coordinate of point C.

Some identified questions to promote posing of conjectures are: Is there any relationships between the area of the inscribed rectangle and the area of triangle *ABC, in particular when the area is maximized*? What is the ratio between the lengths of segments *AE* and *EC* and between the segments *CF* and *FB*? Are there any similar triangles in Fig. 3? Students could be oriented to recognize that some relevant ratios between lengths or areas are ½, 1/3, 2, 3, $\sqrt{2}, \frac{1}{\sqrt{2}}$, etc., and that significant relations have to do with midpoints, equality of lengths, areas or perimeters and so on. Observing the previous relationships can also be useful in the phase of justification of conjectures. In this case the main conjecture is that the maximum area of a rectangle is half of the area

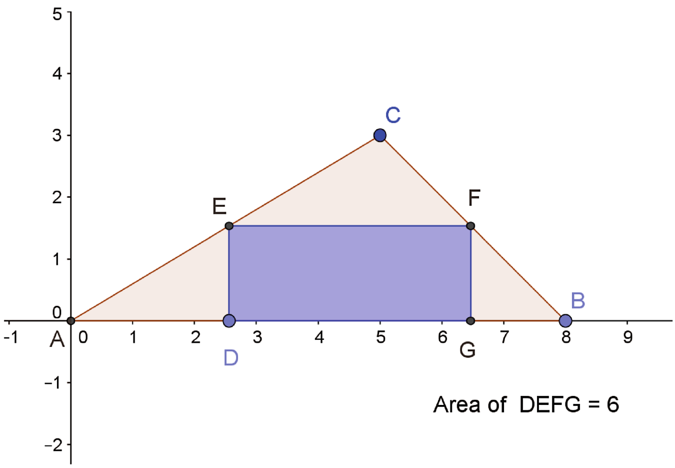

Fig. 2. Solution path based on particular cases.

of triangle *ABC*. Other related conjectures are that *E* is the midpoint of *AC*, *F* is the midpoint of *CB*, and that $AD + GB = (AB)/2$.

Continuing the analysis of particular cases, teachers considered as relevant that students relate graphically some variable quantities. For example, using the locus tool students can be oriented to observe that the relation between the area of the inscribed rectangles and length AD, is described by a parabola that opens downwards and that the vertex of the parabola provides information regarding the point where the inscribed rectangle reaches the maximum area. Teachers recognized that graphical representation highlights some relationships that are not readily observable in the numeric register, for example, that the maximum area is reached in a position such that the x-coordinate of point D is a half of the x-coordinate of point C, and that relation has to do with symmetry properties of a parabola.

Third Episode: Generalizing Results and Searching for Arguments. Teachers proposed that from observations about particular cases, students could be oriented to detect what are the similarities with the aim to generalize the results and to express those results in symbolic language. Teachers recognized as an important element that students should pay attention on relations between symbolic expressions and its concrete referents as well as to which letters represent fixed quantities in the context of the particular problem or which represents variable quantities.

Searching for arguments involves relating factual knowledge with results developed during problem solving processes. For developing this activity teachers consider relevant to look over digital content related to the concepts emerging during previous episodes (similarity of triangles, properties of a parabola, ratios and proportions) and those that are available on sites such as Wikipedia, GeoGebra tube or Khan Academy as a mean to promote connections between mathematical concepts and procedures, and develop a conception of mathematical content as tools to solve problems.

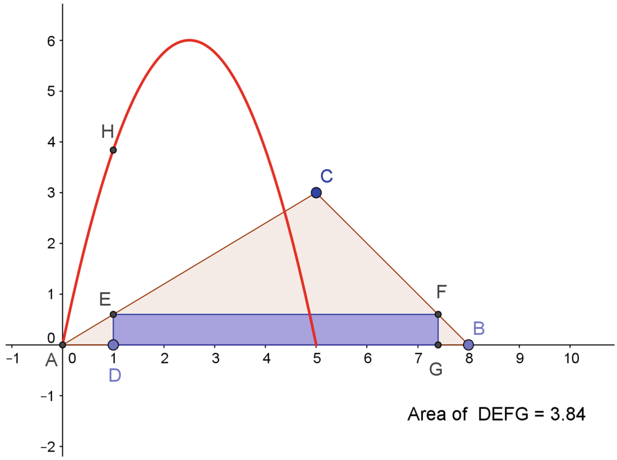

Fig. 3. Connecting several representations.

Fourth Episode: Searching for Alternative Solution Paths and Formulating New Problems. Searching for or discussing multiple solution paths has been recognized as a powerful strategy for students to engage in mathematical thinking and learning [13]. Teachers considered that an alternative route to solve the problem without employing a coordinate system is the following: the distance AD can be denoted by the expression r *(AH)*, such that r is a real number in the interval (0,1) and H is the foot of the altitude of the triangle passing through vertex C (Fig. 4). Then, by similarity of triangles ABC and EFC, $EF = (1\text{-}r)(AB)$, and $ED = r(CH)$.

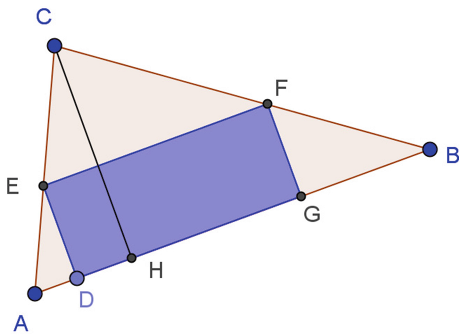

Fig. 4. Another solution route.

Based on the previous relations, the area of rectangle DEFG can be calculated to be (EF)(ED) = $r(1-r)(AB)(CH)$, since $(AB)(CH)$ is two times the area of the triangle ABC, which is a fixed quantity, then the maximum area is reached when f(r) = $r-r^2$, reaches its maximum value. The graph of this function is a parabola whose vertex has coordinates (½, ¼). Then, the maximum area is reached when D is the midpoint of segment AH, and the area of the rectangle is half of the triangle's area.

4 Final Remarks

Analyzing teachers' activities during the seminar allowed us to identify some elements that should be considered when designing technology-based tasks oriented to foster mathematical understanding.

First of all, interaction among member of a professional community is important to identify the core concepts that guide construction of conceptual mathematical structures (function concept and covariation). Secondly, teachers must define the possible contexts that can help students' to give meaning to those concepts (uniform motion, filling liquid containers, population growth, simple and compound interest or problems of maxima and minima). At this stage it is necessary to think about problems in a specific context that could contribute to structure core mathematical ideas, concepts and procedures. Textbooks constitute a relevant source to find problems; however the problems must be adapted in order to represent an intellectual challenge for students as pointed out by the problem solving approach [14].

Another important activity during tasks design involves defining the minimum resources needed to approach the problem (concepts of triangle and inscribed rectangle and area calculation), the mental images useful to structure the prior and new knowledge (quantities changing together), and the sensorial experiences or reflection activities useful to construct those images, based on potentialities and limitations of available tools (construction of a family of inscribed rectangles whose area depend of the position of a point that can be dragged).

An important resource for tasks design is the basic cycle to develop conceptual understanding (Fig. 1), since it describes important processes that allow students to connect previous knowledge with the new core concepts. We would like to point out specifically the interaction between action and observation phases in which technology affordances play a crucial role. In the discussed task, different dynamical visual representations (changing images and changing numerical data) help to identify some patterns (the rectangle of maximum area is reached when two vertices are located on the midpoints of two sides of the triangle).

A central point during the interaction between observation and posing conjectures phases is related to identifying general results and their formulation in mathematical symbols, which possess meaning through the particular cases explored.

We would like to highlight that designing technology-based tasks involve several complex elements that should be explored and reflected within a professional community in order to provide a wider range of opportunities to promote students learning.

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